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Dynamics on the graph of the torus parametrization

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Abstract. Model sets are projections of certain lattice subsets. It was realized by Moody [Uniform distribution in model sets. *Canad. Math. Bull.* **45**(1) (2002), 123–130] that dynamical properties of such a set are induced from the torus associated with the lattice. We follow and extend this approach by studying dynamics on the graph of the map that associates lattice subsets to points of the torus and then we transfer the results to their projections. This not only leads to transparent proofs of known results on model sets, but we also obtain new results on so-called weak model sets. In particular, we prove pure point dynamical spectrum for the hull of a weak model set of maximal density together with the push forward of the torus Haar measure under the torus parametrization map, and we derive a formula for its pattern frequencies.

1. *Introduction*

Let G and H be locally compact second countable abelian groups, for example, $\mathbb{Z}^d$ or $\mathbb{R}^d$. Each pair $(\mathcal{L}, W)$, consisting of a lattice $\mathcal{L} \subset G \times H$ and a relatively compact subset W of H, also called the window, defines a weak model set $\Lambda(\mathcal{L}, W)$ as the set of all points $x_G \in G$ for which there exists a point $x_H \in W$ such that $(x_G, x_H) \in \mathcal{L}$. Under additional assumptions on $\mathcal{L}$ and W, the structure of such sets has attracted much attention in the mathematics and physics literature.

Model sets, which satisfy $\mathrm{int}(W) \neq \emptyset$, have been introduced by Meyer [**38**, **39**] in the context of number theory and harmonic analysis. After the discovery of quasicrystal alloys, model sets, as a mathematical abstraction of these structures, have been advertised and developed by Moody (see, e.g., [**41**, **42**]). We also mention the recent comprehensive monograph by Baake and Grimm [**2**]. The name *weak model set* was coined by Moody [**43**]. In fact, weak model sets were initially studied by Schreiber [**55**, **56**] (see also [**26**], for background).

For a given cut-and-project scheme $(G, H, \mathcal{L})$ with window W, the torus $\hat{X}$ associated with the lattice $\mathcal{L}$ may be used to parametrize model sets arising from translations in G and shifts of the window. This is named the *torus parametrization* in [**3**, **24**], which

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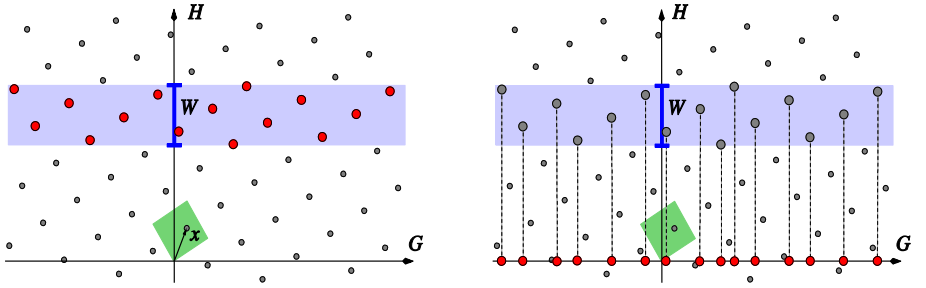
was also studied in [**7**, **12**, **21**, **48**, **50**, **54**]. For weighted model sets with continuous compactly supported weight functions, it was investigated in [**36**]. Some results about the torus parametrization of general weak model sets appear in [**43**].

In this paper, we consider weak model sets with compact windows, along with their associated dynamical systems, which are also called their hulls. The hull is the vague translation orbit closure of the weak model set if one identifies the point set with a measure that puts unit mass to every point. In contrast to previous approaches, we will study hulls in the space $\mathcal{M}$ of locally finite measures on $G \times H$, instead of G, and in an even larger space, namely, on the graph of the torus parametrization, which takes values in $\hat{X} \times \mathcal{M}$. This highlights connections with the dynamics of skew-product dynamical systems with monotone fibre maps, because the latter system is of this type. Our approach simplifies many arguments and makes the origin of certain standard assumptions on model sets more transparent, which only enter when projecting from $G \times H$ to G. Extensions to relatively compact windows are discussed as well.

A central result in the theory of model sets with topologically regular windows is almost automorphy of their hull, if the window boundaries have empty intersection with the projected lattice (see Robinson [**48**–**50**] and also [**1**, **21**]). We give an alternative proof of this result by our abstract dynamical systems approach. We will, however, not investigate, in this paper, how almost automorphy characterizes certain classes of model sets (compare [**1**, Theorem 3.16]). On the measure theoretic side, a central result by Schlottmann [**54**] relates to so-called regular model sets. These are weak model sets whose window boundaries have vanishing Haar measure. Schlottmann's result expresses that the hull of a regular model set has pure point dynamical spectrum. This, in turn, implies that any regular model set has pure point diffraction spectrum [**6**, **35**]. A generalization for weak model sets was obtained by Moody [**43**]. His result, when restricted to weak model sets with compact window, states that 'almost all' weak model sets have pure point diffraction spectrum and so-called maximal density. The importance of weak model sets of maximal density was realized recently by Strungaru [**58**], who argued that such model sets have pure point diffraction spectrum. Within our set-up, we can prove that the hull of any weak model set of maximal density, equipped with the push forward of the torus Haar measure by the torus parametrization, has pure point dynamical spectrum. Our arguments follow from structural assumptions on the torus parametrization and from a careful revision of Moody's arguments. They rely crucially on dynamical properties of weak model sets of maximal density. This will be compared to the recent work by Baake *et al* [**5**], who obtain similar dynamical results for weak model sets with a non-empty relatively compact window, which are of maximal or minimal density. Their approach is approximation with regular model sets.

There are many examples of point sets in G that have a—sometimes hidden—description as weak model set (see, e.g., the monograph [**2**]). We do not consider here the important question of reconstructing the internal space, the lattice and the window from the point set, but refer to the results of Meyer [**39**, Ch. II.5], Schreiber [**56**, Theorem 2], Schlottmann [**53**, Theorem 6] and Aujogue [**1**, Theorem 3.16]. Compare this with the approach taken by Baake and Moody [**8**], which is based on the autocorrelation of the point set, and which has recently been revisited by Strungaru [**59**].

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Red/dark dots: support of $\nu_W(\hat{x})$ Red/dark dots: support of $\nu_W^G(\hat{x})$

FIGURE 1. Shifted lattice $\mathcal{L}$ (black dots), fundamental domain of $\mathcal{L}$ (green/grey) and the supports of $\nu_W(\hat{x})$ and $\nu_W^G(\hat{x})$.

In fact certain such weak model sets of maximal density have recently attracted attention due to their intimate connection with the Möbius function from number theory. These are the k-free lattice points and, more generally, the $\mathcal{B}$-free systems, see [**4**, **20**, **51**] and the references therein. For the visible lattice points, pure point diffraction spectrum was shown already in [**9**]. Pure point dynamical spectrum for the hull of the square-free integers was shown only recently in [**17**] without referring to weak model sets, and in [**4**] using weak model sets. Whereas these results were obtained by explicit calculation, pure point dynamical spectrum follows from structural arguments within our approach. This indicates that one may take advantage of the underlying weak model set structure in order to further analyse the simplex of invariant probability measures for these systems.

The main contribution of this paper is the systematic use of the dynamical system that arises from the graph of the torus parametrization $\nu_W : \hat{X} \to \mathcal{M}$, where $\nu_W(\hat{x})$ is the configuration on $G \times H$ defined by the window W and the lattice $\mathcal{L}$ shifted by $\hat{x} = x + \mathcal{L}$. It has support $\operatorname{supp}(\nu_W(\hat{x})) = (x + \mathcal{L}) \cap (G \times W)$, as illustrated in Figure 1. This point of view separates most dynamical considerations from purely model dependent technical problems that are unavoidable when passing from configurations in $G \times H$ to their projections in G, see Figure 1†.

On the general dynamical level (that is, before passing to configurations on G):

- we obtain, as a warm-up, some relatives to well-known topological results on maximal equicontinuous factors and the lack of weak mixing (Theorem 1);
- we can introduce a general notion of Mirsky measures, namely, the push forward of Haar measure on $\hat{X}$ under the map ν_W, compare [**17**, **19**, **34**, **51**];
- we show that the systems equipped with this Mirsky measure have pure point dynamical spectrum (Theorem 2);
- we prove strict ergodicity when $m_H(\partial W) = 0$ (Theorem 2);
- we identify the Mirsky measure as the unique invariant measure with maximal density for typical configurations (Theorem 4);

† In the existing literature, the term torus parametrization is mostly used for the map ν_W^G illustrated in Figure 1 on p. 1050 (or for its inverse, when it exists), so that the purely dynamical aspects are always intertwined with problems solely due to the passage from $G \times H$ to G.

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- we show that the configurations with maximal density are precisely the generic points for the Mirsky measure (Theorem 5); and
- we deduce from this a formula for the pattern frequencies of configurations with maximal density (Remark 3.12), which is also discussed in [**5**, Remark 5].

While the measure theoretic assertions from this list 'survive' the passage to configurations on G even if the relevant projection is not one-to-one, some finer information can be transferred in this way only if that projection is one-to-one when restricted to sufficiently large subsets. This issue is discussed in §4 for model sets with interval windows, topologically regular windows and $\mathcal{B}$-free systems.

In the next section, we define the basic objects of this paper and describe the relationships between the various dynamical systems entering the scene. Section 3 contains the main theorems of this paper as well as a number of auxiliary results that elucidate our approach to study the dynamics on the graph of ν_W. The proofs of most results are deferred to §§5–8. We finish with an outlook to further perspectives of our approach in §9.

2. *The setting*

2.1. *Assumptions and notation.* Certain spaces and mappings are needed for the construction of weak model sets. The following assumptions will be in force in any of our statements below.

(1) G and H are *locally compact second countable abelian groups* with Haar measures m_G and m_H. Then the product group $G \times H$ is locally compact second countable abelian as well, and we choose $m_{G\times H} = m_G \times m_H$ as Haar measure on $G \times H$.

(2) $\mathcal{L} \subseteq G \times H$ is a *cocompact lattice*, that is, a discrete subgroup whose quotient space $(G \times H)/\mathcal{L}$ is compact. Thus $(G \times H)/\mathcal{L}$ is a compact second countable abelian group. Denote by $\pi^G : G \times H \to G$ and $\pi^H : G \times H \to H$ the canonical projections. We assume that $\pi^G|_{\mathcal{L}}$ is one-to-one and that $\pi^H(\mathcal{L})$ is dense in H†.

(3) G acts on $G \times H$ by translation: $T_g x = (g, 0) + x$.

(4) Let $\hat{X} := (G \times H)/\mathcal{L}$. As we assumed that $\hat{X}$ is compact, there is a measurable relatively compact fundamental domain $X \subseteq G \times H$ such that $x \mapsto x + \mathcal{L}$ is a bijection between X and $\hat{X}$. Elements of $G \times H$ (and hence also of X) are denoted by $x = (x_G, x_H)$, elements of $\hat{X}$ by $\hat{x}$ or by $x + \mathcal{L} = (x_G, x_H) + \mathcal{L}$, when a representative x of $\hat{x}$ is to be stressed. We normalize the Haar measure $m_{\hat{X}}$ on $\hat{X}$ such that $m_{\hat{X}}(\hat{X}) = 1$. Thus $m_{\hat{X}}$ is a probability measure.

(5) The *window* W is a compact subset of H. This is a more restrictive assumption than the one made originally for weak model sets in [**43**], where only relative compactness and measurability of W are assumed. We choose to assume compactness of W because this guarantees strong structural results and a more coherent presentation. Results for relatively compact windows are, however, discussed in Remark 3.16. (Recall that (full) model sets according to [**43**] are weak model sets where $\operatorname{int}(W) \neq \emptyset$.)

† Denseness of $\pi^H(\mathcal{L})$ can be assumed, without loss of generality, by passing from H to the closure of $\pi^H(\mathcal{L})$. In that case, m_H must be replaced by $m_{\overline{\pi^H(\mathcal{L})}}$.

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2.2. *Consequences of the assumptions.* We list a few facts from topology and measure theory that follow from the above assumptions. We will call any neighbourhood of the neutral element in an abelian topological group a *zero neighbourhood*.

(1) Being locally compact second countable abelian groups, G, H and $G \times H$ are metrizable with a translation invariant metric with respect to which they are complete metric spaces. In particular, they have the Baire property. As such groups are σ-compact, m_G, m_H and $m_{G\times H}$ are σ-finite.

(2) As $G \times H$ is σ-compact, the lattice $\mathcal{L} \subseteq G \times H$ is at most countable. Note that $G \times H$ can be partitioned by shifted copies of the relatively compact fundamental domain X. This means that $\mathcal{L}$ has a positive finite point density $\mathrm{dens}(\mathcal{L}) = 1/m_{G\times H}(X)$. We thus have $m_{\hat{X}}(\hat{A}) = \mathrm{dens}(\mathcal{L}) \cdot m_{G\times H}(X \cap (\pi^{\hat{X}})^{-1}(\hat{A}))$ for any measurable $\hat{A} \subseteq \hat{X}$, where $\pi^{\hat{X}} : G \times H \to \hat{X}$ denotes the quotient map. As a factor map between topological groups, $\pi^{\hat{X}}$ is open.

(3) As $\mathcal{L}$ is a discrete group, there is an open zero neighbourhood $V \subseteq G \times H$ whose closure is compact and for which all sets $\overline{V} + x$ $(x \in \mathcal{L})$ are pairwise disjoint.

(4) $\mathcal{L}$ acts on (H, m_H) by $h \mapsto \ell_H + h$ metrically transitively: that is, for every measurable $A \subseteq H$ such that $m_H(A) > 0$, there exist at most a countable number of $\ell^i \in \mathcal{L}$ such that $m_H((\bigcup_i (\ell^i_H + A))^c) = 0$ (see [**33**, Ch. 16, Example 1]).

(5) The action $\hat{T}_g : \hat{x} \mapsto (g, 0) + \hat{x}$ of G on $\hat{X}$ is minimal. Indeed, let $x + \mathcal{L}$, $y + \mathcal{L} \in \hat{X}$ be arbitrary. Choose a sequence $(\ell_n)_n$ from $\mathcal{L}$ such that $\ell_{n,H} \to y_H - x_H$ and let $g_n = y_G - \ell_{n,G} - x_G$. Then

$$\begin{aligned}\hat{T}_{g_n}(x + \mathcal{L}) &= (g_n, 0) + x + \mathcal{L} = (g_n, 0) + x + \ell_n + \mathcal{L}\\ &= (y_G, \ell_{n,H} + x_H) + \mathcal{L} \to y + \mathcal{L}.\end{aligned}$$

This shows that the G-orbit of $x + \mathcal{L}$ is dense in $\hat{X}$: that is minimality of the action $\hat{T}$. This implies that $\hat{X}$ with its natural action is uniquely ergodic (see, e.g., [**43**, Proposition 1]).

(6) Denote by $\mathcal{M}$ and $\mathcal{M}^G$ the spaces of all locally finite measures on $G \times H$ and G, respectively. They are endowed with the topology of vague convergence. As G and $G \times H$ are complete metric spaces, this is a Polish topology (see [**29**, Theorem A.2.3]).

2.3. *The objects of interest.* The pair $(\mathcal{L}, W)$ assigns to each point $\hat{x} \in \hat{X}$ a discrete point set in $G \times H$. We will identify such point set P with the measure $\sum_{y\in P} \delta_y \in \mathcal{M}$ and call these objects *configurations*. The following states this more precisely.

