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A theorem on Kampé de Fériet function

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1. Kampé de Fériet (1) has defined a generalized hypergeometric function of two variables as

$$F\left(\begin{matrix}\lambda \\ \mu \\ \nu \\ \rho\end{matrix}\middle|\begin{matrix}\alpha_1, \ldots, \alpha_\lambda: \\ \beta_1, \beta_1'; \ \ldots \beta_\mu, \beta_\mu'; \\ \gamma_1, \ldots, \gamma_\nu: \\ \delta_1 \delta_1'; \ \delta_\rho, \delta_\rho';\end{matrix}\middle| x, y\right) \tag{1·1}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Pi(\alpha_\lambda)_{m+n} \, \Pi[(\beta_\mu)_m (\beta_\mu')_n]}{m! \, n! \, \Pi(\gamma_\nu)_{m+n} \, \Pi[(\delta_\rho)_m (\delta_\rho')_n]} x^m y^n,$$

where $\Pi(\sigma_p)_s$ stands for the product $(\sigma_1)_s (\sigma_2)_s \ldots (\sigma_p)_s$.

Pandey and Saran (4) and Carlitz (2, 3) recently obtained the sum of the above series for $\lambda = \nu = \rho = 1, \mu = 2$, for different conditions on the parameters. In this paper, we have summed the above series for these values of λ, μ, ν, ρ, but with conditions different from the above writers and then extended it for higher parameters. In the end we have made use of the sum to evaluate a definite integral.

For the sake of brevity, the L.H.S. of (1·1) has been denoted by

$$F_{\nu, \rho}^{\lambda, \mu}\left[\begin{matrix}|\alpha|_\lambda: \ |\beta, \beta'|_\mu; \\ |\gamma|_\nu: \ |\delta, \delta'|_\rho;\end{matrix} x, y\right].$$

2. The result to be proved is:

$$F_{1,1}^{1,2}\left[\begin{matrix}\alpha: \rho, -n; \ \beta, \alpha+\gamma+n-1; \\ \alpha+\gamma: \ \delta, \alpha;\end{matrix}\right] = \frac{(-)^n (\beta)_n (\gamma)_n (\rho)_n}{(\alpha+\gamma)_{2n} (\delta)_n} \, {}_3F_2\left[\begin{matrix}\alpha+n, \beta+n, \rho+n; \\ \alpha+\gamma+2n, \delta+n;\end{matrix}\right]. \tag{2·1}$$

Proof.

$$F_{1,1}^{1,2} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\alpha)_{p+q} (\rho)_p (-n)_q (\beta)_p (\alpha+\gamma+n-1)_q}{p! q! \, (\alpha+\gamma)_{p+q} (\delta)_p (\alpha)_q}$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha)_p (\rho)_p (\beta)_p}{p! \, (\alpha+\gamma) \, p (\delta)_p} \, {}_3F_2\left[\begin{matrix}-n, \alpha+p, \alpha+\gamma+n-1; \\ \alpha+\gamma+p, \alpha;\end{matrix}\right].$$

The series ${}_3F_2$ is Saalschützian and is, therefore, equal to

$$\frac{(\gamma)_n (1+p-n)_n}{(\alpha+\gamma+p)_n (1-\alpha-n)_n}$$

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therefore

$$F_{1,1}^{1,2} = \frac{(-)^n(\gamma)_n}{(\alpha)_n}\sum_{p=n}^{\infty}\frac{(\alpha)_p(\beta)_p(\rho)_p(1+p-n)_n}{(\alpha+\gamma)_{p+n}(\delta)_p\,p!}$$

$$= \frac{(-)^n(\gamma)_n}{(\alpha)_n}\sum_{s=0}^{\infty}\frac{(\alpha)_{n+s}(\beta)_{n+s}(\beta)_{n+s}(\rho)_{n+s}(1+s)_n}{(\alpha+\gamma)_{2n+s}(\delta)_{n+s}(n+s)!}$$

$$= \frac{(-)^n(\beta)_n(\gamma)_n(\rho)_n}{(\alpha+\gamma)_{2n}(\delta)_n}\,{}_3F_2\left[\begin{matrix}\alpha+n,\beta+n,\rho+n;\\ \alpha+\gamma+2n,\delta+n;\end{matrix}\right].$$

The ${}_3F_2$ on the right can be summed by well-known theorems, say, Saalschützian, to find out the sum in terms of Gamma-functions. We can also reduce ${}_3F_2$ into ${}_2F_1$, by choosing the parameters suitably to sum it by Gauss's theorem. A few of the important reduced cases are given at the end.

From (2·1) it is easy to establish the following general result.

$$F_{1,\lambda+1}^{1,\lambda+2}\left[\begin{matrix}\alpha:\rho,-n;\ \beta,\alpha+\gamma+n-1;\ |\mu,\nu|_\lambda;\\ \alpha+\gamma:\delta,\alpha;\ |\sigma,\nu|_\lambda;\end{matrix}\ z,\ 1\right]$$

$$= \frac{(-)^n(\beta)_n(\gamma)_n(\rho)_n\,\Pi(\mu_\lambda)_n}{(\alpha+\gamma)_{2n}(\delta)_n\,\Pi(\sigma_\lambda)_n}\,{}_{\lambda+3}F_{\lambda+2}\left[\begin{matrix}\alpha+n,\beta+n,\rho+n,|\mu|_\lambda+n;\\ \alpha+\gamma+2n,\delta+n,|\sigma|_\lambda+n;\end{matrix}\ z\right]. \quad (2\cdot2)$$

Here $|\mu|_\lambda+n$ stands for the sequence $\mu_1+n,\mu_2+n,\ldots,\mu_\lambda+n$.

