Math. Proc. Camb. Phil. Soc. (1978), **83**, 73
With 1 *text-figure*
Printed in Great Britain

A constrained isoperimetric problem

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(*Received* 16 *May* 1977)

1. *Introduction.* Of all sets lying within a given convex plane set S, which one gives the greatest ratio of area A to perimeter P? This was posed to one of us by E. Bombieri for the case when S is a square. He asserted that T. Bang used an easy upper bound on A/P in an elementary proof of the Prime Number Theorem. After solving this problem from scratch, we learned from M. Silver that Besicovitch (1) had solved a closely related problem, which we call Besicovitch's problem and which will be discussed shortly. Later, we found that the problem of maximizing A/P had occurred to Garvin (4, 5) and that he solved the problem when S is a triangle. Garvin inspired DeMar (2) to consider and solve Besicovitch's problem for the triangle and for polygons. DeMar found that Steiner (8, p. 166) had solved Besicovitch's problem for triangles, but Steiner was only interested in subsets which touched all sides of a polygon (8, p. 168). More recently, J. Wills has referred us to Herz and Kaapke (6) who use Besicovitch's results to solve our problem of maximizing A/P for polygons circumscribed about a circle.

Let S be a compact convex set in the plane having perimeter $P(S)$. Besicovitch showed that, of all closed convex subsets lying in S and having a given perimeter P satisfying $P_0 \leqslant P \leqslant P(S)$, where P_0 will be defined later, the set with the maximum area A is the union S_r of all closed discs of some common radius r which lie in S. Otherwise viewed, S_r is the given set of S 'rounded off' by circular arcs of radius r and tangent to the boundary of S. The value of r is not readily determined from P unless S is a rather simple region such as a polygon circumscribed about a circle. In this note, we show that the maximal value of A/P occurs for the set S_r corresponding to the unique value r such that $A(S_r)/P(S_r) = r$. We also discuss some points concerning Besicovitch's problem and some consequences of our result.

2. *Besicovitch's problem.* By a convex region, we mean a compact, convex, planar set with non-empty interior. We may call this simply a convex set. We denote the boundary of such a region S by δS. It is clear that any solution of our problem of maximizing A/P over subsets of S must be convex, else (the closure of) its convex hull would have larger A and smaller P, hence greater A/P. However, for Besicovitch's problem maximizing A for given P, it is not clear that the solution must be convex. Besicovitch restricted his work to convex subsets of S, but DeMar assumed the maximal subset was convex, without proof. Now, when $P > P(S)$, there is no convex subset T of S with $P(T) = P$ (9, p. 146), but there are non-convex subsets T with $P(T) = P$ and $A(T) = A(S) - \epsilon$ for any $\epsilon > 0$, so convexity is not obvious, but is desirable. The situation $P > P(S)$ leads to an unsatisfactory solution, so we suppose $P \leqslant P(S)$, which is implicit in Besicovitch's restriction to convex T. (In this section the context shows that closed sets are taken to have rectifiable boundaries.)

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Now consider a non-convex, compact subset T of S, such that T is not one-dimensional and $P(T) = P \leqslant P(S)$. Let T' be the convex hull of T, so we have $S \supseteq T' \supset T$, $A(S) \geqslant A(T') > A(T)$ (since T' is closed and convex and contains the closed set T which is not one-dimensional) and $P(S) \geqslant P \geqslant P(T')$. Consider a convex combination $U = \lambda S + (1-\lambda)T'$. This is a convex set with $P(U) = \lambda P(S) + (1-\lambda)P(T')$. Since P lies between $P(S)$ and $P(T')$, there is a value λ, $1 \geqslant \lambda \geqslant 0$, such that $P(U) = P$. Further $S \supseteq T'$ implies $U \supseteq T'$, so $A(U) \geqslant A(T') > A(T)$. (See (3, pp. 79–89; 9, pp. 152–156; 10, pp. 39–50) for the basic facts on sums of sets and their perimeter and area.) If T is a one-dimensional subset of S, then $A(T') = A(T)$, but $A(T) = 0$, so this case can be safely excluded. Hence we have shown that any value of $A(T)$ achieved by a non-convex closed subset T with $P(T) = P$, $0 < P \leqslant P(S)$, can be exceeded by such a subset U which is also convex. Hence the maximum of A over all closed subsets of S, for a given P, must be the same as the maximum over all convex subsets of S, i.e. we can replace Besicovitch's restriction to convex subsets with the restriction to closed subsets. It does not seem reasonable to extend this further.