(1) For $\hat{x} = x + \mathcal{L} \in \hat{X}$, define the configuration

$$\nu_W(\hat{x}) := \sum_{y\in(x+\mathcal{L})\cap(G\times W)} \delta_y. \tag{1}$$

It is important to understand ν_W as a map from $\hat{X}$ to $\mathcal{M}$. The canonical projection $\pi^G : G \times H \to G$ projects measures $\nu \in \mathcal{M}$ to measures $\pi^G_* \nu$ on G, defined by $\pi^G_* \nu(A) := \nu((\pi^G)^{-1}(A))$. We abbreviate to

$$\nu^G_W := \pi^G_* \circ \nu_W : \hat{X} \to \mathcal{M}^G. \tag{2}$$

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The configurations $\nu_W^G(\hat{x})$ on G are sometimes called *Dirac combs* in the literature. See Figure 1 for a visualization of these two maps. The set $\nu_W^G(\hat{X})$ is called 'the model set family associated to the window W' and denoted by $\mathcal{M}(W)$ in [**50**, Equation (4–2)].

(2) Denote the graph of $\nu_W : \hat{X} \to \mathcal{M}$ by $\mathcal{G}\nu_W = \{(\hat{x}, \nu_W(\hat{x})) : \hat{x} \in \hat{X}\}$ and that of $\nu_W^G : \hat{X} \to \mathcal{M}^G$ by $\mathcal{G}\nu_W^G = \{(\hat{x}, \nu_W^G(\hat{x})) : \hat{x} \in \hat{X}\}$.

(3) Denote by:

- $\mathcal{M}_W$ the vague closure of $\nu_W(\hat{X})$ in $\mathcal{M}$;
- $\mathcal{G}\mathcal{M}_W$ the vague closure of $\mathcal{G}\nu_W$ in $\hat{X} \times \mathcal{M}$;
- $\mathcal{M}_W^G$ the vague closure of $\nu_W^G(\hat{X})$ in $\mathcal{M}^G$; and
- $\mathcal{G}\mathcal{M}_W^G$ the vague closure of $\mathcal{G}\nu_W^G$ in $\hat{X} \times \mathcal{M}^G$.

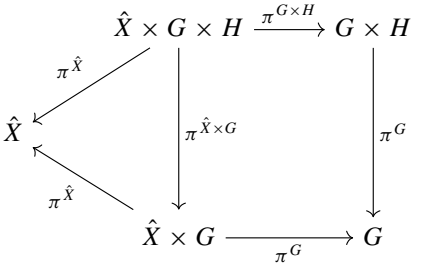
The group G acts continuously by translations on all these spaces: that is,

$$(S_g\nu)(A) := \nu(T_g^{-1}A) = \nu(T_{-g}A), \quad S_g(\hat{x}, \nu) := (\hat{T}_g\hat{x}, S_g\nu).$$

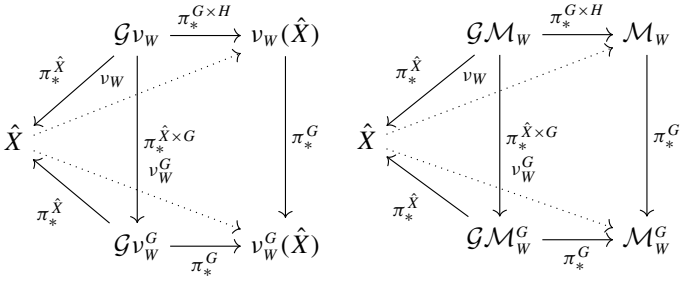
Here we used the same notation S_g for translations on $\mathcal{M}_W$, $\mathcal{M}_W^G$, $\mathcal{G}\mathcal{M}_W$ and $\mathcal{G}\mathcal{M}_W^G$, as the meaning will always be clear from the context.

(4) As $\nu_W(\hat{x})(T_{-g}A) = (S_g\nu_W(\hat{x}))(A) = \nu_W(\hat{T}_g\hat{x})(A)$, by Lemma 5.1, it is obvious that all $\nu_W(\hat{x})$ are uniformly translation bounded, and it follows, from [**6**, Theorem 2], that all four spaces from item (3) are compact.

2.4. *Spaces and factor maps.* On the group level, we will work with the sets and canonical projections shown by



The reader may notice the minor ambiguity in the use of π^G and $\pi^{\hat{X}}$. All these projections commute with the respective actions of T and $\hat{T}$. On the level of the spaces $\mathcal{M}_W$, $\mathcal{M}_W^G$, $\mathcal{G}\mathcal{M}_W$ and $\mathcal{G}\mathcal{M}_W^G$, this translates to



where we use the standard notation $\pi_*(\nu)(A) := \nu(\pi^{-1}(A))$.

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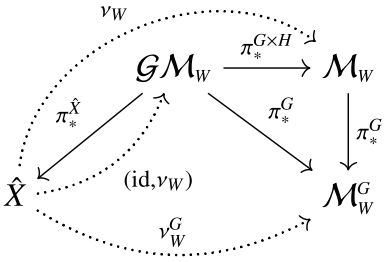


FIGURE 2. The dynamical systems under study.

LEMMA 2.1. *Each solid arrow in the right-hand diagram represents a factor between compact dynamical systems: that is, the maps are continuous, surjective and commute with the actions S, respectively $\hat{T}$. The maps ν_W and ν_W^G, represented by the dotted arrow, are neither onto nor continuous, in general, yet they commute with the other maps.*

Proof. All arrows in the left-hand diagram represent onto maps, by definition, and all solid arrows represent continuous maps. As all closures involved in the right-hand diagram are compact metric spaces and, as $\pi_*^{\hat{X}\times G}(\mathcal{G}\nu_W) = \mathcal{G}\nu_W^G$, $\pi_*^{G\times H}(\mathcal{G}\nu_W) = \nu_W(\hat{X})$, $\pi_*^G(\mathcal{G}\nu_W^G) = \mathcal{M}_W^G$ and $\pi_*^G(\nu_W(\hat{X})) = \nu_W^G(\hat{X})$, also all solid arrows on that side represent onto maps. As they all commute with the respective actions of S and $\hat{T}$, they all represent factor maps of compact dynamical systems. □

We will see, in Lemma 5.3, that $\pi_*^{\hat{X}\times G} : \mathcal{G}\mathcal{M}_W \to \mathcal{G}\mathcal{M}_W^G$ is always a homeomorphism because of the injectivity of $\pi^G|_{\mathcal{L}}$. Therefore the diagram simplifies to the one in Figure 2.

Observe also that:

- $\pi_*^{G\times H} : \mathcal{G}\mathcal{M}_W \to \mathcal{M}_W$ is one-to-one on $(\pi_*^{G\times H})^{-1}(\mathcal{M}_W\backslash\{\underline{0}\})$, where $\underline{0}$ denotes the zero configuration; and
- if the window W has non-empty interior, then $\pi_*^{G\times H}$ is a homeomorphism (Proposition 3.5).

In particular, the map $\hat{\pi} := \pi_*^{\hat{X}} \circ (\pi_*^{G\times H})^{-1} : \mathcal{M}_W\backslash\{\underline{0}\} \to \hat{X}$, which associates to each non-zero configuration its torus parameter, is well defined (see Definition 5.5).

The factor maps π_*^G may have a more complicated structure and, for some results, we will have to make the additional assumption that $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is one-to-one, which implies immediately that π_*^G is a homeomorphism. For windows $W \subset \mathbb{R}^d$ with $W = \overline{\mathrm{int}(W)}$, we will show that this assumption is satisfied, so that the diagram from Figure 2 simplifies further to $\hat{X} \xleftarrow{\pi_*^{\hat{X}}} \mathcal{G}\mathcal{M}_W \xleftrightarrow{\pi_*^G} \mathcal{M}_W^G$.

For $\mathcal{B}$-free systems and those that are similar, we will show that the restriction of π_*^G to a rather large subset $\mathcal{Y}$ of $\mathcal{M}_W$ (or $\mathcal{G}\mathcal{M}_W$) is one-to-one. This set was identified in [**45**] (see also [**4**, **34**]).

The advantages of the general picture shown in the above diagrams include the following.

- The structure of the map $\pi_*^{\hat{X}} : \mathcal{G}\mathcal{M}_W \to \hat{X}$ can be conveniently analysed under quite general assumptions so that the special properties of particular systems enter only the analysis of the map π_*^G.

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- The orbit closures of $\nu_W(\hat{x})$ and $\nu_W^G(\hat{x})$ for different points $\hat{x} \in \hat{X}$ can all be studied as subsets of $\mathcal{M}_W$ and $\mathcal{M}_W^G$, respectively. Indeed, it will turn out that these orbit closures are identical for $m_{\hat{X}}$-almost all $\hat{x}$.

Remark 2.2. In some publications on model sets, such as [**3**, **43**], the compact factor group $\hat{X}$ is called a torus and is denoted by $\mathbb{T}$. The set $\mathcal{M}_W^G$ contains all orbit closures $\mathbb{X}_{W'}$ of weak model sets produced from shifted copies W' of the window W.

3. *Main results*

We start with some purely topological results about the dynamical system on (the closure of) the graph of the torus parametrization and their consequences for identifying maximal equicontinuous factors and the lack of weak mixing. In a number of specific situations these consequences are well known (see, e.g., [**1**, **4**, **50**]).

3.1. *Topological results.*

Our main topological results will show that $(\hat{X}, \hat{T})$ is the maximal equicontinuous factor of $(\mathcal{GM}_W, S)$ and how close to or far from being homeomorphisms $\pi_*^{G\times H}$ and π_*^G are. We will start the proof with the observation that $\mathcal{GM}_W$ is the closure of the graph of an upper semicontinuous function that is invariant under the skew-product action $\hat{T} \times S$ on $\hat{X} \times \mathcal{M}$. To state the relevant proposition, we give the following definitions.

Definition 3.1. (Zeros and continuity points of ν_W)

(a) $C_W \subseteq \hat{X}$ is the set of all continuity points of the map ν_W: that is, the set of all points $\hat{x} \in \hat{X}$ such that, for each sequence $(\hat{x}^n)_n$ in $\hat{X}$ which converges to $\hat{x}$, the sequence of configurations $\nu_W(\hat{x}^n)$ converges vaguely to $\nu_W(\hat{x})$.

(b) $Z_W \subseteq \hat{X}$ is the set of all points $\hat{x} \in \hat{X}$ for which $\nu_W(\hat{x})$ is the zero configuration that we denote by $\underline{0}$.

Remark 3.2.

(a) $\hat{x} \in C_W$ if and only if $(\pi_*^{\hat{x}})^{-1}\{\hat{x}\} \cap \mathcal{GM}_W = \{(\hat{x}, \nu_W(\hat{x}))\}$.

(b) C_W is also the set of continuity points of $\nu_W^G = \pi_*^G \circ \nu_W$ and $Z_W = \{\hat{x} \in \hat{X} : \nu_W^G(\hat{x}) = \underline{0}\}$. The latter identity is obvious; the first one follows from the fact that the graph closures $\mathcal{GM}_W$ of ν_W and $\mathcal{GM}_W^G$ of ν_W^G are fibre-wise homeomorphic (see Lemma 5.3).

(c) $\hat{X} = Z_W$ if and only if $W = \emptyset$.

(d) The sets C_W and Z_W are $\hat{T}$-invariant.

PROPOSITION 3.3. (Upper semicontinuity of ν_W)

(a) *Suppose that* $\lim_{n\to\infty} \hat{x}^n = \hat{x}$. *Then* $\nu \leqslant \nu_W(\hat{x})$ *for all vague limit points* ν *of the sequence* $(\nu_W(\hat{x}^n))_n$ *and* $d\nu/d\nu_W(\hat{x})$ *takes only values zero and one. (This is the upper semicontinuity of* ν_W *with respect to the natural order relation on* $\mathcal{M}$.*)*

(b) $\hat{x} \in C_W$ *if and only if* $\{\nu \in \mathcal{M}_W : \nu \leqslant \nu_W(\hat{x})\} = \{\nu_W(\hat{x})\}$.

(c) C_W *is a dense* G_δ*-subset of* $\hat{X}$.

(d) *If* $\overline{\text{int}(W)} = \emptyset$, *then* $Z_W = C_W$ *is a dense* G_δ*-subset of* $\hat{X}$. *If, in addition,* $W \neq \emptyset$, *then* $\underline{0} \in \nu_W(\hat{X}\backslash Z_W)$.

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(e) *If* $\mathrm{int}(W) \neq \emptyset$, *then* $Z_W = \emptyset$ *and, even more,* $\underline{0} \notin \mathcal{M}_W$.

These assertions remain valid if W *is only assumed to be closed (and not compact).*

Remark 3.4. Proposition 3.3 shows that the set $\mathcal{GM}_W$ is upper bounded by the graph of the upper semicontinuous function $\nu_W : \hat{X} \to \mathcal{M}$. In a similar way, it can be lower bounded by the lower semicontinuous function $\mu_W := \nu_H - \nu_{\mathrm{int}(W)^c}$. This function is lower semicontinuous because $\nu_H(\hat{x}) = \sum_{\ell \in \mathcal{L}} \delta_{x+\ell}$ is clearly continuous as a function of $\hat{x} = x + \mathcal{L}$ and, as $\mathrm{int}(W)^c$ is closed, $\nu_{\mathrm{int}(W)^c}$ is upper semicontinuous. Furthermore, as $\nu_W + \nu_{\mathrm{int}(W)^c} = \nu_{W \cup \mathrm{int}(W)^c} + \nu_{W \cap \mathrm{int}(W)^c} = \nu_H + \nu_{\partial W}$, we have $\mu_W = \nu_W - \nu_{\partial W} \leqslant \nu_W$. Hence, as $\mathcal{GM}_W$ is the vague closure of $\mathcal{G}\nu_W$, this set is lower bounded by the graph of μ_W. (For weak model sets with $\mathrm{int}(W) = \emptyset$, $\mu_W(\hat{x}) = \underline{0}$ for all $\hat{x} \in \hat{X}$.) Observe that, for general $\nu \in \mathcal{M}$, the condition $\mu_W(\hat{x}) \leqslant \nu \leqslant \nu_W(\hat{x})$ for some $\hat{x} \in \hat{X}$ does not imply $\nu \in \mathcal{M}_W$. However, see the $\mathcal{B}$-free systems from [**34**] discussed in §4.4 for a situation where this implication does hold.

Some pieces of information on how to pass information about the system $(\mathcal{GM}_W, S)$ to its various factor systems are collected in the following proposition.

PROPOSITION 3.5.

(a) $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ *is continuous.*

(b) $\pi_*^{G \times H} : \mathcal{GM}_W \to \mathcal{M}_W$ *is one-to-one on* $(\pi_*^{G \times H})^{-1}(\mathcal{M}_W \setminus \{\underline{0}\})$ *and, if* $\mathrm{int}(W) \neq \emptyset$, *then* $\pi_*^{G \times H}$ *is a homeomorphism.*

(c) $\pi_*^{\hat{X}} : (\mathcal{GM}_W, S) \to (\hat{X}, \hat{T})$ *is a topological almost one-to-one extension of its maximal equicontinuous factor.*

Our first main result provides topological information on the various systems under study. Its proof, as well as those of the other results of this subsection, is provided in §§5–7.

