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The Radon–Nikodym problem for approximately proper equivalence relations

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Abstract. We study the Radon–Nikodym problem for approximately proper equivalence relations and more specifically the uniqueness of certain Gibbs states. One of our tools is a variant of the dimension group introduced in the study of approximately finite-dimensional (AF) algebras. As applications, we retrieve sufficient conditions for the uniqueness of traces on AF algebras and parts of the Perron–Frobenius–Ruelle theorem.

1. *Introduction*

The motivation of this work is a problem in the theory of C^*-algebras, namely the study of the Kubo–Martin–Schwinger (KMS) states of some automorphism groups of the Cuntz algebras and their generalizations studied in [**12**]. While the crucial role of the Perron–Frobenius theorem in this problem has been noticed from the beginning (see, for example, [**8, 9**]), the application of Ruelle's version of this theorem is more recent [**11, 22, 24, 28**]. The existing proofs of the Perron–Frobenius–Ruelle theorem, in particular [**15, 33**], heavily rely on the use of a sequence of expectations associated with the asymptotic algebra. The purpose of this work is to use the formalism of groupoids (cf. [**27**]), approximately proper equivalence relations and dimension groups (cf. [**19**]) to provide a convenient setting for these proofs. A non-commutative version of approximately proper (AP) equivalence relations appears in [**13**] with a similar motivation.

Given a groupoid G on a space X and a cocycle $D \in Z^1(G, \mathbb{R}_+^*)$, the Radon–Nikodym problem mentioned in the title is the study of the probability measures on X which are quasi-invariant with respect to G (the definition is recalled in §3) and which admit this cocycle as the Radon–Nikodym derivative. When $G = R$ is an AP equivalence relation, i.e. an increasing union of proper equivalence relations R_n, the cocycle defines a sequence of expectations E_n with range $C(X/R_n)$. The solutions of the Radon–Nikodym problem are exactly the measures which factor through E_n for all n. A classical example of this situation is provided by statistical mechanics on a lattice Λ. The sequence (R_n) is defined by an increasing sequence of finite sets Λ_n with union Λ. The cocycle is the energy

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cocycle, as in [**27**, §II.5]. In this setting, the solutions of the Radon–Nikodym problem are called Gibbs states (this definition was introduced by Capocaccia in [**5**]).

Our main concern is the uniqueness of the solution of the Radon–Nikodym problem on an AP equivalence relation. We give sufficient conditions for uniqueness in two cases. First we consider quasi-product cocycles on AF equivalence relations. The data consist of a Bratteli diagram and a labeling of its edges. Then a convenient condition on this labeling (Corollary 4.3(ii)) guarantees uniqueness. Our result covers (and was inspired by) Török's work [**29**] (pointed out to me by Bratteli) on the uniqueness of traces on AF C^*-algebras. The second case is the classical setting of Ruelle's Perron–Frobenius theorem and gives the part of Walters' Theorem 2.16 in [**33**] concerning the transpose of the Ruelle operator (Theorem 6.1 and Proposition 7.2). The eigenvalue problem for this operator amounts to a Radon–Nikodym problem on the semi-direct groupoid $G(X, T)$. Our method, outlined in §4.1 of [**28**], is first to solve the Radon–Nikodym problem on the asymptotic equivalence relation $R(X, T)$ which is approximately proper. If it has a unique solution, this is also a solution of the initial problem.

This work is organized as follows. The definition and some examples of AP equivalence relations are given in §2. Section 3 deals with cocycles. We are only concerned here with cocycles with values in the multiplicative group $\mathbb{R}_+^*$ of strictly positive real numbers. Let D be a cocycle defined on the AP equivalence relation $R = \bigcup R_n$ on the compact space X. Then its restriction to R_n can be written as a coboundary. This provides a normalized potential ρ_n and an expectation E_n from $C(X)$ to $C(X/R_n)$, where $C(X)$ denotes the space of real-valued continuous functions on the compact space X. This is the sequence of expectations mentioned earlier. The main result is Theorem 3.10, which characterizes unique ergodicity of D in terms of the additive and multiplicative variations of the sequence $(E_n(f))$. The theory of dimension groups (this means here inductive limits of $C(X_n)$, viewed as ordered vector spaces, under positive linear maps), as developed by Goodearl in [**19**], is well suited to our problem. Indeed, the solutions of the Radon–Nikodym problem are the states of the dimension group associated to the sequence $(E_{n,n-1})$ (where $E_n = E_{n,n-1} \circ E_{n-1}$). Therefore, we recall some results of this theory in Appendix A. One can consider sequences of positive maps which do not preserve the units, for example the stationary sequence $(A_n = A)$, where A is a positive $d \times d$ matrix or the Ruelle operator $\mathcal{L}_g$ (see §7). When the vector spaces $C(X_n)$ have finite dimension, there is a straightforward sharp estimate of the rate at which Markovian (i.e. unital) operators contract the variation (Lemma A.5) (this is an improvement of an estimate given by Török in [**29**]). This contraction rate admits a simple estimate in terms of non-unital positive linear maps (Lemma A.6). This gives an elementary proof of the Perron–Frobenius theorem for primitive matrices (Example A.1). This estimate is also used in §4 devoted to quasi-product cocycles on AF equivalence relations. This section is very close to [**10**, §§3 and 4], where the emphasis was on measures rather than on cocycles. The main result is Corollary 4.3, which gives a sufficient condition for unique ergodicity as mentioned earlier. Section 5 deals with the general framework of continuous cocycles on AP equivalence relations. When X is a compact space, we find it to be more convenient to use the entourages associated to finite open covers rather than those defined by a compatible metric. The main result is Theorem 5.6. Section 6 studies the case of a

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stationary system defined by a single surjective local homeomorphism $T : X \to X$, where X is compact. The main result (Theorem 6.1) is known: it gives the unique ergodicity of the cocycle D defined by a potential $g \in C(X, \mathbb{R}_+^*)$ under the usual assumptions on the dynamical system (T is assumed to be expansive and $R(X, T)$ to be minimal) and on the potential (Bowen's condition). The elementary proof is essentially Walters' in [**33**]. In §7, these results are applied to the transfer operator.

2. *AP equivalence relations*

In this section, X is a locally compact second countable Hausdorff space and R is an equivalence relation on X. For the sake of simplicity, we only consider equivalence relations with countable equivalence classes. We also denote by $R \subset X \times X$ its graph.

Definition 2.1. The equivalence relation R on X will be called *proper and étale* if its quotient space is Hausdorff and its quotient map $\pi : X \to X/R$ is a local homeomorphism.

Endowed with the product topology of $X \times X$, R is a locally compact groupoid which is étale. This simply means that the projection maps $r, s : R \to X$ are local homeomorphisms. Every open cover $\mathcal{U}$ of X by open sections of π gives a cover $\{(U \times V) \cap R : U, V \in \mathcal{U}\}$ of R by open bisections (i.e. simultaneous sections of the projection maps r, s). The construction of [**27**] applies and gives the C^*-algebra $C^*(R)$. Moreover this groupoid is proper in the sense of [**1**]. It is known, (see [**26**]) that $C^*(R)$ has continuous trace and that it is Rieffel-Morita equivalent to $C_0(X/R)$. Since our study is limited to étale equivalence relations and groupoids, we shall often omit the word étale and say 'proper' rather than 'proper and étale'.

Definition 2.2. The equivalence relation R on X will be called *AP* if there exists a sequence

$$X_0 \xrightarrow{\pi_{1,0}} X_1 \to \cdots \to X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \to \cdots$$

where $X_0 = X$ and for each $n \geq 1$, X_n is a Hausdorff space and $\pi_{n,n-1}$ is a surjective local homeomorphism such that

$$R = \{(x, y) \in X \times X : \exists n \in \mathbb{N} : \pi_n(x) = \pi_n(y)\}$$

where $\pi_n = \pi_{n,n-1} \circ \pi_{n-1,n-2} \circ \cdots \circ \pi_{1,0}$.

In the context of the above definition, we let R_n be the equivalence relation on X defined by π_n.

PROPOSITION 2.1. *Let R be the equivalence relation defined by the sequence (π_n) as above.*

(i) *(R_n) is an increasing sequence of subsets of $X \times X$ and $R = \bigcup R_n$.*

(ii) *For $m < n$, R_m is a closed and open subset of R_n.*

(iii) *Endowed with the inductive limit topology, R is an étale locally compact groupoid.*

Proof. The assertion (i) is obvious. For the assertion (ii), we introduce the equivalence relation S on X_m defined by the map $\pi_{n,m} = \pi_{n,n-1} \circ \cdots \circ \pi_{m+1,m}$. The diagonal Δ_{X_m} is closed and open in S. Therefore, $R_m = (\pi_m \times \pi_m)^{-1}(\Delta_{X_m})$ is open and closed in $R_n = (\pi_m \times \pi_m)^{-1}(S)$. For the assertion (iii), according to [**27**], it suffices to construct

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a cover of R consisting of open locally compact bisections. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be a cover of X. Then, the family $\{(U \times V) \cap R_n\}$, where $U, V \in \mathcal{U}_n$, is an open cover of R_n and the union of these covers is the desired cover of R. □

COROLLARY 2.2. *Let $R = \bigcup R_n$ be an AP equivalence relation as above. Then, $C^*(R)$ is the inductive limit of the $(C^*(R_n))$. More precisely, $C^*(R_n)$ can be identified with a sub C^*-algebra; these subalgebras are increasing and their union is dense.*

On the other hand, for all $n \in \mathbb{N}$, $C(X_n)$ can be identified with the subalgebra $(\pi_n^* C(X_n))$ of $C(X)$ and this sequence of subalgebras is decreasing. Its intersection is the fixed point subalgebra, as defined below.

Definition 2.3. Let (X, R) be an étale equivalence relation. The *fixed point subalgebra* is the subalgebra of $C(X)$:

$$C(X)^R = \{f \in C(X) : (x, y) \in R \Rightarrow f(x) = f(y)\}.$$

Its spectrum will be denoted by X_∞. One says that (X, R) is *irreducible* if $C(X)^R$ consists only of constant functions.

Thus, when (X, R) is an AP equivalence relation defined by a sequence (X_n), the fixed point subalgebra $C(X)^R$ is the intersection of the $\pi_n^*(C(X_n))$. The inclusion $C(X)^R \subset \pi_n^* C(X_n)$ defines a continuous surjective map $\pi_{\infty,n} : X_n \to X_\infty$. In particular, we write $\pi_\infty = \pi_{\infty,0} : X \to X_\infty$. These maps identify X_∞ to the inductive limit of the sequence $(\pi_{n,n-1} : X_{n-1} \to X_n)$. In the irreducible case, this space reduces to a point while the equivalence relation R is non-trivial.

Example 2.1. (AF equivalence relations; see [**28, 18**]) Let (V, E) be a Bratteli diagram. Recall that this means an oriented graph, where the vertices are stacked on levels $n = 0, 1, 2, \ldots$ and the edges run from a vertex of level $n-1$ to a vertex of level n. We denote by $V(n)$ the set of vertices of the level n and by $E(n)$ the set of edges from level $n-1$ to level n. We assume that every vertex v emits finitely many, but at least one, edges and that every vertex on a level $n \geq 1$ receives at least one edge. An infinite path is a sequence of connected edges $x = x_1 x_2 \ldots$, where x_1 starts from level 0. The space X of infinite paths has a natural totally disconnected topology, with the cylinder sets $Z(x_1 x_2 \ldots x_n)$ as a basis. We similarly define the space X_n of infinite paths starting from level n and we have the obvious projection maps $\pi_{n,n-1} : X_{n-1} \to X_n$. This defines an AP equivalence relation R on X called the tail equivalence relation of the Bratteli diagram. In the sequel, following [**18**], we use the notation $(X = X(V, E), R = R(V, E))$ to designate this AP equivalence relation; we call it the tail equivalence relation of the Bratteli diagram (V, E). An explicit construction of matrix units in $C^*(R)$ shows that it is an AF C^*-algebra admitting (V, E) as the Bratteli diagram. Conversely, it was shown in [**18**, Theorem 3.9] (and in [**28**, Theorem 3.1] when the space X is compact) that every AP equivalence relation on a space X which is locally compact and totally disconnected is the tail equivalence relation of a Bratteli diagram. Such an equivalence relation is called an AF equivalence relation. Properties of AF equivalence relations were studied in [**18**], where an equivalent definition was used.

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Example 2.2. (Infinite lattices; see [**27**, §III.1.6]) The following equivalence relation, which is an example of an AF equivalence relation as above, is relevant to statistical mechanics. Let $\Lambda = \mathbb{Z}^d$ be a lattice and let F be a finite set. On the compact space $X = F^\Lambda$, we define the equivalence relation $(x, y) \in R$ if and only if x and y agree outside a finite subset of Λ. If we choose an increasing sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of finite subsets of Λ such that $\Lambda_0 = \emptyset$ and $\bigcup \Lambda_n = \Lambda$, then R is the equivalence relation defined by the sequence $(\pi_{n,n-1} : X_{n-1} \to X_n)$, where $X_n = X^{\Lambda\setminus\Lambda_n}$ and $\pi_{n,n-1}$ is the projection. This equivalence relation is minimal in the sense that all of its equivalence classes are dense.

