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On dynamical coherence

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Abstract. A partially hyperbolic diffeomorphism is dynamically coherent if its center, center-stable and center-unstable invariant distributions are integrable, i.e. tangent to foliations. Dynamical coherence is a key assumption in the theory of stable ergodicity. The main result: a partially hyperbolic diffeomorphism $f: M \to M$ is dynamically coherent if the strong stable and unstable foliations are quasi-isometric in the universal cover $\widetilde{M}$, i.e. for any two points in the same leaf, the distance between them in $\widetilde{M}$ is bounded from below by a linear function of the distance along the leaf.

Let M be an m-dimensional differentiable manifold. A k-dimensional distribution E on M is a continuous (in $x \in M$) family of k-planes $E(x) \subset T_x M$.

By a k-dimensional *foliation* W of M we mean a partition of M into k-dimensional, complete, connected, C^1-submanifolds $W(x) \ni x$ (called *leaves*) which (as C^1-submanifolds) depend continuously (in the compact–open topology) on $x \in M$. Let D^n denote the open unit ball in $\mathbb{R}^n$. For each point $x \in M$ there is a *coordinate chart* (U, ϕ) of W at x, i.e. a neighborhood U and a homeomorphism $\phi : D^k \times D^{m-k} \to U$ such that for each $p \in D^{m-k}$, the set $W_U(\phi(0, p)) := \phi(D^k, p)$ (called the *local leaf*) is contained in $W(\phi(0, p))$ and $\phi(\cdot, p) : D^k \to W(\phi(p))$ is a C^1 diffeomorphism which depends continuously on $p \in D^{m-k}$ in the C^1-topology.

If W is a k-dimensional foliation on M, then the family of tangent planes $\{E(x) = T_x W(x)\}_{x \in M}$ is a k-dimensional distribution on M. A distribution E is called *integrable* if there is a foliation W whose leaves are tangent to E, i.e. $E(x) = T_x W(x)$ for each $x \in M$. It is not clear (to the author) to what extent the existence of such an integral foliation (globally or locally) implies its uniqueness.

If W is a foliation, denote by d_W the distance along the leaves of W and by $W_\delta(x)$ the ball of radius δ in $W(x)$ centered at x.

A diffeomorphism f of a compact Riemannian manifold M is called *partially hyperbolic* if there are constants $0 < \lambda_1 \leq \lambda_2 < \gamma_1 \leq 1 \leq \gamma_2 < \mu_2 \leq \mu_1$, $C_{\mathrm{ph}} \geq 1$ and, for every $x \in M$, subspaces $E^s(x)$, $E^c(x)$, $E^u(x)$ (called *stable, center* and *unstable*, respectively) such that for every $x \in M$:

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1. $T_x M = E^s(x) \oplus E^c(x) \oplus E^u(x)$;
2. the distributions E^s, E^c and E^u are invariant under the derivative df, i.e. $df(x)E^\nu(x) = E^\nu(f(x))$, for $\nu = u, c, s$;
3. $C_{\text{ph}}^{-1}\lambda_1^n\|v^s\| \le \|df^n(x)v^s\| \le C_{\text{ph}}\lambda_2^n\|v^s\|$ for each $v^s \in E^s(x)$ and $n > 0$;
4. $C_{\text{ph}}^{-1}\gamma_1^n\|v^c\| \le \|df^n(x)v^c\| \le C_{\text{ph}}\gamma_2^n\|v^c\|$ for each $v^c \in E^c(x)$ and $n > 0$;
5. $C_{\text{ph}}^{-1}\mu_1^n\|v^u\| \le \|df^n(x)v^u\| \le C_{\text{ph}}\mu_2^n\|v^u\|$ for each $v^u \in E^u(x)$ and $n > 0$.

It is assumed that at least one of the two distributions E^s or E^u is non-trivial. The stable E^s and unstable E^u distributions are known to be Hölder continuous [**BP01**] and integrable, i.e. there exist foliations W^s and W^u, called *(strong) stable* and *unstable*, respectively, such that $T_x W^s(x) = E^s(x)$ and $T_x W^u(x) = E^u(x)$ for each $x \in M$ [**BP74**].

