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UNIVARIATE AND MULTIVARIATE STOCHASTIC COMPARISONS AND AGEING PROPERTIES OF THE GENERALIZED PÓLYA PROCESS

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Abstract

In this work we consider the generalized Pólya process with baseline intensity function r and parameters α and β recently studied by Cha (2014). The aim of this work is to provide both univariate and multivariate stochastic comparisons between two generalized Pólya processes with different baseline intensity functions and the same parameters α and β for the epoch and inter-epoch times of the two processes. The comparisons are analogous to stochastic comparisons in Belzunce *et al.* (2001) for two nonhomogeneous Poisson or pure-birth processes with different intensity functions. Moreover, we study both univariate and multivariate ageing properties of the epoch and inter-epoch times of the generalized Pólya process.

Keywords: Generalized Pólya process; epoch time; stochastic order; ageing property; generalized order statistics

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1. Introduction

The generalized Pólya process is a counting process with many applications for modeling real phenomena. Generalized Pólya processes can be regarded as an interesting and useful practical alternative to the nonhomogeneous Poisson process. The generalized Pólya process was introduced by Konno [31] and characterized by Cha [18]. In [18], a generalized Pólya process with baseline intensity function r (nonnegative function) and parameters α ($\alpha \geq 0$) and β ($\beta > 0$) was defined as a counting process $\{N(t): t \geq 0\}$ with $N(0) = 0$, and with a stochastic intensity λ_t given by

$$\lambda_t = (\alpha N(t^-) + \beta) r(t).$$

The stochastic intensity is defined as

$$\lambda_t = \lim_{h \to 0} \frac{\mathbb{P}(N(t, t+h) = 1 \mid H_{t^-})}{h},$$

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where $H_{t^-} = \{N(s)\colon 0 \le s < t\}$ is the internal filtration (history) of the process in $[0, t)$. The main difference between nonhomogeneous Poisson and generalized Pólya processes is that the latter counting process does not have independent increments. It is notable that specific choices of the baseline intensity function $r(\cdot)$ can be made in order to obtain suitable growth models of interest in stochastic modeling; see, e.g. [21] and [46]. As β tends to 0, the generalized Pólya process tends to a nonhomogeneous Yule process assuming that $N(0) = 1$, and when $\alpha = 0$ and $\beta = 1$, it becomes a nonhomogeneous Poisson process with intensity function $r(t)$. The generalized Pólya process for $\beta = 1$ has been considered in several papers in the literature under the name of the linear extension of a nonhomogeneous Yule process; see, e.g. [2], [3], [33], and [34].

The generalized Pólya process is a particular case of a nonhomogeneous pure-birth process (NHPBP) where an NHPBP is a counting process $\{N(t)\colon t \ge 0\}$ with a stochastic intensity given by

$$\lambda_t = r_{N(t^-)}(t)$$

with r_j, $j = 0, 1, \ldots$, being nonnegative functions which are failure rates of nonnegative random variables.

Our aim in this paper is to describe various conditions on the parameters of pairs of generalized Pólya processes under which the corresponding epoch or inter-epoch times are stochastically ordered in various senses. Moreover, we analyze ageing properties and multivariate dependence concepts for the epoch and inter-epoch times of generalized Pólya processes. Stochastic comparisons for pairs of generalized Pólya processes in this paper are analogous to the comparisons of Belzunce *et al.* [16] for the nonhomogeneous Poisson or pure-birth processes. For more information on the stochastic comparison of nonhomogeneous Poisson processes, we refer the reader to Belzunce *et al.* [16] and the references therein. Recall also that nonhomogeneous Poisson processes can appear as record values or as a particular case of generalized order statistics (GOS). For some interesting results on stochastic comparison of GOSs, see, e.g. [14], [17], [24], and [48]. Bassan and Spizzichino [7] offer an interesting look at multivariate dependence, ageing, and stochastic comparison for the residual lifetimes of different components of different age.

The stochastic comparison of various ageing measures associated with lifetimes has been an interesting and important topic in reliability theory and related fields. A useful tool employed in such studies is the theory of stochastic orders. Stochastic comparisons of system lifetimes, mixture distributions, order statistics, and so on have been studied intensively in the literature. For some recent surveys on the topics, see [27], [39], and [41] for system lifetimes, [28] for mixture distributions, [38] for generalized mixtures, and [14] and [29] for order statistics.

The general framework for the stochastic comparison of the epoch or inter-epoch times of two generalized Pólya processes is as follows. Let $\{N(t)\colon t \ge 0\}$ and $\{N^*(t)\colon t \ge 0\}$ be two generalized Pólya processes with parameters $(r(t), \alpha, \beta)$ and $(r^*(t), \alpha, \beta)$, respectively. Let X and X^* be nonnegative random variables with failure rates r and r^*, respectively. The conditions for r (r^*) to be a failure rate is that the function is nonnegative and

$$\int_0^\infty r(t)\,\mathrm{d}t = \infty \left(\int_0^\infty r^*(t)\,\mathrm{d}t = \infty \right).$$

Let $\Delta\,(F, \bar{F}, f)$ and $\Delta^*\,(F^*, \bar{F}^*, f^*)$ be the cumulative failure rate (CFR), cumulative distribution function (CDF), survival function (SF), probability density function (PDF)) of X and X^*, respectively. Let $S_1, S_2, \ldots$ and $S_1^*, S_2^*, \ldots$ ($T_1, T_2, \ldots$ and $T_1^*, T_2^*, \ldots$) be the epoch (inter-epoch) times of $\{N(t)\colon t \ge 0\}$ and $\{N^*(t)\colon t \ge 0\}$, respectively. Our study will show the

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following result: if $X \leq X^*$ in some stochastic order and the parameters of generalized Pólya processes satisfy some assumptions, then $S_n(T_n) \leq S_n^*(T_n^*)$ or $(S_1, \ldots, S_n)((T_1, \ldots, T_n)) \leq (S_1^*, \ldots, S_n^*)((T_1^*, \ldots, T_n^*))$ for some univariate or multivariate stochastic order.

The stochastic comparison of epoch and inter-epoch times of a nonhomogeneous Poisson process was studied in [16] and extended to other stochastic orders in [9]–[11]. Stochastic comparisons of inter-epoch times of nonhomogeneous Poisson processes can be found in [8] and [30]. Most of the results of stochastic comparisons of nonhomogeneous Poisson process can be found in [45] and the references therein. Univariate ageing properties related with the log concavity of epoch times of nonhomogeneous Poisson processes were studied in [42] and in the multivariate setting in [13].

The paper is organized as follows. In Section 2 we introduce the definitions of univariate ageing classes, multivariate positive dependence concepts, and univariate and multivariate stochastic orders. We also present some relevant auxiliary results for the rest of the paper. In Section 3 the univariate and multivariate distributions of epoch and inter-epoch times of a generalized Pólya process are developed. The ageing properties and stochastic comparisons of univariate epoch and inter-epoch times of a generalized Pólya process are provided in Section 4. In Section 5 the multivariate positive dependence and ageing properties of epoch and inter-epoch times of a generalized Pólya process together with a multivariate stochastic comparison are provided. Finally, in Section 6, the results obtained in this paper are applied to a shock model, the Pólya process, a repair process, and generalized order statistics.

2. Preliminaries

Throughout this paper we will use the following notation. For vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, we denote $\boldsymbol{x} \wedge \boldsymbol{y} = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$ and $\boldsymbol{x} \vee \boldsymbol{y} = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$, where $a \wedge b$ and $a \vee b$ are the minimum and maximum of two real numbers a and b, respectively. For vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $\boldsymbol{x} \leq (\geq) \boldsymbol{y}$ means that $x_j \leq (\geq) y_j$, $j = 1, 2, \ldots, n$. Henceforth, 'increasing' and 'decreasing' mean 'nondecreasing' and 'nonincreasing', respectively.

Totally positive functions play a significant role in reliability. A function $f: I_1 \times I_2 \to [0, \infty)$, with I_1 and I_2 being two ordered sets in $\mathbb{R}$, is said to be totally positive of order two (TP2) if $f(x_1, y_1) f(x_2, y_2) \geq f(x_1, y_2) f(x_2, y_1)$ for $x_1 < x_2$ and $y_1 < y_2$.

The next definition refers to some multivariate functions which are important in the context of this paper.

Definition 1. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function. We say that f is

(a) super modular if $f(\boldsymbol{x}) + f(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{y}) + f(\boldsymbol{x} \vee \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$;

(b) MTP2 (multivariate totally positive of order two) if f is nonnegative and $f(\boldsymbol{x}) f(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{y}) f(\boldsymbol{x} \vee \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ or, equivalently, $\ln f$ is super modular ($\ln 0 = -\infty$);

(c) log-convex (log-concave) if f is nonnegative and $f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq (\geq) f(\boldsymbol{x})^{\alpha} f(\boldsymbol{y})^{1-\alpha}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\alpha \in [0, 1]$ or, equivalently, $\ln f$ is convex (concave);

(d) directionally convex if f is super modular and componentwise convex.

Remark 1. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function. It is well known that when the second partial derivatives of f exist, f is super modular if and only if $\partial^2 f(\boldsymbol{x}) / \partial x_i \partial x_j \geq 0$, $1 \leq i < j \leq n$. It is also immediate that f is MTP2 if the previous condition holds for $\ln f$ on a convex open subset $A \subseteq \mathbb{R}^n$ being a lattice and $f(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin A$.

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Stochastic comparisons in this paper are based on the univariate and multivariate stochastic orders considered in the following definitions. We refer the reader to [37] and [45] for a comprehensive analysis of stochastic orders.