Example 2.3. (Stationary equivalence relations) Let X be a locally compact space and $T : X \to X$ a local homeomorphism which is onto. We form the stationary sequence

$$X \xrightarrow{T} X \xrightarrow{T} X \xrightarrow{T} \cdots .$$

The corresponding AP equivalence relation

$$R = \{(x, y) \in X \times X : \exists n \in \mathbb{N} : T^n x = T^n y\}$$

plays an essential role in the study of T when it is large enough. This example is developed later.

The definition of an AP equivalence relation makes an implicit reference to a defining sequence

$$X_0 \xrightarrow{\pi_{1,0}} X_1 \to \cdots \to X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \to \cdots .$$

Definition 2.4. We say that two sequences

$$X_0 \to X_1 \to \cdots \to X_m \to \cdots$$
$$Y_0 \to Y_1 \to \cdots \to Y_n \to \cdots$$

as in Definition 2.2 are equivalent if there are subsequences (m_k) and (n_k) and surjective local homeomorphisms $X_{m_k} \to Y_{n_k}$ and $Y_{n_k} \to X_{m_{k+1}}$ making commutative diagrams.

This is an equivalence relation. Another way to obtain the same definition is to define first the contraction of a sequence (X_n): it is the new sequence $(\underline{X}_k)$ defined by a subsequence (n_k); explicitly, $\underline{X}_k = X_{n_k}$ and $\underline{\pi}_{k,k-1} = \pi_{n_k,n_{k-1}}$. Then we say that the original sequence is a dilation of the new sequence. Two sequences are equivalent if and only if they admit contractions which have a common dilation.

PROPOSITION 2.3. *Let X be a locally compact space.*

(i) *Equivalent defining sequences (X_n) and (Y_n) as above with $X_0 = Y_0 = X$ define the same equivalence relation R on X and the same topology on R.*

(ii) *Conversely, if X is compact, two sequences (X_n) and (Y_n) as above with $X_0 = Y_0 = X$ which define the same equivalence relation R are equivalent.*

Proof. Let us call (R_n) (respectively (S_n)) the sequence of equivalence relations on X defined by (X_n) (respectively (Y_n)). If (X_n) and (Y_n) are equivalent and if we have

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subsequences (m_k) and (n_k) as in the definition, we have the inclusions $R_{m_k} \subset S_{n_k} \subset R_{m_{k+1}}$ for all k. We also know from Proposition 2.4(ii) that these inclusion maps are open. Therefore, the sequences (R_n) and (S_n) have the same union R and the inductive limit topology is the same. Suppose now that X is compact and that the sequences (R_n) and (S_n) have the same union R. Let us endow R with the inductive limit topology of the (S_n). Let us fix m. Since R_m is closed in $X \times X$, it is a compact subset of R. Since it is covered by the union of the open sets S_n, it is contained in some S_n. Similarly, any S_n is contained in some R_m. Therefore, one can construct subsequences (m_k) and (n_k) such that $R_{m_k} \subset S_{n_k} \subset R_{m_{k+1}}$ for all k. These inclusions give the desired maps $X_{m_k} \to Y_{n_k}$ and $Y_{n_k} \to X_{m_{k+1}}$. □

3. *Cocycles*

We only consider cocycles with values in the group $A = \mathbb{R}$ or equivalently $A = \mathbb{R}_+^*$. If R is an equivalence relation on X, a cocycle with values in A is a map $c : R \to A$ satisfying $c(x, y) + c(y, z) = c(x, z)$ for all $(x, y), (y, z) \in R$. A cocycle c is a coboundary if there is a map $b : X \to A$, called a *potential*, such that $c(x, y) = b(x) - b(y)$. Two cocycles are cohomologous if their difference is a coboundary. In our setting, R is a topological groupoid, A is a topological group and we assume that c and b as above are continuous. We denote by $Z^1(R, A)$ the set of continuous A-valued cocycles. We use the convention $c \in Z^1(R, \mathbb{R})$ and $D \in Z^1(R, \mathbb{R}_+^*)$.

One similarly defines cocycles on arbitrary groupoids. Let G be a locally compact groupoid with a continuous Haar system λ (cf. [**27**, Ch. 1]). A measure μ on its unit space $X = G^{(0)}$ is said to be quasi-invariant if the measures $\mu \circ \lambda$ and $(\mu \circ \lambda)^{-1}$ on G are equivalent. Then, the Radon–Nikodym derivative $D_\mu = d(\mu \circ \lambda)/d(\mu \circ \lambda)^{-1}$ satisfies the cocycle identity almost everywhere. When G is an étale groupoid, for example a proper and étale equivalence relation or an AP equivalence relation as above, it carries the canonical continuous Haar system consisting of counting measures λ^x on the fibers $G^x = r^{-1}(x)$ of the range map r. A measure μ on $X = G^{(0)}$ is quasi-invariant if and only if for all open bisections $S \subset G$, the measures $\sigma(S)_*(\mu_{|s(S)})$ and $\mu_{|r(S)}$ are equivalent, where $\sigma(S) : s(S) \to r(S)$, such that $\sigma(S)(x) = r(Sx)$, is the map induced by S. Moreover $d\sigma(S)_*\mu(y) = D_\mu^{-1}(yS)\, d\mu(y)$.

The *Radon–Nikodym problem* for $D \in Z^1(G, \mathbb{R}_+^*)$, where G is a locally compact groupoid with Haar system λ and compact unit space X, is to determine the set $S_{G,D}(X)$ (or $S_D(X)$ if there is no ambiguity on G) of probability measures μ on X which are quasi-invariant and admit D as their Radon–Nikodym derivative. We denote by $S(X)$ the set of probability measures on X.

3.1. *Cocycles on proper equivalence relations.*

Let us first look at cocycles on proper equivalence relations.

PROPOSITION 3.1. *Let R be a proper equivalence relation on X defined by $\pi : X \to \Omega$ and let $c \in Z^1(R, A)$, where $A = \mathbb{R}$ or $\mathbb{R}_+^*$. Then:*

(i) *c is a coboundary;*

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(ii) *if b and b' are two potentials for c, they differ by a function of the form $h \circ \pi$, where $h : \Omega \to A$ is continuous;*

(iii) *if X is compact and $D \in Z^1(R, \mathbb{R}_+^*)$, there is a unique potential $\rho \in C(X, \mathbb{R}_+^*)$ such that $\sum_\omega \rho(x) = 1$ for all $\omega \in \Omega$.*

Proof. For (i) and (ii), we assume that $A = \mathbb{R}$. To prove the first assertion, we choose a locally finite open cover $\{V_j\}$ of Ω, continuous sections $\sigma_j : V_j \to X$ of π and a partition of unity $\{h_j\}$ subordinate to $\{V_j\}$. We define $b_j : \pi^{-1}(V_j) \to \mathbb{R}$ by $b_j(x) = c(x, \sigma_j(\pi(x)))$ and $b : X \to \mathbb{R}$ by $b(x) = \sum_j h_j(\pi(x)) b_j(x)$. Then $c(x, y) = b(x) - b(y)$. For the second assertion, we notice that if b and b' satisfy $b'(x) - b'(y) = b(x) - b(y)$ for all $(x, y) \in R$, then $b' - b$ is constant on the equivalence classes, hence of the form $h \circ \pi$. For (iii), we pick an arbitrary potential $g \in C(X, \mathbb{R}_+^*)$ and define $Z(\omega) = \sum_\omega g(x)$. Then $Z \in C(\Omega, \mathbb{R}_+^*)$ and $\rho = g/Z \circ \pi$ is the desired potential. Because of (ii), two potentials satisfying this normalization agree. □

Therefore, via the introduction of this normalized potential, we relate a cocycle D with a family of probability measures along the fibers of the map $\pi : X \to \Omega$ or equivalently, a conditional expectation as follows.

Definition 3.1. Given a proper equivalence relation R on a compact space X and $D \in Z^1(R, \mathbb{R}_+^*)$, the potential $\rho = \rho_D \in C(X, \mathbb{R}_+^*)$ satisfying the normalization $\sum_\omega \rho(x) = 1$ for all $\omega \in \Omega$ is called the *normalized potential* of D (we always reserve the letter ρ for it). It defines a Markovian (Definition A.1) operator $E = E_D : C(X) \to C(\Omega)$ according to $E(f)(\omega) = \sum_\omega \rho(x) f(x)$, called the *expectation* relative to D.

Remark 3.1.

- Our definition carries an abuse of language. It is not E itself but $\pi^* \circ E$ which is a conditional expectation from $C(X)$ to $C(X)$.
- The following observation is due to R. Exel. The above expectation E has finite index. Conversely, given a continuous surjection $\pi : X \to \Omega$, there exists a conditional expectation $E : C(X) \to C(\Omega)$ of finite index if and only if π is a local homeomorphism. Then, there exists $D \in Z^1(R, \mathbb{R}_+^*)$ such that $E = E_D$.

The Radon–Nikodym problem is easily solved for proper equivalence relations. The following proposition states that the solutions are the measures on X admitting E_D as conditional expectation.

PROPOSITION 3.2. *Let R be a proper equivalence relation on a compact space X and $D \in Z^1(R, \mathbb{R}_+^*)$. Then*

$$S_D(X) = \{\Lambda \circ E_D : \Lambda \in S(\Omega)\},$$

where E_D is the expectation relative to D. More precisely, $E_D^* : C(\Omega)^* \to C(X)^*$ identifies $S_D(X)$ and $S(\Omega)$ as compact convex sets.

Proof. Let us first show that, for $\Lambda \in S(\Omega)$, $\mu = \Lambda \circ E \in S_D(X)$. We denote by α the system of counting measures on the fibers of $\pi : X \to \Omega$, by λ_r the system of counting measures on the fibers of $r : R \to X$ (the first projection) and by λ_s the system

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of counting measures on the fibers of $s : R \to X$ (the second projection). Then, we have for $f \in C_c(R)$,

$$\mu \circ \lambda_r(f) = \Lambda \circ \alpha(\rho\lambda_r(f)) = \Lambda \circ \alpha \circ \lambda_r((\rho \circ r)f),$$

$$\mu \circ \lambda_s(f) = \Lambda \circ \alpha(\rho\lambda_s(f)) = \Lambda \circ \alpha \circ \lambda_s((\rho \circ s)f).$$

The obvious equality $\alpha \circ \lambda_r = \alpha \circ \lambda_s$ gives the result. Conversely, let us suppose that $\mu \in S_D(X)$. Then, for every $f \in C(X \times X)$, we have

$$\int \sum_{\pi(y)=\pi(x)} f(x, y)\, d\mu(x) = \int \sum_{\pi(x)=\pi(y)} f(x, y) D(x, y)\, d\mu(y).$$

If we choose $f(x, y) = \rho(y) f(y)$, where $f \in C(X)$, we get

$$\int \sum_{\pi(y)=\pi(x)} \rho(y) f(y)\, d\mu(x) = \int f(y)\, d\mu(y),$$

which says that $\pi_*\mu \circ E = \mu$. □

LEMMA 3.3. *Let $X \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_{2,1}} X_2$ be surjective local homeomorphisms. Let R_1 (respectively R_2) be the equivalence relation defined by π_1 (respectively $\pi_2 = \pi_{2,1} \circ \pi_1$). Let $D_2 \in Z^1(R_2, \mathbb{R}_+^*)$ with potential $g_2 \in C(X, \mathbb{R}_+^*)$ and $D_1 = D_{|R_1}$ with potential g_1. Then, there exists a unique $g_{2,1} \in C(X_1, \mathbb{R}_+^*)$ such that $g_2 = g_1(g_{2,1} \circ \pi_1)$. Moreover, if g_2 and g_1 are normalized, then so is $g_{2,1}$, i.e. $\sum_{\pi_{2,1}(x_1)=x_2} g_{2,1}(x_1) = 1$ for all $x_2 \in X_2$.*

Proof. The uniqueness of $g_{2,1}$ results from the surjectivity of $\pi_{2,1}$. Since g_2 is also a potential for D_1, its existence results from Proposition 3.1(ii). Let us assume that g_2 and g_1 are normalized. Then for $x_2 \in X_2$, we have

$$\begin{aligned} 1 &= \sum_{\pi_2(x)=x_2} g_2(x) \\ &= \sum_{\pi_{2,1}(x_1)=x_2} \sum_{\pi_1(x)=x_1} g_1(x) g_{2,1} \circ \pi_1(x) \\ &= \sum_{\pi_{2,1}(x_1)=x_2} g_{2,1}(x_1) \sum_{\pi_1(x)=x_1} g_1(x) \\ &= \sum_{\pi_{2,1}(x_1)=x_2} g_{2,1}(x_1). \end{aligned}$$

□

3.2. *Cocycles on AP equivalence relation.* Let us consider now the case of an AP equivalence relation R on a compact space X with a defining sequence (X_n). We use the notation of Definition 2.2; in particular, we denote by R_n the proper equivalence relation defined by $\pi_n : X \to X_n$. Given $D \in Z^1(R, \mathbb{R}_+^*)$, we consider the sequence of its restrictions $D_n = D_{|R_n} \in Z^1(R_n, \mathbb{R}_+^*)$. For each n, we pick a potential g_n for D_n. From Lemma 3.3, we obtain $g_{n,n-1} \in C(X_{n-1}, \mathbb{R}_+^*)$ such that $g_n = g_{n-1}(g_{n,n-1} \circ \pi_{n-1})$. Equivalently, $g_n(x)/g_{n-1}(x)$ depends only on $\pi_{n-1}(x)$. In particular, we can choose the sequence of normalized potentials; in that case, we write ρ_n instead of g_n and $\rho_{n,n-1}$ is also normalized.