THEOREM 1. (Topological factors and extensions)

(a) *If* $\mathrm{int}(W) \neq \emptyset$, *that is, if* $Z_W = \emptyset$, *then*

$$\pi_*^{\hat{X}} : (\mathcal{GM}_W, S) \to (\hat{X}, \hat{T}) \quad \text{and} \quad \pi_*^{\hat{X}} \circ (\pi_*^{G \times H})^{-1} : (\mathcal{M}_W, S) \to (\hat{X}, \hat{T}) \tag{3}$$

are topological almost one-to-one extensions of maximal equicontinuous factors. In particular, none of the systems is topologically weakly mixing if $\hat{X}$ *is non-trivial. Furthermore, the restrictions of* S *to the subsystems* $\overline{\mathcal{G}(\nu_W|_{C_W})} \subseteq \mathcal{GM}_W$ *and* $\overline{\nu_W(C_W)} \subseteq \mathcal{M}_W$ *are almost automorphic extensions*† *of* $(\hat{X}, \hat{T})$, *and they are the only minimal subsystems of* $(\mathcal{GM}_W, S)$ *and* $(\mathcal{M}_W, S)$, *respectively.*

(b) *If* $\mathrm{int}(W) = \emptyset$, *that is, if* $Z_W \neq \emptyset$, *then* $\pi_*^{\hat{X}} : (\mathcal{GM}_W, S) \to (\hat{X}, \hat{T})$ *is a topological almost one-to-one extension of its maximal equicontinuous factor, whereas* $(\mathcal{M}_W, S)$ *and* $(\mathcal{M}_W^G, S)$ *do not have a non-trivial minimal equicontinuous factor. On the other hand,* $(\mathcal{M}_W, S)$ *and* $(\mathcal{M}_W^G, S)$ *are never topologically weakly mixing, except if* $\mathrm{card}(\mathcal{M}_W^G) = 1$.

† This means that both systems are minimal topological almost one-to-one extensions of $(\hat{X}, \hat{T})$.

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COROLLARY 1.

(a) *All assertions of the theorem remain valid for the system* $(\mathcal{GM}_W^G, S)$ *(see Lemma 5.3) and, if* $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ *is one-to-one (and hence a homeomorphism), the same is true for* $(\mathcal{M}_W^G, S)$.

(b) *If* $\mathrm{int}(W) \neq \emptyset$ *and if the window* $W' := \overline{\mathrm{int}(W)}$ *is* aperiodic*, that is, if* $h + W' = W'$ *implies* $h = 0$*, then the restriction of* S *to the subsystem* $\overline{\nu_{W'}^G(C_{W'})} = \overline{\nu_W^G(C_W)} \subseteq \mathcal{M}_W^G$ *is an almost automorphic extension of* $(\hat{X}, \hat{T})$*, and it is the only minimal subsystem of* $(\mathcal{M}_W^G, S)$.

While the first part of this corollary is obvious, the second part is proved in §7. How to get rid of the aperiodicity assumption is discussed in Remark 4.9.

3.2. *Measure theoretic results.* On the measure theoretic side, we have results related to the Mirsky measure [**17**, **19**, **34**, **51**] and its spectral properties, to configurations with maximal density [**43**, **58**] and to the denseness of individual orbits.

PROPOSITION 3.6. (Haar measure of Z_W and C_W)

(a) *The sets* C_W *and* Z_W *have either full or vanishing Haar measure.*

(b) $m_{\hat{X}}(Z_W) = 0$ *if and only if* $m_H(W) > 0$.

(c) $m_{\hat{X}}(C_W) = 1$ *if and only if* $m_H(\partial W) = 0$.

Definition 3.7. (Mirsky measures) Denote by:

- $Q_{\mathcal{M}} := m_{\hat{X}} \circ \nu_W^{-1}$ the lift of Haar measure on $\hat{X}$ to $\mathcal{M}_W$;
- $Q_{\mathcal{GM}} := m_{\hat{X}} \circ (\mathrm{id}_{\hat{X}}, \nu_W)^{-1}$ the lift of Haar measure on $\hat{X}$ to $\mathcal{GM}_W$;
- $Q_{\mathcal{M}^G} := m_{\hat{X}} \circ (\nu_W^G)^{-1}$ the lift of Haar measure on $\hat{X}$ to $\mathcal{M}_W^G$; and
- $Q_{\mathcal{GM}^G} := m_{\hat{X}} \circ (\mathrm{id}_{\hat{X}}, \nu_W^G)^{-1}$ the lift of Haar measure on $\hat{X}$ to $\mathcal{GM}_W^G$.

Remark 3.8. In some cases, as for example the square-free integers and the $\mathcal{B}$-free systems [**34**], the measure $Q_{\mathcal{M}^G}$ is called the *Mirsky measure*. Note also that, in the trivial case where $m_H(W) = 0$, the measures $Q_{\mathcal{M}}$ and $Q_{\mathcal{M}^G}$ are the point masses in the zero configuration $\underline{0}$.

THEOREM 2. (Measurable factors and extensions) *Consider the case* $m_H(W) > 0$.

(a) *The systems* $(\hat{X}, m_{\hat{X}}, \hat{T})$, $(\mathcal{GM}_W, Q_{\mathcal{GM}}, S)$ *and* $(\mathcal{M}_W, Q_{\mathcal{M}}, S)$ *are measure theoretically isomorphic, and the system* $(\mathcal{M}_W^G, Q_{\mathcal{M}^G}, S)$ *is a factor of* $(\hat{X}, m_{\hat{X}}, \hat{T})$ *via the measurable factor map* ν_W^G*. In particular,* $(\mathcal{M}_W^G, Q_{\mathcal{M}^G}, S)$ *has pure point dynamical spectrum*†*, and its group of eigenvalues is a subgroup of the group of eigenvalues of* $(\hat{X}, m_{\hat{X}}, \hat{T})$.

(b) *If* $\nu_W^G : \hat{X} \to \mathcal{M}_W^G$ *is not constant* $m_{\hat{X}}$*-almost everywhere, then* $(\hat{X} \times \mathcal{M}_W^G, Q_{\mathcal{GM}}, \hat{T} \times S)$ *is a non-trivial joining of the systems* $(\hat{X}, m_{\hat{X}}, \hat{T})$ *and* $(\mathcal{M}_W^G, Q_{\mathcal{M}^G}, S)$.

(c) *If* $m_H(\partial W) = 0$*, then* $Z_W = \emptyset$ *and* $Q_{\mathcal{GM}}$*,* $Q_{\mathcal{M}}$ *and* $Q_{\mathcal{M}^G}$ *are the only invariant measures for* $(\mathcal{GM}_W, \hat{T} \times S)$*,* $(\mathcal{M}_W, S)$ *and* $(\mathcal{M}_W^G, S)$*, respectively. In particular, these systems are uniquely ergodic.*

These results are proved in §7.

† In ergodic theory, the term *discrete spectrum* may be more common (see, e.g., [**61**]).

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Remark 3.9. Baake and Lenz [**6**, Theorem 7] proved in a very general context, which encompasses the setting chosen in this paper, that (with our specialized notation) the diffraction measure of $Q_{\mathcal{M}^G}$ is purely atomic if and only if the dynamical system $(\mathcal{M}_W^G, Q_{\mathcal{M}^G}, S)$ has pure point spectrum. However, the latter property is guaranteed by Theorem 2(a) for all weak model sets with compact window of positive Haar measure.

Remark 3.10. (Entropy) An interesting question concerns the topological entropy of the above systems in the case when $m_H(\partial W) > 0$. It is bounded by $\mathrm{dens}(\mathcal{L}) \cdot m_H(\partial W) \cdot \log 2$, as may be proved using relative entropy theory for skew-product systems. Recently, a combinatorial proof for the corresponding pattern entropy bound has been given in [**26**]. More generally, if $m_H(\partial W) > 0$, the above systems may have many invariant probability measures, all extending the Haar measure on $\hat{X}$. They all have entropy at most $\mathrm{dens}(\mathcal{L}) \cdot m_H(\partial W) \cdot \log 2$.

For the example of $\mathcal{B}$-free systems, which is discussed below, the upper bound is attained. This follows from the hereditary property of that system: that is, for any given configuration $\nu \in \mathcal{M}_W$, any of its subconfigurations $\nu' \leq \nu$ is also in $\mathcal{M}_W$. For this particular example, its simplex of invariant probability measures is quite well understood [**34**, **45**].

A certain kind of sequence $(A_n)_{n\in\mathbb{N}_0}$ of compact subsets of G that typically exhaust G and have 'nice' boundaries are called *tempered van Hove sequences*. They are discussed by Moody in [**43**]†. Here it suffices for the moment to note that $A_n = [-n, n]^d$ defines tempered van Hove sequences in $\mathbb{Z}^d$ and in $\mathbb{R}^d$, and that tempered van Hove sequences exist in every σ-compact locally compact abelian group.

THEOREM 3. (Moody [**43**, Theorem 1]) *Let $(A_n)_{n\in\mathbb{N}_0}$ be a tempered van Hove sequence of subsets of G. Then*

$$\lim_{n\to\infty} \frac{\nu_W(\hat{x})(A_n \times H)}{m_G(A_n)} = \mathrm{dens}(\mathcal{L}) \cdot m_H(W) \quad \textit{for } m_{\hat{X}}\textit{-almost every } \hat{x} \in \hat{X}.$$

As W is compact, the convergence is indeed semi-uniform: for each $\epsilon > 0$ there is an $n_0 \in \mathbb{N}_0$ such that, for all $n \geqslant n_0$ and all $\hat{x} \in \hat{X}$,

$$\frac{\nu_W(\hat{x})(A_n \times H)}{m_G(A_n)} \leqslant \mathrm{dens}(\mathcal{L}) \cdot m_H(W) + \epsilon. \tag{4}$$

Proof. The almost everywhere convergence is stated explicitly (and proved) in [**43**, Theorem 1], with a particular normalization of the Haar measure $m_{G\times H}$ such that $\mathrm{dens}(\mathcal{L}) = 1$. The argument is based on a version of the generalized Birkhoff ergodic theorem, which may be replaced by [**37**]. At the end of that proof, additional arguments are provided to prove uniform convergence when $m_H(\partial W) = 0$. The same arguments yield the semi-uniform convergence for general compact windows W (see also [**44**, **60**]). □

† (Generalized) van Hove sequences were introduced by Schlottmann [**54**], where also their existence is discussed. For compact $A, K \subseteq G$, let $\partial^K A = ((K + A)\setminus \mathrm{int}(A)) \cup ((-K + \overline{G\setminus A}) \cap A)$ denote the K-boundary of A. A sequence $(A_n)_n$ of compact subsets of G of positive Haar measure is van Hove if $\lim_{n\to\infty} m_G(\partial^K A_n)/m_G(A_n) = 0$ for any compact $K \subseteq G$. The sequence $(A_n)_n$ is tempered [**37**] if there is a constant $C > 0$ such that $m_G(\bigcup_{k<n}(A_k - A_k)) \leq C m_G(A_n)$ for every n. For further background, see [**6**, **36**, **44**] and the references therein.

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COROLLARY 3. *The same statements hold for the ratios* $(\nu_W^G(\hat{x})(A_n))/(m_G(A_n))$, *because* $\nu_W^G(\hat{x})(A_n) = \nu_W(\hat{x})(A_n \times H)$.

We extend Theorem 3 to cover also measures ν dominated by $\nu_W(\hat{x})$.

THEOREM 4. (Ergodic point densities) *Let* $(A_n)_{n\in\mathbb{N}_0}$ *be a tempered van Hove sequence of subsets of* G *as before. Let* P *be an ergodic* S*-invariant probability measure on* $\mathcal{M}_W$. *Then, for* P*-almost every* $\nu \in \mathcal{M}_W$, *the inequality*

$$D_P := \lim_{n\to\infty} \frac{\nu(A_n \times H)}{m_G(A_n)} \quad \textit{exists and is } \leqslant \operatorname{dens}(\mathcal{L}) \cdot m_H(W)$$

is satisfied, with equality P*-almost everywhere if and only if* $P = Q_{\mathcal{M}}$. *So, in particular,* $D_{Q_{\mathcal{M}}} = \operatorname{dens}(\mathcal{L}) \cdot m_H(W)$.

For each sufficiently small compact zero neighbourhood $B \subseteq G$

$$D_P = \frac{1}{m_G(B)} \int_{\mathcal{M}_W} \nu(B \times H)\, dP(\nu) \tag{5}$$

holds so that, in particular, the map $P \mapsto D_P$ *is upper semicontinuous with respect to the weak topology on probability measures on* $\mathcal{M}_W$.

The proof of this theorem, which follows closely the proof of Moody's theorem, is given in §8.

COROLLARY 4. *Also the assertions of Theorem 4 remain valid if* $\mathcal{M}_W$, $Q_{\mathcal{M}}$ *and* $\nu(\cdot \times H)$ *are replaced by* $\mathcal{M}_W^G$, $Q_{\mathcal{M}^G}$ *and* $\nu(\cdot)$, *respectively.*

This is obvious for measures P on $\mathcal{M}_W^G$ that can be represented as $\tilde{P} \circ (\pi_*^G)^{-1}$ with some ergodic S-invariant probability $\tilde{P}$ on $\mathcal{M}_W$. That all ergodic S-invariant P on $\mathcal{M}_W^G$ can be represented in this way is proved in §8.

For the rest of this section, we fix one tempered van Hove sequence $(A_n)_{n\in\mathbb{N}_0}$. Our final theorem highlights the exceptional role played by configurations $\nu_W(\hat{x})$ with maximal density with respect to $(A_n)_{n\in\mathbb{N}_0}$: that is, by configurations $\nu_W(\hat{x})$ with

$$\hat{x} \in \hat{X}_{\max} := \left\{ \hat{x} \in \hat{X} : \lim_{n\to\infty} \nu_W(\hat{x})(A_n \times H)/m_G(A_n) = D_{Q_{\mathcal{M}}} \right\}. \tag{6}$$

These points occur already in [**43**], and their importance was stressed more recently by Strungaru [**58**]. Here we show that the set $\nu_W(\hat{X}_{\max}) \subseteq \mathcal{M}_W$ coincides with the set of generic points for the S-invariant probability measure $Q_{\mathcal{M}}$ on $\mathcal{M}_W$. Observe that

$$\hat{X}_{\max} = \hat{X}_{\max}^G := \left\{ \hat{x} \in \hat{X} : \lim_{n\to\infty} \nu_W^G(\hat{x})(A_n)/m_G(A_n) = D_{Q_{\mathcal{M}}} \right\}. \tag{7}$$

THEOREM 5. (Configurations of maximal density are generic for the Mirsky measure)

(a) $\hat{X}_{\max}$ *is* $\hat{T}$*-invariant and* $m_{\hat{X}}(\hat{X}_{\max}) = 1$.

(b) *For each* $\hat{x} \in \hat{X}_{\max}$, *the empirical measures*

$$Q_{n,\hat{x}} := \frac{1}{m_G(A_n)} \int_{A_n} \delta_{S_g \nu_W(\hat{x})}\, dm_G(g)$$

converge weakly to $Q_{\mathcal{M}}$.

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(c) *For $\hat{x} \in \hat{X}$, denote by $\mathcal{M}_W(\hat{x})$ the orbit closure of $\nu_W(\hat{x})$ under the action of S in $\mathcal{M}_W$. Then $Q_{\mathcal{M}}(\mathcal{M}_W(\hat{x})) = 1$ for all $\hat{x} \in \hat{X}_{\max}$: that is,* $\operatorname{supp}(Q_{\mathcal{M}}) \subseteq \mathcal{M}_W(\hat{x}) \subseteq \mathcal{M}_W$ *for all $\hat{x} \in \hat{X}_{\max}$.*

(d) $\operatorname{supp}(Q_{\mathcal{M}}) = \mathcal{M}_W(\hat{x})$ *for all* $\hat{x} \in \hat{X}_{\max} \cap \nu_W^{-1}(\operatorname{supp}(Q_{\mathcal{M}}))$ *and* $m_{\hat{X}}(\hat{X}_{\max} \cap \nu_W^{-1}(\operatorname{supp}(Q_{\mathcal{M}}))) = 1$.

(e) $\overline{\nu_W(C_W)} \subseteq \operatorname{supp}(Q_{\mathcal{M}})$.