Definition 2. Let X and X^* be nonnegative random variables with CDFs F and F^*, SFs $\bar{F}$ and $\bar{F}^*$, PDFs f and f^*, failure rates r and r^*, reversed failure rates q and q^*, and mean residual lifetimes m and m^*, respectively. We say that X is less than or equal to X^* in

(a) usual stochastic order (written $X \leq_{\text{st}} X^*$) if $\bar{F}(x) \leq \bar{F}^*(x)$ for all $x \geq 0$ or, equivalently, $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(X^*)]$ for all increasing function ϕ;

(b) increasing convex order (written $X \leq_{\text{icx}} X^*$) if $\int_x^\infty \bar{F}(y)\,\mathrm{d}y \leq \int_x^\infty \bar{F}^*(y)\,\mathrm{d}y$ for all $x \geq 0$;

(c) hazard rate stochastic order (written $X \leq_{\text{hr}} X^*$) if $r(x) \geq r^*(x)$ for all $x \geq 0$ or, equivalently, $\bar{F}^*/\bar{F}$ is an increasing function;

(d) reversed hazard rate stochastic order (written $X \leq_{\text{rhr}} X^*$) if $q(x) \leq q^*(x)$ for all $x \geq 0$ or, equivalently, F^*/F is an increasing function;

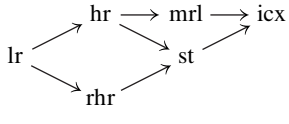
(e) likelihood ratio stochastic order (written $X \leq_{\text{lr}} X^*$) if f^*/f is an increasing function;

(f) mean residual lifetime stochastic order (written $X \leq_{\text{mrl}} X^*$) if $m(x) \leq m^*(x)$ for all $x \geq 0$;

(g) dispersive stochastic order (written $X \leq_{\text{disp}} X^*$) if $F^{*-1}F(x) - x$ is an increasing function of $x \geq 0$;

(h) stochastic variability order (written $X \leq_{\text{st:icx}} X^*$) if $X \leq_{\text{st}} X^*$ and $\text{var}[h(X)] \leq \text{var}[h(X^*)]$ for all increasing convex function h such that the variances exist.

Remark 2. The chain of relationships for the univariate stochastic orders in definition 2 is as follows.



Remark 3. The relationship between the dispersive stochastic order and other stochastic orders can be found in Theorems 2.6.5 and 2.6.6 of Belzunce *et al.* [12] for the hazard rate stochastic order and for the usual stochastic order, respectively.

Definition 3. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be random vectors having joint PDFs (SFs, CDFs) $f_{\boldsymbol{X}}$ ($\bar{F}_{\boldsymbol{X}}$, $F_{\boldsymbol{X}}$) and $f_{\boldsymbol{Y}}$ ($\bar{F}_{\boldsymbol{Y}}$, $F_{\boldsymbol{X}}$), respectively. We say that $\boldsymbol{X}$ is less than or equal to $\boldsymbol{Y}$ in

(a) upper orthant stochastic order (written $\boldsymbol{X} \leq_{\text{uo}} \boldsymbol{Y}$) if $\bar{F}_{\boldsymbol{X}}(\boldsymbol{x}) \leq \bar{F}_{\boldsymbol{Y}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$;

(b) lower orthant stochastic order (written $\boldsymbol{X} \leq_{\text{lo}} \boldsymbol{Y}$) if $F_{\boldsymbol{X}}(\boldsymbol{x}) \geq F_{\boldsymbol{Y}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$;

(c) multivariate stochastic order (written $\boldsymbol{X} \leq_{\text{st}} \boldsymbol{Y}$) if $\mathbb{E}[\phi(\boldsymbol{X})] \leq \mathbb{E}[\phi(\boldsymbol{Y})]$ for all functions ϕ increasing;

(d) dynamic hazard stochastic order (written $\boldsymbol{X} \leq_{\text{dyn-hr}} \boldsymbol{Y}$) if the conditional multivariate hazard rate of $\boldsymbol{X}$ is greater or equal than the multivariate conditional failure rate of $\boldsymbol{Y}$;

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(e) multivariate likelihood ratio stochastic order (written $\boldsymbol{X} \leq_{\mathrm{lr}} \boldsymbol{Y}$) if $f_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{Y}}(\boldsymbol{y}) \leq f_{\boldsymbol{X}}(\boldsymbol{x} \wedge \boldsymbol{y}) f_{\boldsymbol{Y}}(\boldsymbol{x} \vee \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$;

(f) increasing directionally convex stochastic order (written $\boldsymbol{X} \leq_{\mathrm{cdx}} \boldsymbol{Y}$) if $\mathbb{E}[\phi(\boldsymbol{X})] \leq \mathbb{E}[\phi(\boldsymbol{Y})]$ for all functions ϕ increasing directionally convex;

(g) multivariate dispersive stochastic order (written $\boldsymbol{X} \leq_{\mathrm{disp}} \boldsymbol{Y}$) if $\| \boldsymbol{Q}_{\boldsymbol{X}}(\boldsymbol{p}) - \boldsymbol{Q}_{\boldsymbol{X}}(\boldsymbol{q}) \|_2 \leq \| \boldsymbol{Q}_{\boldsymbol{Y}}(\boldsymbol{p}) - \boldsymbol{Q}_{\boldsymbol{Y}}(\boldsymbol{q}) \|_2$ for all $\boldsymbol{p}, \boldsymbol{q} \in (0, 1)^n$, where $\| \cdot \|_2$ is the Euclidian norm in $\mathbb{R}^n$ and $\boldsymbol{Q}_{\boldsymbol{X}}$ and $\boldsymbol{Q}_{\boldsymbol{Y}}$ are the standard constructions (see [12]) of the random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively.

Remark 4. The chain of relationships for the multivariate stochastic orders in Definition 3 is as follows.

$$\mathrm{lr} \longrightarrow \text{dyn-hr} \longrightarrow \mathrm{st} \begin{array}{l} \nearrow \mathrm{uo} \\ \searrow \mathrm{lo} \end{array}$$

The univariate ageing classes and multivariate dependence concepts together with some of their properties are stated in the following definitions and remarks. A useful reference for them is [32].

Definition 4. Let X be a nonnegative random variable and let F, $\bar{F}$, f, r, q, and m be its CDF, SF, PDF, failure rate, reversed failure rate, and mean residual lifetime, respectively. We say that X is

(a) increasing failure rate (IFR) if r is increasing or, equivalently, $\bar{F}$ is log-concave;

(b) decreasing failure rate (DFR) if r is decreasing or, equivalently, $\bar{F}$ is log-convex;

(c) decreasing reversed failure rate (DRFR) if q is decreasing or, equivalently, F is log-concave;

(d) increasing reversed failure rate (IRFR) if q is increasing or, equivalently, F is log-convex;

(e) log-concave (convex) if f is a log-concave (convex) function;

(f) decreasing mean residual lifetime (DMRL) if m is decreasing.

Remark 5. The log-concave, log-convex, IFR, DFR, and DRFR ageing classes of nonnegative random variables satisfy the following chain of relationships.

$$\text{log-concave (log-convex)} \begin{array}{l} \nearrow \text{IFR (DFR)} \\ \searrow \text{DRFR (IRFR)} \end{array}$$

The SF ($\bar{F}$), CFR (Δ), and PDF (f) of a random variable X can be written in terms of its failure rate r as

$$\begin{aligned} \bar{F}(x) &= \exp\left(- \int_0^x r(u)\, \mathrm{d}u \right) = \exp(-\Delta(x)), \\ f(x) &= r(x) \exp\left(- \int_0^x r(u)\, \mathrm{d}u \right) = r(x) \exp(-\Delta(x)), \\ \Delta(x) &= \int_0^x r(u)\, \mathrm{d}u = -\ln \bar{F}(x). \end{aligned} \tag{1}$$

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For the following definition we will use the notation $\boldsymbol{X} \uparrow_{\mathrm{st}} \boldsymbol{Y}$ if the conditional distribution of $\boldsymbol{X}$ given $\boldsymbol{Y} = \boldsymbol{y}$ is increasing in $\boldsymbol{y}$ (in the usual stochastic order).

Definition 5. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a random vector with $f_{\boldsymbol{X}}$ and $\bar{F}_{\boldsymbol{X}}$ its joint PDF and SF, respectively. We say that $\boldsymbol{X}$ is

(a) MTP2 if $f_{\boldsymbol{X}}$ is an MTP2 function;

(b) right corner set increasing (RCSI) if $\bar{F}_{\boldsymbol{X}}$ is MTP2;

(c) conditionally increasing in sequence (CIS) if $X_i \uparrow_{\mathrm{st}} (X_1, \ldots, X_{i-1})$, for all $i = 2, \ldots, n$;

(d) conditionally increasing (CI) if $\boldsymbol{X}_\pi = (X_{\pi(1)}, \ldots, X_{\pi(n)})$ is CIS for all permutation π on the set $\{1, \ldots, n\}$;

(e) associated if $\mathrm{cov}(g_1(\boldsymbol{X}), g_2(\boldsymbol{X})) = \mathbb{E}[g_1(\boldsymbol{X})g_2(\boldsymbol{X})] - \mathbb{E}[g_1(\boldsymbol{X})]\mathbb{E}[g_2(\boldsymbol{X})] \geq 0$ being $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$, both increasing (decreasing);

(f) positive upper orthant dependent (PUOD) if $\bar{F}_{\boldsymbol{X}}(x_1, \ldots, x_n) \geq \prod_{i=1}^n \bar{F}_{X_i}(x_i)$ for all $\boldsymbol{x} \in \mathbb{R}^n$;