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Definition 3.2. Let R be an AP equivalence relation on a compact space X defined by a sequence (X_n).

(i) A *compatible sequence of potentials* is a sequence $(g_n \in C(X, \mathbb{R}_+^*))$ such that $g_n(x)/g_{n-1}(x)$ depends only on $\pi_{n-1}(x)$. Equivalently, a compatible sequence of potentials is defined by an initial potential $g_0 \in C(X, \mathbb{R}_+^*)$ and a sequence of local potentials $g_{n,n-1} \in C(X_{n-1}, \mathbb{R}_+^*)$; then g_n is given by

$$g_n = (g_0)(g_{1,0})(g_{2,1} \circ \pi_1) \dots (g_{n,n-1} \circ \pi_{n-1}).$$

If for all n and for all $x_n \in X_n$, $\sum_{x_n} g_n(x) = 1$, we speak of a sequence of normalized potentials and we write ρ_n rather than g_n.

(ii) Given $D \in Z^1(R, \mathbb{R}_+^*)$, a sequence of potentials g_n of $D_n = D_{|R_n}$ is called a sequence of *potentials* for D. The associated sequence of functions $(g_{n,n-1})$ is called a sequence of *local potentials* for D. In particular, we can consider the sequence of *normalized potentials* (ρ_n) for D.

Note that a compatible sequence of potentials (g_n) defines a cocycle $D \in Z^1(R, \mathbb{R}_+^*)$ such that $D(x, y) = g_n(x)/g_n(y)$ for $(x, y) \in R_n$ and that every cocycle $D \in Z^1(R, \mathbb{R}_+^*)$ is defined by a compatible sequence of potentials.

Remark 3.2. The sequence of normalized potentials for a cocycle $D \in Z^1(R, \mathbb{R}_+^*)$ is unique. However, it is sometimes advantageous to consider compatible sequences of unnormalized potentials. For example, a change of initial potential amounts to replacing the cocycle by a cohomologous cocycle. Although we restrict the discussion in the following to normalized potentials, the theory of dimension groups and their state space also applies to the general case.

PROPOSITION 3.4. *The correspondence which associates to a cocycle D its sequence of normalized potentials (ρ_n) (respectively its sequence of normalized local potentials $(\rho_{n,n-1})$) is a bijection from $Z^1(R, \mathbb{R}_+^*)$ onto the set of compatible sequences of normalized potentials (respectively onto the set of sequences of normalized local potentials).*

The normalized potentials ρ_n define conditional expectations $E_n : C(X) \to C(X_n)$ and the normalized local potentials $(\rho_{n,n-1})$ define conditional expectations

$$E_{n,n-1} : C(X_{n-1}) \to C(X_n).$$

For $m \leq n$, we set

$$E_{n,m} = E_{n,n-1} E_{n-1,n-2} \dots E_{m+1,m}.$$

Then, we have $E_n = E_{n,0}$.

PROPOSITION 3.5. *Let $R = \bigcup R_n$ be an AP equivalence relation on a compact space X, $D \in Z^1(R, \mathbb{R}_+^*)$ and $D_n = D_{|R_n} \in Z^1(R_n, \mathbb{R}_+^*)$ as above. Then, the sequence of conditional expectations $(P_n = \pi_n^* E_n)$ is a (reversed) martingale, i.e. it satisfies $P_m P_n = P_n P_m = P_n$ for $m \leq n$.*

Proof. This is an immediate consequence from the fact that $E_m \pi_m^*$ is the identity map of $C(X_m)$ and of the factoriations $\pi_n = \pi_{n,m} \circ \pi_m$ and $E_n = E_{n,m} E_m$. □

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Thus, we obtain a sequence of Markovian operators

$$\underline{E} = (E_{n,n-1} : C(X_{n-1}) \to C(X_n))$$

and we can apply the results of Appendix A. We denote by $\mathcal{E} = \mathcal{E}(\underline{E})$ its dimension group and by $S = S(\underline{E})$ its state space. Recall that it is a non-empty compact convex Choquet simplex. We denote by $E_{\infty,n} : C(X_n) \to \mathcal{E}$ the canonical morphisms; they are surjective. In particular, we write $E_\infty = E_{\infty,0} : C(X) \to \mathcal{E}$. We have a convenient description of the state space $S(\underline{E})$. Indeed, given a reversed martingale (P_n), Brown and Dooley defined in [**4**] (see also [**17**, §1.2]) a G-measure as a probability measure μ on X satisfying $\mu = (P_n)^*\mu$ for all n. On the other hand, by Definition A.2, $S(\underline{E})$ is the set of sequences (μ_n), where $\mu_n \in S(X_n)$ and $\mu_{n-1} = \mu_n \circ E_{n,n-1}$. For such a sequence, $\mu = \mu_0 = E_n^*\mu_n$ does not depend on n and is a G-measure. Conversely, one recovers the sequence (μ_n) from the G-measure μ by setting $\mu_n = (\pi_n)_*\mu$. Thus we have the following.

LEMMA 3.6. *The map $(\mu_n) \mapsto \mu_0$ identifies the state space $S(\underline{E})$ to the set of G-measures.*

Moreover, we have seen previously that the measures of the form $E_n^*\mu_n$, where $\mu_n \in S(X_n)$, are exactly the probability measures admitting D_n as Radon–Nikodym derivative. This gives the following proposition. In the context of equilibrium states in statistical mechanics, the equivalence of the definitions of a Gibbs measure as a G-measure or as a quasi-invariant measure is well known [**20**, Theorem 5.2.4.(a)].

PROPOSITION 3.7. *Let (X, R), $D \in Z^1(R, \mathbb{R}_+^*)$, D_n, E_n and $\underline{E}$ as above.*

(i) *The sequence $(S_{D_n}(X))$ is decreasing and its intersection is $S_D(X)$.*

(i) *The isomorphisms $E_n^* : S(X_n) \to S_{D_n}(X)$ induce an isomorphism of compact convex sets from $S = S(\underline{E})$ onto $S_D(X)$. In other words, the quasi-invariant probability measures admitting D as Radon–Nikodym derivative are exactly the G-measures of the martingale $(P_n = \pi_n^* E_n)$.*

Proof. (i) A measure $\mu \in S(X)$ belongs to $S_{D_n}(X)$ if and only if it is quasi-invariant with respect to R_n and admits D_n as its Radon–Nikodym derivative. Since $R_{n-1} \subset R_n$ and $D_{n-1} = D_{n|R_{n-1}}$, a measure $\mu \in S(X)$ which belongs to $S_{D_n}(X)$ also belongs to $S_{D_{n-1}}(X)$. Moreover, $\mu \in S(X)$ belongs to $S_D(X)$ if and only if, for all n, it is quasi-invariant with respect to R_n with Radon–Nikodym derivative D_n. Therefore, the intersection of the $(S_{D_n}(X))$ is $S_D(X)$.

(ii) Let $(\mu_n) \in S(\underline{E})$ and let $\mu = E_n^*\mu_n$ the associated G-measure. Then μ belongs to $S_{D_n}(X)$ for all n and therefore belongs to $S_D(X)$. Conversely, if μ belongs to $S_D(X)$, we define $\mu_n = \pi_{n*}\mu \in S(X_n)$ for all n. Then the sequence (μ_n) belongs to $S(\underline{E})$ and $\mu = E_n^*\mu_n$. Therefore, the map which sends $(\mu_n) \in S(\underline{E})$ into $\mu_0 \in S(X)$ is an isomorphism of $S(\underline{E})$ onto $S_D(X)$. □

We reformulate Lemma A.2 of Appendix A in this setting.

COROLLARY 3.8. *Let (X, R), $D \in Z^1(R, \mathbb{R}_+^*)$, E_n be as above. For $f \in C(X)$, the following conditions are equivalent:*

(i) *$\|E_n(f)\|$ tends to 0 as n tends to ∞;*

(ii) *$\mu(f) = 0$ for all $\mu \in S_D(X)$.*

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With the notation of Appendix A, we denote by $\mathrm{Aff}(S)$ the space of continuous affine functions on the compact convex space S and by $\theta : \mathcal{E} \to \mathrm{Aff}(S_D(X))$ the evaluation map. Then $\theta_0 = \theta \circ E_\infty : C(X) \to \mathrm{Aff}(S_D(X))$ is the evaluation map $\theta_0(f)(\mu) = \mu(f)$ for $\mu \in S_D(X)$.

PROPOSITION 3.9. *Let (X, R) be an AP equivalence relation and let D be a continuous cocycle in $Z^1(R, \mathbb{R}_+^*)$. Then the evaluation map θ_0 induces an isomorphism of ordered Banach spaces with order-unit from $C(X)/\operatorname{Ker}\theta_0$ onto $\mathrm{Aff}(S_D(X))$.*

Proof. This results from Proposition A.8. □

The study of the convergence of the expectations $(\pi_n^* E_n(f))$ can be reduced to the irreducible case, i.e. when X_∞ reduces to one point by considering the fibers of the map $\pi_\infty : X \to X_\infty$ defined earlier. Before stating the main result about uniqueness of the solution of the Radon–Nikodym problem, we introduce the additive and multiplicative variations. The multiplicative variation defined in (ii) is (the exponential of) the projective metric (see [**2**]).

Definition 3.3.

(i) Given $\Delta \subset X \times X$ and $\varphi \in C(X)$, we define the *additive variation* of φ over Δ as

$$\mathrm{var}(\varphi, \Delta) = \sup_{\Delta} |\varphi(x) - \varphi(x')|.$$

Similarly, given $g \in C(X, \mathbb{R}_+^*)$, we define the *multiplicative variation* of g over Δ as

$$\mathrm{var}_*(g, \Delta) = \sup_{\Delta} \frac{g(x)}{g(x')}.$$

When $\Delta = X \times X$, we simply write $\mathrm{var}(\varphi)$ and $\mathrm{var}_*(g)$.

(ii) More generally, if $(\mathcal{E}, u)$ is an ordered real vector space with order-unit, we define for $a \in \mathcal{E}^+$,

$$\max_u(a) = \inf\{\lambda \in \mathbb{R}_+^* : a \le \lambda u\}.$$

If, moreover, a is also an order-unit, we define

$$\mathrm{var}_*(a) = \mathrm{var}_*(a, u) = \max_u(a)\max_a(u).$$

Given $f \in C(X)$ (respectively $C(X, \mathbb{R}_+^*)$), $E_n(f)$ belongs to $C(X_n)$ (respectively $C(X_n, \mathbb{R}_+^*)$) and the sequence $\mathrm{var}(E_n(f))$ (respectively $\mathrm{var}_*(E_n(f))$) is decreasing. The equivalence of unique ergodicity and uniform convergence to a constant of the averages appears in [**15**, Proposition 1] (see also [**4**]) in the setting of transfer operators on one-sided subshifts of finite type. The criterion (v), which is based on [**15**, Proposition 1], appears implicitly in [**33**, Theorem 2.16]. It is equivalent to condition (c) of [**19**, Theorem 4.18], which is given there for an arbitrary partially ordered abelian group with order-unit.

THEOREM 3.10. *Let $\left(X, R = \bigcup R_n\right)$ and $D \in Z^1(R, \mathbb{R}_+^*)$ be as above. The following assertions are equivalent and imply that (X, R) is irreducible.*

(i) *D is* uniquely ergodic*, i.e. $S_D(X)$ contains one element.*

(ii) *For all $f \in C(X)$, $(\pi_n^* E_n(f))$ converges uniformly to a constant function.*

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(ii) *For all* $f \in C(X)$, $\mathrm{var}(E_n(f))$ *tends to* 0.
(iv) *For all* $f \in C(X, \mathbb{R}_+^*)$, $\mathrm{var}_*(E_n(f))$ *tends to* 1.
(v) *There exists a finite* c *such that for all* $f \in C(X, \mathbb{R}_+^*)$, *there exists* n *(depending on* f*) such that* $\mathrm{var}_*(E_n(f)) \leq c$.

Proof. The equivalence of (i), (ii) and (iii) is a particular case of Corollary A.3. The implication (iii) $\Rightarrow$ (iv) is clear because $E_n(f)$ is bounded from below by the minimum of f. The implication (iv) $\Rightarrow$ (v) is obvious. Let us prove that (v) $\Rightarrow$ (i). We fix $f \in C(X, \mathbb{R}_+^*)$. For n sufficiently large, we have

$$E_n(f)(x) \leq cE_n(f)(y)$$

for all $(x, y) \in X_n \times X_n$. By integrating this inequality first with respect to x then with respect to y, we obtain that $\mu'(f) \leq c\mu(f)$ for all $\mu, \mu' \in S_D(X)$. This implies $\mu' \leq c\mu$ and the equality $\mu' = \mu$ if μ is an extremal element of $S_D(X)$. According to the Krein–Milman theorem, $S_D(X)$ has a single element.

Finally, since $\pi_n^* E_n$ acts as the identity on $C(X)^R$, (ii) implies $C(X)^R = \mathbb{R}1$. □

Remark 3.3. Condition (v) has a neat formulation in terms of the dimension group $\mathcal{E}$: it states that the multiplicative variation is uniformly bounded on the order-units of $\mathcal{E}$.