COROLLARY 5. *All assertions of this theorem remain valid if $\mathcal{M}_W$, $\hat{X}_{\max}$, $\nu_W(\hat{x})$, $Q_{\mathcal{M}}$ and $\mathcal{M}_W(\hat{x})$ are replaced by the corresponding objects $\mathcal{M}_W^G$, $\hat{X}_{\max}^G$, $\nu_W^G(\hat{x})$, $Q_{\mathcal{M}^G}$ and $\mathcal{M}_W^G(\hat{x})$, respectively. For (a) and (b) this is obvious; for (c) and (d) one notes that $\pi_*^G(\mathcal{M}_W(\hat{x})) = \mathcal{M}_W^G(\hat{x})$ and $\pi_*^G(\operatorname{supp}(Q_{\mathcal{M}})) = \operatorname{supp}(Q_{\mathcal{M}^G})$, because the involved spaces are compact. Finally, (e) follows from Remark 3.2.*

Combining Theorem 2(a), Theorem 5(c) and the previous corollary, we arrive at the following result.

COROLLARY 6. *Consider the case $m_H(W) > 0$. For any $\hat{x} \in \hat{X}_{\max}$, the systems $(\hat{X}, m_{\hat{X}}, \hat{T})$ and $(\mathcal{M}_W(\hat{x}), Q_{\mathcal{M}}, S)$ are measure theoretically isomorphic, and the system $(\mathcal{M}_W^G(\hat{x}), Q_{\mathcal{M}^G}, S)$ is a factor of $(\hat{X}, m_{\hat{X}}, \hat{T})$ via the factor map ν_W^G. In particular, $(\mathcal{M}_W^G(\hat{x}), Q_{\mathcal{M}^G}, S)$ has pure point dynamical spectrum, and its group of eigenvalues is a subgroup of the group of eigenvalues of $(\hat{X}, m_{\hat{X}}, \hat{T})$.*

Remark 3.11. In view of Theorem 5(e), it may be worth noting that the two sets $\overline{\nu_W(C_W)}$ and $\operatorname{supp}(Q_{\mathcal{M}})$ are the topological, respectively, measure theoretic result of the process to rid the set $\mathcal{M}_W$ of 'negligible' parts. Indeed, as the topology on $\mathcal{M}_W$ has a countable base and as C_W is a dense G_δ-subset of $\hat{X}$, it is not hard to show that:

(a) $\overline{\nu_W(C_W)}$ is the intersection of all sets $\overline{\nu_W(R)}$ where R ranges over all dense G_δ-subsets of $\hat{X}$;

(b) $\operatorname{supp}(Q_{\mathcal{M}})$ is the intersection of all sets $\overline{\nu_W(F)}$ where F ranges over all full measure subsets of $\hat{X}$; and

(c) $\overline{\nu_W(C_W)} = \operatorname{supp}(Q_{\mathcal{M}})$ if $m_H(\partial W) = 0$.

The same statements hold for ν_W^G and $Q_{\mathcal{M}^G}$. The proofs are provided in §8.

The next corollary is, in conjunction with Theorem 1(a) and Theorem 2(c), an immediate consequence of the previous remark.

COROLLARY 7. (Strict ergodicity) *Whenever $m_H(\partial W) = 0$ and $\operatorname{int}(W) \neq \emptyset$, the restrictions of S to $\operatorname{supp}(Q_{\mathcal{M}})$ and to $\operatorname{supp}(Q_{\mathcal{M}^G})$ are strictly ergodic, that is, minimal and uniquely ergodic.*

Remark 3.12. (Pattern frequencies) Let us call any non-empty, finite $\mathcal{L}_1 \subset \mathcal{L}$ a local configuration or pattern. We are interested in how frequently shifted copies of a given pattern appear in some fixed configuration $\nu \in \mathcal{M}_W$. For some van Hove sequence $(A_n)_{n\in\mathbb{N}_0}$ in G, we may thus consider relative frequencies

$$f_n(\mathcal{L}_1, \nu) = \frac{\operatorname{card}\{(g, h) \in A_n \times H : -(g, h) + \mathcal{L}_1 \subseteq \operatorname{supp}(\nu)\}}{m_G(A_n)}$$

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and ask whether a limiting frequency $f(\mathcal{L}_1, \nu)$ exists as $n \to \infty$. As a corollary to Theorem 5(b), we prove, in §8, that, for configurations $\nu_W(\hat{x})$ with maximal density, this is indeed the case: namely,

$$\lim_{n\to\infty} f_n(\mathcal{L}_1, \nu_W(\hat{x})) = f(\mathcal{L}_1, \nu_W(\hat{x})) = \operatorname{dens}(\mathcal{L}) \cdot m_H\Bigg(\bigcap_{\ell\in\mathcal{L}_1} (W - \ell_H)\Bigg) \quad \text{for all } \hat{x} \in \hat{X}_{\max}. \tag{8}$$

As $\pi^G_*|_{\mathcal{L}}$ is one-to-one, it is easily checked that the projected configurations in $\mathcal{M}^G_W$ have the same pattern frequencies, that is,

$$f_n(\mathcal{L}_1, \nu) = \frac{\operatorname{card}\{g \in A_n : -g + \pi^G(\mathcal{L}_1) \subseteq \operatorname{supp}(\pi^G_*\nu)\}}{m_G(A_n)}.$$

3.3. *Remarks and comments.*

Remark 3.13. (Generic windows and repetitive model sets) It follows from a characterization of the set C_W in Lemma 6.1, that a point $\hat{x} = (x + \mathcal{L})$ belongs to C_W if and only if $\pi^H(\mathcal{L}) \cap ((\partial W) - x_H) = \emptyset$: that is, if and only if the window $W - x_H$ is generic in the sense of Schlottmann [**53**]. If $\operatorname{int}(W) \neq \emptyset$, then Theorem 1(a) shows that the subsystem $\overline{\nu_W(C_W)}$ is almost automorphic and, in particular, minimal. Observe that $\overline{\nu_W(C_W)} = \mathcal{M}_W(\hat{x})$ and hence also that $\overline{\nu^G_W(C_W)} = \mathcal{M}^G_W(\hat{x})$ for each point $\hat{x} \in C_W$. But minimality is equivalent to almost periodicity which translates to repetitivity in the traditional language of model sets. Hence we rediscover the fact that generic windows with non-empty interior generate repetitive model sets/configurations. For topologically regular windows, this was also proved by Robinson [**50**, Proposition 5.18 and Corollary 5.20]. Note, however, that when $m_H(\partial W) > 0$, then $m_{\hat{X}}(C_W) = 0$, so that the set of these configurations has Mirsky measure zero.

Remark 3.14. (Some open problems related to $\operatorname{supp}(Q_{\mathcal{M}})$) Theorem 5 suggests that the 'relevant' dynamical system to look at is not $(\mathcal{M}_W, S)$ but $(\operatorname{supp}(Q_{\mathcal{M}}), S)$, because it has the property that the orbit of $\nu_W(\hat{x})$ is dense in this space for all $\hat{x} \in \hat{X}_{\max} \cap \nu_W^{-1}(\operatorname{supp}(Q_{\mathcal{M}}))$. This imposes the following questions.

(i) Which points belong to $\mathcal{M}_W \backslash \operatorname{supp}(Q_{\mathcal{M}})$? A very partial answer is that $\underline{0} \in \operatorname{supp}(Q_{\mathcal{M}})$ whenever $\underline{0}$ belongs to $\mathcal{M}_W$ (combine Proposition 3.3(d,e) with Theorem 5(c)).

(ii) Which points belong to $\nu_W(\hat{X}_{\max}) \backslash \operatorname{supp}(Q_{\mathcal{M}})$? Observe that if $\hat{x} \in \hat{X}_{\max}$ but $\nu_W(\hat{x}) \notin \operatorname{supp}(Q_{\mathcal{M}})$, then there is a (locally specified) open neighbourhood U of $\nu_W(\hat{x})$ in $\mathcal{M}_W$ such that $Q_{\mathcal{M}}(U) = 0$. As $\nu_W(\hat{x})$ is generic for $Q_{\mathcal{M}}$, it follows that the local pattern of $\nu_W(\hat{x})$, specified by U, can repeat in $\nu_W(\hat{x})$ only with density zero.

These questions will be studied for the examples in §4.1. The following proposition indicates that the general case is subtle. A proof can be found at the end of §8.

PROPOSITION 3.15. *Let $\hat{x} = (x_G, x_H) + \mathcal{L} \in \hat{X}$. Then $\nu_W(\hat{x}) \in \operatorname{supp}(Q_{\mathcal{M}})$ if and only if*

$$m_H\{h \in H : (-x_H + W) \cap \pi^H(\mathcal{L}') = (-h + W) \cap \pi^H(\mathcal{L}')\} > 0 \quad \textit{for all finite } \mathcal{L}' \subseteq \mathcal{L}. \tag{9}$$

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Remark 3.16. (Results for non-compact windows) Suppose that $W_0 \subseteq H$ is only relatively compact and that ∂W_0 is nowhere dense. This setting encompasses the Fibonacci chain (see Example 4.2). It extends slightly the approaches used in [**7**, **21**, **50**, **54**], which restrict to topologically regular windows for most of their results—a stronger assumption than our $\mathrm{int}(\partial W_0) = \emptyset$. Let $W = \overline{W_0}$. Then $\partial W \subseteq \partial W_0$ and $\underline{0} \leqslant \nu_W - \nu_{W_0} \leqslant \nu_{\partial W_0}$, where $Z_{\partial W_0} = C_{\partial W_0}$ is a dense G_δ-set, by Proposition 3.3.

3.3.1. *Topological results.* It follows that $C_{W_0} \cap C_{\partial W_0} = C_W \cap C_{\partial W_0}$, so that the set of continuity points of ν_{W_0} contains a dense G_δ-set. Moreover, $\overline{\nu_{W_0}(C_{W_0})} = \overline{\nu_{W_0}(C_{\partial W_0})} = \overline{\nu_W(C_{\partial W_0})} = \overline{\nu_W(C_W)}$ and, similarly, also $\overline{\nu^G_{W_0}(C_{W_0})} = \overline{\nu^G_W(C_W)}$ (compare Remark 3.11). This shows also that Remark 3.13 holds for W_0.

In fact, also our main Theorem 1 and its Corollary 1 remain true, with W_0 replacing W. (To check the proofs, one should observe that, for each $\nu \in \mathcal{M}_{W_0} \setminus \{\underline{0}\}$, there is a unique $\hat{x} \in \hat{X}$ such that $\nu \leqslant \nu_W(\hat{x})$, although $\nu \leqslant \nu_{W_0}(\hat{x})$ need not hold. For Corollary 1(b) note also that $W' = \overline{\mathrm{int}(\overline{W_0})} = \overline{\mathrm{int}(W_0)}$.) A sufficient criterion for $\pi^G_* : \mathcal{M}_{W_0} \to \mathcal{M}^G_{W_0}$ to be one-to-one is that $\overline{W_0}$ is aperiodic and that W_0 is topologically regular, that is, $\overline{W_0} = \overline{\mathrm{int}(W_0)}$, the latter condition implying that ∂W_0 is nowhere dense. For its proof, note that aperiodicity of $\overline{W_0}$ implies that $(W_0, \mathcal{L})$ is strongly uniquely coding (compare the proof of Lemma 4.6) and that Lemma 4.5 remains valid, with W_0 relatively compact replacing W. Of course, ν_{W_0} will generally no longer be upper semicontinuous.

3.3.2. *Measure theoretic results.* Assume that $W_0 \subseteq H$ is relatively compact and measurable. Then the map $\nu_{W_0} : \hat{X} \to \mathcal{M}_{W_0}$ is measurable, which is checked with methods from the proof of Remark 3.12. As a consequence, the resulting Mirsky measures in Definition 3.7 are well defined, and Theorem 2, Corollary 7 and Proposition 3.15 hold, all with W_0 replacing W. Denote the resulting Mirsky measure on $\mathcal{M}$ by $Q^0_{\mathcal{M}}$ and fix a tempered van Hove sequence $(A_n)_n$. As also the first part of Moody's Theorem 3 holds, there is a full measure set $\hat{X}_1 \subseteq \hat{X}$ such that configurations are generic for $Q^0_{\mathcal{M}}$. These configurations are then pure point diffractive, as follows from Theorem 2, Remark 3.9 and the ergodic theorem. (In [**43**, Corollary 1], pure point diffractivity is shown in a different way.) Since $Q^0_{\mathcal{M}}$-almost sure density might be less than maximal density $D_{Q_{\mathcal{M}}} = \mathrm{dens}(\mathcal{L}) \cdot m_H(W)$, Theorem 5 will no longer be valid, in general. An example having an open window with fat Cantor set boundary has been discussed in [**43**].

In order to recover the setting and results of [**5**], let us compare configurations from $\mathcal{M}_{W_0}$ to configurations from $\mathcal{M}_W$. For any $\hat{x} \in \hat{X}$, $0 \leqslant \nu_{W_0}(\hat{x}) \leqslant \nu_W(\hat{x})$ so, in particular, the density of $\nu_{W_0}(\hat{x})$ along the sequence $(A_n)_n$ is bounded by that of $\nu_W(\hat{x})$ and hence by $D_{Q_{\mathcal{M}}}$. If $\nu_{W_0}(\hat{x})$ achieves the maximal density $D_{Q_{\mathcal{M}}}$, then $\hat{x} \in \hat{X}_{\max}$ and the density of $\nu_W(\hat{x}) - \nu_{W_0}(\hat{x})$ is clearly zero. Consequently, the empirical measures $Q^0_{n,\hat{x}} := (1/m_G(A_n)) \int_{A_n} \delta_{S_g \nu_{W_0}(\hat{x})}\, dm_G(g)$ are asymptotically equivalent to the measures $Q_{n,\hat{x}}$ in the sense that both sequences do have the same weak limit points, and Theorem 5 implies that the measures $Q^0_{n,\hat{x}}$ converge weakly to $Q_{\mathcal{M}}$. It follows that statistical properties of $\nu_{W_0}(\hat{x})$ and $\nu_W(\hat{x})$, like pattern frequencies and especially their autocorrelation coefficients, coincide for such $\hat{x}$ (compare [**8**]). Observe, however, that such $\nu_{W_0}(\hat{x})$ need not be an element of $\mathcal{M}_W$.

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3.3.3. *Combined results.* If $m_H(\partial W_0) = 0$, then ∂W_0 is nowhere dense and $0 \leqslant \nu_W - \nu_{W_0} \leqslant \nu_{\partial W_0}$ implies that $\nu_{W_0} = \nu_W$ $m_{\hat{X}}$-almost everywhere and on a dense G_δ-set. Hence the corresponding Mirsky measures $Q^0_{\mathcal{M}}$ and $Q_{\mathcal{M}}$ coincide, and the measures $Q^0_{n,\hat{x}}$ converge weakly to $Q^0_{\mathcal{M}}$ for $Q^0_{\mathcal{M}}$-almost every $\hat{x}$. Moreover, $\operatorname{supp}(Q^0_{\mathcal{M}}) = \overline{\nu_{W_0}(C_{W_0})}$ and $\operatorname{supp}(Q^0_{\mathcal{M}^G}) = \overline{\nu^G_{W_0}(C_{W_0})}$ and, if $\operatorname{int}(W_0) \neq \emptyset$, then both subsystems are strictly ergodic (see Corollary 7).

Note that these results apply, in particular, to various subclasses of so-called inter model sets, which are discussed in [**1**].