(g) positive lower orthant dependent (PLOD) if $F_{\boldsymbol{X}}(x_1, \ldots, x_n) \geq \prod_{i=1}^n F_{X_i}(x_i)$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Remark 6. The following relations hold for multivariate dependence concepts in Definition 5(a)–(g).

$$\begin{array}{ccccccc} & & \mathrm{RCSI} & & & & \mathrm{PUOD} \\ \nearrow & & & & & \nearrow & \\ \mathrm{MTP2} & \longrightarrow & \mathrm{CI} & \longrightarrow & \mathrm{CIS} & \longrightarrow \mathrm{Association} & \longrightarrow \mathrm{PLOD} \end{array}$$

Remark 7. In parts (c), (e), and (f) it is easy to define negative dependence concepts. If, in part (c), we replace $\uparrow_{\mathrm{st}}$ by $\downarrow_{\mathrm{st}}$ we obtain the concept of conditionally decreasing in sequence (CDS). Similarly, if, in parts (e) and (f), we invert the sense of the inequalities we obtain the concepts of negative upper orthant dependent (NUOD) and negative lower orthant dependent (NLOD). It is known that CDS implies NLOD; see, e.g. [44]. It is also easy to see that '$\boldsymbol{X}$ is CDS' implies '$-\boldsymbol{X}$ is CDS'. Thus, CDS also implies NUOD since, if $\boldsymbol{X}$ is absolutely continuous, $\bar{F}_{\boldsymbol{X}}(x_1, \ldots, x_n) = F_{-\boldsymbol{X}}(-x_1, \ldots, -x_n)$. Thus, we have

$$\mathrm{CDS} \longrightarrow \mathrm{NUOD\ and\ NLOD}.$$

The following lemma plays a key role in deriving the results in Section 4.

Lemma 1. *Let r (r^*), Δ (Δ^*), and f (f^*) be the failure rate, CFR, and PDF of a random variable X (X^*).*

(a) *If X is log-convex and $c \geq 1$, then $r(x)\mathrm{e}^{-c\Delta(x)}$ is log-convex.*

(b) *If X is log-concave and $c \geq 1$, then $r(x)\mathrm{e}^{-c\Delta(x)}$ is log-concave and $1 - \mathrm{e}^{-c\Delta(x)}$ is log-concave.*

(c) *If $X \leq_{\mathrm{hr}} X^*$ then $\mathrm{e}^{-c\Delta^*(x)}/\mathrm{e}^{-c\Delta(x)}$ is increasing in x for $c \geq 0$.*

(d) *If $X \leq_{\mathrm{lr}} X^*$ and $c \geq 1$ then $r^*(x)\mathrm{e}^{-c\Delta^*(x)}/r(x)\mathrm{e}^{-c\Delta(x)}$ and $(1-\mathrm{e}^{-c\Delta^*(x)})/(1-\mathrm{e}^{-c\Delta(x)})$ are increasing in x.*

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Proof. (a) Since the condition that X is log-convex implies that X is DFR (see Remark 5), f and $\mathrm{e}^{-\Delta(x)}$ are log-convex functions. The result then follows by log-convexity of the previous functions and by the identity

$$r(x)\mathrm{e}^{-c\Delta(x)} = f(x)\mathrm{e}^{-(c-1)\Delta(x)}.$$

(b) The proof of the first part of this case is derived by the identity in the proof of case (a), taking into account that if X is log-concave then X is IFR (see Remark 5), and, as a result, f and $\mathrm{e}^{-\Delta(x)}$ are log-concave functions.

The second part is derived from the first one taking into account that $1-\mathrm{e}^{-c\Delta(x)}$ is the CDF of a random variable with PDF $cr(x)\mathrm{e}^{-c\Delta(x)}$, and the condition that X is log-concave implies that X is DRFR; see Remark 5.

(c) The function in this case is increasing if $cr(x) - cr^*(x) \geq 0$, and this property holds if $X \leq_{\mathrm{hr}} X^*$.

(d) The ratio $r^*(x)\mathrm{e}^{-c\Delta^*(x)}/r(x)\mathrm{e}^{-c\Delta(x)}$ can be written as

$$\frac{r^*(x)\mathrm{e}^{-c\Delta^*(x)}}{r(x)\mathrm{e}^{-c\Delta(x)}} = \frac{f^*(x)}{f(x)}\frac{\mathrm{e}^{-(c-1)\Delta^*(x)}}{\mathrm{e}^{-(c-1)\Delta(x)}}.$$

This identity is the product of two increasing functions, since $X \leq_{\mathrm{lr}} X^*$ and since case (c) of this lemma holds. Hence, the fact that $r^*(x)\mathrm{e}^{-c\Delta^*(x)}/r(x)\mathrm{e}^{-c\Delta(x)}$ is increasing in x is proved. The fact that $1-\mathrm{e}^{-c\Delta^*(x)}/1-\mathrm{e}^{-c\Delta(x)}$ is increasing in x is derived from the preceding result, taking into account that likelihood ratio stochastic order implies reversed failure rate stochastic order; see Remark 2. □

Remark 8. In the proof of Lemma 1(b), $\mathrm{e}^{-c\Delta(x)}$ and $cr(x)\mathrm{e}^{-c\Delta(x)}$ are the SF and PDF of a random variable of the proportional failure rate model with the baseline SF and failure rate $\mathrm{e}^{-\Delta(x)}$ and $r(x)$, respectively.

The log-convexity property of the integrals of log-convex functions is proved based on this auxiliary result.

Lemma 2. *Let $h\colon I^{n+m} \to [0,\infty)$ be a nonnegative function defined over the product space I^{n+m}, where I is an interval in the set of real numbers. Assume that $h(x_1,\ldots,x_n,y_1,\ldots,y_m)$ is a log-convex function in $(x_1,\ldots,x_n)$. For any function $g\colon I^m \to [0,\infty)$, it holds that*

$$\int_I \cdots \int_I h(x_1,\ldots,x_n,y_1,\ldots,y_m)g(y_1,\ldots,y_m)\,\mathrm{d}y_1\cdots\mathrm{d}y_m$$

is a log-convex function in $(x_1,\ldots,x_n)$.

Proof. Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in I^n$ and let $0 \leq \lambda \leq 1$. As h is log-convex in $(x_1,\ldots,x_n)$, we have

$$h(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, y_1,\ldots,y_m) \leq h(\boldsymbol{x}_1, y_1,\ldots,y_m)^{\lambda}h(\boldsymbol{x}_2, y_1,\ldots,y_m)^{1-\lambda}.$$

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Then, applying the Hölder inequality, we can derive

$$\begin{aligned}
&\int_I \cdots \int_I h(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, y_1, \ldots, y_m) g(y_1, \ldots, y_m)\, \mathrm{d}y_1 \cdots \mathrm{d}y_m \\
&\quad \le \int_I \cdots \int_I (h(\boldsymbol{x}_1, y_1, \ldots, y_m) g(y_1, \ldots, y_m))^{\lambda} (h(\boldsymbol{x}_2, y_1, \ldots, y_m) g(y_1, \ldots, y_m))^{1-\lambda} \\
&\qquad\qquad \times \mathrm{d}y_1 \cdots \mathrm{d}y_m \\
&\quad \le \left(\int_I \cdots \int_I h(\boldsymbol{x}_1, y_1, \ldots, y_m) g(y_1, \ldots, y_m)\, \mathrm{d}y_1 \cdots \mathrm{d}y_m \right)^{\lambda} \\
&\qquad \times \left(\int_I \cdots \int_I h(\boldsymbol{x}_2, y_1, \ldots, y_m) g(y_1, \ldots, y_m)\, \mathrm{d}y_1 \cdots \mathrm{d}y_m \right)^{1-\lambda}
\end{aligned}$$

and the conclusion follows. □

Finkelstein and Eusalova [23] introduced the concept of the bivariate weak IFR (DFR) random vector. In the multivariate case this concept is defined as follows. A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ it said to be multivariate weak IFR (DFR) if $\bar{F}_{\boldsymbol{X}}(x+t, \ldots, x+t)/\bar{F}_{\boldsymbol{X}}(x, \ldots, x)$ is decreasing (increasing) in x for all $t \ge 0$ with $\bar{F}_{\boldsymbol{X}}$ the joint SF of $\boldsymbol{X}$. Analogously, we say that a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is multivariate weak IRFR (DRFR) if $F_{\boldsymbol{X}}(x+t, \ldots, x+t)/F_{\boldsymbol{X}}(x, \ldots, x)$ is increasing (decreasing) in x for all $t \ge 0$ with $F_{\boldsymbol{X}}$ the joint CDF of $\boldsymbol{X}$. The multivariate weak IFR and DRFR (DFR and IRFR) concept is related to the log-concavity (convexity) of $\bar{F}_{\boldsymbol{X}}$ as we show in the following lemma.