4. *Quasi-product cocycles on AF relations*

Let (V, E) be a Bratteli diagram. We denote by $X = X(V, E)$ its infinite path space, by X_n the space of infinite paths starting at level n and by $R = R(V, E) = \bigcup R_n$ the tail equivalence relation on X. A function $\Phi : E \to \mathbb{R}_+^*$ (called a *labeling*) defines a compatible sequence of potentials (g_n) such that for $x \in X$,

$$g_n(x) = \Phi(x_1)\Phi(x_2)\ldots\Phi(x_n)$$

and therefore a cocycle $D \in Z^1(R, \mathbb{R}_+^*)$.

Definition 4.1. Given a Bratteli diagram (V, E), a cocycle D on the tail equivalence relation $R(V, E)$ of its infinite path space $X(V, E)$ is called a *quasi-product cocycle* if it is of the above form.

Remark 4.1. It is shown in [**28**, Theorem 3.1] that every continuous cocycle on an AP equivalence relation on a compact totally disconnected space is cohomologous to a quasi-product cocycle relative to some Bratteli diagram. However, this result gives little information about the Bratteli diagram, in particular about its growth. We characterize the cocycles which are cohomologous to a quasi-product cocycle relative to some contraction of a given Bratteli diagram later.

We assume from now on that the set of vertices $V(n)$ of each level n is finite (and, as before, that every vertex emits finitely many, but at least one, edges and that every vertex of a level $n \geq 1$ receives at least one edge). Then the space of infinite paths X of the diagram is compact. Let D be a quasi-product cocycle given by $\Phi : E \to \mathbb{R}_+^*$. Let us apply the general approximation procedure to D. The normalized potential of $D_n = D_{|R_n}$ is the function $\rho_n \in C(X)$ given by

$$\rho_n(x) = \Phi(x_1)\Phi(x_2)\ldots\Phi(x_n)/Z_n(r(x_n)),$$

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where we have introduced the normalization factor

$$Z_n(v) = \sum \Phi(y_1)\Phi(y_2)\dots\Phi(y_n)$$

where $v \in V(n)$ and the sum is over all the finite paths $y_1 y_2 \dots y_n$ of length n ending at v. The sequence of normalized local potentials of D is given by

$$\rho_{n,n-1}(y_n x_{n+1}\dots) = Z_n \circ r(y_n)^{-1}\Phi(y_n)Z_{n-1}\circ s(y_n).$$

Note that $\rho_{n,n-1}$ depends only on y_n. Let us point out this elementary property in the following proposition.

PROPOSITION 4.1. *Let (X, R) be the AF equivalence relation defined by the Bratteli diagram (V, E) and let $D \in Z^1(R, \mathbb{R}_+^*)$. Then, the following conditions are equivalent.*

(i) *D is a quasi-product cocycle relative to (V, E).*

(ii) *For all n, its normalized local potential $\rho_{n,n-1}$ depends only on the first variable x_n.*

(iii) *For all n, its normalized potential ρ_n depends only on the first n variables: $x_1 x_2 \dots x_n$.*

(iv) *D admits an initial potential $\rho_0 \equiv 1$ and a sequence of local potentials $(\rho_{n,n-1})$ such that $\rho_{n,n-1}(x_n\dots)$ depends only on $x_n \in E(n)$.*

Proof. The equivalence of (i) and (iv) is clear. It is also clear that (iv) implies (ii), and that (ii) implies (iii), because

$$\rho_n(x) = \rho_{1,0}(x)\dots\rho_{n,n-1}\circ\pi_{n-1}(x).$$

Finally, (iii) implies (iv) because $\rho_{n,n-1}\circ\pi_{n-1}(x) = \rho_n(x)/\rho_{n-1}(x)$. □

The conditional expectation $E_n : C(X) \to C(X_n)$ is given by

$$E_n(f)(x_{n+1}\dots) = \sum \rho_n(y_1\dots y_n x_{n+1}\dots) f(y_1\dots y_n x_{n+1}\dots)$$

where the sum is over all the finite paths $y_1 y_2 \dots y_n$ of length n ending at $s(x_{n+1})$. The conditional expectation $E_{n,n-1} : C(X_{n-1}) \to C(X_n)$ is given by

$$E_{n,n-1}(f)(x_{n+1}\dots) = \sum \rho_{n,n-1}(y_n x_{n+1}\dots) f(y_n x_{n+1}\dots)$$

where the sum is over all the $y_n \in E(n)$ ending at $s(x_{n+1})$. Thus, as usual we have a sequence $\underline{E} = (E_{n,n-1} : C(X_{n-1}) \to C(X_n))$ of Markovian operators. On the other hand, Φ defines a sequence of matrices $\underline{A} = (A_n : C(V(n-1)) \to C(V(n)))$, where the coefficients of A_n are $A_n(w, v) = \sum \Phi(e)$, the sum being over all the $e \in E(n)$ starting at $v \in V(n-1)$ and ending at $w \in V(n)$. The state space $S(\underline{A})$ of this sequence is defined in Appendix A. It is the state space of the inductive limit dimension group. By definition, it consists of sequences (ρ_n), where ρ_n belongs to $C(V(n))^*$ and is positive, such that

$$\rho_{n-1}(v) = \sum_{s(e)=v} \Phi(e)\rho_n\circ r(e) \quad (n = 1, 2, \dots, v \in V(n-1)) \tag{1}$$

$$1 = \sum_{V(0)} \rho_0(v). \tag{2}$$

Such a sequence (ρ_n) defines a *Markov measure* μ such that

$$\mu(Z(x_1\dots x_n)) = \rho_0(s(x_1))p_1(x_1)\dots p_n(x_n)$$

where $p_n(e) = (\rho_{n-1}\circ s(e))^{-1}\Phi_n(e)\rho_n\circ r(e)$ for $n = 1, 2, \dots, e \in E(n)$.

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PROPOSITION 4.2. [**28**, Proposition 3.3] *Let D be a quasi-product cocycle. The above construction defines an isomorphism from the state space $S(\underline{A})$ of the sequence of matrices $\underline{A} = (A_n : C(V(n-1)) \to C(V(n)))$ to the space $S_D(X)$ of solutions of the (Radon–Nikodym) equation $D_\mu = D$. In other words, the quasi-invariant probability measures admitting D as Radon–Nikodym derivative are exactly the above Markov measures.*

Proof. We refer the reader to [**28**]. It is easy to check that the Markov measure μ constructed above belongs to $S_D(X)$. It was shown in [**28**] that conversely every $\mu \in S_D(X)$ is of that form. The maps so defined are continuous and preserve convex combinations. □

Note that the maps $J_n : C(V(n)) \to C(X_n)$ defined by $J_n(f)(x_{n+1}\dots) = Z_n \circ s(x_{n+1})^{-1} f \circ s(x_{n+1})$ yield a morphism from $\underline{A}$ to $\underline{E}$, hence a morphism of their dimension groups $J : \mathcal{E}(\underline{A}) \to \mathcal{E}(\underline{E})$. This morphism induces the above isomorphism of their state spaces. However, J itself is not necessarily an isomorphism. For example, let the Bratteli diagram (V, E) be a tree (and $\Phi : E \to \mathbb{R}_+^*$ be identically one). Then $\pi_{n.n-1} : X_{n-1} \to X_n$ is a bijection and $E_{n,n-1} : C(X_{n-1}) \to C(X_n)$ is the transposed map $\pi^*_{n.n-1}$. The inductive limit $\underline{E}$ can be identified with $C(X)$. On the other hand, the image of J consists of locally constant functions.

Taking into account this proposition, Corollaries A.3 and A.7 become the following.

COROLLARY 4.3. *Let R be the tail equivalence relation on the infinite path space X of a Bratteli diagram (V, E) and let $D \in Z^1(R, \mathbb{R}_+^*)$ be a quasi-product cocycle defined by $\Phi : E \to \mathbb{R}_+^*$. Define the matrix $A_n(w, v) = \sum \Phi(e)$, the sum being over all the $e \in E(n)$ starting at $v \in V(n-1)$ and ending at $w \in V(n)$ as above.*

(i) *A necessary and sufficient condition for $S_D(X)$ to have exactly one element is that, for any fixed m and $v \in V(m)$, the variation of the function $w \in V(n) \mapsto B_n \dots B_{m+1}(v, w)$ goes to 0 when n goes to infinity, where $u_n(w) = \sum_{v \in V(0)} A_n \dots A_1(w, v)$ and $B_n(w, v) = u_n(w)^{-1} A_n(w, v) u_{n-1}(v)$ for $w \in V(n)$ and $v \in V(n-1)$.*

(ii) *A sufficient condition for $S_D(X)$ to have exactly one element is that the series $\sum \epsilon_n$ diverges, where ϵ_n is the ratio of the smallest element of A_n over its largest element.*

The most studied quasi-product cocycle is $D \equiv 1$. Then, the elements of $S_D(X)$ are the invariant (with respect to the tail equivalence relation) probability measures on the path space X of the Bratteli diagram (V, E) (they are called central measures in [**30**]). They correspond to the tracial states of the AF algebra of the Bratteli diagram. We choose $\Phi \equiv 1$. The matrices A_n are the adjacency matrices of the graph. They have coefficients in $\mathbb{N}$ and the inductive limit of the $(A_n : C(V(n-1, \mathbb{Z})) \to C(V(n), \mathbb{Z}))$ is the usual dimension group of the Bratteli diagram. Then Proposition 4.2 gives the well-known correspondence between invariant probability measures and states of the dimension group. In that case, Corollary 4.3(ii) improves condition (2) of [**29**, Corollary, p. 130]. The necessary and sufficient condition (i) also appears in [**30**]. We now give a few examples.

Example 4.1. (An example of Fack and Maréchal) In their work [**14**] on the symmetries of uniformly hyperfinite (UHF) algebras, Fack and Maréchal studied the dimension group

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defined by the sequence of matrices $\underline{A} = (A_n)$, where

$$A_n = \begin{pmatrix} p_n & r_n \\ r_n & p_n \end{pmatrix}$$

and (p_n) and (r_n) are two sequences of integers such that $0 < r_n \leq p_n$ for all n. Let $\epsilon_n = r_n/p_n$. According to the above, a sufficient condition for $\mathcal{E}(\underline{A})$ having a unique state is that $\sum \epsilon_n = \infty$. Fack and Maréchal show by an explicit computation of the dimension group that this condition is necessary. This can also be deduced from Corollary 4.3(i).

Example 4.2. (Pascal's triangle) The simple random walk on $\mathbb{Z}$ provides another example. We let $X = \prod_1^\infty \{0, 1\}$ be the space of increments. We introduce $X_n = \{0, 1, \ldots, n\} \times \prod_{n+1}^\infty \{0, 1\}$ and $\pi_n : X \to X_n$ defined by

$$\pi_n(x_1 x_2 \ldots) = (x_1 + \cdots + x_n, x_{n+1} x_{n+2} \ldots).$$

Its first coordinate is the position of the walker at time n (assuming that his initial position is 0). We let R be the AP equivalence relation on X defined by the π_n. Note that (X, R) admits the infinite Pascal triangle as Bratteli diagram and that $D \equiv 1$ is the quasi-product cocycle defined by the function $\phi \equiv 1$. It is known that the state space S_1 of invariant probability measures is isomorphic to the space of probability measures on $[0, 1]$; see, for example, [**16**, §VII.4], [**21**] or [**27**, Appendix], which contains an explicit computation of the dimension group of the infinite Pascal triangle. The following comment of the proof is inspired by [**34**, §5]. First, it is immediate to check that for $t \in [0, 1]$, the product measure

$$\mu_t = \prod_1^\infty ((1 - t)\delta_0 + t\delta_1)$$

is invariant and that $\mu_t(Z(n, k)) = C_n^k t^k (1-t)^{n-k}$, where $Z(n, k)$ is the set of paths having position k at time n. An elementary estimate using the expansion of a polynomial of degree not greater than n in the basis of Bernstein polynomials $\{t^k(1 - t)^{n-k} : k = 0, 1, \ldots, n\}$ shows that for all $f \in C(X)$,

$$\lim_{n\to\infty} \sup_{(k,x)\in X_n} |E_n(f)(k, x) - \hat{f}(k/n)| = 0,$$

where $E_n(f)(k, x)$ is the average of the $f(a_1 \ldots a_n x)$ over all $a_1 \ldots a_n$ such that $a_1 + \cdots + a_n = k$ and $\hat{f}(t) = \mu_t(f)$. Using Corollary 3.8, one deduces that $\hat{f} = 0$ if and only if $\mu(f) = 0$ for all invariant probability measures μ. Therefore, the measures $\mu_t, t \in [0, 1]$ are exactly the extremal elements of S_1. The equivalence relation R is irreducible (it has dense orbits) and the equivalent conditions of Proposition 3.10 are not realized. One can see directly that for $x \in X$, $\pi_n^* E_n(f)(x)$ converges to $\hat{f}(t)$ for all $f \in C(X)$ if and only if the path $x = x_1 x_2 \ldots$ has the property that $(x_1 + \cdots + x_n)/n$ tends to t. The strong law of large numbers (or the martingale convergence theorem) states that, with respect to the measure μ_t, this set is conull.