The center distribution E^c is not always integrable [**Wil98**]. Its integrability was established by Hirsch, Pugh and Shub for small perturbations of partially hyperbolic diffeomorphisms whose center distribution is integrable and is either Lipschitz continuous or has an integrable foliation with compact leaves [**HPS77**].

A partially hyperbolic diffeomorphism is called *dynamically coherent* if the distributions E^c, E^{cs} and E^{cu} are integrable. The corresponding foliations are called the *center, center-stable* and *center-unstable* foliations, respectively [**BPSW01**].

Dynamical coherence is a key assumption in the theory of stable ergodicity for partially hyperbolic diffeomorphisms which was created by Grayson, Pugh and Shub [**GPS94, PS97, PS00**] and further developed with Burns and Wilkinson [**BPSW01**].

The following property of a distribution implies its 'unique' integrability and is similar to the uniqueness of solutions of ordinary differential equations.

A continuous k-dimensional distribution E on a smooth Riemannian manifold M is called *locally uniquely integrable* if for each $x \in M$ there are a k-dimensional C^1-submanifold $W_{\text{loc}}(x)$ and $\alpha(x) > 0$ such that every piecewise C^1-curve $\sigma : [0, 1] \to M$ satisfying (i) $\sigma(0) = x$, (ii) $\dot\sigma(t) \in E(\sigma(t))$ for $t \in [0, 1]$, and (iii) $\text{length}(\sigma) < \alpha(x)$, is contained in $W_{\text{loc}}(x)$. Obviously, if E is locally uniquely integrable, then it is integrable and the integral foliation is unique.

If E is a smooth distribution, its unique local integrability can be established by checking the Frobenius condition—the Lie brackets of vector fields tangent to E must be tangent to E.

The main result of this paper is the integrability of the center distribution (and its joint integrability with the stable and unstable distributions) for partially hyperbolic diffeomorphisms whose stable and unstable foliations are quasi-isometric.

A foliation W of a simply connected Riemannian manifold $\widetilde M$ is called *quasi-isometric* [**Fen92**] if there are positive constants a and b such that for any two points x and y which lie in the same leaf of W,

$$d_W(x, y) \le a \cdot d(x, y) + b.$$

THEOREM 1. *Let f be a partially hyperbolic diffeomorphism of a compact m-dimensional Riemannian manifold M. Suppose the stable W^s and unstable W^u foliations of f are quasi-isometric in the universal cover $\widetilde M$. Then the distributions E^c, $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$ are locally uniquely integrable; in particular, f is dynamically coherent.*

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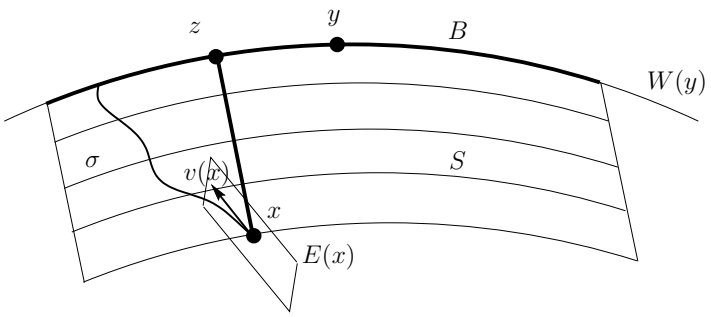


FIGURE 1. Construction of σ.

Proof. We will prove the local unique integrability of E^{cs}. The local unique integrability of E^{cu} and the existence of the center-unstable foliation W^{cu} follow by reversing the time. The local unique integrability of E^c and the existence of the center foliation W^c are established by observing that every C^1-curve starting at $x \in M$ and tangent to E^c is also tangent to E^{cs} and to E^{cu} and therefore is contained in $W^{cs}(x) \cap W^{cu}(x)$.