Lemma 3. *Let $\boldsymbol{X}$ be a random vector with SF $\bar{F}_{\boldsymbol{X}}$ and CDF $F_{\boldsymbol{X}}$.*

(a) *If $\bar{F}_{\boldsymbol{X}}$ is log-concave (log-convex) then $\boldsymbol{X}$ is multivariate weak IFR (DFR).*

(b) *If $F_{\boldsymbol{X}}$ is log-concave (log-convex) then $\boldsymbol{X}$ is multivariate weak DRFR (IRFR).*

Proof. (a) Let $x \le y$ and $t \ge 0$. Then $(x+t, \ldots, x+t) = \alpha(x, \ldots, x) + (1-\alpha)(y+t, \ldots, y+t)$ and $(y, \ldots, y) = (1-\alpha)(x, \ldots, x) + \alpha(y+t, \ldots, y+t)$ with $\alpha = (y-x)/(y-x+t) \in [0, 1]$. Since $\bar{F}_{\boldsymbol{X}}$ is log-concave (log-convex), we have the following inequalities:

$$\begin{aligned}
\bar{F}_{\boldsymbol{X}}(x+t, \ldots, x+t) &\ge (\le) \bar{F}_{\boldsymbol{X}}(x, \ldots, x)^{\alpha} \bar{F}_{\boldsymbol{X}}(y+t, \ldots, y+t)^{1-\alpha}, \\
\bar{F}_{\boldsymbol{X}}(y, \ldots, y) &\ge (\le) \bar{F}_{\boldsymbol{X}}(x, \ldots, x)^{1-\alpha} \bar{F}_{\boldsymbol{X}}(y+t, \ldots, y+t)^{\alpha}.
\end{aligned}$$

Multiplying the terms in both inequalities, we obtain the desired conclusion, that is,

$$\bar{F}_{\boldsymbol{X}}(x+t, \ldots, x+t) \bar{F}_{\boldsymbol{X}}(y, \ldots, y) \ge (\le) \bar{F}_{\boldsymbol{X}}(x, \ldots, x) \bar{F}_{\boldsymbol{X}}(y+t, \ldots, y+t).$$

(b) This follows from case (a) by replacing the SF with the CDF. □

For other interesting properties related to the log concavity of multivariate SF and CDF, see [40].

3. Basic properties of epoch and inter-epoch times of a generalized Pólya process

The generalized Pólya process $\{N(t) : t \ge 0\}$ with baseline intensity function r and parameters α and β introduced in the previous section is a counting process such that, for $\alpha = 0$, it is a nonhomogeneous Poisson process with intensity function βr. Using the restarting property

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for the generalized Pólya process (see [18, Proposition 1]), it is straightforward to see that the joint density function of vector $(S_1, \dots, S_n)$ of the first n arrivals is given by

$$\begin{aligned} f_{S_1,S_2,\dots,S_n}(s_1, s_2, \dots, s_n) &= \prod_{i=1}^{n} (\beta + \alpha(i-1)) r(s_i) \mathrm{e}^{-(\beta+\alpha(i-1))(\Delta(s_i)-\Delta(s_{i-1}))} \\ &= \prod_{j=0}^{n-1} (\beta + j\alpha) \prod_{i=1}^{n-1} r(s_i) \mathrm{e}^{\alpha\Delta(s_i)} r(s_n) \mathrm{e}^{-(\beta+\alpha(n-1))\Delta(s_n)}, \end{aligned} \tag{2}$$

$0 = s_0 \le s_1 \le \cdots \le s_n$. Henceforth, $s_0 = S_0 = 0$ by convention.

The PDF of S_n can be written as

$$f_{S_n}(x) = r(x) g_n(\Delta(x)), \tag{3}$$

where g_n is the PDF of a nonnegative random variable Z_n given by

$$g_n(x) = \alpha \frac{(\beta/\alpha)_n}{(n-1)!} \mathrm{e}^{-\beta x} (1 - \mathrm{e}^{-\alpha x})^{n-1},$$

where $(a)_n = a(a+1)\cdots(a+(n-1))$. Indeed,

$$\begin{aligned} f_{S_n}(x) &= \int_0^x \int_0^{s_{n-1}} \cdots \int_0^{s_2} f_{S_1,S_2,\dots,S_n}(s_1, s_2, \dots, s_{n-1}, x) \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1} \\ &= \prod_{j=0}^{n-1} (\beta + j\alpha) r(x) \mathrm{e}^{-(\beta+(n-1)\alpha)\Delta(x)} \\ &\quad \times \int_0^x r(s_{n-1}) \mathrm{e}^{\alpha\Delta(s_{n-1})} \int_0^{s_{n-1}} r(s_{n-2}) \mathrm{e}^{\alpha\Delta(s_{n-2})} \cdots \int_0^{s_2} r(s_1) \mathrm{e}^{\alpha\Delta(s_1)} \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1}. \end{aligned}$$

The conclusion follows if the following identity holds:

$$\begin{aligned} &\int_0^x r(s_{n-1}) \mathrm{e}^{\alpha\Delta(s_{n-1})} \int_0^{s_{n-1}} r(s_{n-2}) \mathrm{e}^{\alpha\Delta(s_{n-2})} \cdots \int_0^{s_2} r(s_1) \mathrm{e}^{\alpha\Delta(s_1)} \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1} \\ &\quad = \frac{(\mathrm{e}^{\alpha\Delta(x)} - 1)^{n-1}}{\alpha^{n-1}(n-1)!}. \end{aligned}$$

The previous identity is a particular case, with $\varphi(x) = r(x)\mathrm{e}^{\alpha\Delta(x)}$ and $v = 0$, of the identity

$$\int_v^x \int_v^{t_n} \int_v^{t_{n-1}} \cdots \int_v^{t_2} \prod_{i=1}^{n} \varphi(t_i) \, \mathrm{d}t_1 \, \mathrm{d}t_2 \cdots \mathrm{d}t_n = \frac{(\int_v^x \varphi(t) \, \mathrm{d}t)^n}{n!},$$

which is derived straightforwardly by induction.

We denote by G_n and $\bar{G}_n$ the CDF and SF of Z_n. Since Z_n is log-concave, Z_n is both IFR ($\bar{G}_n$ log-concave) and DRFR (G_n log-concave); see Remark 5. Furthermore, the CDF and SF of S_n are $F_{S_n}(x) = G_n(\Delta(x))$ and $\bar{F}_{S_n}(x) = \bar{G}_n(\Delta(x))$. The SF of Z_n can be written as

$$\bar{G}_n(x) = \alpha \frac{(\beta/\alpha)_n}{(n-1)!} \mathrm{e}^{-\beta x} \int_0^\infty \mathrm{e}^{-\beta u} (1 - \mathrm{e}^{-\alpha(x+u)})^{n-1} \, \mathrm{d}u. \tag{4}$$

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Indeed,

$$\bar{G}_n(x) = \int_x^\infty g_n(y)\,\mathrm{d}y = \alpha \frac{(\beta/\alpha)_n}{(n-1)!} \int_x^\infty \mathrm{e}^{-\beta y}(1-\mathrm{e}^{-\alpha y})^{n-1}\,\mathrm{d}y$$

and (4) follows by the change of variable $u = y - x$. In the case of $\alpha = \beta$, the SF is given by

$$\bar{G}_n(x) = 1 - (1-\mathrm{e}^{-\alpha x})^n,$$

which is the SF of the maximum of n independent and identically distributed exponential random variables with failure rate α.

Finally, using again the restarting property of [18], it is easy to show that the conditional distribution of T_{n+1}, given that $S_n = u$, is

$$\mathbb{P}(T_{n+1} > x \mid S_n = u) = \mathrm{e}^{-(\beta+n\alpha)(\Delta(u+x)-\Delta(u))}.$$

Analogously,

$$\mathbb{P}(T_{n+1} > x \mid (T_1,\ldots,T_n) = (t_1,\ldots,t_n)) = \mathrm{e}^{-(\beta+n\alpha)(\Delta(s_n+x)-\Delta(s_n))}, \tag{5}$$

where $s_n = t_1 + \cdots + t_n$.

4. Stochastic comparisons of the nth epoch and inter-epoch times of generalized Pólya processes

In this section we analyze ageing properties (see Subsection 4.1) and stochastic comparisons (see Subsection 4.2) of univariate epoch and inter-epoch times of a generalized Pólya process. In what follows, it is interesting to have in mind the shape properties for compositions of convex (concave) functions; see, e.g. [36, p. 689].

4.1. Ageing properties

In the following theorem, we show ageing properties of univariate epoch and inter-epoch times of generalized Pólya processes under the usual conditions common in the literature dealing with ageing properties; see, e.g. [47, Theorem 4.2] and [20, Theorem 4.5] in the case of GOS.

Theorem 1. *Let $\{N(t)\colon t \geq 0\}$ be a generalized Pólya process with baseline intensity function $r(x)$ and parameters α and β. Let X be the associated random variable having its failure rate $r(x)$. Let S_n and T_n be the nth epoch time and nth inter-epoch time of the generalized Pólya process. Let T be a random time independent from the generalized Pólya process.*

(a) *If X is IFR then S_n is IFR.*

(b) *If X is DFR then S_n is DRFR.*

(c) *If X is log-concave and $\alpha, \beta \geq 1$, then S_n is log-concave.*

(d) *If X is DFR then T_n is DFR.*

(e) *If X is log-convex and $\beta + n\alpha \geq 1$, then T_{n+1} is log-convex, $n = 0, 1, \ldots$.*

(f) $S_n \leq_{\mathrm{lr}} S_{n+1}$.

(g) *If X is IFR then $T_{n+1} \geq_{\mathrm{st}} T_{n+2}$, $n = 0, 1, \ldots$.*

(h) *If T is DRFR and X is IFR then $N(T)$ is DRFR.*

(i) *If T is DFR and X is IFR then $N(T)$ is DFR.*

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Proof. (a) Since X is IFR ($\Delta(x)$ is convex) and Z_n is IFR ($\bar{G}_n$ is decreasing and log-concave), we deduce that $\bar{F}_{S_n}(x) = \bar{G}_n(\Delta(x))$ is log-concave and the property has been proved.

(b) As X is DFR ($\Delta(x)$ concave) and Z_n is DRFR (G_n is increasing and log-concave), hence, $F_{S_n}(x) = G_n(\Delta(x))$ is log-concave and the property has been proved.