For any P, $0 < P \leqslant P(S)$, there is a convex subset $T = [P/P(S)]S$ of S such that $P(T) = P$. (Strictly, to make $T \subseteq S$, we must assume the origin is in S, which we are free to do.) Since the collection of convex subsets T with $P(T) = P$ is non-empty, the Blaschke Selection Theorem (3, pp. 64–67; 9, pp. 37–39; 10, pp. 100–104) assures us that a convex subset with maximal area A exists and hence it is a maximum over all closed subsets T.

Suppose now that $P(T) = P$, $0 < P \leqslant P(S)$, and that T has the largest area A of all such closed subsets of S. Besicovitch's Lemma 1 asserts that $\delta T \sim \delta S$ is a set of circular arcs. By this he means that each component of $\delta T \sim \delta S$ is a circular arc. He simply cites the classical fact that the area, contained between a given chord and an arc of given length, is maximized for the circular arc. DeMar makes the same assertion and Steiner (8, p. 125) asserts that any 'free' arc of a maximal figure between two points is a circular arc. In our work from scratch, this point bothered us since we did not see that the result of making a component of $\delta T \sim \delta S$ into a circular arc of the same length would necessarily lie within S, and indeed it might not. That is, a component of $\delta T \sim \delta S$ is not a free arc in the sense of Steiner, since it is restricted to remain within S. Therefore it may be worth pointing out how this problem can be resolved.

Let a be a point of $\delta T \sim \delta S$. Then a is interior to S, so there is a ball $B(a, r)$ of some radius r about a which is entirely within S. Let b, c be the two points of δT which are at distance $r/3$ from a, measured along δT. It is a simple geometric exercise to verify that any circular arc, of length $2r/3$ and passing through b and c, is entirely within $B(a, r)$ and hence S. Hence we can get a larger area for the same perimeter, unless the arc bac is already circular. For our maximal T, we let $f(a)$ be the radius of the circular arc through a. The above shows that f is locally constant and hence continuous on each component of $\delta T \sim \delta S$. Thus $f^{-1}(f(a))$ is both open and closed in its component of $\delta T \sim \delta S$, hence must be the whole component. That is, $f(a)$ is constant on each component, or each component is a circular arc.

Besicovitch then shows the following.

(1) All the circular arcs of $\delta T \sim \delta S$ have the same radius, say r.

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(2) The upper curvature is $\leqslant 1/r$ everywhere on δT.

(3) If $0 < P < P(S)$, then δT has a tangent line at each of its points.

(4) If $r < r_0$, the inradius of S, then each arc of $\delta T \sim \delta S$ has angle $< \pi$.

(5) Let S_r be the union of all closed discs of radius r which lie in S and let $P_0 = P(S_{r_0})$.

If $P \geqslant P_0$, then $T = S_r$ for some r, $0 \leqslant r \leqslant r_0$.

If $P_0 \geqslant P \geqslant 2\pi r_0$, then T is a union of some circles of radius r_0 and is a rectangle with semicircular ends of radius r_0. (For an example, consider a rectangular S.)

If $2\pi r_0 \geqslant P > 0$, then T is a closed disc of radius $P/2\pi$.

3. *Preliminaries to our problem.* We now turn to our problem of finding the subset T of S such that $A(T)/P(T)$ is maximal. We already know that T must be convex if it exists and we can further take it as closed. The Blaschke Selection Theorem then shows that a maximal convex subset exists and so it is a maximum over all subsets. Further, this subset must have the maximum area for all subsets of the same perimeter, so it must be among the solutions of Besicovitch's problem.

When $P(T) = P$, $0 < P \leqslant 2\pi r_0$, then the Besicovitch solution is a closed disc of radius $r = P/2\pi$, $0 < r \leqslant r_0$. Then $A = \pi r^2$, $P = 2\pi r$ and $A/P = r/2$, which is increasing with r, so the maximum of A/P does not occur for $P < 2\pi r_0$. When $2\pi r_0 \leqslant P \leqslant P_0$, then the Besicovitch solution is a rectangle of some length $a \geqslant 0$, with semicircular ends of radius r_0. Then $A = \pi r_0^2 + 2r_0 a$, $P = 2\pi r_0 + 2a$ and $A/P = r_0 - \pi r_0^2/(2\pi r_0 + 2a)$, which is increasing with a, so the maximum of A/P does not occur with $P < P_0$. (If S is an infinite strip of width $2r_0$, this shows that the maximal subset is S itself.) Hence T must be S_r for some r, $0 \leqslant r \leqslant r_0$.