Example 4.3. (Stationary cocycles) The Bratteli diagram (V, E) is called stationary if for all $n \geq 1$, $V(n) = V(0)$ and $E(n) = E(1)$. In that case, the one-sided shift

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$T(x_1x_2\dots) = x_2x_3\dots$ acts on the infinite path space X of the diagram. Then, $\Phi : E \to \mathbb{R}_+^*$ is called stationary if it does not depend on the level n. The associated quasi-product cocycle is also called stationary. The sufficient condition (ii) of Corollary 4.3 is always satisfied for a stationary quasi-product cocycle D. Therefore, $S_D(X)$ reduces to one element.

Example 4.4. (The Ising model; see [**27**, §III.1.6].) Let (X, R) be the AF equivalence relation constructed as in Example 2.2 from the lattice $\Lambda = \mathbb{Z}^d$ and $F = \{-1, 1\}$. The Ising model is described by the energy cocycle $c \in Z^1(R, \mathbb{R})$ given by the expression

$$c(x, y) = \sum J(i, j)(x_i x_j - y_i y_j)$$

where $J : \Lambda \times \Lambda \to \mathbb{R}$ depends on the nature of the interaction. When the interaction has a finite range, c (or rather $D = e^{-c}$) can be written as a quasi-product cocycle. When $d = 1$ and $J(i, j) = 0$ if $|i - j| \neq 1$ and $J(i, j) = J$ if $|i - j| = 1$, the quasi-product cocycle is stationary and D is uniquely ergodic.

5. *Beyond quasi-product cocycles*

The purpose of this section is to extend the previous results on quasi-product cocycles in two directions. First, we wish to consider a larger class of cocycles on AF equivalence relations. Although every continuous cocycle $D \in Z^1(R, \mathbb{R}_+^*)$ of an AF equivalence relation R on a compact space X is cohomologous to a quasi-product cocycle with respect to some Bratteli diagram, we usually do not have sufficient information on the Bratteli diagram in order to apply Corollary 4.3. When the cocycle D satisfies an appropriate condition of equicontinuity relative to a given Bratteli diagram, it is approximately a quasi-product cocycle with respect to the same Bratteli diagram (or a contraction of it); then one can use the technique of Corollary 4.3 to obtain unique ergodicity. Secondly, we wish to consider cocycles on arbitrary AP equivalence relations. The definitions below are adapted from [**28**].

Definition 5.1. Let $\pi : X \to Y$ be a surjective local homeomorphism. We say that an open subset V of Y is *well-covered* if $\pi^{-1}(V)$ is the disjoint union of a family of open sets $\{U_i, i \in I\}$ which all map homeomorphically onto V. By definition, a π*-cover* will consist of a cover $\mathcal{V}$ of Y by well-covered open subsets and for each $V \in \mathcal{V}$ a partition of $\pi^{-1}(V)$ by open subsets U of X which all map homeomorphically onto V. We denote by $\mathcal{U}$ the cover of X by these open sets U.

When X is compact, π admits finite π-covers. Moreover, if an open cover $\mathcal{W}$ of X is given, we can construct our π-cover such that $\mathcal{U}$ strictly refines $\mathcal{W}$ (notation: $\mathcal{U} \prec \mathcal{W}$), in the sense that for each $U \in \mathcal{U}$, there exists $W \in \mathcal{W}$ such that $\overline{U} \subset W$.

In the sequel, we always assume that the space X is compact and only consider finite π-covers. Recall from [**3**, §II.4] that a finite open cover $\mathcal{U}$ of a compact space X defines the entourage

$$\Delta_{\mathcal{U}} = \bigcup_{\mathcal{U}} U \times U$$

of the canonical uniform structure of X. The additive variation $\mathrm{var}(\varphi, \Delta)$ and the multiplicative variation $\mathrm{var}_*(g, \Delta)$ used in this section have been defined in Definition 3.3.

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LEMMA 5.1. *Let $\pi : X \to Y$ be a surjective local homeomorphism. Let $(\mathcal{V}, \mathcal{U})$ be a π-cover with associated entourages $\delta = \Delta_{\mathcal{V}}$, $\Delta = \Delta_{\mathcal{U}}$. Consider a potential $g \in C(X, \mathbb{R}_+^*)$ and its normalized potential $\rho = (Z \circ \pi)^{-1} g$, where $Z(y) = \sum_{\pi^{-1}(y)} g(x)$ for $y \in Y$. If $\mathrm{var}_*(g, \Delta) < C$, then $\mathrm{var}_*(\rho, \Delta) < C^2$.*

Proof. For all $(x, x') \in \Delta$, we have $g(x) < Cg(x')$. Let $(y, y') \in \delta$. We have by construction a bijection $x \in \pi^{-1}(y) \mapsto x' \in \pi^{-1}(y')$ such that $(x, x') \in \Delta$. Summing above inequality over $\pi^{-1}(y)$, we obtain $Z(y) < CZ(y')$. Therefore, for all $(x, x') \in \Delta$, we have $\rho(x) < C^2 \rho(x')$. □

We keep the same notation as above. Given $g \in C(X, \mathbb{R}_+^*)$, we introduce the positive linear map $\mathcal{L}_g : C(X) \to C(Y)$ such that

$$\mathcal{L}_g(f)(y) = \sum_{\pi^{-1}(y)} g(x) f(x).$$

When $g = \rho$ is a normalized potential, we write E instead of $\mathcal{L}_\rho$.

LEMMA 5.2. (Cf. [**31**, Lemma 1]) *Let $\pi : X \to Y$ be a surjective local homeomorphism. Let $(\mathcal{V}, \mathcal{U})$ be a π-cover with associated entourages δ, Δ.*

(i) *Let $\rho \in C(X, \mathbb{R}_+^*)$ be a normalized potential. Then, for all f in $C(X)$,*

$$\mathrm{var}(E(f), \delta) \leq \mathrm{var}(f, \Delta) + \|f\|[\mathrm{var}_*(\rho, \Delta) - 1].$$

(ii) *Let $g \in C(X, \mathbb{R}_+^*)$ be an unnormalized potential. Then, for all $f \in C(X, \mathbb{R}_+^*)$,*

$$\mathrm{var}_*(\mathcal{L}_g(f), \delta) \leq \mathrm{var}_*(g, \Delta)\mathrm{var}_*(f, \Delta).$$

Proof. We only prove (i), the proof of (ii) is similar. Let $(y, y') \in \delta$. By construction, we have a bijection $x \in \pi^{-1}(y) \mapsto x' \in \pi^{-1}(y')$ such that $(x, x') \in \Delta$. Then,

$$E(f)(y) - E(f)(y') = \sum_{\pi^{-1}(y)} \rho(x)[f(x) - f(x')] + [\rho(x) - \rho(x')]f(x')$$

$$|E(f)(y) - E(f)(y')| \leq \max_{\pi(x)=y} |f(x) - f(x')| + \|f\| \max_{\pi(x)=y} \left| \frac{\rho(x)}{\rho(x')} - 1 \right|$$

$$\leq \mathrm{var}(f, \Delta) + \|f\|[\mathrm{var}_*(\rho, \Delta) - 1].$$ □

Let us consider now an AP equivalence relation R on a compact space X defined by a sequence

$$X_0 \xrightarrow{\pi_{1,0}} X_1 \to \cdots \to X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \to \cdots$$

where $X_0 = X$ and for each $n \geq 1$, X_n is a Hausdorff space and $\pi_{n,n-1}$ is a surjective local homeomorphism. We can construct inductively $\pi_{n,n-1}$-covers $(\mathcal{V}^n, \mathcal{U}^n)$ such that $\mathcal{U}^n \prec \mathcal{V}^{n-1}$.

Definition 5.2. Such a sequence $(\mathcal{V}^n, \mathcal{U}^n)$ of $\pi_{n,n-1}$-covers will be called a *tower* relative to the sequence (X_n).

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Given $m < n$ and a sequence $a = (U_{m+1}, U_{m+2}, \dots, U_n)$, where $U_k \in \mathcal{U}^k$, such that $\overline{U}_{k+1} \subset \pi_{k,k-1}(U_k)$ for $k = m+1, \dots, n-1$, we define the following subset of X_m:

$$U_a = U_{m+1} \cap \pi_{m+1,m}^{-1}(U_{m+2}) \cap \cdots \cap \pi_{n-1,m}^{-1}(U_n).$$

To avoid redundancies, we implicitly choose once and for all, for $n = 1, 2, \dots$ and $U \in \mathcal{U}_n$, a set $V = V(U) \in \mathcal{V}_{n-1}$ such that $\overline{U} \subset V$. We only consider sequences $a = (U_{m+1}, U_{m+2}, \dots, U_n)$ with $U_k \in \mathcal{U}^k$ such that $\pi_{k,k-1}(U_k) = V(U_{k+1})$ for $m < k < n$.

LEMMA 5.3. *Let $(\mathcal{V}^n, \mathcal{U}^n)$ be a tower. Given $m < n$, we define $\mathcal{U}^{m,n}$ as the family of open sets U_a, where*

$$a = (U_{m+1}, \dots, U_n), \quad U_k \in \mathcal{U}^k, \quad \pi_{k,k-1}(U_k) = V(U_{k+1}), \quad m < k < n.$$

Then $(\mathcal{V}^n, \mathcal{U}^{m,n})$ is a $\pi_{n,m}$-cover.

Proof. Let $V \in \mathcal{V}^n$. Let us show that $\pi_{n,m}^{-1}(V)$ is the disjoint union of the family of U_a, where $a = (U_{m+1}, U_{m+2}, \dots, U_n)$, $U_k \in \mathcal{U}^k$, $\pi_{k,k-1}(U_k) = V(U_{k+1})$ for $m < k < n$ and $V = \pi_{n,n-1}(U_n)$ and that $\pi_{n,m}$ maps each U_a homeomorphically onto V. Let $x_m \in \pi_{n,m}^{-1}(V)$. We define $x_k = \pi_{k,m}(x_m)$ for $m < k \leq n$. Since $x_n \in V$, there is a unique $U_n \in \mathcal{U}_n$ mapping homeomorphically onto V and containing x_{n-1}. We define $V_{n-1} = V(U_n) \in \mathcal{V}_{n-1}$ and proceed by induction to construct $U_{n-1} \in \mathcal{U}_{n-1}, \dots, U_{m+1} \in \mathcal{U}_{m+1}$. Then $x_m \in U_a$ where $a = (U_{m+1}, U_{m+2}, \dots, U_n)$. If $a = (U_{m+1}, U_{m+2}, \dots, U_n)$ and $a' = (U'_{m+1}, U'_{m+2}, \dots, U'_n)$ are distinct, there exists a larger $k \leq n$ such that $U_k \neq U'_k$. Then, U_k and U'_k are disjoint and so are U_a and $U_{a'}$. The restriction of $\pi_{n,m}$ to U_a is a composition of homeorphisms and therefore maps U_a homeomorphically onto V. □

Thus, if we contract the initial sequence of spaces by means of a subsequence (n_k), our initial tower $(\mathcal{V}^n, \mathcal{U}^n)$ provides a tower $(\underline{\mathcal{V}}^k, \underline{\mathcal{U}}^k)$ relative to the sequence $(\underline{X}_k = X_{n_k})$: we set $\underline{\mathcal{V}}^k = \mathcal{V}^{n_k}$ and $\underline{\mathcal{U}}^k = \mathcal{U}^{n_{k-1}, n_k}$.

In particular, we denote $\mathcal{U}_n = \mathcal{U}^{0,n}$. Note that $(\mathcal{V}^n, \mathcal{U}_n)$ is a π_n-cover and that $\mathcal{U}_n \prec \mathcal{U}_{n-1}$.

Given a tower $(\mathcal{V}^n, \mathcal{U}^n)$ as above for the defining sequence (X_n), we define the following entourages in $X \times X$:

$$\Delta_n = \Delta_{\mathcal{U}_n} = \bigcup \{U \times U : U \in \mathcal{U}_n\}$$

and in $X_n \times X_n$:

$$\delta_n = \Delta_{\mathcal{V}^n} = \bigcup \{V \times V : V \in \mathcal{V}^n\}.$$

Note that (Δ_n) is a sequence of neighborhoods of the diagonal Δ_X of $X \times X$ such that $\Delta_n \subset \overline{\Delta}_n \subset \Delta_{n-1}$.

Definition 5.3. We say that the tower $(\mathcal{V}^n, \mathcal{U}^n)$ for the defining sequence (X_n) is a *generator* if $\bigcap \Delta_n$ reduces to the diagonal Δ_X of $X \times X$. In other words, a tower $(\mathcal{V}^n, \mathcal{U}^n)$ is a generator if and only if (Δ_n) is a fundamental system of neighborhoods of Δ_X.

Example 5.1. (AF equivalence relation defined by a Bratteli diagram) Let (V, E) be a Bratteli diagram, let X be its infinite path space and X_n the space of paths starting at level n.