Remark 3.17. (The Mirsky measure as a zero temperature limit) The measure $Q_{\mathcal{M}^G}$ is characterized by the variational formula

$$D_{Q_{\mathcal{M}^G}} = \sup_P D_P \quad \text{where } D_P = \int_{\mathcal{M}^G_W} \chi_B \, dP \ \text{ with } \ \chi_B(\nu) = \frac{\nu(B)}{m_G(B)}.$$

Here B is any sufficiently small compact zero neighbourhood in G, and the supremum extends over all S-invariant probability measures on $\mathcal{M}^G_W$ (see Corollary 4). As B is compact, $\chi_B : \mathcal{M}^G_W \to \mathbb{R}$ is upper semicontinuous and, as any sufficiently small compact zero neighbourhood can be used, one can replace the indicator function of B actually by a continuous approximation. In this sense, $Q_{\mathcal{M}^G}$ is the unique maximizing measure for each such approximation. In the special case of $\mathbb{Z}$-actions, there is some general theory for such measures that asserts, among other things, that, for generic continuous observables, there is a unique maximizing measure and that this measure has zero entropy; also that each ergodic measure is maximizing for some continuous function (see, e.g., [**15**, **28**]).

When there is even a unique maximizing measure, then a very brief argument shows that this measure is the temperature zero limit of equilibrium states associated to the observable. The prerequisite for the argument is a suitable version of the thermodynamic formalism; in particular, the existence of equilibrium states†. Whenever this is available, one can argue as follows. For an S-invariant probability measure P on $\mathcal{M}^G_W$, denote by $h_S(P)$ the Kolgomorov–Sinai entropy of the dynamical system $(\mathcal{M}^G_W, S, P)$. Then for each $\beta > 0$, there is at least one S-invariant probability measure P_β that maximizes the functional $P \mapsto h_S(P) + \beta \cdot \int \chi_B(\nu)\, dP(\nu)$. (In other words, P_β is an equilibrium state for $\beta\, \chi_B$.) Let Q be any weak limit of such measures P_β along a sequence $\beta_n \to \infty$. Then for each S-invariant probability measure P on $\mathcal{M}^G_W$,

$$\begin{aligned}\int \chi_B \, dP &= \lim_{n\to\infty} \frac{1}{\beta_n}\left(h_S(P) + \beta_n \int \chi_B \, dP\right) \leqslant \limsup_{n\to\infty} \frac{1}{\beta_n}\left(h_S(P_{\beta_n}) + \beta_n \int \chi_B \, dP_{\beta_n}\right) \\ &= \limsup_{n\to\infty} \int \chi_B \, dP_{\beta_n} \leqslant \int \chi_B \, dQ,\end{aligned}$$

because $h_S(P_{\beta^n})$ is bounded by the (finite!) topological entropy of $(\mathcal{M}^G_W, S)$. This means that Q maximizes χ_B and, as $Q_{\mathcal{M}^G}$ is the only such measure, we conclude that $Q = Q_{\mathcal{M}^G}$ is the weak limit of the P_β as $\beta \to \infty$. When the maximizing measure is not unique, one can show (with a similar reasoning) that all temperature zero limit measures have maximal entropy among the maximizing measures.

† When $G = \mathbb{Z}^d$, this is covered by [**40**] (see also [**32**] and, for general discrete abelian groups, [**16**] could be used).

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4. *Examples*

In this section, we first illustrate our main results with the golden and silver mean chains, continue by providing some general facts about topologically regular windows and we finally discuss how also $\mathcal{B}$-free systems also fit our framework.

4.1. *Interval windows.* One of the most elementary non-trivial settings for model sets is probably the case where $G = H = \mathbb{R}$, $W = [\alpha, \beta]$ is a compact interval and where the lattice $\mathcal{L} \subset \mathbb{R}^2$ is spanned by two vectors $v = (v_G, v_H)$ and $w = (w_G, w_H)$. Two classical examples are the golden (Fibonacci) and the silver mean chain, which are discussed in some detail in the monograph [**2**]. Before we look at their peculiarities, we collect some facts that apply to both of them. First, observe that $\mathcal{L} = \{mv + nw : m, n \in \mathbb{Z}\}$. From now on, we assume that v_G and w_G are rationally independent and that the same is true for v_H and w_H. Then $\pi^G|_{\mathcal{L}}$ is one-to-one and $\pi^H(\mathcal{L})$ is dense in $H = \mathbb{R}$.

Note next that π^G_* is a homeomorphism (see Lemma 4.7), so we can restrict our discussion to the set $\mathcal{M}_W$ and need not consider $\mathcal{M}^G_W$ and its subsystems. Further general facts are that $(\mathcal{M}_W, S)$ is uniquely ergodic, by Theorem 2(c), because $m_H(\partial W) = 0$, and that its subsystem $\overline{\nu_W(C_W)} = \mathrm{supp}(Q_{\mathcal{M}})$ is minimal, by Remark 3.11 and Theorem 1(a). Because of the unique ergodicity, all points $\nu \in \mathcal{M}_W$ are generic for $Q_{\mathcal{M}}$ and, in particular, $\nu_W(\hat{X}_{\max}) = \mathcal{M}_W$.

In order to determine $\mathrm{supp}(Q_{\mathcal{M}})$, we have a closer look at the set C_W of continuity points of ν_W. It follows, from Lemma 6.1, that $\hat{x} = x + \mathcal{L} \notin C_W$ if and only if $\alpha \in x_H + \pi^H(\mathcal{L})$ or $\beta \in x_H + \pi^H(\mathcal{L})$. We distinguish the following two cases.

(i) $\beta - \alpha \notin \pi^H(\mathcal{L})$. Then, for each x_H, at most one of the points α and β can belong to $x_H + \pi^H(\mathcal{L})$, and Proposition 3.15 implies at once that $\hat{x} \in \mathrm{supp}(Q_{\mathcal{M}})$. Hence $\mathcal{M}_W = \mathrm{supp}(Q_{\mathcal{M}})$ so that $(\mathcal{M}_W, S)$ is minimal: that is, it coincides with the orbit closures of all its points.

(ii) $\beta - \alpha \in \pi^H(\mathcal{L})$. Then, for each x_H, the point α belongs to $x_H + \pi^H(\mathcal{L})$ if and only if also β belongs to this set. Hence Proposition 3.15 implies that those $\hat{x}$ for which this happens do not belong to $\mathrm{supp}(Q_{\mathcal{M}})$. Indeed, a moment's reflection shows that $\mathcal{M}_W \backslash \mathrm{supp}(Q_{\mathcal{M}})$ is a translate of $\nu_W(\mathcal{L} + (G \times \{0\}))$.
Traditionally, one is mostly interested in $\hat{x} = \hat{0}$, or, more precisely, in $\nu_W(\hat{0})$ and its orbit closure $\mathcal{M}_W(\hat{0})$. (Note that the support of $\nu^G_W(\hat{0}) = \pi^G_*(\nu_W(\hat{0}))$ is the point set, which is often denoted by $\curlywedge(W)$ in the literature.) The following two classical examples show that $\nu_W(\hat{0})$ can be contained in $\mathrm{supp}(Q_{\mathcal{M}})$ or not.

Example 4.1. (Silver mean chain (see [**2**, §7.1])) Let $v = (1, 1)$, $w = (\sqrt{2}, -\sqrt{2})$ and $W = [\alpha, \beta] = [-(\sqrt{2}/2), (\sqrt{2}/2)]$. Then $\beta - \alpha = \sqrt{2} \in \pi^H(\mathcal{L}) = \mathbb{Z} + \sqrt{2}\,\mathbb{Z}$, but $\alpha, \beta \notin \pi^H(\mathcal{L})$, so that $\hat{0} \in C_W$ and therefore $\mathcal{M}_W(\hat{0}) = \mathrm{supp}(Q_{\mathcal{M}})$.

Example 4.2. (Golden mean (Fibonacci) chain (see [**2**, Example 7.3])) Let $v = (1, 1)$ and $w = (\tau, \tau')$, where $\tau = (1 + \sqrt{5})/2$ and $\tau' = (1 - \sqrt{5})/2$. Let $W = [\alpha, \beta] = [-1, -\tau']$. Then $\beta - \alpha = 1 - \tau' \in \pi^H(\mathcal{L}) = \mathbb{Z} + \tau'\mathbb{Z}$ and $\alpha, \beta \in \mathbb{Z} + \tau'\mathbb{Z}$, so that $\hat{0} \notin C_W$ and $\nu_W(\hat{0}) \notin \mathrm{supp}(Q_{\mathcal{M}})$. This problem is sometimes addressed by considering half-open windows W', for which $\hat{x} = \hat{0}$ is still a point of discontinuity of $\nu_{W'}$, but for which $\nu_{W'}(\hat{0}) \in \overline{\nu_{W'}(C_{W'})} = \overline{\nu_W(C_W)} = \mathrm{supp}(Q_{\mathcal{M}})$ (see, e.g., the discussion at the end of [**2**, Example 7.3]).

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4.2. *Injectivity properties of π_*^G.* In order to characterize situations where the factor map $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is 'nearly' one-to-one, we introduce one more concept that we will use to study the examples in this section. We kept it separate from the main results in §3, because it is not as universal as the results presented there.

Definition 4.3. ((Strong) unique coding)

(a) The pair $(W, \mathcal{L})$ is *uniquely coding* if, for all $h, h' \in H$ holds: $(-h + W) \cap \pi^H(\mathcal{L}) = (-h' + W) \cap \pi^H(\mathcal{L}) \neq \emptyset$ implies $h = h'$.

(b) The pair $(W, \mathcal{L})$ is *strongly uniquely coding* if $h, h', h_n, h'_n \in H$ are such that $h_n \to h$, $h'_n \to h'$ and if

$$\text{for all } \ell \in \mathcal{L}, \text{ there exists } n_\ell \in \mathbb{N}_0, \text{ for all } n \geqslant n_\ell : 1_W(h_n + \ell_H) = 1_W(h'_n + \ell_H),$$

where this common value is 1 for at least one $\tilde{\ell} \in \mathcal{L}$ for all $n \geqslant n_{\tilde{\ell}}$; then $h = h'$.

LEMMA 4.4. *$(W, \mathcal{L})$ is uniquely coding if and only if $\pi_*^G|_{v_W(\hat{X})}$ is one-to-one.*

LEMMA 4.5. *$(W, \mathcal{L})$ is strongly uniquely coding if and only if $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is a homeomorphism.*

The slightly technical proofs of these two lemmas are provided at the end of §7.

4.3. *Topologically regular windows.* We call a window $W \subseteq H$ *topologically regular* if $W = \overline{\text{int}(W)}$ (see [**50**, Definition 4.11]). Dynamical properties of model sets with such windows have been studied, for example, in [**7**, **21**, **50**, **54**]. A window $W \subseteq H$ is *aperiodic* [**50**, Definition 5.12] or *irredundant* [**7**, Definition 1] if $h + W = W$ implies $h = 0$.

LEMMA 4.6. *Let W be a topologically regular window. Then the following are equivalent.*

(i) *W is aperiodic.*

(ii) *$W \neq \emptyset$ and $(W, \mathcal{L})$ is strongly uniquely coding.*

(iii) *$W \neq \emptyset$ and $(W, \mathcal{L})$ is uniquely coding.*

Proof. (i) $\Rightarrow$ (ii). $W \neq \emptyset$ is a consequence of aperiodicity. Let h, h', h_n, h'_n, n_ℓ and $\tilde{\ell}$ be as in the definition of strong unique coding (Definition 4.3) and let $\mathcal{L}_0 = \{\ell \in \mathcal{L} : \exists n_\ell \in \mathbb{N}_0 \ \forall n \geqslant n_\ell : 1_W(h_n + \ell_H) = 1\}$. Then $\tilde{\ell} \in \mathcal{L}_0$, by assumption, and $h + \ell_H, h' + \ell_H \in W$ for all $\ell \in \mathcal{L}_0$ because W is closed. As $\ell \in \mathcal{L}_0$ whenever $h + \ell_H \in \text{int}(W)$, it follows that

$$W = \overline{\text{int}(W)} = \overline{\text{int}(W) \cap (h + \pi^H(\mathcal{L}))} \subseteq \overline{h + \pi^H(\mathcal{L}_0)}.$$

Let $d := h' - h$. Then $d + h + \ell_H = h' + \ell_H \in W$ for each $\ell \in \mathcal{L}_0$, so that $d + W \subseteq d + \overline{h + \pi^H(\mathcal{L}_0)} = \overline{d + h + \pi^H(\mathcal{L}_0)} = \overline{h' + \pi^H(\mathcal{L}_0)} \subseteq \overline{W} = W$, and the aperiodicity of W implies $d = 0$: that is, $h' = h$.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). Fix $h' \in H$ such that $(-h' + W) \cap \pi_H(\mathcal{L}) \neq \emptyset$, which is possible since $W \neq \emptyset$. Consider $h \in H$ such that $h + W = W$. Then $(-h' + h + W) \cap \pi_H(\mathcal{L}) = (-h' + W) \cap \pi_H(\mathcal{L}) \neq \emptyset$. As $(W, \mathcal{L})$ is uniquely coding, this implies that $-h' + h = -h'$: that is, $h = 0$. Hence W is aperiodic. □

Example 4.7. If $H = \mathbb{R}^d$ and if W is topologically regular, then $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is a homeomorphism. This is a special case of the subsequent proposition.

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PROPOSITION 4.8. *Assume that the only compact subgroup of H is the trivial one. If W is topologically regular, then $(W, \mathcal{L})$ is strongly uniquely coding and $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is a homeomorphism.*

Proof. If $W = \emptyset$, the statement is obvious. Hence assume that $W \neq \emptyset$. In view of Lemmas 4.5 and 4.6, we only have to show that W is aperiodic. So let $H_W := \{h \in H : h + W = W\}$. H_W is a closed subgroup of H and, because $H_W + W = W$ is compact and W is non-empty, H_W is compact. Hence H_W is the trivial subgroup. □

Remark 4.9. This proposition suggests that one can assume irredundancy of the window without loss of generality by passing from $(G, H, \mathcal{L})$ with window W to $(G, H', \mathcal{L}')$ with window $W' = W/H_W$, where the period group $H_W = \{h \in H : h + W = W\}$ of W has been factored out. For non-empty topologically regular windows, this is described in detail in [**7**, Lemma 7], where it is shown, among other things, that W' is again topologically regular and that $m_{H/H_W}(\partial W') = 0$ if $m_H(\partial W) = 0$.

Example 4.10. A prominent example are the rhombic Penrose tilings, which can be realized as regular model sets in $G = \mathbb{R}^2$ with a topologically regular window in $H = \mathbb{R}^2 \times \mathbb{Z}/5\mathbb{Z}$ (see [**41**, §3.2] and [**2**, Example 7.11]). In fact, this description goes back to de Bruijn (see also the discussion in [**2**, §7.5.2]). In that, case we have $\hat{0} \notin C_W$ and the cardinalities one, two and ten of the fibres $\hat{\pi}^{-1}\{\hat{x}\}$ for $\hat{x} \in \hat{X}$ (compare Definition 5.5) are discussed in [**48**].

4.4. *$\mathcal{B}$-free systems.* Let $(b_k)_{k\in\mathbb{N}_0}$ be an increasing sequence of pairwise coprime integers greater than one such that

$$\sum_{k\in\mathbb{N}_0} \frac{1}{b_k} < \infty. \tag{10}$$

Writing $\mathcal{B} = \{b_k : k \in \mathbb{N}_0\} \subset \{2, 3, \ldots\}$, the set $V_{\mathcal{B}}$ of $\mathcal{B}$-free integers consists of all integers having no factor in $\mathcal{B}$, that is,

$$V_{\mathcal{B}} = \mathbb{Z} \Big\backslash \bigcup_{b\in\mathcal{B}} b\mathbb{Z}.$$

Such sets have been studied in [**20**]. With $\mathcal{B} = \{p^2 : p \text{ prime}\}$, the set $V_{\mathcal{B}}$ are the square-free integers [**17**, **45**, **51**]. As remarked in [**4**], the $\mathcal{B}$-free integers and their lattice generalizations are weak model sets. Indeed, consider $G = \mathbb{Z}$, the product group $H = \prod_{k\in\mathbb{N}_0} \mathbb{Z}/b_k\mathbb{Z}$, and write $h = (h_k)_{k\in\mathbb{N}_0}$ for $h \in H$. The map $\iota : G \to H$, defined by $\iota(g) = (g \mod b_k\mathbb{Z})_{k\in\mathbb{N}_0}$, is a continuous embedding of G into H and $\iota(G)$ is dense in H, because the b_k are pairwise coprime. Define the lattice $\mathcal{L} \subseteq G \times H$ to be the diagonal embedding $\mathcal{L} = \{(g, \iota(g)) : g \in G\}$. Then $(G, H, \mathcal{L})$ is a cut-and-project scheme. The $\mathcal{B}$-free integers are the weak model set defined by the compact window

$$W = \prod_{k\in\mathbb{N}_0} \mathbb{Z}/b_k\mathbb{Z}\backslash\{0_k\},$$

where 0 denotes the neutral element in H.