In a previous version of this work, we directly attacked the problem of comparing A/P for the subsets S_r for any convex set, by a geometrical approach which took 12 manuscript pages, and then we considered the easy case when S is a polygon. J. Paul has pointed out that our proof could be simplified by considering the polygonal case first and then treating any convex set as a limit of inscribed polygons. We now proceed along this line.

4. *The polygonal case.* We now consider the sets S_r in a bounded convex polygon S. S_{r_0} is a rectangle of some length a, $a \geqslant 0$, with semicircular ends. But the length a is maximal and S is a polygon, so the semicircular ends must meet δS at points other than the end-points of the semicircles. Thus all arcs of $\delta S_{r_0} \sim \delta S$ have angle $< \pi$, i.e. property 4 of section 2 holds for all our sets S_r, $0 \leqslant r \leqslant r_0$.

It may be that S_r does not touch all the sides of S. Let $S' = S'_r$ be the polygon formed by the sides of S which S_r touches, extended as necessary. Since each arc of S_r has angle $< \pi$, S' is a bounded convex polygon and it clearly contains S. We call $S' = S'_r$ the *contact polygon* of S_r. Thus S_r is S' with *all* of its corners rounded by tangent circular arcs of radius r and we have $S_r = (S')_r$. Let $2\psi_i$ be the angle of the ith arc of S_r. Fig. 1 shows a corner of S' with the corresponding arc of S_r.

We readily find

$$\left.\begin{aligned} A(S_r) &= A(S'_r) - r^2 \Sigma_i \tan \psi_i + \pi r^2, \\ P(S_r) &= P(S'_r) - 2r \Sigma_i \tan \psi_i + 2\pi r, \end{aligned}\right\} \tag{1}$$

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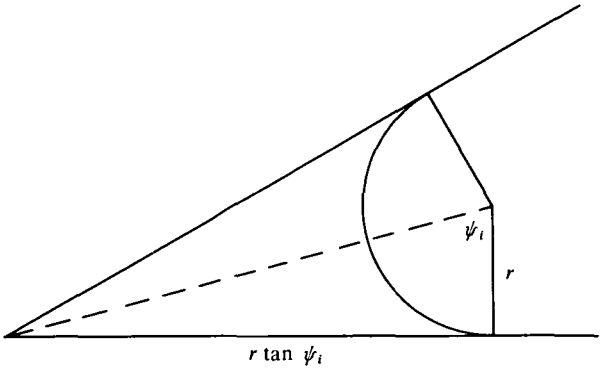


Fig. 1

using the fact that $\Sigma_i \psi_i = \pi$. From equations (1), we directly obtain that $dA/dP = r$, over the range of r where the contact polygon S'_r is fixed. At a value r_1 where S_{r_1} meets a side of S in just one point, we have a *transition* from one contact polygon to another. $A(S_r)$ and $P(S_r)$ are continuous at this transition but dA/dr and dP/dr are discontinuous. However, $dA/dP = r$ holds as $r \to r_1$, both from above and below, so $dA/dP = r$ is continuous at such transitions. Further the sets S_r are continuous in the Hausdorff metric (3, p. 60; 9, p. 36), even at transitions. We can even include r_0 as a transition point at which dA/dP and S_r are continuous. Note that $P = P(S_r)$ is strictly monotone increasing as S_r increases, i.e. as r decreases.

Now consider maximizing A/P over all the subsets S_r. Then A is a differentiable function of P, so a maximum will occur when

$$\frac{d}{dP}\frac{A}{P} = \frac{P\,dA/dP - A}{P^2} = 0,$$

i.e. when $dA/dP = A/P$, hence when $r = A/P$. We know that some S_r gives the maximal value of A/P, so there is at least one point where $r = A/P$. We assert that there is only one such value of r. This can be verified by direct consideration of the graphs of A/P and r for each possible contact polygon and then piecing together the behaviour across the transitions. Alternatively, suppose $r = A(S_r)/P(S_r)$ holds for some value $r = r_2$ and let $r_0 \geqslant r_2 > r_3 \geqslant 0$. Let P_i, A_i denote the perimeter and area of S_{r_i}. By the Mean Value Theorem, we have $(A_3 - A_2)/(P_3 - P_2) = dA/dP(r_4) = r_4$, where $r_2 > r_4 > r_3$, using the fact that r is a strictly monotonic decreasing function of P. Hence $A_3 - A_2 > r_3 P_3 - r_3 P_2$. Now $r_3 < r_2 = A_2/P_2$ so $A_2 > r_3 P_2$ and we get $A_3 > r_3 P_3$ or $A_3/P_3 > r_3$. Symmetrically, if $r_0 \geqslant r_3 > r_2 \geqslant 0$, we get $A_3/P_3 < r_3$. Thus there are unique values of r, $r_0 \geqslant r \geqslant 0$, and of P, $P_0 \leqslant P \leqslant P(S)$, giving a unique maximum of A/P for the sets S_r, corresponding to the unique point where $A(S_r)/P(S_r) = r$. (We actually have $r > 0$.)