(c) The conclusion for $n = 2, 3, \ldots$ follows from (3) and Lemma 1(b). For $n = 1$, note that $S_1 = T_1$, and the density function, applying (3), is given by $r(x)\mathrm{e}^{-\beta\Delta(x)}$. The conclusion follows again by Lemma 1(b) (only the condition $\beta \geq 1$ is needed).

(d) If X is DFR ($\Delta(u+x)$ is concave in x) then $\mathrm{e}^{-(\beta+n\alpha)(\Delta(u+x)-\Delta(u))}$ is log-convex in x. The SF of T_{n+1} can be written, using (5), as

$$\bar{F}_{T_{n+1}}(x) = \mathbb{E}[\mathrm{e}^{-(\beta+n\alpha)(\Delta(S_n+x)-\Delta(S_n))}] = \int_0^\infty \mathrm{e}^{-(\beta+n\alpha)(\Delta(u+x)-\Delta(u)} f_{S_n}(u)\,\mathrm{d}u. \tag{6}$$

Hence, by Lemma 2, $\bar{F}_{T_{n+1}}$ is a log-convex function and then T_{n+1} is DFR for $n = 1, 2, \ldots$. Note that as $\Delta(x)$ is concave, $T_1 = S_1$ is DFR since its SF is $\mathrm{e}^{-\beta\Delta(x)}$.

(e) The PDF of T_{n+1} derived from (6) is given by

$$f_{T_{n+1}}(x) = \int_0^\infty (\beta + \alpha n) r(u+x)\mathrm{e}^{-(\beta+n\alpha)(\Delta(u+x)-\Delta(u))} f_{S_n}(u)\,\mathrm{d}u.$$

Hence, by Lemma 2, the conclusion holds since $r(u+x)\mathrm{e}^{-(\beta+n\alpha)(\Delta(u+x)-\Delta(u))}$ is a log-convex function in x if X is log-convex by Lemma 1(a), and, thus, the $n = 2, 3, \ldots$ case is proven. The log-convexity of T_1 follows from Lemma 1(a) since its density function is $r(x)\mathrm{e}^{-\beta\Delta(x)}$.

(f) The ratio of the PDF of S_{n+1} and S_n from (3) is given by

$$\frac{f_{S_{n+1}}(x)}{f_{S_n}(x)} = \frac{g_{n+1}(\Delta(x))}{g_n(\Delta(x))} = \frac{(\beta+\alpha n)}{\alpha n}(1 - \mathrm{e}^{-\alpha\Delta(x)}),$$

which is increasing and the conclusion holds.

(g) If X is IFR then $\Delta(u+x) - \Delta(u)$ is increasing in u. Thus, applying (f), we have

$$\mathbb{E}[\mathrm{e}^{-(\beta+n\alpha)(\Delta(S_n+x)-\Delta(S_n))}] \geq \mathbb{E}[\mathrm{e}^{-(\beta+n\alpha)(\Delta(S_{n+1}+x)-\Delta(S_{n+1}))}].$$

Moreover, $\mathbb{E}[\mathrm{e}^{-(\beta+n\alpha)(\Delta(S_{n+1}+x)-\Delta(S_{n+1}))}] \geq \mathbb{E}[\mathrm{e}^{-(\beta+(n+1)\alpha)(\Delta(S_{n+1}+x)-\Delta(S_{n+1}))}]$. Hence, the conclusion holds using previous inequalities together with (6).

(h) The distribution of T_{n+1} given $S_n = u$ (see (5)) is decreasing in n in the stochastic order and also is decreasing in u when X is IFR. Moreover, $S_n \leq_{\mathrm{hr}} S_{n+1}$ since $S_n \leq_{\mathrm{lr}} S_{n+1}$ by case (f) in this theorem; see also Remark 2. Therefore, by [4, Theorem 2], it follows that for a random time independent of Pólya process T DRFR, $N(T)$ is discrete DRFR.

(i) This is consequence of [5, Theorem 4.3]. We have to check that the so-called condition 'Cdec' for a counting process holds. Condition Cdec is enunciated as follows. Let $\mathcal{F}_n$ be the past information of the arrival times S_i, $i = 1, 2, \ldots, n$. Then, for every $g\colon \mathbb{R}_+^{n+1} \to \mathbb{R}$, $n = 0, 1, \ldots$, which is increasing in the last component, we have (almost surely)

$$\mathbb{E}[g(T_2) \mid \mathcal{F}_1] \leq \mathbb{E}[g(T_1)], \qquad n = 0, \tag{7}$$

$$\mathbb{E}[g(T_1, \ldots, T_n, T_{n+2}) \mid \mathcal{F}_{n+1}] \leq \mathbb{E}[g(T_1, \ldots, T_n, T_{n+1}) \mid \mathcal{F}_n], \qquad n = 1, 2, \ldots. \tag{8}$$

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First note that the SF of T_2 given S_1 is, by (5),

$$\mathbb{P}(T_2 > x \mid S_1 = u) = \mathrm{e}^{-(\beta+\alpha)(\Delta(u+x)-\Delta(u))},$$

whereas the SF of T_1 is $\mathrm{e}^{-\beta\Delta(x)}$. As X is IFR then Δ is a convex function and, therefore, $\Delta(u+x)-\Delta(u) \geq \Delta(u)$. Thus, we easily observe that the SF of T_2 given S_1 is less than the SF of T_1 and, therefore, T_2 given S_1 is stochastically less than T_1 so that (7) holds. The proof of (8) is similar. Denote by $H(S_n, z)$, $z \geq 0$, the distribution function of T_{n+1} given S_n. We can write

$$\mathbb{E}[g(T_1, \ldots, T_n, T_{n+2}) \mid \mathcal{F}_{n+1}] = \int_{\mathbb{R}_+} g(T_1, \ldots, T_n, z)\, \mathrm{d}H(S_{n+1}, z), \tag{9}$$

$$\mathbb{E}[g(T_1, \ldots, T_n, T_{n+1}) \mid \mathcal{F}_n] = \int_{\mathbb{R}_+} g(T_1, \ldots, T_n, z)\, \mathrm{d}H(S_n, z). \tag{10}$$

By (5), we have $\bar{H}(S_n, z) = \mathrm{e}^{-(\beta+n\alpha)(\Delta(S_n+x)-\Delta(S_n))}$. As $S_n \leq S_{n+1}$ (almost surely) and Δ is a convex function, $\Delta(S_{n+1}+x)-\Delta(S_{n+1}) \geq \Delta(S_n+x)-\Delta(S_n))$, so that, trivially, we have $\bar{H}(S_{n+1}, z) \leq \bar{H}(S_n, z)$ (that is, T_{n+2} given S_{n+1} is stochastically less than T_{n+1} given S_n), which shows (8) using (9) and (10). Thus, the conclusion follows. □

4.2. Stochastic comparison

Stochastic comparisons of univariate epoch and inter-epoch times of generalized Pólya processes with different baseline intensity functions are listed in the following theorem. They are similar to other univariate stochastic comparisons performed for nonhomogeneous Poisson processes (see [16]) and GOS (see [24]).

Theorem 2. *Let $\{N(t): t \geq 0\}$ and $\{N^*(t): t \geq 0\}$ be two generalized Pólya processes with baseline intensity functions r and r^*, respectively, sharing the same parameters α and β. Let X (S_n, T_{n+1}) and X^* (S_n^*, T_{n+1}^*) be the associated random variables having failure rate r and r^*, (nth epoch time, $(n+1)$th inter-epoch time) of the processes $\{N(t): t \geq 0\}$ and $\{N^*(t): t \geq 0\}$, respectively. Then we have the following.*

(a) *If $X \leq_{\mathrm{st}} X^*$ then $S_n \leq_{\mathrm{st}} S_n^*$.*

(b) *If $X \leq_{\mathrm{hr}} X^*$ then $S_n \leq_{\mathrm{hr}} S_n^*$.*

(c) *If $X \leq_{\mathrm{lr}} X^*$ and $\alpha, \beta \geq 1$, then $S_n \leq_{\mathrm{lr}} S_n^*$.*

(d) *If $X \leq_{\mathrm{disp}} X^*$ then $S_n \leq_{\mathrm{disp}} S_n^*$.*

(e) *If $X \leq_{\mathrm{mrl}} X^*$, $m(0) = m^*(0)$, X or X^* DMRL and $0 \leq \beta \leq 1$, then $S_n \leq_{\mathrm{icx}} S_n^*$.*

(f) *If $X \leq_{\mathrm{hr}} X^*$ and X or X^* is DFR, then $T_{n+1} \leq_{\mathrm{st}} T_{n+1}^*$.*

(g) *If $X \leq_{\mathrm{lr}} X^*$, $\alpha, \beta \geq 1$, X, and X^* are DFR and $r(x) - r^*(x)$ is increasing, then $T_{n+1} \leq_{\mathrm{hr}} T_{n+1}^*$.*

(h) *If $X \leq_{\mathrm{st}} X^*$ then $N^*(t) \leq_{\mathrm{hr}} N(t)$ and $N^*(t) \leq_{\mathrm{rhr}} N(t)$ for all $t \geq 0$.*

Proof. (a) In this case $\Delta(x) \geq \Delta^*(x)$ since $X \leq_{\mathrm{st}} X^*$ (recall (1)). As a result

$$\bar{F}_{S_n}(x) = \bar{G}_n(\Delta(x)) \leq \bar{G}_n(\Delta^*(x)) = \bar{F}_{S_n^*}(x)$$

using the fact that $\bar{G}_n$ is decreasing and the claim is proved.