LEMMA 4.11. *$Q_{\mathcal{M}}$ has full support: that is,* $\operatorname{supp}(Q_{\mathcal{M}}) = \mathcal{M}_W$.

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Proof. We show $\nu_W(\hat{X}) \subseteq \mathrm{supp}(Q_{\mathcal{M}})$, using Proposition 3.15. Then the claim follows since $\mathrm{supp}(Q_{\mathcal{M}})$ is closed. Fix a fundamental domain X of $\mathcal{L}$ and take arbitrary $x \in X$. We must show that

$$m_H\{h \in H : (-x_H + W) \cap \pi^H(\mathcal{L}') = (-h + W) \cap \pi^H(\mathcal{L}')\} > 0 \quad \text{for all finite } \mathcal{L}' \subseteq \mathcal{L}.$$

We can, of course, restrict to sets $\mathcal{L}' = \{(n, \iota(n)) : n \in \{-N, \dots, N\}\}$ for $N \in \mathbb{N}_0$. Let $\mathcal{L}'_x = \{\ell \in \mathcal{L}' : \exists k = k(\ell) \in \mathbb{N}_0 \text{ such that } x_{H,k} + \ell_{H,k} = 0 \mod b_k\}$. Given $\ell \in \mathcal{L}'_x$, we choose the smallest k with this property as $k(\ell)$. Then, for any $\ell \in \mathcal{L}'$, the condition $\ell_H \in (-x_H + W) \cap \pi^H(\mathcal{L}')$ is equivalent to $\ell \in \mathcal{L}' \backslash \mathcal{L}'_x$. Let $k_0 = \max\{k(\ell) : \ell \in \mathcal{L}'_x\}$ and fix $k_1 > k_0$ such that $b_{k_1} > 2N$. Observe that $\ell_{H,k} = \iota(n)_k \in \{0, \dots, N\} \cup \{b_k - N, \dots, b_k - 1\}$ for all $\ell \in \mathcal{L}'$ and $k > k_1$, and define

$$H_x := \{h \in H : h_k = x_{H,k} \text{ if } k \leqslant k_1,\ h_k \in \{N+1, \dots, b_k - N - 1\} \text{ if } k > k_1\}.$$

Then $m_H(H_x) \geqslant \prod_{k \leqslant k_1} (1/b_k) \cdot \prod_{k > k_1} (1 - ((2N+1)/b_k)) > 0$, and it remains to be shown that $(-x_H + W) \cap \pi^H(\mathcal{L}') = (-h + W) \cap \pi^H(\mathcal{L}')$ for every $h \in H_x$. So let $h \in H_x$ and $\ell \in \mathcal{L}'$. If $\ell \in \mathcal{L}'_x$, then $\ell_H \notin -x_H + W$ and $h_{k(\ell)} + \ell_{H,k(\ell)} = x_{H,k(\ell)} + \ell_{H,k(\ell)} = 0 \mod b_{k(\ell)}$, so that $\ell_H \notin -h + W$ too.

If $\ell \in \mathcal{L}' \backslash \mathcal{L}'_x$, then $\ell_H \in -x_H + W$, and we show that $\ell_H \in -h + W$ as well. Suppose, for a contradiction, that this is not the case. Then there is some k such that $h_k + \ell_{H,k} = 0 \mod b_k$. But as $x_{H,k} + \ell_{H,k} \neq 0 \mod b_k$, by assumption, $k > k_1$ such that $h_k \in \{N+1, \dots, b_k - N - 1\}$. Together with $\ell_{H,k} \in \{0, \dots, N\} \cup \{b_k - N, \dots, b_k - 1\}$, a case analysis implies that $h_k + \ell_{H,k} \neq 0 \mod b_k$, which is a contradiction. □

We study the domain of injectivity of the projection $\pi^G_* : \mathcal{M}_W \to \mathcal{M}^G_W$†. To this end, we introduce the following notation. For $A \subseteq \mathbb{Z}$ and $k \in \mathbb{N}_0$, we write $\langle A \rangle_k = \{\iota(g)_k : g \in A\}$. Define $\mathcal{Y} \subseteq \mathcal{M}_W$ by

$$\mathcal{Y} = \{\nu \in \mathcal{M}_W : \mathrm{card}\, \langle \mathrm{supp}(\pi^G_* \nu) \rangle_k = b_k - 1\ \forall k \in \mathbb{N}_0\}.$$

The set $\mathcal{Y} \subseteq \mathcal{M}_W$ is measurable and consists of all measures such that 'every b_k-reduction misses exactly one coset'. The set $\pi^G_*(\mathcal{Y})$ was studied for square-free integers by Peckner [**45**] (called X_1 in his paper); for $\mathcal{B}$-free systems it is the set Y of [**34**], and for visible lattice points the set $\mathbb{A}_1$ of [**4**]. The next lemmas have close analogues in these three publications.

LEMMA 4.12. *π^G_* is one-to-one on $\mathcal{Y}$ and $\pi^G_*|_{\mathcal{Y}} : \mathcal{Y} \to \pi^G_*(\mathcal{Y})$ is a Borel isomorphism.*

Proof. Choose a fundamental domain X of $\mathcal{L}$ satisfying $\pi^G(X) = \{0\}$ so that each $x \in X$ is of the form $(0, x_H)$. Consider $\nu, \nu' \in \mathcal{Y}$ such that $\pi^G_* \nu = \pi^G_* \nu'$. Since $\nu, \nu' \neq \underline{0}$, there exist unique $x, x' \in X$ such that

$$\nu \leq \nu_W(x + \mathcal{L}), \quad \nu' \leq \nu_W(x' + \mathcal{L}).$$

Then

$$\pi^G_* \nu \leq \pi^G_* \nu_W(x + \mathcal{L}) \quad \text{and} \quad \pi^G_* \nu' \leq \pi^G_* \nu_W(x' + \mathcal{L}).$$

† Note that $(W, \mathcal{L})$ is not uniquely coding. Consider, for example, $h, h' \in H$ given by $h_1 = h'_1 = 1$, $h_2 = h'_2 = b_2 - 1$, $h_3 = 1$, $h'_3 = b_3 - 1$, $h_{2k} = h'_{2k} = b_{2k} - k$ and $h_{2k+1} = h'_{2k+1} = k$ for $k \geqslant 2$. Then $h \neq h'$ and $h, h' \in W$, and it is not hard to check that $h + \iota(k), h' + \iota(k) \notin W$ for all $k \in \mathbb{Z} \backslash \{0\}$.

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Note that $h + W = \prod_k (\mathbb{Z}/b_k\mathbb{Z}\backslash\{h_k\})$ for any $h \in H$. Thus

$$\langle \mathrm{supp}(\pi^G_* \nu)\rangle_k \subseteq \langle \mathrm{supp}(\pi^G_* \nu_W(x + \mathcal{L}))\rangle_k \subseteq \langle \iota^{-1}(-x_H + W)\rangle_k = \mathbb{Z}/b_k\mathbb{Z}\backslash\{-x_{H,k}\}$$

for all $k \in \mathbb{N}_0$ and $\langle \mathrm{supp}(\pi^G_* \nu')\rangle_k \subseteq \mathbb{Z}/b_k\mathbb{Z}\backslash\{-x'_{H,k}\}$ follows in the same way. Since, by assumption, $\mathrm{card}\langle \mathrm{supp}(\pi^G_* \nu)\rangle_k = b_k - 1$ and $\pi^G_* \nu = \pi^G_* \nu'$, we must have $x_{H,k} = x'_{H,k}$ for all $k \in \mathbb{N}_0$. We thus conclude that $x = x'$, which implies that $\nu = \nu'$ because $\pi^G|_{\mathcal{L}}$ is one-to-one. Hence $\pi^G_*|_{\mathcal{Y}}$ is one-to-one and it follows, from classical results by Lusin and Souslin [**30**, Corollary 15.2], that $\pi^G_*|_{\mathcal{Y}} : \mathcal{Y} \to \pi^G_*(\mathcal{Y})$ is a Borel isomorphism. □

LEMMA 4.13. *$\nu_W(\hat{X}_{\max}) \subseteq \mathcal{Y}$. As a consequence, $\mathcal{Y}$ has full Mirsky measure.*

Proof. Take a van Hove sequence $(A_n)_n$ associated to $\hat{X}_{\max}$. Then, for any $\hat{x} \in \hat{X}_{\max}$, the density of $\nu_W(\hat{x})$ is given by

$$\lim_{n\to\infty} \frac{\nu_W(\hat{x})(A_n \times H)}{m_G(A_n)} = 1 \cdot m_H(W) = \prod_{k\in\mathbb{N}_0} \left(1 - \frac{1}{b_k}\right).$$

This implies that $\mathrm{card}\langle \mathrm{supp}(\pi^G_* \nu_W(\hat{x}))\rangle_k = b_k - 1$ for every $k \in \mathbb{N}_0$. Indeed, otherwise, for some b_k, at least two cosets will be missed. But then $\nu_W(\hat{x})$ has an upper density less than maximal. We thus conclude that $\nu_W(\hat{x}) \in \mathcal{Y}$. As $\hat{x} \in \hat{X}_{\max}$ was arbitrary, $\nu_W(\hat{X}_{\max}) \subseteq \mathcal{Y}$. Since $\hat{X}_{\max} \subseteq \hat{X}$ has full Haar measure, $\mathcal{Y}$ must have full Mirsky measure. □

For $\mathcal{B}$-free systems, we have the following strengthening of Corollary 6.

PROPOSITION 4.14. (Discrete spectrum for $\mathcal{B}$-free systems) *For any $\hat{x} \in \hat{X}_{\max}$, $\mathcal{M}^G_W(\hat{x}) = \mathcal{M}^G_W$, and the measure theoretic dynamical systems $(\mathcal{M}^G_W(\hat{x}), Q_{\mathcal{M}^G}, S)$ and $(\hat{X}, m_{\hat{X}}, \hat{T})$ are isomorphic. The configuration $\nu_W(\hat{0})$ has maximal density.*

Proof. Observe that $\mathrm{supp}(Q_{\mathcal{M}}) = \mathcal{M}_W$, by Lemma 4.11. Hence $\mathcal{M}_W(\hat{x}) = \mathcal{M}_W$ for all $\hat{x} \in \hat{X}_{\max}$, by Theorem 5(d), so $(\mathcal{M}_W(\hat{x}), Q_{\mathcal{M}}, S)$ and $(\hat{X}, m_{\hat{X}}, \hat{T})$ are isomorphic, by Theorem 2(a). That these systems are also isomorphic to $(\mathcal{M}^G_W(\hat{x}), Q_{\mathcal{M}^G}, S)$ follows now from Lemmas 4.12 and 4.13. The density of $\nu_W(\hat{0})$ is, for example, computed in [**4**]. □

Remark 4.15. The Mirsky measure $Q_{\mathcal{M}}$ is not the only invariant probability on $\mathcal{M}_W$ that assigns full mass to the uniqueness set $\mathcal{Y}$. In [**45**] and [**34**], it is proved that $\pi^G_*(\mathcal{Y})$ has full measure also under the measure of maximal entropy. Therefore this measure also can be transferred to $\mathcal{Y} \subseteq \mathcal{M}_W$ unambiguously.

Remark 4.16. Recently, the authors of [**11**] studied the dynamics of more general $\mathcal{B}$-free systems. They investigated the situation when the two assumptions that all b_k are pairwise coprime and that $\sum_{k\in\mathbb{N}_0} 1/b_k < \infty$ are weakened or skipped, and they obtained a wealth of topological, measure theoretic and number theoretic results. With regards to the present setting, the lack of coprimeness means that $\iota(G)$ is no longer dense in H so that H must be replaced by $\overline{\iota(G)}$. Note, however, that the measure m_H must be replaced by the Haar measure on $\overline{\iota(G)}$. The summability assumption was only used to show that the window has positive Haar measure and that $\mathrm{supp}(Q_{\mathcal{M}}) = \mathcal{M}_W$ in Lemma 4.11. Our Theorems 1, 3, 4 and 5 apply without this assumption. So we obtain results similar to some of those in [**11**, Theorems A, B and E], and also our definition of the Mirsky measure $Q_{\mathcal{M}^G}$ corresponds to the one in [**11**, Definition 2.30].

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5. *Basic observations*

LEMMA 5.1.

(a) $S_g(\nu_W(\hat{x})) = \nu_W(\hat{T}_g\hat{x})$ *for all* $\hat{x} \in \hat{X}$ *and* $g \in G$.

(b) $S_g(\mathcal{M}_W) = \mathcal{M}_W$ *for all* $g \in G$.

Proof. For each measurable $A \subseteq G \times H$,

$$\begin{aligned} S_g(\nu_W(\hat{x}))(A) &= \sum_{y \in (x+\mathcal{L}) \cap (G \times W)} S_g(\delta_y)(A) = \sum_{y \in (x+\mathcal{L}) \cap (G \times W)} \delta_{T_g y}(A) \\ &= \sum_{y \in T_g((x+\mathcal{L}) \cap (G \times W))} \delta_y(A) = \sum_{y \in (T_g x + \mathcal{L}) \cap (G \times W)} \delta_y(A) \\ &= \nu_W(\hat{T}_g\hat{x})(A). \end{aligned}$$

As $S_g^{-1} = S_{-g}$ it suffices to prove that $S_g(\mathcal{M}_W) \subseteq \mathcal{M}_W$. So let $\nu \in \mathcal{M}_W$ be the vague limit of the sequence $(\nu_W(\hat{x}^n))_n$. Then $\nu_W(\hat{T}_g\hat{x}^n) = S_g\nu_W(\hat{x}^n)$ converges vaguely to $S_g\nu$ because S_g is continuous. □

LEMMA 5.2. *Suppose that a sequence* $(\nu_W(x^n + \mathcal{L}))_{n \in \mathbb{N}_0}$, $x^n \in G \times H$ *converges vaguely to some* $\nu \in \mathcal{M}_W$.

(a) *If* $\lim_{n\to\infty}(x^n + \mathcal{L}) = \hat{x} = (x + \mathcal{L})$, *then* $\nu = \varphi \cdot \nu_W(\hat{x})$ *with*

$$\varphi(z) = \begin{cases} 0 & \text{if } z \notin x + \mathcal{L}, \\ 0 & \text{if } z \in x + \mathcal{L} \text{ and } z_H \notin W, \\ \lim\limits_{n\to\infty} 1_W((x^n + \ell)_H) & \text{if } z = x + \ell \text{ for some } \ell \in \mathcal{L} \text{ and } z_H \in \partial W, \\ 1 & \text{if } z \in x + \mathcal{L} \text{ and } z_H \in \operatorname{int}(W). \end{cases} \tag{11}$$

(b) *If* $\lim_{n\to\infty}(x^n + \mathcal{L}) = \hat{x} = (x + \mathcal{L})$ *and* $\nu_W(\hat{x})(G \times \partial W) = 0$, *then* $\nu = \nu_W(\hat{x})$.

(c) *If* ν *is not the zero measure, then there exists* $\hat{x} \in \hat{X}$ *such that* $\lim_{n\to\infty}(x^n + \mathcal{L}) = \hat{x}$.