LEMMA 2. *Suppose W is a k-dimensional foliation of M and E is an l-dimensional distribution such that $k + l = m$ and E is (uniformly) transverse to W, i.e. $T_x W(x) \cap E(x) = \{0\}$ for each $x \in M$.*

There is $C_0 > 0$ and for every $\delta > 0$ there is $\epsilon_0 > 0$ such that if $d(x, y) < \epsilon_0$ then there is a C^1-curve $\sigma : [0, 1] \to M$ with the property that $\sigma(0) = x$, $\sigma(1) \in W_\delta(y)$, $\dot\sigma(t) \in E(\sigma(t))$ for $t \in [0, 1]$, $\text{length}(\sigma) \le C_0 d(x, y)$ and $d_W(y, \sigma(1)) \le C_0 d(x, y)$.

Proof. By choosing a small enough ϵ_0 we may assume that a neighborhood of x under consideration is identified by a coordinate chart (with uniformly bounded derivative) with the unit ball in $\mathbb{R}^m$. For a small enough $\delta > 0$, the ball $B = W_\delta(y)$ is contained in a local leaf and is identified with a k-dimensional C^1-submanifold in $\mathbb{R}^m$. Assuming that ϵ_0 is small enough, there is a point $z \in B$ for which $d(x, z) = d(x, B)$ and $d_W(z, y) \le C d(x, y)$, where C depends only on W. In particular, $d(x, z) \le d(x, y)$ and $x - z \perp T_z B$. The union

$$S = \bigcup_{0 \le t \le 1} (B + (1 - t)(x - z))$$

of the parallel translates of the local leaf B along the segment connecting x to z is a $(k + 1)$-dimensional C^1-surface which is a Riemannian product of the interval $[0, \|x - z\|]$ and B; see Figure 1. We refer to B and $x - z$ as the horizontal and vertical components, respectively. For $s \in S$, let $L(s) = E(s) \cap T_s S$. Since E is uniformly transverse to B, if δ and ϵ_0 are small enough, then L is a continuous line field on S which is uniformly (in x, y and s) transverse to the codimension 1 (in S) submanifolds $B + (1 - t)(x - z)$. Therefore, for each $s \in S$ there is a unique vector $v(s) \in L(s)$ whose vertical component is $z - x$. The uniform transversality of E and $B + (1 - t)(x - z)$ implies that v is a uniformly (in x, y and s) bounded, continuous vector field on S. For each $s \in S$, the vertical component of $v(s)$ is $z - x$ and the horizontal component is bounded by a constant C' which depends

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on ϵ_0 and δ but not on x, y and s. By the existence theorem for ordinary differential equations, there is a (possibly not unique) solution $\sigma(\cdot)$ of $\dot{s} = v(s)$ with $\sigma(0) = x$. The solution stays within the C'-cone near the vertical segment in S. Therefore, if ϵ_0 and δ are small enough, its interval of definition can be extended to $[0, 1]$ so that $\sigma(1) \in B$, $\text{length}(\sigma) \leq \sqrt{1 + C'^2} d(x, z) \leq C_0 d(x, y)$ and $d_W(y, \sigma(1)) \leq C_0 d(x, y)$. □

To prove the unique integrability of E^{cs}, lift all distributions and foliations to the universal cover $\widetilde{M}$. Fix $r \in (0, \epsilon_0)$. For $x \in \widetilde{M}$ denote by $W_r^{cs}(x)$ the set of ends of all piecewise C^1-curves $\sigma : [0, 1] \to M$ such that $\sigma(0) = x$, $\dot{\sigma}(t) \in E^{cs}(\sigma(t))$ and $\text{length}(\sigma) < r$.

LEMMA 3. *Let $y, z \in \widetilde{M}$ be such that $y, z \in W_r^{cs}(x)$ and $z \in W^u(y)$. Then $z = y$.*

Proof. By assumption, there are C^1-curves σ_y and σ_z tangent to E^{cs} and connecting x to y and z, respectively. Consider the points $f^n(x)$, $f^n(y)$ and $f^n(z)$. From the partial hyperbolicity inequalities we have

$$\text{length}(f^n(\sigma_y)) \leq C_{\text{ph}} \gamma_2^n \cdot \text{length}(\sigma_y), \quad \text{length}(f^n(\sigma_z)) \leq C_{\text{ph}} \gamma_2^n \cdot \text{length}(\sigma_z)$$

and therefore

$$d(f^n(y), f^n(z)) \leq C_{\text{ph}} \gamma_2^n (\text{length}(\sigma_y) + \text{length}(\sigma_z)).$$