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The Bratteli diagram defines a tower $(\mathcal{V}^n, \mathcal{U}^n)$, where $\mathcal{V}^n$ is the partition of X_n by the cylinder sets

$$Z^n(v) = \{x_{n+1}x_{n+2}\cdots \in X_n : s(x_{n+1}) = v\},$$

where $v \in V(n)$ and $\mathcal{U}^n$ is the partition of X_{n-1} by the cylinder sets

$$Z^{n-1}(e) = \{x_n x_{n+1}\cdots \in X_{n-1} : x_n = e\},$$

where $e \in E(n)$. Then $\mathcal{U}_n$ is the partition of X by the cylinder sets $Z(a)$, where $a = a_1 \ldots a_n$ is a path from level 0 to level n. By definition of the path space X, this tower is a generator. We have, for $x, y \in X$,

$$(x, y) \in \Delta_n \Leftrightarrow x_1 = y_1, \ldots, x_n = y_n$$

and for $x, y \in X_n$,

$$(x, y) \in \delta_n \Leftrightarrow s(x) = s(y),$$

i.e x and y start from the same vertex.

Let $D \in Z^1(R, \mathbb{R}_+^*)$ and let (ρ_n) be the sequence of its normalized potentials. We have

$$\mathrm{var}_*(\rho_n, \Delta_n) = \sup \frac{\rho_n(ax)}{\rho_n(ay)}$$

where the supremum is taken over all paths a from level 0 to level n and all infinite paths x, y starting at $r(a)$. Note that if D is a quasi-product cocycle relative to (V, E), then for all n and for all paths a from level 0 to level n, we have $\rho_n(ax)/\rho_n(ay) = 1$ and, therefore, $\mathrm{var}_*(\rho_n, \Delta_n) = 1$. The estimates of this section can be used to extend the results of Corollary 4.3 concerning quasi-product cocycles to cocycles for which there is a good control of $\mathrm{var}_*(\rho_n, \Delta_n)$.

Let us return to the general case of an AP equivalence relation R with a given generator $(\mathcal{V}^n, \mathcal{U}^n)$. A natural condition on a cocycle $D \in Z^1(R, \mathbb{R}_+^*)$ is that $\lim \mathrm{var}_*(g_n, \Delta_n) = 1$, where $\Delta_n = \Delta_{\mathcal{U}_n}$ and where (g_n) is a sequence of potentials for D. It results from Lemma 5.1 that if this condition is satisfied by a sequence of (unnormalized) potentials (g_n), it is also satisfied by the sequence of normalized potentials (ρ_n). Let us also observe that this condition is invariant under cohomology. Indeed, if

$$\underline{D}(x, y) = b(x)D(x, y)b(y)^{-1}$$

for some $b \in C(X, \mathbb{R}_+^*)$, it admits the sequence of potentials (bg_n). The condition $\lim \mathrm{var}_*(g_n, \Delta_n) = 1$ and the uniform continuity of b imply the same condition for (bg_n). Thus, a cocycle D on an AF equivalence relation given by a Bratteli diagram (V, E) which is cohomologous to a quasi-product cocycle with respect to the diagram must satisfy $\lim \mathrm{var}_*(g_n, \Delta_n) = 1$. One can prove a weak converse: if $\liminf \mathrm{var}_*(g_n, \Delta_n) = 1$, then D is cohomologous to a quasi-product cocycle relative to a contraction of (V, E). The proof is similar to that of [**28**, Theorem 3.1].

We next consider a weaker condition on the sequence of potentials (g_n). We use the following crude estimate.

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LEMMA 5.4. (Cf. [**33**, Theorem 2.14]) *Let $\pi : X \to Y$ be a surjective local homeomorphism, with X compact. Let δ be an entourage of X such that for all $(x, y') \in X \times Y$, there exists $x' \in \pi^{-1}(y')$ such that $(x, x') \in \delta$. Then, for all f in $C(X, \mathbb{R}_+^*)$ and all $(y, y') \in Y \times Y$, we have*

$$\sum_{\pi(x)=y} f(x) \leq C(\pi, \delta)\mathrm{var}_*(f, \delta) \sum_{\pi(x')=y'} f(x'),$$

where $C(\pi, \delta)$ is a bound for the cardinality of the sets $\{x \in \pi^{-1}(y) : (x, x') \in \delta\}$ when (x', y) runs over $X \times Y$.

Proof. We have

$$\sum_{\pi^{-1}(y)} f(x) \leq \sum_{(\pi\times\pi)^{-1}(y,y')\cap\delta} f(x) \tag{3}$$

$$\leq \mathrm{var}_*(f, \delta) \sum_{(\pi\times\pi)^{-1}(y,y')\cap\delta} f(x') \tag{4}$$

$$\leq C(\pi, \delta)\mathrm{var}_*(f, \delta) \sum_{\pi^{-1}(y')} f(x'). \qquad \square$$

PROPOSITION 5.5. (Cf. [**33**, Theorem 2.14]) *Let R be an AP equivalence relation on a compact space X with a defining sequence (X_n, π_n). Let $(\mathcal{V}^n, \mathcal{U}^n)$ be a generator with associated entourages δ_n, Δ_n. Assume that for all n and for all $V \in \mathcal{V}^n$, $\pi_{n+1,n}(V) = X_{n+1}$. Let (g_n) be compatible sequence of potentials, defining the operators $L_n = \mathcal{L}_{g_n} : C(X) \to C(X_n)$ and $\rho_n = g_n/Z_n \circ \pi_n$ the normalized potentials defining the expectations $E_n : C(X) \to C(X_n)$. Let $f \in C(X, \mathbb{R}_+^*)$. Then, we have the following estimates:*

(i) $\mathrm{var}_*(L_{n+1}(f)) \leq C(\pi_{n+1,n}, \delta_n)\mathrm{var}_*(g_{n+1,n}, \delta_n)\mathrm{var}_*(g_n, \Delta_n)\mathrm{var}_*(f, \Delta_n)$;

(ii) $\mathrm{var}_*(Z_{n+1}) \leq C(\pi_{n+1,n}, \delta_n)\mathrm{var}_*(g_{n+1,n}, \delta_n)\mathrm{var}_*(g_n, \Delta_n)$;

(iii) $\mathrm{var}_*(E_{n+1}(f)) \leq [C(\pi_{n+1,n}, \delta_n)\mathrm{var}_*(g_{n+1,n}, \delta_n)\mathrm{var}_*(g_n, \Delta_n)]^2\mathrm{var}_*(f, \Delta_n)$.

Proof. For (i), we have $L_{n+1} = L_{n+1,n} \circ L_n$ with $L_{n+1,n} = \mathcal{L}_{g_{n+1,n}} : C(X_n) \to C(X_{n+1})$. We apply Lemma 5.4 with $\pi_{n+1,n}$, δ_n and $g_{n+1,n}L_n(f)$. We obtain

$$\begin{aligned}\mathrm{var}_*(L_{n+1}(f)) &\leq C(\pi_{n+1,n}, \delta_n)\mathrm{var}_*(g_{n+1,n}L_n(f), \delta_n)\\ &\leq C(\pi_{n+1,n}, \delta_n)\mathrm{var}_*(g_{n+1,n}, \delta_n)\mathrm{var}_*(L_n(f), \delta_n).\end{aligned}$$

Then we apply Lemma 5.2(ii) to get

$$\mathrm{var}_*(L_n(f), \delta_n) \leq \mathrm{var}_*(g_n, \Delta_n)\mathrm{var}_*(f, \Delta_n).$$

Assertion (ii) results from (i) with $f = 1$. Since $E_n(f) = L_n(f)/Z_n$, we deduce (iii). $\square$

THEOREM 5.6. *Let R be an AP equivalence relation on a compact space X with a defining sequence (X_n, π_n). Let (g_n) be compatible sequence of potentials. Let $(\mathcal{V}^n, \mathcal{U}^n)$ be a generator with associated entourages δ_n, Δ_n. Assume that:*

(i) *for all n and for all $V \in \mathcal{V}^n$, $\pi_{n+1,n}(V) = X_{n+1}$;*

(ii) *the sequence $(\mathrm{var}_*(g_n, \Delta_n))$ is bounded;*

(iii) *the sequence $(C(\pi_{n+1,n}, \delta_n)\mathrm{var}_*(g_{n+1,n}, \delta_n))$ as above is bounded.*

Then the cocycle $D \in Z^1(R, \mathbb{R}_+^*)$ defined by (g_n) is uniquely ergodic.

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Proof. Since $(\mathcal{V}^n, \mathcal{U}^n)$ be a generator, $\mathrm{var}_*(f, \Delta_n)$ tends to 1 for all $f \in C(X, \mathbb{R}_+^*)$. According to Proposition 5.5, Theorem 3.10(v) is satisfied when c is a bound for $[C(\pi_{n+1,n}, \delta_n)\mathrm{var}_*(g_{n+1,n}, \delta_n)\mathrm{var}_*(g_n, \Delta_n)]^2$. □

Remark 5.1.

- Condition (i) implies that R is minimal, i.e. there are no non-trivial invariant open sets in X. Conversely, if R is minimal, condition (i) is satisfied for a subsequence of (X_n).
- In the case of an AF equivalence relation defined by a Bratteli diagram, condition (i) means that every pair of vertices of consecutive levels is connected by an edge. Moreover, in the case of a quasi-product cocycle, condition (ii) is always satisfied while condition (iii) is more stringent than the sufficient conditions given in Corollary 4.3 and Lemma A.5.
- When for all n, $\mathcal{V}_n$ reduces to $\{X_n\}$ ($\pi_{n,n-1}$ is a covering map), Lemma 5.4 is not needed and condition (iii) can be dropped.

6. *Stationary systems*

We now apply the above results to stationary systems, in the following sense. We say that the sequence

$$X_0 \xrightarrow{\pi_{1,0}} X_1 \to \cdots \to X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \to \cdots$$

is *stationary* if for all $n \in \mathbb{N}$, $X_n = X$ and for all $n \geq 1$, $\pi_{n,n-1} = T$. As usual, we assume that X is compact and that T is a local homeomorphism and a surjection. The AP equivalence relation defined by this sequence is

$$R = R(X, T) = \{(x, y) \in X \times X : \exists n \in \mathbb{N} : T^n x = T^n y\}.$$

Then R is a subgroupoid of the semi-direct product groupoid (see, for example, [**28**]) of the dynamical system

$$G(X, T) = \{(x, m - n, y) \in X \times \mathbb{Z} \times X : m, n \in \mathbb{N}, T^m x = T^n y\}.$$

We say that a cocycle $D \in Z^1(R, \mathbb{R}_+^*)$ is *stationary* if it is the restriction of a cocycle in $Z^1(G(X, T), \mathbb{R}_+^*)$. As $G(X, T)$ is singly generated (see [**28**, §4.1]), cocycles in $Z^1(G(X, T), \mathbb{R}_+^*)$ are in a one-to-one correspondence with functions $g \in C(X, \mathbb{R}_+^*)$. However, the cocycles defined by g and λg, where $\lambda \in \mathbb{R}_+^*$ (or more generally $\lambda \in C(X, \mathbb{R}_+^*)^R$), will have the same restriction to R. More explicitly, a cocycle $D \in Z^1(R, \mathbb{R}_+^*)$ is stationary if and only if it admits a sequence of potentials (g_n) of the form:

$$g_n(x) = g(x)g(Tx)\ldots g(T^{n-1}x)$$

where g is a given function in $C(X, \mathbb{R}_+^*)$. Then the normalized potentials are given by

$$\rho_n = \frac{g_n}{Z_n \circ T^n} \quad \text{where } Z_n(x) = \sum_{T^n y = x} g_n(y)$$

are the partition functions. The normalized local potentials are given by

$$\rho_{n,n-1}(x) = (Z_n(Tx))^{-1} g(x) Z_{n-1}(x).$$

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Given a stationary system $(X_n = X, \pi_{n,n-1} = T)$, a tower $(\mathcal{V}^n, \mathcal{U}^n)$ will be called stationary if it does not depend on n. Thus, a stationary tower is given by a T-cover $(\mathcal{V}, \mathcal{U})$ such that $\mathcal{U} \prec \mathcal{V}$. As before, we define $\mathcal{U}_n$ as the cover of X by the open sets

$$U_a = U_1 \cap T^{-1}(U_2) \cap \cdots \cap T^{-(n-1)}(U_n),$$

where $a = (U_1, \ldots, U_n)$, $U_1, \ldots, U_n \in \mathcal{U}$, $\overline{U}_{k+1} \subset T(U_k)$. We also define the entourages $\Delta_n = \Delta_{\mathcal{U}_n} \subset X \times X$ and $\delta_n = \Delta_{\mathcal{V}^n} \subset X_n \times X_n$. Note that δ_n does not depend on n; we call it δ. Remember that the tower is called a generator if $\bigcap \Delta_n$ reduces to the diagonal Δ_X of $X \times X$. We then say that the corresponding T-cover $(\mathcal{V}, \mathcal{U})$ is a generator. In the classical terminology of [**32**], the cover $\mathcal{U}$ itself is called a generator if $\bigcap \Delta_n$ reduces to the diagonal Δ_X.

Definition 6.1. Let $T : X \to X$ be a continuous map on a compact space and let Δ be an entourage of the uniform structure of X such that the sequence

$$\Delta_n = \{(x, y) \in X \times X : \forall k = 1, \ldots, n-1, (T^k x, T^k y) \in \Delta\}$$

is a fundamental system of neighborhoods of the diagonal. One says that $g \in C(X, \mathbb{R}_+^*)$ satisfies *Bowen's condition* with respect to T if the sequence $(\mathrm{var}_*(g_n, \Delta_n))$ is bounded.