Proof. (a) As $\lim_{n\to\infty}(x - x^n + \mathcal{L}) = \mathcal{L}$ (in $\hat{X}$), there exist $\ell^n \in \mathcal{L}$ such that $\lim_{n\to\infty}(x - x^n + \ell^n) = 0$ (in $G \times H$). Replacing all x^n by $x^n + \ell^n$ in the lemma does not change its assumptions nor its assertions, because they are all formulated in terms of the $x^n + \mathcal{L}$. Therefore we may assume that all $\ell^n = 0$ and hence $\lim_{n\to\infty} x^n = x$.

Let $V \subseteq G \times H$ be an open zero neighbourhood whose closure is compact and for which all sets $\overline{V} + \ell$ ($\ell \in \mathcal{L}$) are pairwise disjoint. For each $z \in G \times H$ we fix some open neighbourhood $V_z \subseteq V + z$ of z in the following way.

- If $z \notin x + \mathcal{L}$, then $\overline{V_z} \cap (x + \mathcal{L}) = \emptyset$.
- If $z \in x + \mathcal{L}$ and $z_H \notin W$, then $\overline{V_z} \cap (G \times W) = \emptyset$.
- If $z \in x + \mathcal{L}$ and $z_H \in \operatorname{int}(W)$, then $\overline{V_z} \subseteq (G \times \operatorname{int}(W))$.

Finally, we fix compact z-neighbourhoods $C_z \subseteq V_z$, which is always possible in a locally compact space. Vague convergence implies that

$$\nu(V_z) \leqslant \liminf_{n\to\infty} \nu_W(x^n + \mathcal{L})(V_z) \quad \text{and} \quad \nu(C_z) \geqslant \limsup_{n\to\infty} \nu_W(x^n + \mathcal{L})(C_z).$$

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It follows that

$$\begin{aligned}\nu(V_z) &\leqslant \liminf_{n\to\infty} \sum_{y\in(x^n+\mathcal{L})\cap(G\times W)} \delta_y(V_z),\\ &= \begin{cases} 0 & \text{if } z\notin x+\mathcal{L},\\ 0 & \text{if } z\in x+\mathcal{L} \text{ and } z_H\notin W,\\ 1 & \text{if } z\in x+\mathcal{L} \text{ and } z_H\in \operatorname{int}(W),\\ \liminf\limits_{n\to\infty} 1_W((x^n+\ell)_H) & \text{if } z=x+\ell \text{ for some } \ell\in\mathcal{L} \text{ and } z_H\in\partial W,\end{cases}\\ &\leqslant 1_{\{x+\mathcal{L}\}}(z)\cdot 1_{G\times W}(z)\\ &= \nu_W(x+\mathcal{L})(V_z)\end{aligned}$$

and

$$\begin{aligned}\nu(C_z) &\geqslant \limsup_{n\to\infty} \nu_W(x^n+\mathcal{L})(C_z) = \limsup_{n\to\infty} \sum_{y\in(x^n+\mathcal{L})\cap(G\times W)} \delta_y(C_z)\\ &= \begin{cases} 0 & \text{if } z\notin x+\mathcal{L},\\ 0 & \text{if } z\in x+\mathcal{L} \text{ and } z_H\notin W,\\ 1 & \text{if } z\in x+\mathcal{L} \text{ and } z_H\in \operatorname{int}(W),\\ \limsup\limits_{n\to\infty} 1_W((x^n+\ell)_H) & \text{if } z=x+\ell \text{ for some } \ell\in\mathcal{L} \text{ and } z_H\in\partial W.\end{cases}\end{aligned}$$

As $\nu_W(x+\mathcal{L})$ is a sum of isolated unit point masses in the set $x+\mathcal{L}$, the combination of both inequalities implies that $\nu=\varphi\cdot\nu_W(x+\mathcal{L})$ with φ from (11).

(b) Consider any $z=x+\ell$, $\ell\in\mathcal{L}$, with $\nu_W(\hat{x})(\{z\})=1$. We only need to show that $\varphi(z)=1$. But $\nu_W(\hat{x})(\{z\})=1$ implies, by definition of ν_W, that $z_H\in W$. From $\nu_W(\hat{x})(G\times\partial W)=0$, we conclude that $z_H\notin\partial W$. Hence $z_H\in\operatorname{int}(W)$ and $\varphi(z)=1$, in view of (11).

(c) As $\hat{X}$ is compact, the sequence $(x^n+\mathcal{L})_n$ has a subsequence that converges to some $\hat{x}\in\hat{X}$. Applying assertion (a) to such a subsequence, we conclude that $\nu\leqslant\nu_W(\hat{x})$. As ν is different from the zero measure, this determines $\hat{x}$ uniquely and independently of the initially chosen subsequence. □

LEMMA 5.3. *$\pi_*^{\hat{X}\times G}:\mathcal{GM}_W\to\mathcal{GM}_W^G$ is a homeomorphism (that respects the fibres $(\pi_*^{\hat{X}})^{-1}\{\hat{x}\}$).*

Proof. As $\pi_*^{\hat{X}\times G}$ is a continuous surjective map between the compact space $\mathcal{GM}_W$ and the Hausdorff space $\mathcal{GM}_W^G$ (Lemma 2.1), only its injectivity needs to be shown. So let $(\hat{x}_1,\nu_1),(\hat{x}_2,\nu_2)\in\mathcal{GM}_W$ and assume that $\pi_*^{\hat{X}\times G}(\hat{x}_1,\nu_1)=\pi_*^{\hat{X}\times G}(\hat{x}_2,\nu_2)$, that is, $(\hat{x}_1,\pi_*^G\nu_1)=(\hat{x}_2,\pi_*^G\nu_2)$, and, in particular, $\hat{x}:=\hat{x}_1=\hat{x}_2=x+\mathcal{L}$. By Lemma 5.2(a), $\nu_i=\varphi_i\cdot\nu_W(\hat{x})$ $(i=1,2)$, so there exist $A_1,A_2\subseteq\mathcal{L}$ such that $\nu_i=\sum_{\ell\in A_i}\delta_{x+\ell}$ and hence $\pi_*^G\nu_i=\sum_{\ell\in A_i}\delta_{\pi^G x+\pi^G\ell}=\sum_{g\in\pi^G(A_i)}\delta_{\pi^G x+g}$. As $\pi_*^G\nu_1=\pi_*^G\nu_2$, it follows that $\pi^G(A_1)=\pi^G(A_2)$ and, as $\pi^G|_{\mathcal{L}}$ is injective, we conclude that $A_1=A_2$ and hence $\nu_1=\nu_2$. □

LEMMA 5.4. *For each $\nu\in\mathcal{M}_W\setminus\{\underline{0}\}$, there is a unique $\hat{x}\in\hat{X}$ such that $\nu\leqslant\nu_W(\hat{x})$.*

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Proof. Each $\nu \in \mathcal{M}_W$ is the limit of a sequence of measures $\nu_W(\hat{x}^n)$. If $\nu \neq \underline{0}$, there is an $\hat{x} \in \hat{X}$ such that $\lim_{n\to\infty} \hat{x}^n = \hat{x}$ and $\nu \leqslant \nu_W(\hat{x})$ by Lemma 5.2. This $\hat{x}$ is obviously unique. □

Definition 5.5. Given $\nu \in \mathcal{M}_W \backslash \{\underline{0}\}$, we denote the unique $\hat{x} \in \hat{X}$ with $\nu \leqslant \nu_W(\hat{x})$ by $\hat{\pi}(\nu)$. This defines a map $\hat{\pi} : \mathcal{M}_W \backslash \{\underline{0}\} \to \hat{X} \backslash Z_W$.

LEMMA 5.6. *$\hat{\pi}$ is continuous on $\mathcal{M}_W \backslash \{\underline{0}\}$ and $\hat{\pi} \circ S_g = \hat{T}_g \circ \hat{\pi}$ for all $g \in G$.*

Proof. Suppose, for a contradiction, that ν, ν^n are in $\mathcal{M}_W \backslash \{\underline{0}\}$ and $\nu^n \to \nu$ vaguely, but there exists $U \subseteq G \times H$ such that its image $\hat{U} \subseteq \hat{X}$ under the quotient map is an open neighbourhood of $\hat{\pi}(\nu)$ satisfying $\hat{\pi}(\nu^n) \notin \hat{U}$ for all n. Then $\nu^n(U) = 0$ for all n, but $\nu(U) > 0$, which contradicts the vague convergence. Hence $\hat{\pi}$ is continuous. Now let $\nu \in \mathcal{M}_W \backslash \{\underline{0}\}$ and $g \in G$. As $\nu \leqslant \nu_W(\hat{\pi}(\nu))$, $S_g \nu \leqslant S_g \nu_W(\hat{\pi}(\nu)) = \nu_W(\hat{T}_g \hat{\pi}(\nu))$. Hence $\hat{\pi}(S_g \nu) = \hat{T}_g \hat{\pi}(\nu)$. □

6. *Zeros and continuity: proof of Proposition 3.3*

Some of the observations of this section appear in various disguises in the literature (see [**2**, Remark 7.4] and the references given there). For example, the Baire argument as in the proof of Proposition 3.3 was used, for example, in [**7**, **8**, **10**, **26**, **52**, **54**]. In the literature on invariant graphs for skew-product transformations, the Baire argument is traditionally used to prove that the set of zeros or the set of continuity points of the graph is a dense G_δ (see, e.g., [**27**, **31**, **57**]). Implicitly, this observation is already contained in [**23**].

LEMMA 6.1. *Let $x \in G \times H$. Then $x_H \in \bigcap_{\ell\in\mathcal{L}}((\partial W)^c - \ell_H)$ if and only if $(x + \mathcal{L}) \in C_W$. Equivalently,*

$$C_W = \pi^{\hat{X}}\left((\pi^H)^{-1}\left(\bigcap_{\ell\in\mathcal{L}}((\partial W)^c - \ell_H)\right)\right).$$

Proof. Let $x + \mathcal{L}$, $x^n + \mathcal{L} \in \hat{X}$, $\lim_{n\to\infty}(x^n + \mathcal{L}) = (x + \mathcal{L})$. Assume that

$$x_H \in \bigcap_{\ell\in\mathcal{L}}((\partial W)^c - \ell_H),$$

that is, $(x + \ell)_H \notin \partial W$ for all $\ell \in \mathcal{L}$. Then $\nu_W(x + \mathcal{L})(G \times \partial W) = 0$, and Lemma 5.2 implies that $\lim_{n\to\infty} \nu_W(x^n + \mathcal{L}) = \nu_W(x + \mathcal{L})$. Hence $x + \mathcal{L} \in C_W$.

Conversely, if $x_H \notin \bigcap_{\ell\in\mathcal{L}}((\partial W)^c - \ell_H)$, then there is an $\ell \in \mathcal{L}$ such that $x_H + \ell_H \in \partial W$. Hence there exist $x_H^n \in W^c - \ell_H$ such that $x_H^n \to x_H$ as $n \to \infty$. Let $x^n = (x_G, x_H^n)$. Then, for each sufficiently small open neighbourhood U of $x + \ell$ in $G \times H$ and sufficiently large n, $\nu_W(x^n + \mathcal{L})(U) = 1_W(x_H^n + \ell_H) \cdot \delta_{x_n+\ell}(U) = 0$, while $\nu_W(x + \mathcal{L})(U) = 1_W(x_H + \ell_H) \cdot \delta_{x+\ell}(U) = 1$, so that $\nu_W(x^n + \mathcal{L}) \not\to \nu_W(x + \mathcal{L})$. In particular, $(x + \mathcal{L}) \notin C_W$. □

LEMMA 6.2. *$\operatorname{int}(C_W) = \emptyset$ if and only if $\partial W \neq \emptyset$.*

Proof. If $\partial W = \emptyset$, then $C_W = \hat{X}$, by Lemma 6.1. Conversely, let $\partial W \neq \emptyset$ and suppose, for a contradiction, that $\operatorname{int}(C_W) \neq \emptyset$. Then $(\pi^{\hat{X}})^{-1}(C_W) \subseteq G \times H$ has non-empty interior, so $\pi^H((\pi^{\hat{X}})^{-1}(C_W))$ contains a non-empty open set U, as π^H is open. Because of Lemma 6.1,

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$U \subseteq \bigcap_{\ell\in\mathcal{L}}((\partial W)^c - \ell_H)$. This implies that $\bigcup_{\ell\in\mathcal{L}}(U + \ell_H) \subseteq (\partial W)^c$. But as $\{\ell_H : \ell \in \mathcal{L}\} = \pi^H(\mathcal{L})$ is dense in H and as $\partial W \neq \emptyset$, this is impossible. □

LEMMA 6.3. *If $A \subseteq \mathcal{GM}_W$ is a non-empty closed S-invariant subset, then $A \supseteq \overline{\mathcal{G}(\nu_W|_{C_W})}$, which is the S-invariant closure of the graph of the restriction of ν_W to the set C_W.*

Proof. The S-invariance of $\overline{\mathcal{G}(\nu_W|_{C_W})}$ holds since C_W is $\hat{T}$-invariant (compare the proof of Lemma 5.1(b)). As $A \neq \emptyset$ is S-invariant, the set $\pi^{\hat{X}}_*(A) \neq \emptyset$ is $\hat{T}$-invariant and, as $\hat{T}$ is minimal, $\pi^{\hat{X}}_*(A) = \hat{X} \supseteq C_W$. Because of Remark 3.2, $(\pi^{\hat{X}}_*)^{-1}\{\hat{x}\} = \{(\hat{x}, \nu_W(\hat{x}))\}$ for each $\hat{x} \in C_W$. Hence $\mathcal{G}(\nu_W|_{C_W}) \subseteq A$ and, as A is closed, the lemma is proved. □

Proof of Proposition 3.3 (a) This assertion is contained in Lemma 5.2.

(b) Because of (a), $\hat{x} \in C_W$ if and only if $\{\nu \in \mathcal{M}_W : \nu \leqslant \nu_W(\hat{x})\} = \{\nu_W(\hat{x})\}$.

(c) We show that C_W is a dense G_δ-set. As $\operatorname{int}(\partial W) = \emptyset$, the closed set ∂W is nowhere dense and so are all translates $(\partial W) - \ell_H$ for $\ell \in \mathcal{L}$. As H is a Baire space, the set $\bigcap_{\ell\in\mathcal{L}}((\partial W)^c - \ell_H)$ is a dense G_δ-set in H. As π^H is continuous and open, $(\pi^H)^{-1}(\bigcap_{\ell\in\mathcal{L}}((\partial W)^c - \ell_H))$ is a dense G_δ-set in $G \times H$. As the quotient map onto $\hat{X}$ is continuous open, we conclude, with Lemma 6.1, that C_W is a dense G_δ-set in $\hat{X}$†.

(d) Assume now that $\operatorname{int}(W) = \emptyset$. Then $W^c = (\partial W)^c$. By definition, $\hat{x} = x + \mathcal{L} \in Z_W$ if and only if $(x + \mathcal{L}) \cap (G \times W) = \emptyset$, so $\hat{x} = x + \mathcal{L} \in Z_W$ if and only if $x_H \in \bigcap_{\ell\in\mathcal{L}}(W^c - \ell_H) = \bigcap_{\ell\in\mathcal{L}}((\partial W)^c - \ell_H)$. Hence $Z_W = C_W$, by Lemma 6.1. It remains to show that $\underline{0} \in \nu_W(\hat{X}\backslash Z_W)$ if $W \neq \emptyset$. Pick some $\hat{x} \in Z_W$, which is possible since $C_W = Z_W$ is dense in the non-empty set $\hat{X}$. As $\operatorname{int}(W) = \emptyset$, $\partial W = W \neq \emptyset$. Therefore Lemma 6.2 shows that $\operatorname{int}(Z_W) = \operatorname{int}(C_W) = \emptyset$. Hence we find $\hat{x}_n \in \hat{X}\backslash Z_W$ with $\hat{x}_n \to \hat{x}$. It follows that $\underline{0} = \nu_W(\hat{x}) = \lim_{n\to\infty} \nu_W(\hat{x}_n)$: that is, $\underline{0} \in \nu_W(\hat{X}\backslash Z_W)$.