We can use this result actually to compute this value of r. We begin either at $r = r_0$ or $r = 0$ and consider the corresponding contact polygon S'_r. Then compute the unique r_1 satisfying equations (1) such that $A/P = r$ with $P > 0$. If this r_1 gives a set S_{r_1} with the same contact polygon S'_r, then we are done. Otherwise proceed to the next possible contact polygon and repeat the process.

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5. *The general case.* Let S now be any convex set in our sense, i.e. compact, convex, planar and with non-empty interior. Let S_i be an increasing sequence of convex polygons which converges to S in the Hausdorff metric (3, pp. 67–71; 9, pp. 143–147). We write $(S_i)_r$ for the set S_i rounded off by arcs of radius r. Let $T_i = (S_i)_{r_i}$ be the rounding of S_i which gives the maximal $A/P = r_i$ over all subsets of S_i. Now $T_i \subseteq S_i \subseteq S_{i+1}$, so $r_i = A(T_i)/P(T_i) \leqslant A(T_{i+1})/P(T_{i+1}) = r_{i+1}$. Thus r_i increases and r_i is bounded by the inradius r_0 of S, so r_i converges to some value r. We wish to deduce $T_i \to S_r$ in the Hausdorff metric. This requires a series of technical results which will also lead to the result that $dA/dP = r$ for the sets S_r. At first we thought this convergence was obvious, but the example after Theorem 6 shows that care is needed. Initially we consider any sequence $r_i \uparrow r$ and any r, but later we return to the particular r_i's and r obtained above. Throughout, convergence or continuity of sets will be with respect to the Hausdorff metric h. We let $B(r, x)$ be the open ball of radius r about the point x and we let $\bar{B}(r, x)$ be its closure. The propositions have reasonably straightforward proofs which are left to the reader.

PROPOSITION 1. (*a*) *A monotonic increasing family of compact convex subsets of a compact plane set converges to the closure of its union.*

(*b*) *A monotonic decreasing family of compact convex plane sets converges to its intersection.*

THEOREM 2. *S_r is a continuous function of r for $r_0 \geqslant r \geqslant 0$. (Of course we only have one-sided continuity at r_0 and 0.)*

Proof. (*a*) For $r < r_0$, the family $S_{r+\alpha}, r_0 - r \geqslant \alpha > 0$, is a monotonic increasing family (as α decreases) of compact convex subsets of S_r, so it converges to $\mathrm{cl}(\cup_\alpha S_{r+\alpha}) = A$, say, and $A \subseteq S_r$. Let $a \in S_r$. Then there is a b such that $a \in \bar{B}(r, b) \subseteq S$. Let $\bar{B}(r_0, c)$ be an incircle of S. The convex hull of $\bar{B}(r, b) \cup \bar{B}(r_0, c)$ is then in S. Its interior is contained in the union of the balls $\bar{B}(r+\alpha, b+\alpha(c-b)/(r_0-r))$ for $0 < \alpha \leqslant r_0 - r$ and each of these balls is in S. Thus each interior point of $\bar{B}(r, b)$ is in $\cup_\alpha S_{r+\alpha}$. Hence $\bar{B}(r, b) \subseteq A$, so $a \in A$, $S_r \subseteq A$ and $S_r = A$, i.e. S_r is continuous from above.

(*b*) For $r > 0$, the family $S_{r-\alpha}, r \geqslant \alpha > 0$, is a monotonic decreasing family (as α decreases) of compact convex supersets of S_r, so it converges to $\cap_\alpha S_{r-\alpha} = A$, say, and $A \supseteq S_r$. Let $a \in A$. Then for each of our α, there is a b_α such that $a \in \bar{B}(r-\alpha, b_\alpha) \subseteq S$. Now the set of closed balls in S is a closed set in the Hausdorff metric topology, hence is compact. (Consider the complement or see (10, pp. 50, 182–184) or the less explicit (3, p. 76).) Further, the set of closed balls in S containing a is a closed set (again consider the complement) and hence is compact. Hence there is a limiting closed ball containing a and in S and it must have radius $r = \sup\{r-\alpha | r \geqslant \alpha > 0\}$ – since

$$h(\bar{B}(s, x), \bar{B}(t, y)) = |s-t| + d(x, y).$$

That is, there is a c such that $a \in \bar{B}(r, c) \subseteq S$, so $a \in S_r$, $A \subseteq S_r$ and $A = S_r$, i.e. S_r is continuous from below.