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(b) The failure rate of S_n is given by

$$r(x)r_n(\Delta(x)),$$

where r_n is the failure rate of Z_n. The facts that r_n is increasing (as Z_n is IFR), $r(x) \geq r^*(x)$, and $\Delta(x) \geq \Delta^*(x)$ (as $X \leq_{\mathrm{hr}} X^*$) imply that the failure rate of S_n is greater than

$$r^*(x)r_n(\Delta^*(x)),$$

which is the failure rate of S_n^* and the conclusion holds.

(c) The conclusion follows since the ratio $f_{S_n^*}(x)/f_{S_n}(x)$ is increasing in x taking into account (3) and Lemma 1(d).

(d) Since $F_{S_n} = G_n \circ \phi \circ F$ with $\phi(x) = -\ln(1-x)$, $0 \leq x \leq 1$, we derive

$$F_{S_n^*}^{-1} \circ F_{S_n} = F^{*-1} \circ F$$

and the conclusion follows obviously from the above identity.

(e) The SF of X in terms of the mean residual lifetime m can be written as (see, e.g. [36, p. 18])

$$\bar{F}(x) = \exp(-\Delta(x)) = \frac{m(0)}{m(x)} \exp\left(-\int_0^x \frac{1}{m(z)}\,\mathrm{d}z\right).$$

Therefore, applying (4) and by the change of the variable in the integral $w = H_m(y) = \exp(-\int_0^y 1/m(z))\,\mathrm{d}z)$, we have

$$\begin{aligned}
&\int_x^\infty \bar{F}_{S_n}(y)\,\mathrm{d}y \\
&\quad = \int_x^\infty \bar{G}_n(\Delta(u))\,\mathrm{d}u \\
&\quad = C_n \int_0^\infty \mathrm{e}^{-\beta u} \int_x^\infty \mathrm{e}^{-\beta\Delta(y)}(1-\mathrm{e}^{-\alpha(\Delta(y)+u)})^{n-1}\,\mathrm{d}y\,\mathrm{d}u \\
&\quad = C_n \int_0^\infty \mathrm{e}^{-\beta u} \int_x^\infty \left(\frac{m(0)}{m(y)}\right)^\beta \exp\left(-\beta\int_0^y \frac{1}{m(z)}\,\mathrm{d}z\right) \\
&\qquad\qquad \times \left(1-\left(\frac{m(0)}{m(y)}\right)^\alpha \exp\left(-\alpha\int_0^y \frac{1}{m(z)}\,\mathrm{d}z\right)\exp(-\alpha u)\right)^{n-1}\mathrm{d}y\,\mathrm{d}u \\
&\quad = C_n \int_0^\infty \mathrm{e}^{-\beta u} \int_0^{\exp(-\int_0^x (1/m((z)\,\mathrm{d}z)} \frac{m(0)^\beta}{m(H_m^{-1}(w))^{\beta-1}} w^{\beta-1} \\
&\qquad\qquad \times \left(1-\left(\frac{m(0)}{m(H_m^{-1}(w))}\right)^\alpha w^\alpha \mathrm{e}^{-\alpha u}\right)^{n-1}\mathrm{d}w\,\mathrm{d}u,
\end{aligned}$$

where $C_n = \alpha(\beta/\alpha)_n/(n-1)!$ and the conclusion follows since $\int_x^\infty \bar{F}_{S_n^*}(y)\,\mathrm{d}y$ is an upper bound for the previous expression based on the following facts:

- $m(x) \leq m^*(x)$, and, as a result, $H_m(x) \leq H_{m^*}(x)$ and $H_m^{-1}(w) \geq H_{m^*}^{-1}(w)$,
- $H_m(y)$ is decreasing in y, and
- m or m^* decreasing, $m(0) = m^*(0)$, $m(x) \leq m^*(x)$, and $0 \leq \beta \leq 1$.

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(f) If $X \leq_{\mathrm{hr}} X^*$, $\Delta(u+x)-\Delta(u) \geq \Delta^*(u+x)-\Delta^*(u)$, $u, x \geq 0$, then

$$\int_0^\infty \mathrm{e}^{-(\beta+\alpha n)(\Delta(u+x)-\Delta(u))} f_{S_n}(u)\,\mathrm{d}u \leq \int_0^\infty \mathrm{e}^{-(\beta+\alpha n)(\Delta^*(u+x)-\Delta^*(u))} f_{S_n}(u)\,\mathrm{d}u. \tag{11}$$

Let us assume without loss of generality that X^* is DFR, then $\phi(u) = \mathrm{e}^{-(\beta+\alpha n)(\Delta^*(u+x)-\Delta^*(u))}$ is increasing in u and since $S_n \leq_{\mathrm{st}} S_n^*$ by case (a) of this theorem, we have $\mathbb{E}[\phi(S_n)] \leq \mathbb{E}[\phi(S_n^*)]$ and the conclusion follows from (6) and (11).

(g) Denote $r_i(x) = r(x)(r^*(x))$, $\Delta_i(x) = \Delta(x)$ $(\Delta^*(x))$, $\bar{F}_i = \bar{F}$ $(\bar{F}^*)$, $\bar{F}_{T_{n+1,i}} = \bar{F}_{T_{n+1}}$ $(\bar{F}_{T_{n+1}^*})$, and $f_{n,i} = f_{S_n}(f_{S_n^*})$, $i = 1(i = 2)$. With this notation, and taking into account (6), we can write

$$\bar{F}_{T_{n+1,i}}(x) = \int_0^\infty \mathrm{e}^{-(\beta+\alpha n)(\Delta_i(u+x)-\Delta_i(u))} f_{n,i}(u)\,\mathrm{d}u.$$

We need to show that $\bar{F}_{T_{n+1,i}}$ is TP2 in (i, x). To this end, we will apply Theorem 5.1 of Karlin [26, p. 123]. Before doing so, we have to check the following conditions.

- $f_{n,i}(u)$ is TP2 in (i, u). This follows since $X \leq_{\mathrm{lr}} X^*$ and part (c).
- $\mathrm{e}^{-(\beta+\alpha n)(\Delta_i(u+x)-\Delta_i(u))}$ is TP2 in (i, x). This follows as $X \leq_{\mathrm{hr}} X^*$.
- $\mathrm{e}^{-(\beta+\alpha n)(\Delta_i(u+x)-\Delta_i(u))}$ is TP2 in (i, u). This follows since $r_1 - r_2$ is increasing.
- $\mathrm{e}^{-(\beta+\alpha n)(\Delta_i(u+x)-\Delta_i(u))}$ is TP2 in (u, x). This follows since r_i is decreasing for $i = 1, 2$.

(h) As both processes are counting processes, it is well known that $\mathbb{P}(N(t) \geq n) = \mathbb{P}(S_n \leq t) = G_n(\Delta(t))$ (similarly, $\mathbb{P}(N^*(t) \geq n) = \mathbb{P}(S_n^* \leq t) = G_n(\Delta^*(t))$ and $\mathbb{P}(N(t) \leq n) = \mathbb{P}(S_{n+1} > t) = \bar{G}_{n+1}(\Delta(t))$ (similarly, $\mathbb{P}(N^*(t) \leq n) = \mathbb{P}(S_{n+1}^* > t) = \bar{G}_{n+1}(\Delta^*(t))$) considering the notation in Section 3. Thus, the stochastic ordering in the hr (rhr) order between the two counting processes is equivalent to

$$\begin{aligned} G_n(\Delta(t))G_{n+1}(\Delta^*(t)) &\leq G_n(\Delta^*(t))G_{n+1}(\Delta(t)), \\ \bar{G}_{n+1}(\Delta(t))\bar{G}_{n+2}(\Delta^*(t)) &\leq \bar{G}_{n+1}(\Delta^*(t))\bar{G}_{n+2}(\Delta(t)), \end{aligned} \qquad n = 0, 1, \ldots.$$

The previous inequalities hold since $\Delta(t) \geq \Delta^*(t)$ $(X \leq_{\mathrm{st}} X^*)$, G_{n+1}/G_n is increasing, and $\bar{G}_{n+2}/\bar{G}_{n+1}$ is increasing. The monotonicity of the ratio of both CDFs and reliability functions follows as g_{n+1}/g_n is increasing and the likelihood ratio order implies hazard rate and reversed hazard rate stochastic orders; see Remark 2. Therefore, the conclusions in this case are proven, completing the proof. □

Remark 9. The hypothesis in Theorem 2(g) holds for $r^*(x) = 1/(a+x)(a \geq 2)$ and $r(x) = 1$.

5. Multivariate epoch and inter-epoch times

In this section we study multivariate dependence concepts (see Subsection 5.1) and stochastic comparisons (see Subsection 5.2) of multivariate epoch and inter-epoch times of the generalized Pólya process.

5.1. Dependence concepts

As in the case of the multivariate dependence properties for GOS in [17], in the following theorem we develop analogous results for the random vector of epoch or inter-epoch times of the generalized Pólya process.

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Theorem 3. *Let $\{N(t)\colon t \geq 0\}$ be a generalized Pólya process with baseline intensity function $r(x)$ and parameters α and β. Let X be the associated random variable having its failure rate $r(x)$ and let $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_{n+1})$ be the random vectors of the n epoch times and of the $n+1$ inter-epoch times of the Pólya process, respectively. Then*

(a) *$(S_1, \ldots, S_n)$ is MTP2;*

(b) *$\bar{F}_{(S_1,\ldots,S_n)}(F_{(S_1,\ldots,S_n)})$ is log-concave if X is IFR (DFR);*

(c) *$(T_1, \ldots, T_n)$ is CIS if X is DFR and CDS if X is IFR.*

Proof. (a) The joint density function of $(S_1, \ldots, S_n)$ described in (2) verifies, for $1 \leq i < j \leq n$, that

$$\frac{\partial^2 \ln f_{(S_1,\ldots,S_n)}(s_1, \ldots, s_n)}{\partial s_i \partial s_j} = 0, \qquad s_1 < s_2 < \cdots < s_n.$$

As $A = \{(s_1, \ldots, s_n)\colon s_1 < s_2 < \cdots < s_n\}$ is a convex open subset of a lattice, $(S_1, \ldots, S_n)$ is MTP2 (recall Remark 2).