On the other hand, let d_u denote the distance along the leaves of W^u and let σ_n be a C^1-curve in $W^u(f^n(y)) = W^u(f^n(z))$ which realizes $d_u(f^n(y), f^n(z))$. Then

$$d_u(f^n(y), f^n(z)) = \text{length}(\sigma_n) \geq C_{\text{ph}}^{-1} \mu_2^n \, \text{length}(f^{-n}(\sigma_n)) \geq C_{\text{ph}}^{-1} \mu_2^n d_u(y, z).$$

Since W^u is a quasi-isometric foliation,

$$d(f^n(y), f^n(z)) \geq (d_u(f^n(y), f^n(z)) - b)/a \geq (C_{\text{ph}}^{-1} \mu_2^n d_u(y, z) - b)/a.$$

Since $d(f^n(x), f^n(y)) \leq \text{length}(f^n(\sigma_y))$, $d(f^n(x), f^n(z)) \leq \text{length}(f^n(\sigma_z))$, and since $\gamma_2 < \mu_2$, we obtain a contradiction with the triangle inequality for $f^n(x)$, $f^n(y)$ and $f^n(z)$ (with n sufficiently large) unless $d_u(y, z) = 0$. □

We need to show that $W_r^{cs}(x)$ is a C^1-submanifold of the same dimension as E^{cs}.

Fix $\delta > 0$ such that for each $x \in \widetilde{M}$, the δ-ball $W_\delta^u(x)$ is contained in a local leaf of W^u. For $\epsilon > 0$ and $y \in W_r^{cs}(x)$, denote by $E_\epsilon^{cs}(y)$ the ϵ-ball in $E^{cs}(y)$ centered at y. Since $W^u(y)$ is transverse to $E^{cs}(y)$ at y, for small enough $0 < \epsilon \ll \delta$ and each $z \in E_\epsilon^{cs}(y)$, the leaf $W^u(z)$ is transverse to $E^{cs}(y)$ at z. By Lemma 2, if r and $\epsilon < r$ are small enough, then each leaf $W_\delta^u(z)$ intersects $W_r^{cs}(x)$ and, by Lemma 3, the intersection consists of a single point $\pi(z)$. Let $z_1, z_2 \in E_\epsilon^{cs}(y)$. Since $W_\delta^u(z)$ depends continuously on z as a C^1-submanifold, $d(\pi(z_1), W_\delta^u(z_2)) \to 0$ as $d(z_1, z_2) \to 0$. Therefore, by Lemma 2, $d(\pi(z_1), \pi(z_2)) \to 0$ as $d(z_1, z_2) \to 0$. It follows that π is continuous and a homeomorphism onto its image which is a neighborhood of $y = \pi(y)$ in $W_r^{cs}(x)$. It follows that $W_r^{cs}(x)$ is a topological submanifold. Obviously, $E^{cs}(y)$ is the tangent plane to $W_\epsilon^{cs}(y)$ at y. Since the distribution E^{cs} is continuous, $W_r^{cs}(x)$ is a C^1 submanifold. Hence E^{cs} is locally uniquely integrable and the leaf $W^{cs}(x)$ of the integral foliation W^{cs} passing through $x \in M$ is the set of ends of piecewise C^1-curves starting at x and tangent to E^{cs}. This completes the proof of Theorem 1. □

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In general, the stable and unstable foliations of a partially hyperbolic diffeomorphism do not have to be quasi-isometric. For example, for the time-1 map of the geodesic flow on a compact manifold of negative sectional curvature, the stable and unstable foliations are not quasi-isometric—the unstable horosphere expands exponentially with time but the distance between points on the horosphere grows linearly in time.