PROPOSITION 3. *If A is closed and $A \subseteq \mathrm{int}\, S$, then $A \subseteq \mathrm{int}\, S_i$ for some i.*

COROLLARY 3·1. *Let ρ_i be the inradius of S_i. Then $\rho_i \uparrow r_0$, the inradius of S. (See (3, p. 77).)*

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PROPOSITION 4. *If X, Y, Z are compact, convex sets and $X \subseteq Y \subseteq Z$, then $h(X, Y) \leqslant h(X, Z)$ and $h(Y, Z) \leqslant h(X, Z)$.* (*See* (3, *p.* 64).)

THEOREM 5. *For $r < r_0$, we have $(S_i)_r \to S_r$.*

Proof. We have $S_i \subseteq S_{i+1} \subseteq S$, so $(S_i)_r \subseteq (S_{i+1})_r \subseteq S_r$, hence

$$\lim_i (S_i)_r = \operatorname{cl} \cup_i (S_i)_r \subseteq S_r.$$

Take any ϵ, $r_0 - r \geqslant \epsilon > 0$. We claim $\operatorname{cl} \cup_i (S_i)_r \supseteq S_{r+\epsilon}$. Let $a \in S_{r+\epsilon}$, so there is a b such that $a \in \bar{B}(r+\epsilon, b) \subseteq S$. If $c \in B(r+\epsilon, b)$, one can easily find a b' such that $c \in \bar{B}(r, b') \subseteq B(r+\epsilon, b)$. Then $\bar{B}(r, b') \subseteq \operatorname{int} S$ and Proposition 3 shows there is an i such that $\bar{B}(r, b') \subseteq S_i$, hence $c \in (S_i)_r$ and $c \in \operatorname{cl} \cup_i (S_i)_r$. Thus $B(r+\epsilon, b) \subseteq \operatorname{cl} \cup_i (S_i)_r$, so $\bar{B}(r+\epsilon, b) \subseteq \operatorname{cl} \cup_i (S_i)_r$ and thus $S_{r+\epsilon} \subseteq \operatorname{cl} \cup_i (S_i)_r$, as claimed. Now the continuity of S_r and Proposition 4 yield $h(S_r, \operatorname{cl} \cup_i (S_i)_r) = 0$ and so $S_r = \lim_i (S_i)_r$.

THEOREM 6. *For $r_i \uparrow r$ and $r < r_0$, we have $(S_i)_{r_i} \to S_r$.*

Proof. If $r = 0$, then all $r_i = 0$ and $(S_i)_0 = S_i \to S = S_0$. Now we assume $r > 0$. Let $\epsilon > 0$ and choose δ, $r \geqslant \delta > 0$, such that $h(S_r, S_{r-\delta}) < \epsilon/3$ (by Theorem 2). Choose N so that $i \geqslant N$ implies $r_i > r - \delta$ and $h((S_i)_r, S_r) < \epsilon/3$ (by Theorem 5). Then $i \geqslant N$ and $(S_i)_r \subseteq (S_i)_{r_i} \subseteq (S_i)_{r-\delta} \subseteq S_{r-\delta}$ imply

$$h((S_i)_{r_i}, S_{r-\delta}) \leqslant h((S_i)_r, S_{r-\delta}) \leqslant h((S_i)_r, S_r) + h(S_r, S_{r-\delta}) \leqslant 2\epsilon/3.$$

Thus $h((S_i)_{r_i}, S_r) \leqslant h((S_i)_{r_i}, S_{r-\delta}) + h(S_{r-\delta}, S_r) < \epsilon$.∎

The above argument easily extends to the case $r_i \to r < r_0$. The Theorem can fail when $r_i \uparrow r_0$ – consider S as a rectangle and let S_i be obtained by turning a long side inward by an angle θ_i, where $\theta_i \downarrow 0$. However, we have $(S_i)_{r_i} \subseteq S_{r_i} \subseteq S_{r_j}$ for all $i > j$. Hence any limiting set of $\{(S_i)_{r_i}\}$ lies in S_{r_j} for any j, hence in $\lim_j S_{r_j}$ which is S_{r_0} when $r_i \uparrow r_0$.