(b) Using (2) and by the relation between PDF and SF, the joint SF function of $(S_1, \ldots, S_n)$ can be expressed as

$$\bar{F}_{(S_1,\ldots,S_n)}(x_1, x_2, \ldots, x_n) = \bar{G}_{1,\ldots,n}(\Delta(x_1), \ldots, \Delta(x_n)) \tag{12}$$

with

$$\begin{aligned}
&\bar{G}_{1,\ldots,n}(x_1, \ldots, x_n) \\
&\quad = \prod_{j=0}^{n-1} (\beta + \alpha j) \int_{x_n}^{\infty} \int_{x_{n-1}}^{y_n} \cdots \int_{x_2}^{y_3} \int_{x_1}^{y_2} \prod_{j=1}^{n-1} \mathrm{e}^{\alpha y_j} \mathrm{e}^{-(\beta+(n-1)\alpha)y_n} \, \mathrm{d}y_1 \, \mathrm{d}y_2 \cdots \mathrm{d}y_n \\
&\quad = \int_0^{\infty} \cdots \int_0^{\infty} \mathbf{1}_A(x_1, \ldots, x_n, y_1, \ldots, y_n) \prod_{j=1}^{n-1} \mathrm{e}^{\alpha y_j} \mathrm{e}^{-(\beta+(n-1)\alpha)y_n} \, \mathrm{d}y_1 \, \mathrm{d}y_2 \cdots \mathrm{d}y_n,
\end{aligned}$$

where $\mathbf{1}_A$ is the indicator function of the set $A = \{(x_1, \ldots, x_n, y_1, \ldots, y_n)\colon x_i \leq y_i \leq y_{i+1}, i = 1, 2, \ldots, n\}$ with $y_{n+1} = \infty$. Since A is a convex set, $\mathbf{1}_A(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a log-concave function and as $\prod_{j=1}^{n-1} \mathrm{e}^{\alpha y_j} \mathrm{e}^{-(\beta+(n-1)\alpha)y_n}$ is a log-concave function in $(y_1, \ldots, y_n)$, we have, by Prekopa's theorem (see [43]) applied to previous integral, $\bar{G}_{1,\ldots,n}$ is log-concave. Prekopa's theorem states that if $h\colon \mathbb{R}^m \times \mathbb{R}^n \to [0, \infty]$ is log-concave on $\mathbb{R}^m \times \mathbb{R}^n$, then

$$g(\boldsymbol{x}) = \int_{\mathbb{R}^m} h(\boldsymbol{x}, \boldsymbol{z}) \, \mathrm{d}\boldsymbol{z} \quad \text{is a log-concave function,}$$

provided that $g(\boldsymbol{x})$ is finite for each $\boldsymbol{x}$. Therefore, by (12), the conclusion holds for the SF function since $\bar{G}_{1,\ldots,n}$ is log-concave and decreasing, and Δ is convex since X is IFR.

Similarly to the previous case, the joint CDF of $(S_1, \ldots, S_n)$ can be written as

$$F_{(S_1,\ldots,S_n)}(x_1, x_2, \ldots, x_n) = G_{1,\ldots,n}(\Delta(x_1), \ldots, \Delta(x_n)),$$

where $G_{1,\ldots,n}$ is the following joint CDF:

$$\int_0^{\infty} \cdots \int_0^{\infty} \mathbf{1}_B(x_1, \ldots, x_n, y_1, \ldots, y_n) \prod_{j=1}^{n-1} \mathrm{e}^{\alpha y_j} \mathrm{e}^{-(\beta+(n-1)\alpha)y_n} \, \mathrm{d}y_1 \, \mathrm{d}y_2 \cdots \mathrm{d}y_n$$

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with B the set defined as $B = \{(x_1, \dots, x_n, y_1, \dots, y_n)\colon 0 \le y_i \le x_i,\ 0 \le y_1 \le \cdots \le y_n,\ i = 1, 2, \dots, n\}$. Then the conclusion holds for the CDF by the same arguments as in the case of the SF except for the fact that $G_{1,\dots,n}$ is increasing.

(c) Recalling (5), we can write

$$\mathbb{P}(T_{n+1} > x \mid (T_1, \dots, T_n) = (t_1, \dots, t_n)) = \mathrm{e}^{-(\beta+n\alpha)(\Delta(s_n+x)-\Delta(s_n))},$$

where $s_n = t_1 + \cdots + t_n$. If X is DFR, $\Delta(x)$ is concave and, thus, the previous expression is increasing in $(t_1, \dots, t_n)$. The conclusion for X IFR is analogous. □

Remark 10. Under the assumptions of Theorem 3(b), $(S_1, \dots, S_n)$ is multivariate weak IFR (DRFR) if X is IFR (DFR) by Lemma 3.

5.2. Stochastic comparison

Belzunce *et al.* [16] established conditions for the stochastic comparison of the random vectors of epoch and inter-epoch times of general pure-birth processes which include generalized Pólya processes. The conditions of [16] are fulfilled for the generalized Pólya process (see Theorems 3.12 and 4.2 of [16]) in the case of the multivariate dynamic hazard rate for the epoch times and in the case of the multivariate stochastic order for the inter-epoch times. In the theorem below we apply the results of [16] for the particular case of a generalized Pólya process.

Theorem 4. *Let $\{N(t)\colon t \ge 0\}$ and $\{N^*(t)\colon t \ge 0\}$ be two generalized Pólya processes with baseline intensity functions r and r^*, respectively, sharing the same parameters α and β. Let X and X^* be the associated random variables of the process $\{N(t)\colon t \ge 0\}$ and $\{N^*(t)\colon t \ge 0\}$ having their failure rate functions r and r^*, respectively. Let $(S_1, \dots, S_n)((T_1, \dots, T_{n+1}))$ and $(S_1^*, \dots, S_n^*)((T_1^*, \dots, T_{n+1}^*))$ be the random vectors of the $n(n+1)$ epoch (inter-epoch) times of $\{N(t)\colon t \ge 0\}$ and $\{N^*(t)\colon t \ge 0\}$, respectively. Then*

(a) *if $X \le_{\mathrm{hr}} X^*$ then $(S_1, S_2, \dots, S_n) \le_{\text{dyn-hr}} (S_1^*, S_2^*, \dots, S_n^*)$;*

(b) *if $r(u) \ge r^*(u+x)$ for all $u, x \ge 0$, then $(T_1, T_2, \dots, T_{n+1}) \le_{\mathrm{st}} (T_1^*, T_2^*, \dots, T_{n+1}^*)$.*

The next theorem concerns the stochastic comparison in the multivariate stochastic order of a pair of generalized Pólya processes and it has weaker assumptions than the corresponding result of Belzunce *et al.* [16]; see Remark 11.

Theorem 5. *Let $\{N(t)\colon t \ge 0\}$ and $\{N^*(t)\colon t \ge 0\}$ be two generalized Pólya processes with baseline intensity functions r and r^*, respectively, sharing the same parameters α and β. Let X and X^* be the associated random variables of the process $\{N(t)\colon t \ge 0\}$ and $\{N^*(t)\colon t \ge 0\}$, having their failure rate functions r and r^*, respectively. Let $(S_1, \dots, S_n)$ and $(S_1^*, \dots, S_n^*)$ be the random vectors of the n epoch times of $\{N(t)\colon t \ge 0\}$ and $\{N^*(t)\colon t \ge 0\}$, respectively. We have the following.*

(a) *If $X \le_{\mathrm{st}} X^*$ then $(S_1, S_2, \dots, S_n) \le_{\mathrm{st}} (S_1^*, S_2^*, \dots, S_n^*)$.*

(b) *If $X \le_{\mathrm{mrl}} X^*$, $m(0) = m^*(0)$, X or X^* DMRL, and $0 \le \beta \le 1$, then we have $(S_1, S_2, \dots, S_n) \le_{\mathrm{icdx}} (S_1^*, S_2^*, \dots, S_n^*)$.*

(c) *If $X \le_{\mathrm{disp}} X^*$ then $(S_1, S_2, \dots, S_n) \le_{\mathrm{disp}} (S_1^*, S_2^*, \dots, S_n^*)$.*

(d) *If $X \le_{\mathrm{st}} X^*$ and $X \le_{\mathrm{disp}} X^*$, then $\phi(S_1, S_2, \dots, S_n) \le_{\text{st:icx}} \phi(S_1^*, S_2^*, \dots, S_n^*)$ for all increasing directionally convex functions ϕ.*

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Proof. (a) Let $(Z_1, \ldots, Z_n)$ be a random vector with SF given by $\bar{G}_{1,\ldots,n}$ in (12). Obviously, for an arbitrary multivariate function ϕ,

$$\mathbb{E}[\phi(S_1, \ldots, S_n)] = \mathbb{E}[\phi(\Delta^{-1}(Z_1), \ldots, \Delta^{-1}(Z_n))],$$
$$\mathbb{E}[\phi(S_1^*, \ldots, S_n^*)] = \mathbb{E}[\phi(\Delta^{*-1}(Z_1), \ldots, \Delta^{*-1}(Z_n))].$$

On the other hand, if $X \leq_{\mathrm{st}} X^*$ we have $\Delta(x) \geq \Delta^*(x)$ for all $x \geq 0$ by (1), hence, $\Delta^{-1}(y) \leq \Delta^{*-1}(y)$ for all y such that $y = \Delta(x)$. Therefore, using the previous identities in this proof, we have, for ϕ multivariate and increasing,

$$\begin{aligned}\mathbb{E}[\phi(S_1, \ldots, S_n)] &= \mathbb{E}[\phi(\Delta^{-1}(Z_1), \ldots, \Delta^{-1}(Z_n))] \\ &\leq \mathbb{E}[\phi(\Delta^{*-1}(Z_1), \ldots, \Delta^{*-1}(Z_n))] \\ &= \mathbb{E}[\phi(S_1^*, \ldots, S_n^*)]\end{aligned}$$

and the conclusion holds.