PROPOSITION 4. *Let W be a k-dimensional foliation of the m-dimensional torus $\mathbb{T}^m$. Suppose there is an $(m-k)$-dimensional plane A such that $T_x W(x) \cap A = \emptyset$ for each $x \in \mathbb{T}^m$. Then the lift $\widetilde{W}$ (of W to $\mathbb{R}^m$) is quasi-isometric.*

Proof. Let $\widetilde{B}$ be the orthogonal complement of the lift $\tilde{A}$ of A in $\mathbb{R}^m$. Fix $x \in \mathbb{R}^m$ and let $\pi : \widetilde{W}(x) \to \widetilde{B}$ be the orthogonal (parallel to $\tilde{A}$) projection of $\widetilde{W}(x)$ to $\widetilde{B}$. By compactness, $\widetilde{W}$ is uniformly transverse to $\tilde{A}$. Therefore there is $\alpha > 0$ such that $\|d\pi(y)v\| \geq \alpha\|v\|$ for each $y \in \widetilde{W}(x)$ and $v \in T_y\widetilde{W}(x)$. Hence, by compactness, there is $\delta > 0$ such that for each $y \in \widetilde{W}(x)$, the map π restricted to the δ-neighborhood $\widetilde{W}_\delta(y)$ of y in $\widetilde{W}(x)$ is a diffeomorphism onto its image which contains the $\alpha\delta$-neighborhood of $\pi(y)$. It follows that π is a covering map. Since $\widetilde{W}(x)$ and $\widetilde{B}$ are simply connected, π is one-to-one.

Let $y, z \in \widetilde{W}(x)$ and let $\sigma : [0,1] \to \widetilde{B}$ be the straight line segment connecting $\pi(y)$ to $\pi(z)$. The lift $\pi^{-1} \circ \sigma$ is a C^1-curve connecting y to z. By uniform transversality,

$$d(\pi(y), \pi(z)) = \text{length}(\sigma) \geq \frac{1}{\alpha}\,\text{length}(\pi^{-1} \circ \sigma) \geq \frac{1}{\alpha} d_{\widetilde{W}}(y, z),$$

where, as before, $d_{\widetilde{W}}(y, z)$ is the distance from y to z in $\widetilde{W}(x)$. □

The next proposition is an application of Proposition 4 to a certain class of partially hyperbolic diffeomorphisms of $\mathbb{T}^3$.

PROPOSITION 5. *Let $f_0 : \mathbb{T}^3 \to \mathbb{T}^3$ act by*

$$f_0(x, y, z) = \left(A \begin{pmatrix} x \\ y \end{pmatrix}, z + \alpha \right) \bmod 1,$$

where $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\alpha \in \mathbb{R}$. Suppose $f_1 = g \circ f_0$, where $g : \mathbb{T}^3 \to \mathbb{T}^3$ is a C^1-diffeomorphism such that $\|dg(p) - \boldsymbol{I}\| < 0.1$ for each $p \in \mathbb{T}^3$. Then f_1 is partially hyperbolic and its weak stable, weak unstable and center distributions are locally uniquely integrable.

Proof. Use a subindex 0 for the invariant distributions of f_0 and a subindex 1 for those of f_1. The stable $E_0^u(x)$, unstable $E_0^s(x)$ and center $E_0^c(x)$ subspaces for f_0 are one-dimensional and mutually perpendicular. The subspaces at different points are also parallel, so we omit the dependence on $x \in \mathbb{T}^3$. The derivative df_0 stretches vectors from E_0^u by a factor of $(3+\sqrt{5})/2 \approx 2.6180$, contracts vectors from E_0^s by a factor of $(3-\sqrt{5})/2 \approx 0.3820$ and acts as an isometry in the direction of E_0^c. For a tangent vector v, let $v = v^u + v^s + v^c$ be the decomposition given by the splitting $E_0^u \oplus E_0^s \oplus E_0^c$.