Now we return to the situation at the beginning of this Section, i.e. $T_i = (S_i)_{r_i}$ and $r_i = A(T_i)/P(T_i)$ is the maximal value of A/P for all closed subsets T of S_i. We have $r_i \uparrow r$. If $r < r_0$, then Theorem 6 gives us $T_i \to S_r$. If $r = r_0$, let V be a limit set of the sequence $\{T_i\}$. By restricting to a subsequence, we can assume $T_i \to V$ without changing our situation. We have $V \subseteq S_{r_0}$ by the previous paragraph.

Let W denote $\lim_i T_i$ in either case. We claim W gives the maximum of $A(T)/P(T)$ for all subsets T of S. We know it suffices to consider that T is convex, closed and with non-empty interior.

PROPOSITION 7. *For such T, we have $S_i \cap T \to T$.* (See (3, pp. 63–64).)

THEOREM 8. *$A(W)/P(W) = r$ is the maximum of $A(T)/P(T)$ over all subsets T of S.*

Proof. Since $T_i \to W$ and A and P are continuous with respect to the Hausdorff metric (3, pp. 87–88; 9, p. 147), we have $r_i = A(T_i)/P(T_i) \to A(W)/P(W)$, so $r_i \uparrow r$ yields $A(W)/P(W) = r$. We consider T to be convex, closed and with non-empty interior. Then $T \cap S_i \subseteq S_i$ gives $A(T \cap S_i)/P(T \cap S_i) \leqslant r_i$ and taking limits gives us

$$A(T)/P(T) \leqslant r.$$

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COROLLARY 8·1. *If $r = r_0$, then $V = S_{r_0}$, i.e. $\lim_i T_i = S_r$ holds for all values of r.*

Proof. When $r = r_0$, $W = V$ gives the maximal A/P for all subsets of S and V is closed. From the preliminaries in section 3, we have $V = S_s$ for some s, $r_0 \geqslant s \geqslant 0$. Hence $V \supseteq S_{r_0}$, but we have already seen that $V \subseteq S_{r_0}$. ∎

Unfortunately the maximum of $A(S_r)/P(S_r)$ can occur for more than one value of r. For example, if S is a circle of radius $r_0 = 1$, then all S_r are equal. The maximizing radius for a regular inscribed n-gon is

$$\frac{\cos \pi/n}{1 + \sqrt{\dfrac{\pi}{n \tan \pi/n}}},$$

which converges to $r = \frac{1}{2}$, giving the correct maximum of A/P and the unique r such that $A(S_r)/P(S_r) = r$, but only one of the possible values of r such that $A(S_r)/P(S_r)$ is maximal.

In our previous approach to the general case, we directly showed $dA(S_r)/dP(S_r) = r$ for $r_0 \geqslant r > r_1$, where r_1 is the greatest r such that $S_{r_1} = S$, by lengthy geometrical considerations. This result has been mostly replaced by the above results, but we still require it to show that there is a unique r such that $A(S_r)/P(S_r) = r$. We now sketch the proof.

When P_0 $(= P(S_{r_0})$, recall$) = P(S)$, then $S_{r_0} = S$ and $A(S)/P(S) = A(S_{r_0})/P(S_{r_0})$ has the value $r = r_0 - \pi r_0^2/(2\pi r_0 + 2a)$ for some $a \geqslant 0$. Hence $r_0 > r \geqslant r_0/2$ and $A(S_r)/P(S_r) = r$ holds for this unique value of r.

Define $R_i(P) = r$ to mean $P((S_i)_r) = P$. R_i is a strictly monotone decreasing, continuous function of P. Let r_1 be the greatest r such that $S_r = S$. Define $R(P) = r$ to mean $P(S_r) = P$, for $P < P(S)$ and set $R(P(S)) = r_1$. Then R is monotone decreasing and continuous. The sequence $\{R_i\}$ is monotone increasing for each P.

PROPOSITION 9. *For $P_0 < Q < P(S)$, $R_i \uparrow R$ on the interval (P_0, Q).*

COROLLARY 9·1. *Hence $R_i \to R$ uniformly on (P_0, Q).*

COROLLARY 9·2. *Let $A_i(P) = A((S_i)_{R_i(P)})$ and $A(P) = A(S_{R(P)})$. Then*

$$A_i \to A \text{ and } dA/dP = \lim_i dA_i/dP = \lim_i R_i = R, \quad \text{on } (P_0, Q).$$

Recasting this back into our earlier notation and letting $Q \to P(S)$, we have $dA(S_r)/dP(S_r) = r$ for $P_0 \leqslant P < P(S)$ or $r_0 \geqslant r > r_1$. Thus we can apply the Mean Value Theorem as in the polygonal case to deduce that there is at most one r, $r_0 \geqslant r \geqslant r_1$, such that $A(S_r)/P(S_r) = r$. We know that the value $r = \lim_i r_i$ has $A(S_r)/P(S_r) = r$. Whether $r \geqslant r_1$ or $r < r_1$, we easily see that this r is the unique value of $r, r_0 \geqslant r \geqslant 0$, such that $A(S_r)/P(S_r) = r$, although there may well be other values of r which give the same S_r.