(b) It holds that $(S_1, \ldots, S_n)$ and $(S_1^*, \ldots, S_n^*)$ share a common copula, by the previous case, which is conditionally increasing since $\bar{G}_{1,\ldots,n}$ is the SF of an MTP2 random vector using Theorem 3(a), thus, the conclusion follows using Theorem 2(e) and applying Theorem 2.4 of [6].

(c) It holds that $(S_1, \ldots, S_n)$ and $(S_1^*, \ldots, S_n^*)$ share a common copula by case (a), which is differentiable since $\bar{G}_{1,\ldots,n}$ is differentiable, hence, the conclusion follows from Theorem 2(d) and applying Theorem 2 of [1].

(d) It holds that $(S_1, \ldots, S_n)$ and $(S_1^*, \ldots, S_n^*)$ share a common copula by case (a), which is positive associated since $\bar{G}_{1,\ldots,n}$ is MTP2 (see Theorem 3(a) and Remark 6), thus, the conclusion follows from Theorem 2(a) and 2(d) and applying Theorem 1 of [15]. □

Remark 11. Theorem 5(a) extends Theorem 3.11 of [16] for an NHPBP in the particular case of a generalized Pólya process, since the condition of [16] in the case of a generalized Pólya process corresponds to $X \leq_{\mathrm{hr}} X^*$, which is stronger than the condition in Theorem 5(a); see Remark 2. Furthermore, under the assumptions in Theorem 5, by Remark 4,

$$(S_1, S_2, \ldots, S_n) \leq_{\mathrm{uo}} (S_1^*, S_2^*, \ldots, S_n^*) \quad \text{and} \quad (S_1, S_2, \ldots, S_n) \leq_{\mathrm{lo}} (S_1^*, S_2^*, \ldots, S_n^*).$$

To the best of the authors' knowledge, the results of Theorem 5(b) and 5(c) have not been reported in the literature and, accordingly, they provide new insights in the particular case of a generalized Pólya process.

The stochastic comparison in the multivariate likelihood ratio does not hold in general. In the next counterexample we show the previous claim. We consider two generalized Pólya processes with the same parameters α and β, and baseline intensity functions $r(x) = \lambda$ and $r^*(x) = \mu$ with $\lambda > \mu$. Obviously, the associated random variables X and X^* are exponentials and verify that $X \leq_{\mathrm{lr}} X^*$. The joint density functions of the first two epoch times for both processes are given by $f_{(S_1,S_2)}(s_1, s_2) = \alpha(\alpha+\beta)\mathrm{e}^{\alpha\lambda s_1}\mathrm{e}^{-(\beta+\alpha)\lambda s_2}$ and $f_{(S_1^*,S_2^*)}(s_1, s_2) = \alpha(\alpha+\beta)\mathrm{e}^{\alpha\mu s_1}\mathrm{e}^{-(\beta+\alpha)\mu s_2}$. In the case that $(S_1, S_2) \leq_{\mathrm{lr}} (S_1^*, S_2^*)$, the ratio $f_{(S_1^*,S_2^*)}(s_1, s_2)/f_{(S_1,S_2)}(s_1, s_2)$ would be increasing in (s_1, s_2) and in this case

$$\frac{f_{(S_1^*,S_2^*)}(s_1, s_2)}{f_{(S_1,S_2)}(s_1, s_2)} = \mathrm{e}^{(\mu-\lambda)s_1}\mathrm{e}^{(\beta+\alpha)(\lambda-\mu)s_2}$$

is increasing in s_2 and decreasing in s_1 and, accordingly, $(S_1, S_2) \not\leq_{\mathrm{lr}} (S_1^*, S_2^*)$.

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6. Applications

In this section we consider the applications of our results to a shock model, the Pólya process, a repair process, and generalized order statistics.

Application 1. A generalized Pólya process has been used for modeling shock processes; see [19]. For instance, the cumulative wear model with shocks following a generalized Pólya process was studied in [19]. In such a model, the cumulative wear at time t is given by $\sum_{i=1}^{N(t)} Y_i$, where Y_i is the wear caused by the ith shock and $N(t)$ is the number of shocks until time t, which follows a generalized Pólya process and is independent from the Y_i. When we consider two generalized Pólya processes $\{N(t),\ t \geq 0\}$ and $\{N^*(t),\ t \geq 0\}$ with parameters $(\alpha, \beta, r(x))$ and $(\alpha, \beta, r^*(x))$, respectively, such that $X \leq_{\mathrm{st}} X^*$, if the Y_i are IFR, then, combining Theorem 2(h) and Theorem 1.B.7 of [45], we have

$$\sum_{i=1}^{N(t)} Y_i \geq_{\mathrm{hr}} \sum_{i=1}^{N^*(t)} Y_i.$$

Application 2. A generalized Pólya process reduces to a Pólya process when $\beta = 1$ and $r(t) = 1/(1+\alpha t)$; see [22] and [31]. Accordingly, our results on stochastic comparisons and ageing properties can be applied to a Pólya process. For instance, given two Pólya processes with parameters α and α^*, respectively, such that $\alpha \leq \alpha^*$ then the corresponding Pólya processes satisfy $\Lambda(t) \geq \Lambda^*(t)$ or, equivalently, $X \leq_{\mathrm{st}} X^*$ and also $r(t) \geq r^*(t+x), t, x \geq 0$. Therefore, the epoch times of Pólya processes with parameters α and α^*, when $\alpha \leq \alpha^*$, are ordered in multivariate stochastic order by Theorem 5(a) and an analogous result holds for the inter-epoch times of both Pólya processes based on Theorem 4(b).

Application 3. In [18], it was mentioned that the generalized Pólya process $(\alpha, \beta, r(x))$ can model a new repair process of a system. In such a case, $\beta r(x)$ is the failure rate of the time until the first failure or initial failure rate of the system, and the conditional failure rate at $x+u$ after the nth failure occurred at time x is given by $(\alpha n + \beta) r(x+u)$, $n = 1, 2, \ldots,\ u \geq 0$. Several examples of this type of repair process can be found in [35]. Comparing this repair process with the classical minimal repair process, it can be seen that this repair process is worse than the minimal repair process; see also [18]. The parameter α in this repair process can be interpreted as the degree of repair and the repair process becomes worse as α increases. Using the results obtained in this work, we can compare the epoch times for failures of two systems which are subject to the new repair process having the same degree of repair and different initial failure rates based on the comparison of the initial failure rates under different stochastic orders.

Application 4. The concept of GOS defined below was introduced by Kamps [25]. This concept has received much attention in the last few decades with a great variety of applications in different fields.

Definition 6. Let $n \in \mathbb{N}$, $k \geq 1$, $m_1, \ldots, m_{n-1} \in \mathbb{R}$, and $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in \{1, \ldots, n-1\}$ and let $\tilde{m} = (m_1, \ldots, m_{n-1})$ if $n \geq 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary for $n = 1$). If the random vector $(U_{(1,n,\tilde{m},k)}, \ldots, U_{(n,n,\tilde{m},k)})$ has the joint PDF given by

$$k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{j=1}^{n-1} (1-u_j)^{m_j}\right)(1-u_n)^{k-1}, \qquad 0 \leq u_1 \leq \cdots \leq u_n \leq 1,$$

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then we call it uniform GOS. Now, given a CDF F, the random vector

$$(X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}) = (F^{-1}(U_{(1,n,\tilde{m},k)}), \ldots, F^{-1}(U_{(n,n,\tilde{m},k)}))$$

is called the GOS based on F. If F has PDF f, the joint PDF of $(X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1}\bar{F}^{m_j}(x_j)f(x_j)\right)\bar{F}(x_n)^{k-1}f(x_n), \qquad F^{-1}(0) \le x_1 \le \cdots \le x_n \le F^{-1}(1).$$

It is straightforward that the random vector of the epoch times of a generalized Pólya process with parameter $(r(t), \alpha, \beta)$ are the GOS based on F with failure rate r and with the corresponding parameters of GOS $k = \alpha(n-1)+\beta$, $m_1 = \cdots = m_{n-1} = -(\alpha+1)$, therefore, $\gamma_r = \alpha(r-1)+\beta$, $1 \le r \le n-1$. Hence, the random vector of epoch times of a generalized Pólya process with parameter $(r(t), \alpha, \beta)$ are the GOS based on F or, equivalently,

$$(S_1, \ldots, S_n) = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}) \quad \Longleftrightarrow \quad \beta \ge 1.$$

Particular cases of GOS are ordinary order statistics, sequential order statistics, kth record value, and Pfeifer record values; see [25]. It is thus immediate that in the case of a generalized Pólya process its epoch times are the epoch times of Pfeifer record values with parameter $\beta_n = \alpha(n-1)+\beta$ in the notation of [25] and, accordingly, the results in this paper apply to this record and inter-record values.

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