This condition does not depend on Δ : if Δ' is another entourage, there exists M such that $\Delta_M \subset \Delta'$. Then for all n, $\Delta_{n+M} \subset \Delta'_n$. Since $g_{n+M} = g_n(g_M \circ T^n)$, $\mathrm{var}_*(g_{n+M}, \Delta_{n+M}) \leq \mathrm{var}_*(g_n, \Delta'_n)\mathrm{var}_*(g_M)$.

THEOREM 6.1. *Let $T : X \to X$ be a surjective local homeomorphism on a compact space and let $g \in C(X, \mathbb{R}_+^*)$. Assume that:*

(i) *the AP equivalence relation $R(X, T)$ is minimal;*

(ii) *there exists an integer $L \geq 1$ such that T^L admits a generator;*

(iii) *there exists an integer $M \geq 1$ such that g_M satisfies Bowen's condition with respect to T^M.*

Then, the stationary cocycle $D \in Z^1(R(X, T), \mathbb{R}_+^*)$ defined by g is uniquely ergodic.

Proof. Let $(\mathcal{V}, \mathcal{U})$ be a generator for T^L. By the compactness of X and the minimality of $R(X, T)$, there exists an integer N such that $T^N(V) = X$ for all $V \in \mathcal{V}$. Let $K = LMN$. Then $(\mathcal{V}, \mathcal{U}_{MN})$ is a generator for T^K. If g_M satisfies Bowen's condition with respect to T^M, g_K satisfies Bowen's condition with respect to T^K. Moreover $R(X, T^K) = R(X, T)$. Therefore, replacing T by T^K, we may assume that $L = M = 1$ and that T has a generator $(\mathcal{V}, \mathcal{U})$ such that $T(V) = X$ for all $V \in \mathcal{V}$. Then, the conditions of Theorem 5.6 are satisfied by the defining stationary sequence (X, T), the sequence of potentials (g_n) and the stationary generator $(\mathcal{V}, \mathcal{U})$. □

Remark 6.1. We refer the reader to [**24**] for conditions equivalent to the minimality of $R(X, T)$. One of them is that for any non-empty open set U, there is an integer $n > 0$ such that $T^n(U) = X$. This condition is also known as weak specification.

Example 6.1. (Subshifts of finite type) Let $\Gamma = (\Gamma^{(0)}, \Gamma^{(1)})$ be a finite graph. We assume that each vertex receives and emits at least one edge. We let X be the space of one-sided infinite paths $x = x_1 x_2 \ldots$ and $T : X \to X$ be the one-sided shift $T(x_1 x_2 \ldots) = x_2 \ldots$.

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For $v \in \Gamma^{(0)}$, we let $Z(v)$ be the set of paths starting at v and for $e \in \Gamma^{(1)}$, we let $Z(e)$ be the set of paths having e as initial edge. We define $\mathcal{V} = \{Z(v) : v \in \Gamma^{(0)}\}$ and $\mathcal{U} = \{Z(e) : e \in \Gamma^{(1)}\}$. Then $(\mathcal{V}, \mathcal{U})$ is a generator for the sequence defined by T. This is a particular case of Example 5.1 where (V, E) is the stationary Bratteli diagram defined by Γ. It is known that $R(X, T)$ is minimal if and only if the graph Γ is primitive (i.e. there exists an integer $L \geq 1$ such that every pair of vertices (v, w) can be joined by a path of length L). This is also equivalent to T being topologically mixing. Bowen's condition for $g \in C(X, \mathbb{R}_+^*)$ is the existence of $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$x_1 \dots x_n = y_1 \dots y_n \Rightarrow g_n(x) \leq c g_n(y).$$

This is also a particular case of the next example.

Example 6.2. (Expansive maps) Let X be a compact metric space and let $T : X \to X$ be surjective and a local homeomorphism. One says that T is (positively) expansive if there exists $\epsilon > 0$ such that for every pair of points of X, $x \neq y$, there is $n \in \mathbb{N}$ such that $d(T^n x, T^n y) \geq \epsilon$.

LEMMA 6.2. *Let $T : X \to X$ be a surjective and expansive local homeomorphism. Then, there exist an integer $L \in \mathbb{N}$ and a T^L-cover $(\mathcal{V}, \mathcal{U})$ which is a generator for T^L.*

Proof. Replacing the metric by a topologically equivalent metric, we may assume that T is locally expanding: there exist $\tau > 0$ and $\lambda < 1$ such that $d(x, y) < \tau \Rightarrow d(x, y) \leq \lambda\, d(Tx, Ty)$. Let $(\mathcal{V}, \mathcal{U})$ be a finite T-cover such that the elements of $\mathcal{U}$ have diameter strictly less than τ. Let c be an upper bound for the diameter of the elements of $\mathcal{V}$. For each $L \in \mathbb{N}$, $(\mathcal{V}, \mathcal{U}_L)$ is a T^L-cover. For L sufficiently large, the diameter of each element of $\mathcal{U}_L$, which is majorized by $\lambda^L c$, will be strictly less than the Lebesgue number of the cover $\mathcal{V}$ and we will have $\mathcal{U}_L \prec \mathcal{V}$. Let $\Delta_n = \Delta_{\mathcal{U}_n}$. Since $d(\Delta_n) \leq \lambda^n c$ tends to 0, $(\mathcal{V}, \mathcal{U}_L)$ is a generator. □

Let $g \in C(X, \mathbb{R}_+^*)$ and let (g_n) be the corresponding sequence of potentials. We fix a compatible metric d on X and define

$$d_n(x, y) = \max\{d(Tx, Ty), \dots, d(T^{n-1}x, T^{n-1}y)\}.$$

and for $\epsilon > 0$,

$$\Delta_n = \{(x, y) \in X \times X : d_n(x, y) < \epsilon\}.$$

If ϵ is an expansiveness constant of T, (Δ_n) is a fundamental system of entourages. Bowen's condition is the boundedness of the sequence $(\mathrm{var}_*(g_n, \Delta_n))$.

7. *Application to the transfer operators*

We show in this section how our above results on the unique ergodicity of cocycles on AP equivalence relations imply parts of the Perron–Frobenius–Ruelle theorem (as stated, for example, in [**31**]) namely the existence and the uniqueness of the Perron eigenvalue and eigenvector of the transpose of the Ruelle operator

$$\mathcal{L}_g^* : C(X)^* \to C(X)^*.$$

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The setting is a dynamical system (X, T), where X is a compact space and T is a surjective local homeomorphism of X onto itself.

An arbitrary continuous function $g \in C(X, \mathbb{R}_+^*)$ defines both a cocycle D_g in $Z^1(G, \mathbb{R}_+^*)$, where the definition of $G = G(X, T)$ has been recalled earlier, and a *Ruelle* (or *transfer*) operator $\mathcal{L}_g : C(X) \to C(X)$ according to

$$\mathcal{L}_g(f)(x) = \sum_{Ty=x} g(y) f(y).$$

The relation between the Ruelle operator $\mathcal{L}_g$ and the cocycle D_g is given by the following elementary result.

PROPOSITION 7.1. [**28**, Proposition 4.2] *Let μ be a probability measure on X. The following conditions are equivalent:*

(i) *μ is quasi-invariant with respect to $G(X, T)$ and admits D_g as Radon–Nikodym derivative;*

(ii) $\mathcal{L}_g^* \mu = \mu$.

Quasi-invariance with respect to $G(X, T)$ is defined in the same way as quasi-invariance with respect to $R(X, T)$: it means that the measures $r^*\mu$ and $s^*\mu$ are equivalent, where $r, s : G(X, T) \to X$ are the projections. Since $R(X, T) \subset G(X, T)$, quasi-invariance with respect to $G(X, T)$ implies quasi-invariance with respect to $R(X, T)$ and the restriction of a Radon–Nikodym derivative relative to $G(X, T)$ is a Radon–Nikodym derivative relative to $R(X, T)$.

PROPOSITION 7.2. *Let (X, T) be a dynamical system as above and let $g \in C(X, \mathbb{R}_+^*)$. Let $G(X, T)$, $R(X, T)$, D_g and $\mathcal{L}_g$ be as above. Assume that the restriction D of D_g to $R(X, T)$ is uniquely ergodic. Then the eigenvalue problem*

$$\mathcal{L}_g^* \mu = \lambda \mu,$$

where μ is a probability measure, admits one and only one solution μ.

Proof. Let us first show the uniqueness of μ. Suppose that the probability measure μ is an eigenvector of the transpose of the Ruelle operator: $\mathcal{L}_g^* \mu = \lambda \mu$. Then $\lambda > 0$ and ${}^t\mathcal{L}_{\lambda^{-1} g} \mu = \mu$. Therefore, μ is quasi-invariant with respect to $G(X, T)$ with Radon–Nikodym derivative $D_{\lambda^{-1} g}$. Since $D_{\lambda^{-1} g}$ and D_g have the same restriction D to $R(X, T)$, μ is solution of the Radon–Nikodym problem for D. This shows that μ, if it exists, is unique.

Let us show that the unique solution μ of the Radon–Nikodym problem for D is a solution of the eigenvalue problem. The cocycle D on the AP equivalence relation $R(X, T)$ admits the sequence (g_n) as (unnormalized) potentials. The corresponding local potentials are $g_{n,n-1} = g$ for all n. In other words, according to Proposition 3.7 and Appendix A, the solutions of the Radon–Nikodym problem for D are in one-to-one correspondence with the states of the inductive limit of the sequence

$$C(X) \xrightarrow{\mathcal{L}_g} C(X) \to \cdots \to C(X) \xrightarrow{\mathcal{L}_g} C(X) \to \cdots .$$

Explicitly, a state is given by a sequence of measures (μ_n) such that μ_0 is a probability measure and $\mu_n = \mathcal{L}_g^* \mu_{n+1}$ for all $n \in \mathbb{N}$. The correspondence is $(\mu_n) \mapsto \mu_0$.

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It is then clear that, if (μ_n) is a state, so is $(\mu'_n = \mathcal{L}_g^* \mu_n/\lambda)$, where $\lambda = \mu_0(\mathcal{L}_g(1))$. By the unique ergodicity of D, we obtain that the above measure μ satisfies $\mu = \mathcal{L}_g^* \mu/\lambda$, where $\lambda = \mu_0(\mathcal{L}_g(1))$. □

A. *Appendix. State space of dimension groups*

Since the work of Elliott [**7**], dimension groups have been used as a convenient tool in the study of AF algebras and topological Markov chains [**23, 25**]. Our focus here is the state space of the dimension group. Classically, it describes the traces of the AF algebra. Our claim is that dimension groups are also useful in the study of Radon–Nikodym cocycles on AP relations. The results quoted here concerning dimension groups can be found in the monograph by Goodearl [**19**], in the general setting of partially ordered abelian groups. Another basic reference on dimension groups is by Effros [**6**]. We also give an improvement of a result of Török [**29**] on the uniqueness of a trace on an AF algebra. The purpose of this appendix is to present the results we need, in a setting which is the most appropriate to the Radon–Nikodym problem. We first introduce some notation.

When X is a compact space, $C(X)$ designates the real vector space of real-valued continuous functions on X endowed with the uniform norm $\|f\| = \sup_X |f(x)|$. We write $f \geq 0$ if $f(x) \geq 0$ for all $x \in X$ and we denote by $1 = 1_X$ the constant function $1(x) = 1$. This turns $(C(X), 1_X)$ into an ordered real vector space with order-unit. Note that $f \in C(X)$ is an order-unit iff $f \in C(X, \mathbb{R}_+^*)$.

Definition A.1. A bounded linear operator $A : C(Y) \to C(X)$, where X, Y are compact spaces is called:

- *positive* if $f \geq 0 \Rightarrow Af \geq 0$;
- *strongly positive* if it preserves order-units;
- *Markovian* if it is positive and $A1_Y = 1_X$.

When $X = \{1, \ldots, p\}$ and $Y = \{1, \ldots, q\}$, $C(X) = \mathbb{R}^p$, $C(Y) = \mathbb{R}^q$ and A is given by a matrix $(a(x, y)) \in M_{p,q}(\mathbb{R})$. Positivity means $a(x, y) \geq 0$ for all x, y and strong positivity means positivity and non-zero rows; it is a weaker condition than the strict positivity of the $a(x, y)$.

Let $(X_n), n \in \mathbb{N}$ be a sequence of compact spaces and let $\underline{A} = (A_n : C(X_{n-1}) \to C(X_n)), n = 1, 2, \ldots$ be a sequence of strongly positive operators. Our problem is to study the set of sequences $\mu = (\mu_n)$, where μ_n is a positive linear functional on $C(X_n)$, i.e. a measure on X_n, satisfying the recurrence relation $\mu_{n-1} = \mu_n A_n$ for all integers $n \geq 1$ as well as the normalization condition $\mu_0(1_{X_0}) = 1$. For $n \geq 1$, we introduce $u_n = A_n A_{n-1} \ldots A_1 1_{X_0}$ and we view $(\mathcal{E}_n = C(X_n), u_n)$ as an ordered real vector space with order-unit. The sequence $\underline{A}$ defines an inductive system of ordered real vector spaces with order-unit and we can consider its inductive limit $\mathcal{E} = \mathcal{E}(\underline{A}) = \varinjlim \mathcal{E}_n$. It is an ordered real vector spaces with order-unit u. We denote by $j_n : \mathcal{E}_n \to \mathcal{E}$ the canonical morphisms.