Since $A(S_r)/P(S_r)$ is increasing at $r = r_0$, we will have the maximal

$$A/P > A(S_{r_0})/P(S_{r_0}) \geqslant r_0/2,$$

except when $r_1 = r_0$, i.e. $S = S_{r_0}$. This occurs if and only if S is a circle or a rectangle with rounded ends. From $A/P = r_0 - \pi r_0^2/(2\pi r_0 + 2a)$, we see that $A/P = r_0/2$ if and

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only if S is a circle of radius r_0. That is, the maximal A/P satisfies $r_0/2 \leqslant A/P < r_0$ with $r_0/2 = A/P$ if and only if S is a circle of radius r_0. Hence the ratio of the maximal A/P to $r_0/2$ (or of r_1 to r_0) is a measure of the circularity or roundness of S. Of course r_1 is just the infimum of the radius of curvature on S, but the maximal A/P does not seem to be related to other measures of roundness. 'Whether each circle is round is a question both deep and profound. In a paper of Erdös, which was written in Kurdish, a counterexample was found.' (7)

6. *Concluding remarks.* Consider again the case when S is a bounded convex polygon. Recall $S' = S'_r$ is the contact polygon generated by the sides of S which S_r touches. For S', we can solve equations (1) and $A = rP$ to get

$$r = \frac{2A(S')/P(S')}{1+\sqrt{(1-4k'A(S')/P(S')^2)}}, \tag{2}$$

where $k' = \Sigma_i \tan \psi'_i - \pi$ with ψ'_i being the angles of S'. (There is a second solution, which is irrelevant.) This gives $P = \sqrt{(P(S')^2 - 4k'A(S'))}$.

Now let r_0 be the inradius of S as before. If S_{r_0} is a circle, then it touches all the sides of S'_{r_0} and so S'_{r_0} has an inscribed circle of radius r_0. If S_{r_0} is not circular but is a rectangle with rounded ends, then S'_{r_0} does not have an inscribed circle but does have inradius r_0. Hence all contact polygons S' have the same inradius as S itself. The only case where S' has an inscribed circle is when S_{r_0} is circular and we consider $S'_{r_0} = S'$ in this case. Then $r_0 \Sigma_i \tan \psi'_i = P(S')/2 = A(S')/r_0$ and (2) simplifies to

$$r = \frac{r_0}{1+\sqrt{(2\pi r_0/P(S'))}} = \frac{r_0}{1+\sqrt{(\pi r_0^2/A(S'))}}. \tag{3}$$

This result is obtained by Herz and Kaapke (6), but in less reduced form. The expressions for r give simple expressions for P, as noted by Garvin (5),

$$P = \sqrt{(2\pi r_0 P(S'))} = \sqrt{(4\pi A(S'))}.$$

In particular, when S itself has an inscribed circle of radius r_0, then there is only one contact polygon $S' = S$ and (3) gives the maximizing r for S. Then the expression for P shows the maximal A/P occurs when $P = P(S_r)$ is the geometric mean of the perimeters of S and of its inscribed circle. This analysis directly applies to all triangles S.

We now consider only polygons S with an inscribed circle of radius 1. Thus the maximal A/P for subsets of S is given by (3) with $r_0 = 1$ and $S' = S$. We let

$$Z(S) = 2A/P = 2r = 2(1+\sqrt{\{\pi/A(S)\}})^{-1}$$

be twice the maximal A/P. Since $A(S) \geqslant \pi$, we have $2 > Z(S) \geqslant 1$. The extreme values of $Z(S)$ correspond to the extreme values of $A(S)$.

Consider the class of n-gons circumscribed about a circle of radius 1. In this class, $A(S)$ is unbounded, so $Z(S)$ approaches 2. But $A(S)$ is minimal when S is regular and then $A(S) = n \tan \pi/n$. Some of the corresponding minimal values Z_n are given in Table 1.

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Table 1

n	3	4	5	6	7
Z_n	1·12514	1·06032	1·03630	1·02443	1·01762
n	8	9	10	20	50
Z_n	1·01333	1·01045	1·00842	1·00207	1·00033

Note that $Z_n \downarrow 1$. It seems clear that Z_n gives the minimum of $Z(S)$ for all n-gons with inradius $r_0 = 1$.