Consider the strong and weak, stable and unstable $\pi/6$-cones

$$K_0^u = \{v : \|v^s + v^c\| \leq 0.5\}, \quad K_0^{cu} = \{v : \|v^s\| \leq 0.5\},$$
$$K_0^s = \{v : \|v^u + v^c\| \leq 0.5\}, \quad K_0^{cs} = \{v : \|v^u\| \leq 0.5\}.$$

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A direct computation shows that the strong and weak unstable cones are invariant under df_1 and the strong and weak stable cones are invariant under df_1^{-1}. It follows that f_1 has invariant distributions E_1^u, E_1^s, E_1^{cu}, E_1^{cs} which lie in the $\pi/6$-cones of the corresponding distributions of f_0.

Again, a direct computation shows that each vector from K_0^u is stretched by df_1 by a factor of at least 2 and each vector from K_0^s is stretched by df_1^{-1} by a factor of at least 2.

Let $v \in E_1^c$. Then $\|v^s\| \leq 0.5\|v\|$ and $\|v^u\| \leq 0.5\|v\|$. A direct computation shows that $1.8^{-1}\|v\| \leq \|df_1 v\| \leq 1.8\|v\|$. Therefore f_1 is partially hyperbolic.

The stable W_1^s and unstable W_1^u foliations of f_1 satisfy the assumptions of Proposition 4 with $A = E_0^{cu}$ for W_1^s and $A = E_0^{cs}$ for W_1^u and the proposition follows from Theorem 1. □

A partially hyperbolic diffeomorphism $f : M \to M$ is *center-isometric* if it acts isometrically in the center direction, i.e. $\|df(x)v\| = \|v\|$ for every $x \in M$ and $v \in E^c(x)$.

PROPOSITION 6. *If a partially hyperbolic diffeomorphism $f : M \to M$ is center-isometric, then the distributions E^c, E^{cs} and E^{cu} are locally uniquely integrable; in particular, f is dynamically coherent.*

Proof. The proof of the proposition is completely analogous to the proof of Theorem 1 with the only exception that Lemma 3 should be replaced by the following lemma. □

LEMMA 7. *There are $r_0 > 0$ and $\delta_0 > 0$ such that for every $r \in (0, r_0]$ and $\delta \in (0, \delta_0]$ if $y, z \in M$, $y, z \in W_r^{cs}(x)$ and $z \in W_\delta^u(y)$, then $z = y$.*

Proof. Since the leaves of W^u are C^1 and depend continuously on the point, there is $\delta_0 > 0$ and $C_1 > 0$ such that if $z' \in W_{\delta_0}^u(y')$, then

$$d_u(y', z') \leq C_1 d(y', z').$$

By the uniform expansion of W^u,

$$d_u(f^n(y), f^n(z)) \geq C_{\text{ph}}^{-1} \mu_2^n d_u(y, z).$$

Assuming that $z \neq y$, choose $n > 0$ so that $d_u(f^k(y), f^k(z)) < \delta_0$ for $k = 0, 1, \ldots, n$ and $d_u(f^{n+1}(y), f^{n+1}(z)) \geq \delta_0$. Then

$$d_u(f^n(y), f^n(z)) \geq \nu\delta_1,$$

where ν is the maximum of the norm of df over M.

By assumption, there are C^1-curves σ_y and σ_z which have length less than or equal to r, are tangent to E^{cs} and connect x to y and z, respectively. Consider the points $f^n(x)$, $f^n(y)$ and $f^n(z)$. By the partial hyperbolicity and center isometry,

$$\text{length}(f^n(\sigma_y)) \leq C_{\text{ph}} \cdot \text{length}(\sigma_y), \quad \text{length}(f^n(\sigma_z)) \leq C_{\text{ph}} \cdot \text{length}(\sigma_z)$$

and therefore

$$d(f^n(y), f^n(z)) \leq C_{\text{ph}}(\text{length}(\sigma_y) + \text{length}(\sigma_z)) \leq 2C_{\text{ph}} r.$$

It follows that

$$C_1^{-1}\delta_0 \leq d(f^n(y), f^n(z)) \leq 2C_{\text{ph}} r$$

which gives a contradiction for a small enough r. □

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Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism (with one-dimensional stable, center and unstable distributions). The following questions arise naturally:

(1) Are the stable and unstable foliations necessarily quasi-isometric?

(2) Is f necessarily dynamically coherent?

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