We introduce the state space of $(\mathcal{E}_n, u_n)$:

$$S_n = \{\tau \in C(X_n)^* : \text{ positive and } \tau(u_n) = 1\}.$$

It is a convex compact subset of $C(X_n)^*$ in the $*$-weak topology and a Choquet simplex.

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We are interested in its projective limit:

$$S = S(\underline{A}) = \text{proj lim}\, S_n = \{(\tau_n) : \tau_n \in S_n;\ \tau_n = \tau_{n+1} A_{n+1},\ n \in \mathbb{N}\}.$$

It is known [**19**, Proposition 6.14] that S is the state space of $(\mathcal{E}, u)$, i.e. the convex set of positive homomorphisms $\tau : \mathcal{E} \to \mathbb{R}$ such that $\tau(u) = 1$. The value of $\tau = (\tau_n) \in S$ on $f = j_n(f_n) \in \mathcal{E}$ is $\tau(f) = \tau_n(f_n)$. It is also known [**19**, Proposition 10.21] that S is a non-empty convex compact set and a Choquet simplex .

Definition A.2. We call $\mathcal{E} = \mathcal{E}(\underline{A}) = \varinjlim \mathcal{E}_n$ the *dimension group* and $S = S(\underline{A})$ the *state space* of the sequence $\underline{A} = (A_n)$.

Remark A.1. When $\underline{A} = (A_n)$ is a sequence of matrices with coefficients in $\mathbb{N}$, the dimension group $\mathcal{E}(\underline{A}) = \varinjlim(A_n : C(X_{n-1}, \mathbb{R}) \to C(X_n, \mathbb{R}))$ is a coarser invariant than the usual dimension group $K_0(\underline{A}) = \varinjlim(A_n : C(X_{n-1}, \mathbb{Z}) \to C(X_n, \mathbb{Z}))$ introduced by Elliott. More precisely $\mathcal{E}(\underline{A})$ is the realification of $K_0(\underline{A})$. However, the state spaces are the same.

Let $\underline{A} = (A_n : C(X_{n-1}) \to C(X_n))$ and $\underline{B} = (B_n : C(Y_{n-1}) \to C(Y_n))$ be two sequences of strongly positive operators as above. A morphism from $\underline{B}$ to $\underline{A}$ is a sequence $\underline{D} = (D_n)$ of positive operators $D_n : C(Y_n) \to C(X_n)$ such that $D_0(1_{Y_0}) = 1_{X_0}$ and for all integers $n \geq 1$, $A_n D_{n-1} = D_n B_n$.

This implies that the D_n are strongly positive. It induces a morphism $D : \mathcal{E}(\underline{B}) \to \mathcal{E}(\underline{A})$ and a morphism of their state spaces $D^* : S(\underline{A}) \to S(\underline{B})$ which sends (τ_n) into $(\tau_n D_n)$.

The following elementary observation reduces the problem to the Markovian case.

PROPOSITION A.1. *Let (A_n) be a sequence of strongly positive operators. Then, there exists an isomorphic sequence consisting of Markovian operators.*

Proof. It suffices to introduce the operator D_n of multiplication by u_n and

$$B_n = D_n^{-1} A_n D_{n-1}. \qquad \square$$

The following result is well known in the theory of dimension groups (see [**6**, Corollary 4.2] and [**19**, Corollary 4.10]), where its proof is based on an ordered group analogue of the Hahn–Banach theorem due to Goodearl and Handelman.

LEMMA A.2. *Let $\underline{B} = (B_n)$ be a sequence of Markovian operators, $\mathcal{E} = \mathcal{E}(\underline{B})$ and $S = S(\underline{B})$ as above. Let $f = j_m(f_m) \in \mathcal{E}$ and let $f_n = B_n B_{n-1} \ldots B_{m+1} f_m$ for $n \geq m+1$. The following conditions are equivalent:*

(i) *the sequence $(\| f_n \|)$ tends to 0;*

(ii) *$\mu(f) = 0$ for all $\mu \in S$.*

Definition A.3. We say that an inductive system $\underline{A} = (A_n)$ is *uniquely ergodic* (or that it has a *unique state*) if $S(\underline{A})$ has only one element.

The following corollary gives a convenient characterization of unique ergodicity.

COROLLARY A.3. *Let $\underline{B} = (B_n)$ be a sequence of Markovian operators. The following conditions are equivalent.*

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(i) *For all* $m \in \mathbb{N}$ *and all* $f \in C(X_m)$, $\mathrm{var}(B_n B_{n-1} \dots B_{m+2} B_{m+1} f)$ *goes to* 0 *as* n *goes to infinity.*

(ii) *For all* $m \in \mathbb{N}$ *and all* $f \in C(X_m)$, *there exists* $\mu_m(f) \in \mathbb{R}$ *such that* $\| B_n B_{n-1} \dots B_{m+2} B_{m+1} f - \mu_m(f) 1_{X_n} \|$ *goes to* 0 *as* n *goes to infinity.*

(iii) *The state space* $S(\underline{B})$ *defined above consists of a single element. If* $\nu = (\nu_n)$ *is this element, then for all* $m \in \mathbb{N}$, ν_m *agrees with* μ_m *defined in (ii).*

Proof. The implication (i) $\Rightarrow$ (ii) results from an elementary property of adjacent sequences. The reverse implication (ii) $\Rightarrow$ (i) is clear. The implication (ii) $\Rightarrow$ (iii) is easy and standard. The implication (iii) $\Rightarrow$ (ii) uses the non-trivial part of Lemma A.2. □

We give now a more practical sufficient condition for unique ergodicity, along the lines of the work of Török [**29**], who studied the uniqueness of traces on AF C^*-algebras. We first give a convenient estimate ensuring condition (i) of Corollary A.3.

COROLLARY A.4. *Let* $\underline{B} = (B_n)$ *be a sequence of Markovian operators. Suppose that there exists a sequence of positive numbers* (ϵ_n) *such that* $\sum \epsilon_n = \infty$ *and* $\mathrm{var}(B_n f) \leq (1 - \epsilon_n)\mathrm{var}(f)$ *for all* n *and all* $f \in C(X_{n-1})$. *Then* $\underline{B}$ *has a unique state.*

It remains to establish an estimate on the variation $\mathrm{var}(B_n f)$. We only consider here the case when the X_n are finite sets (i.e. the B_n are Markovian matrices). We have the following lemma, which is an improvement of [**29**, Lemma 2] and has a similar proof.

LEMMA A.5. *Let* X, Y *be finite sets and let* $B : C(Y) \to C(X)$ *be a Markovian operator defined by a Markovian matrix* $b : X \times Y \to \mathbb{R}$. *Let*

$$\epsilon = \min \left\{ \sum_{y \in Y_1} b(x, y) + \sum_{y \notin Y_1} b(x', y) : x, x' \in X,\ Y_1 \subset Y \right\}.$$

Then, for all $f \in C(I)$, $\mathrm{var}(Bf) \leq (1 - \epsilon)\mathrm{var}(f)$.

Proof. We first note that $\mathrm{var} : C(Y) \to \mathbb{R}_+$ is a semi-norm. It is invariant under translation by $\mathbb{R}1_Y$ and induces a norm on the quotient space $C(Y)/\mathbb{R}1_Y$. Therefore, it suffices to establish the inequality $\mathrm{var}(Bf) \leq (1 - \epsilon)$ for all f in

$$C = \{ f \in C(Y) :\ \forall y \in Y, 0 \leq f(y) \leq 1 \}.$$

By convexity, it suffices to check the inequality on the extremal points of C, which are the characteristic functions $f = 1_{Y_1}$, where $Y_1 \subset Y$. This results from the definition of ϵ. □

Remark A.2. This lemma gives, in the case $X = Y$, an estimate of the spectral gap of the Markovian matrix B and a Perron–Frobenius theorem for such a matrix. As noted in [**29**], the matrix need not be primitive: a necessary and sufficient condition for $\epsilon > 0$ is that B does not have any pair of orthogonal rows.

We deduce from the above lemma the following result, valid for strongly positive matrices rather than Markovian matrices, which will be used to show the convergence of stationary systems.

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LEMMA A.6. *Let X, Y be finite sets and let $A : C(Y) \to C(X)$ be an operator defined by a strongly positive matrix $a : X \times Y \to \mathbb{R}_+$. Let $u \in C(Y)$ be a strictly positive vector and let $B : C(Y) \to C(X)$ be the Markovian operator defined by the matrix*

$$b(x, y) = (Au(x))^{-1} a(x, y)u(y).$$

Then, for all $f \in C(Y)$,

$$\operatorname{var}(Bf) \leq \left(1 - \frac{\min(a)}{\max(a)}\right) \operatorname{var}(f).$$

Proof. The obvious inequalities

$$\sum_{y \in Y_1} a(x, y)u(y) \geq \min(a) \sum_{y \in Y_1} u(y) \quad \text{and} \quad Au(x) \leq \max(a) \sum_{y \in Y} u(y),$$

which are valid for all $Y_1 \subset Y$ and $x \in X$, imply that the constant ϵ of the previous lemma satisfies $\epsilon \geq \min(a)/\max(a)$. □

Thus, one gets a convenient condition on the sequence (A_n) ensuring unique state. It improves condition $\sum \epsilon_n \epsilon_{n-1} = \infty$ of [**29**, Corollary, p. 130].

COROLLARY A.7. *Let $\underline{A} = (A_n : C(I_{n-1}) \to C(I_n))$ be a sequence of strongly positive operators, where the I_n are finite spaces. Let ϵ_n be the ratio of the smallest matrix coefficient of A_n over its largest matrix coefficient. If $\sum \epsilon_n = \infty$, then $\underline{A}$ has a unique state.*

As an application, let us study the case of a stationary sequence $(A_n = A)$.

Example A.1. (The Perron–Frobenius theorem for primitive matrices) Let I be a finite set and let $a : I \times I \to \mathbb{R}_+$ be a primitive matrix; this means that there exists a positive integer L such that a^L is strictly positive. The matrix a defines a strongly positive linear operator $A : C(I) \to C(I)$. Let us first show that the stationary sequence $\underline{A} = (A_n = A)$ has a unique state. Since contracting does not change the state space, we consider instead the stationary sequence $\underline{A}' = (A'_n = A^L)$. Using the above notation, $\epsilon_n = \epsilon = \min(a^L)/\max(a^L)$ is constant and strictly positive; therefore the condition $\sum \epsilon_n = \infty$ is satisfied. This shows that $S(\underline{A})$ has a unique element $\mu = (\mu_n)$ (see [**6**, Theorem 6.1] for a similar proof). Recall that this is a sequence of (positive) measures on I satisfying $\mu_{n-1} = A^* \mu_n$, where A^* is the transpose of A and $\mu_0(1) = 1$. Since the sequence $\nu = (\nu_n)$ where $\nu_n = \lambda \mu_{n+1}$ and $\lambda = 1/\mu_1(1)$ is also a state, we must have $\mu = \nu$. This shows that μ_0 is an eigenvector of A^* for the eigenvalue λ and that $\mu_n = \lambda^{-n} \mu_0$. Conversely, if ν_0 is a probability measure on I which is an eigenvector of A^* for the eigenvalue ρ, then $\nu = (\rho^{-n} \nu_0)$ is a state of $\underline{A}$. By uniqueness of the state, $\nu_0 = \mu_0$ and $\rho = \lambda$. This shows the existence and the uniqueness of the Perron–Frobenius eigenvector and eigenvalue of A^*. This also gives the same result for A since A is primitive if and only if A^* is so. Our proof of the Perron–Frobenius–Ruelle theorem will follow the same pattern: we first establish that the state space of the stationary system defined by the Ruelle operator $\mathcal{L}_g : C(X) \to C(X)$ has a single element and then deduce the existence and the uniqueness of the Perron–Frobenius eigenvector and eigenvalue of the transpose $\mathcal{L}_g^*$.

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To conclude, let us quote from [**19**] some properties of the affine representation of dimension groups. Given an arbitrary sequence $\underline{B} = (B_n)$ of Markovian operators, we want to relate the inductive limit $\mathcal{E} = \mathcal{E}(\underline{B})$ and the state space $S = S(\underline{B})$. As usual in the theory of compact convex sets, we define $\mathrm{Aff}(S)$ as the space of continuous real-valued affine functions on S. It is a closed subspace of $C(S)$. We have the evaluation map $\theta : f \in \mathcal{E} \mapsto \hat{f} \in \mathrm{Aff}(S)$ defined by $\hat{f}(\mu) = \mu(f)$ for $\mu \in S$.

PROPOSITION A.8. (See [**19**, Ch. 7].) *With the above notation:*

(i) *the evaluation map θ has dense range in* $\mathrm{Aff}(S)$*; moreover the image of positive cone of $\mathcal{E}$ is dense in the positive cone of* $\mathrm{Aff}(S)$*;*

(ii) $\mathrm{Aff}(S)$ *is isomorphic as an ordered Banach space with unit to the completion of* $\mathcal{E}/\operatorname{Ker}\theta$ *with respect to the norm*

$$\|f\| = \inf\{\|f_n\| : f_n \in C(X_n),\ j_n(f_n) = f\}.$$

Proof. The statements of (i) are proved in [**19**, Theorem 7.9]. The equality of the norms $\|\hat{f}\|$ and $\|f\|$ in (ii), for $f \in \mathcal{E}$, results from [**19**, Proposition 7.12(e)]. □

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