Fejes Tóth (11) considers a similar problem. Consider the class of all n-gons having a given area. Of all the convex regions T lying in these n-gons and with $P(T) = P$ given, he shows that the maximum area occurs for the S_r with $P(S_r) = P$, in the regular n-gon S. Hence the maximum of the maximal values of A/P for n-gons occurs for the maximizing S_r in the regular n-gon. This probably also holds if we take the class of n-gons having a given perimeter.

For figures other than polygons or polygons with rounded corners, it is difficult to determine the maximizing r explicitly. We tried the case when S was a semicircle of radius 1. We numerically found that the maximum occurs for $r = A/P = 0{\cdot}3170$, $A = 1{\cdot}4863$, $P = 4{\cdot}6881$.

We have considered the problem of extending these results to three dimensions. At first we thought that the subset which maximized the ratio of volume to surface area would be the union of all solid balls of a given radius lying in our given set S, i.e. it would be the set S rounded off in the obvious (?) way. However, the corresponding basic isoperimetric fact for surfaces is that, under appropriate conditions, the surface of given area, passing through a given boundary and maximizing an appropriate volume, is a surface of constant mean curvature. Hence the obvious rounding of a cube, with spherical corners and cylindrical edges, cannot be a maximizing figure. Herz and Kaapke (6) have explicitly verified this by showing that the obvious rounding of a cylinder does not give a maximizing figure. Finding the maximizing figure for the cylinder should be a straightforward calculus of variations problem. It seems clear that a figure T in S which maximizes volume for a given surface area must have each component of $\delta T \sim \delta S$ being of constant mean curvature, that this curvature will be the same for all components and will be an upper bound on the mean curvature on δT, but we do not have proofs.

D. Larman has asked about the dual problem – minimize A/P over all supersets of a convex plane region S. Clearly we can construct a tapering figure of arbitrarily large P and bounded A which contains S, so we can make A/P approach 0. More interestingly, we can look for the minimum A/P for convex supersets of S. If r_0 is the inradius of S, then the inradius of each superset is $\geqslant r_0$. Thus $A \geqslant r_0 P/2$ for any convex superset and so the minimal A/P is $\geqslant r_0/2$. This minimum can be actually achieved by taking the superset generated by the tangents to an incircle of S at the points where it meets S. When this superset is not bounded (e.g. for a rectangle), we can only approach $r_0/2$. This result readily generalizes to non-convex S, for each convex superset of S contains

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its convex hull S', so we can simply let r_0 be the inradius of S'. Further, in happy contrast to the maximization problem this result generalizes immediately to n dimensions – we then get r_0/n instead of $r_0/2$.

We are indebted to the following for discussions or letters which have been of help to us: E. Bombieri, S. Collings, A. Garvin, D. G. Larman, P. McMullen, J. Paul, C. A. Rogers, P. R. Scott, M. Silver, E. G. Straus, J. Wills.

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Addendum. Tung-Po Lin in 'Maximum area under constraint', *Math. Mag.* **50** (1977), 32–34, has recently obtained the results for the case of a polygon circumscribed about a circle – as had Herz and Kaapke (6).

At the end of section 5, we saw that the maximal A/P is less than the inradius r_0. Hence $A(S)/P(S) < r_0$. We now consider only the whole set S, letting $A = A(S)$, $P = P(S)$. The result $A/P < r_0$ is obtainable from one of Bonnesen's inequalities (3, pp. 107–111): $r_0 P \geqslant A + \pi r_0^2$. $A/P = r_0$ holds only for extended cases such as an infinite strip. The n-dimensional version $V/S < r_0$ was shown by Jorg M. Wills in 'Zum Verhältnis von Volumen zu Oberfläche bei konvexen Körpern', *Arch. Math. (Basel)* **21** (1970), 557–560. Osserman (*Bonnesen-style inequalities*, Warwick 1977) shows that this and Wills' conjecture that $r_0 S \geqslant V + (n-1)V_i$, where V_i is the volume of the insphere, follow from Dinghas' inequality – H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie* (Berlin: Springer, 1957, p. 269).

The problem of maximizing V/S for surfaces in a cylinder leads to a surface of constant mean curvature which is a surface of revolution. Osserman informs us that these were considered by Delaunay and there is a discussion of them in D'Arcy Thompson, *On growth and form*, Abridged Edition, Cambridge University Press (1971), pp. 53–61. It would appear that the desired curve must be a part of the nodoid (or hyperbolic catenary), but we hope to investigate this further in the future.

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