3. *Application.* We have proved (5, 1) that,

$$\int_0^1 x^{s-1}(1-x^h)^{\sigma-1}F_{\nu,\rho}^{\lambda,\mu}\left[\begin{matrix}|\alpha|_\lambda:\ |\beta,\beta'|_\mu;\\ |\gamma|_\nu:\ |\delta,\delta'|\rho;\end{matrix}\ ax^{nh},bx^{nh}\right]dx$$

$$= B_h F_{\nu+n,\rho}^{\lambda+n,\mu}\left[\begin{matrix}|\alpha|_\lambda,\{s/h\}_n:\ |\beta,\beta'|_\mu;\\ |\gamma|_\nu,\{s/h+\sigma\}_n:\ |\delta,\delta'|_\rho;\end{matrix}\ a,b\right],$$

where

$$B_h = \frac{\Gamma(\sigma)\sum_{\gamma=0}^{n-1}\Gamma\left(\dfrac{s/h+\gamma}{n}\right)}{hn^\sigma\prod_{\gamma=0}^{n-1}\Gamma\left(\dfrac{s/h+\sigma+\gamma}{n}\right)},$$

and the symbols $|T|_m$ and $\{T\}_n$ stand for the sequences $T_m(m=1,2,\ldots,m)$ and $(T+\gamma)/n\,(\gamma=0,1,\ldots,n-1)$ respectively.

The result is valid if $h>0$, $\Re s>0$, $\Re\sigma>0$ and the series involved are assumed to be convergent.

Putting $\lambda=\nu=0$, $\mu=2$, $\rho=1$, $n=h=1$, $s=\delta_1=\delta_1'=1+p$, $\sigma=1+q$, $\beta_1=\beta_1'=-m$, $\beta_2=\beta_2'=1+p+q+m$, $a=b=1$, $x=\sin^2\phi$,

we have,

$$\int_0^{\frac{1}{2}\pi}\sin^{2p+1}\phi\cos^{2q+1}\phi[2F_1(-m,1+p+q+m;\ 1+p;\ \sin^2\phi)]^2\,d\phi\ \ \Re p>-1,\Re q>-1$$

$$= \frac{\Gamma(1+p)\,\Gamma(1+q)}{2\Gamma(2+p+q)}F_{1,1}^{1,2}\left[\begin{matrix}1+p;-m,-m;1+p+q+m,1+p+q+m;\\ 2+p+q:\ 1+p,1+p;\end{matrix}\right].$$

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Using (2·1) for $\alpha = \delta = 1+p, \rho = -n = -m, \beta = 1+p+q+m, \gamma = 1+q$, we have,

$$\int_0^{\frac{1}{2}\pi} \sin^{2p+1}\phi \cos^{2q+1}\phi [2F_1(-m, 1+p+q+m;\ 1+p;\ \sin^2\phi)]^2 d\phi \quad \Re p > -1, \Re q > -1$$

$$= \frac{m![\Gamma(1+p)]^2\,\Gamma(1+q+m)}{2(1+p+q+2m)\,\Gamma(1+p+q+m)\,\Gamma(1+p+m)},$$

a result due to Jacobi ((6), 12·2, 379).

4. *Particular cases of* (2·1). (i) Let $\rho = -m, \beta = \alpha+\gamma+m-1, \delta = \alpha$, we have,

$$F_{1,1}^{1,2}\left[\begin{matrix}\alpha: -m, -n;\ \alpha+\gamma+m-1, \alpha+\gamma+n-1;\\ \alpha+\gamma: \alpha, \alpha;\end{matrix}\right]$$

$$= 0 \qquad \text{when } m \neq n$$

and

$$= \frac{n!\Gamma(\alpha+\gamma)(\gamma)_n}{(\alpha+\gamma+2n-1)(\alpha)_n}, \quad \text{when } m = n.$$

(ii) Let $\rho = -m, \beta = \alpha+\delta+m-1$, we have

$$F_{1,1}^{1,2}\left[\begin{matrix}\alpha: -m, -n;\ \alpha+\delta+m-1, \alpha+\gamma+n-1;\\ \alpha+\gamma: \delta, \alpha;\end{matrix}\right]$$

$$= \frac{(-)^{m+n} n!\,(\gamma)_n\,(\alpha+\gamma-1)_{m+n}}{(n-m)!\,(\alpha)_n\,(\alpha+\gamma-1)_n\,(\alpha-\delta+1)_{m-n}\,(\alpha+\gamma)_{m+n}}.$$

(iii) Let $\rho = -n, \beta = \alpha+\lambda+n-1, \delta = \alpha$, we have

$$F_{1,1}^{1,2}\left[\begin{matrix}\alpha;\ -n, -n;\ \alpha+\lambda+n-1, \alpha+\gamma+n-1;\\ \alpha+\gamma: \alpha, \alpha;\end{matrix}\right] = \frac{n!(\gamma)_n\,(\alpha+\lambda-1)_{2n}}{(\alpha)_n\,(\alpha+\lambda-1)_n\,(\alpha+\gamma)_{2n}}.$$

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