(e) Let $\operatorname{int}(W) \neq \emptyset$ and assume, for a contradiction, that $\underline{0} \in \mathcal{M}_W$. Then there exist $x^n \in X$ such that $\lim_{n\to\infty} \nu_W(x^n + \mathcal{L}) = \underline{0}$. As $\pi^H(\mathcal{L})$ is dense in H and as $\pi^H(X)$ is relatively compact in H, there is a finite set $\mathcal{L}_0 \subseteq \mathcal{L}$ such that $\pi^H(X) \subseteq \bigcup_{\ell\in\mathcal{L}_0}(W - \ell_H)$. Let Q be a compact subset of $G \times H$ that contains all sets $X + \ell$, $\ell \in \mathcal{L}_0$. As Q is compact, $\limsup_{n\to\infty} \nu_W(x^n + \mathcal{L})(Q) \leqslant 0$, so there is an $n_0 \in \mathbb{N}_0$ such that $\nu_W(x^n + \mathcal{L})(Q) = 0$ for all $n \geqslant n_0$. As $x_n + \ell \in X + \ell \subseteq Q$ for all n and all $\ell \in \mathcal{L}_0$, this implies that $(x^n + \ell)_H \notin W$ for $n \geqslant n_0$ and $\ell \in \mathcal{L}_0$: that is, $\pi^H(x^n) \notin \bigcup_{\ell\in\mathcal{L}_0}(W - \ell_H)$ for all $n \geqslant n_0$. As $x^n \in X$, this contradicts the above choice of $\mathcal{L}_0$.

7. *The projected system*

LEMMA 7.1.

(a) *All measures in $\mathcal{M}^G_W$ are uniformly locally finite.*

(b) *$\pi^G_* : \mathcal{M}_W \to \mathcal{M}^G_W$ is continuous.*

(c) *$\mathcal{M}^G_W$ is vaguely compact.*

(d) *For any $x, x' \in G \times H$, either $\pi^G_*\nu_W(x + \mathcal{L}) \perp \pi^G_*\nu_W(x' + \mathcal{L})$ or $\pi^G(x' - x) \in \pi^G(\mathcal{L})$.*

† The sets $U_\ell := (\pi^H)^{-1}((\partial W)^c - \ell_H)$ are open and dense in $G \times H$. As $\bigcap_\ell U_\ell$ is invariant under translations by $\ell \in \mathcal{L}$ and as $\pi^{\hat{X}}|_X$ is bijective, $\pi^{\hat{X}}(\bigcap_\ell U_\ell) = \pi^{\hat{X}}(X \cap \bigcap_\ell U_\ell) = \bigcap_\ell \pi^{\hat{X}}(X \cap U_\ell) = \bigcap_\ell \pi^{\hat{X}}(U_\ell)$. As $\pi^{\hat{X}}$ is open, this set is G_δ and, as $\pi^{\hat{X}}$ is continuous and onto, it is also dense.

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Proof. (a) This follows from the same property of $\mathcal{M}_W$ together with the compactness of $W \subseteq H$.

(b) Let $f \in C_c(G)$. Then $f \circ \pi^G$ is continuous but, typically, does not have compact support. However, as H is locally compact, there is a function $g_W \in C_c(H)$ such that $1_W \leqslant g_W$. Hence, for any ν^n, $\nu \in \mathcal{M}_W$: $\int f \circ \pi^G \, d\nu^n \to \int f \circ \pi^G \, d\nu$ if and only if $\int f \circ \pi^G \cdot g_W \circ \pi^H \, d\nu^n \to \int f \circ \pi^G \cdot g_W \circ \pi^H \, d\nu^n$. As $f \circ \pi^G \cdot g_W \circ \pi^H$ has compact support, we can conclude that $\pi^G_* \nu^n$ converges vaguely to $\pi^G_* \nu$ whenever ν^n converges vaguely to ν. This proves that π^G_* is continuous.

(c) Because of (b), $\mathcal{M}_W^G$ is the continuous image of a compact set. Hence it is compact.

(d) Let $x, x' \in X$ and suppose that $\pi^G_* \nu_W(x + \mathcal{L}) \not\perp \pi^G_* \nu_W(x' + \mathcal{L})$. Then there is a $g \in G$ such that $\pi^G_* \nu_W(x + \mathcal{L})(\{g\}) > 0$ and $\pi^G_* \nu_W(x' + \mathcal{L})(\{g\}) > 0$. It follows that there exist $\ell, \ell' \in \mathcal{L}$ such that $\pi^G(x + \ell) = g = \pi^G(x' + \ell')$ and hence $\pi^G(x' - x) = \pi^G(\ell - \ell') \in \pi^G(\mathcal{L})$. □

Proof of Proposition 3.5. (a) This is Lemma 7.1(b).

(b) $\pi^{G\times H}_* : \mathcal{GM}_W \to \mathcal{M}_W$ is one-to-one on $(\pi^{G\times H}_*)^{-1}(\mathcal{M}_W \backslash \{\underline{0}\})$, by Lemma 5.4. Suppose now that $\mathrm{int}(W) \neq \emptyset$. Then $\underline{0} \notin \mathcal{M}_W$, because of Proposition 3.3(e), so that $\pi^{G\times H}_*$ is one-to-one. As it is a continuous and surjective map between the compact space $\mathcal{GM}_W$ and the Hausdorff space $\hat{X}$, it is a homeomorphism.

(c) By Proposition 3.3, C_W is a dense G_δ-subset of $\hat{X}$ and $(\pi^{\hat{X}}_*)^{-1}\{\hat{x}\} = \{(\hat{x}, \nu_W(\hat{x}))\}$ for every $\hat{x} \in C_W$. Hence $\pi^{\hat{X}}_* : \mathcal{GM}_W \to \hat{X}$ is an almost one-to-one extension of an equicontinuous factor. This extension can be factorised over the maximal equicontinuous factor of $(\mathcal{GM}_W, S)$; call it (Y, R). Then also (Y, R) is an almost one-to-one extension of $(\hat{X}, \hat{T})$, and Lemma 7.2 below shows that it coincides with $(\hat{X}, \hat{T})$. □

The following lemma is well known when (Y, R) is minimal, but we could not locate this slight generalization in the literature. Recall that a metric G-dynamical system (Y, R) is distal if, for any pair $x \neq y$ in Y, there is an $\epsilon > 0$ such that $d(R_g x, R_g y) \geq \epsilon$ for all $g \in G$. For example, if (Y, R) is equicontinuous, then (Y, R) is distal.

LEMMA 7.2. *Let $\pi : (Y, R) \to (X, S)$ be an extension of compact metrizable G-dynamical systems and suppose that:*

(1) *(Y, R) is distal;*

(2) *(X, S) is minimal; and*

(3) *there exist some $x_0 \in X$ such that* $\mathrm{card}(\pi^{-1}\{x_0\}) = 1$.

Then π is a homeomorphism.

Proof. Let Z be a minimal subset of Y. Then $\pi(Z) = X$, as (X, S) is minimal. Hence $\pi^{-1}\{x_0\} \subseteq Z$. In particular, Y has only one minimal subset. But as (Y, R) is distal, it is the disjoint union of its minimal subsystems [**22**, Proposition II.7]. Hence (Y, R) is minimal. The rest of the proof is classical.

Let y_0 be the unique point in $\pi^{-1}\{x_0\}$. Suppose that $y_1, y_2 \in Y$ are such that $\pi(y_1) = \pi(y_2)$. Then there is a sequence $(g^n)_n$ in G such that $\lim_{n\to\infty} R_{g^n} y_1 = y_0$ and, passing to a subsequence if necessary, we may assume that $y_0' := \lim_{n\to\infty} R_{g^n} y_2$ exists as well. Then $\pi(y_0') = \lim_{n\to\infty} \pi(R_{g^n} y_2) = \lim_{n\to\infty} S_{g^n}(\pi(y_2)) = \lim_{n\to\infty} S_{g^n}(\pi(y_1)) = \lim_{n\to\infty} \pi(R_{g^n} y_1) = \pi(y_0) = x_0$ so that $y_0' = y_0$. As (Y, R) is distal, this implies that

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$y_1 = y_2$. Since π is a one-to-one and surjective map between a compact and a Hausdorff space, it follows that π is a homeomorphism. □

Proof of Theorem 1. (a) The first claim is a corollary to Proposition 3.5(b) and (c). We turn to the subsystems $\overline{\mathcal{G}(\nu_W|_{C_W})} \subseteq \mathcal{GM}_W$ and $\overline{\nu_W(C_W)} \subseteq \mathcal{M}_W$. As the second subsystem is a continuous factor of the first one, and as both are topological almost one-to-one extensions of $(\hat{X}, \hat{T})$, by the first claim, it suffices to note that the restriction of S to $\overline{\mathcal{G}(\nu_W|_{C_W})}$ is the unique minimal subsystem of $(\mathcal{GM}_W, S)$, in view of Lemma 6.3.

(b) The first claim is Proposition 3.5(c). As $Z_W \neq \emptyset$, $\underline{0} \in \mathcal{M}_W$ and, of course, $\underline{0}$ is a fixed point for the action of S on $\mathcal{M}_W$. Therefore, any factor of $(\mathcal{M}_W, S)$ has a fixed point, so any minimal factor must be trivial. As $(\mathcal{M}_W^G, S)$ is a factor of $(\mathcal{M}_W, S)$, the same holds for $(\mathcal{M}_W^G, S)$. □

We only prove the failure of weak mixing for the system $(\mathcal{M}_W^G, S)$ and observe that it is a factor of $(\mathcal{M}_W, S)$. To this end, assume that $(\mathcal{M}_W^G, S)$ is topologically weak mixing, and observe that $\mathcal{GM}_W^G \subseteq \hat{X} \times \mathcal{M}_W^G$ is a topological joining between $(\hat{X}, \hat{T})$ and $(\mathcal{M}_W^G, S)$. As $(\hat{X}, \hat{T})$ is minimal and distal (since it is even equicontinuous), an old disjointness result of Furstenberg [**22**, Theorem II.3] then shows that the joining is trivial: that is, that $\mathcal{GM}_W^G = \hat{X} \times \mathcal{M}_W^G$ (see [**4**] for this argument). As $C_W \neq \emptyset$, Proposition 3.3(c) and Remark 3.2 show that $\mathrm{card}(\mathcal{M}_W^G) = 1$.

Proof of Corollary 1(b). Suppose that $\mathrm{int}(W) \neq \emptyset$ and let $W' := \overline{\mathrm{int}(W)}$. Then $\mathrm{int}(W') = \mathrm{int}(W)$ so that W' is topologically regular (see §4.3) and $\partial W' \subseteq \partial W$. Therefore $C_W \subseteq C_{W'}$ and $\nu_W^G|_{C_W} = \nu_{W'}^G|_{C_W}$. Hence $\overline{\nu_W^G(C_W)} = \overline{\nu_{W'}^G(C_W)} \subseteq \overline{\nu_{W'}^G(C_{W'})}$ and, as $\overline{\nu_{W'}^G(C_{W'})} \subseteq \overline{\nu_{W'}^G(C_W)}$, by Remark 3.11(a), we conclude that $M := \overline{\nu_{W'}^G(C_{W'})} = \overline{\nu_W^G(C_W)} \subseteq \mathcal{M}_W^G \cap \mathcal{M}_{W'}^G$. As W' is assumed to be aperiodic, Lemmas 4.5 and 4.6 show that $\pi_*^G : \mathcal{M}_{W'} \to \mathcal{M}_{W'}^G$ is a homeomorphism. Hence, by Corollary 1, $(\overline{\nu_W^G(C_W)}, S) = (\overline{\nu_{W'}^G(C_{W'})}, S)$ is an almost automorphic extension of $(\hat{X}, \hat{T})$ and, as $(\mathcal{M}_W^G, S)$ is a topological factor of $(\mathcal{M}_W, S)$, $\overline{\nu_W^G(C_W)}$ is the only minimal subsystem of $(\mathcal{M}_W^G, S)$ (see also Theorem 1(a)). □

Proof of Proposition 3.6. (a) The $\hat{T}$-invariance is Remark 3.2; the remaining statement is obvious from (b) and (c) below.

(b) First, suppose that $m_H(W) > 0$. As the action $h \mapsto \ell_H + h$ of $\mathcal{L}$ on (H, m_H) is metrically transitive, it follows easily that the set $\bigcup_{\ell \in \mathcal{L}} (W - \ell_H)$ has conull m_H-measure, so that

$$m_{G\times H}\Big((\pi^H)^{-1}\Big(\bigcap_{\ell\in\mathcal{L}}(W^c - \ell_H)\Big)\Big) = m_{G\times H}\Big(G \times \Big(\bigcup_{\ell\in\mathcal{L}}(W - \ell_H)\Big)^c\Big) = 0.$$

Hence

$$m_{\hat{X}}(Z_W) = \mathrm{dens}(\mathcal{L}) \cdot m_{G\times H}\Big(X \cap (\pi^H)^{-1}\Big(\bigcap_{\ell\in\mathcal{L}}(W^c - \ell_H)\Big)\Big) = 0.$$

Conversely, if $m_H(W) = 0$, then

$$m_{\hat{X}}(\hat{X} \setminus Z_W) \leqslant \mathrm{dens}(\mathcal{L}) \cdot m_{G\times H}\Big(G \times \Big(\bigcup_{\ell\in\mathcal{L}}(W - \ell_H)\Big)\Big) = 0.$$

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(c) Recall that $\hat{x} = x + \mathcal{L} \in C_W$ if and only if $x_H \in \bigcap_{\ell\in\mathcal{L}}((\partial W)^c - \ell_H)$, by Lemma 6.1. Hence the claim follows by the same arguments as in (b), with W replaced by ∂W. □

Proof of Theorem 2. (a) As $m_H(W) > 0$, $m_{\hat{X}}(Z_W) = 0$, because of Proposition 3.6(b). This implies that $Q_{\mathcal{GM}}(\hat{X} \times \{\underline{0}\}) = 0$, so $(\mathcal{GM}_W, Q_{\mathcal{GM}}, S)$ and $(\mathcal{M}_W, Q_{\mathcal{M}}, S)$ are measure theoretically isomorphic, in view of Proposition 3.5(b). Since, by definition, $Q_{\mathcal{GM}} = m_{\hat{X}} \circ (\mathrm{id}, \nu_W)^{-1}$, it follows that $(\mathrm{id}, \nu_W) : \hat{X} \to \mathcal{GM}_W$ is a measure theoretic isomorphism between the systems $(\hat{X}, m_{\hat{X}}, \hat{T})$ and $(\mathcal{GM}_W, Q_{\mathcal{GM}}, S)$ with inverse $\pi^{\hat{X}}_*$.

(b) $(\hat{X} \times \mathcal{M}^G_W, Q_{\mathcal{GM}}, \hat{T} \times S)$ is obviously a joining of $(\hat{X}, m_{\hat{X}}, \hat{T})$ and $(\mathcal{M}^G_W, Q_{\mathcal{M}^G}, S)$. If it is trivial, then $Q_{\mathcal{GM}}$ is a product measure that is supported by the graph of ν^G_W so that ν^G_W is constant $m_{\hat{X}}$-almost everywhere.

(c) By assumption, we must have $\mathrm{int}(W) \neq \emptyset$, which implies that $Z_W = \emptyset$, by Proposition 3.3. Since $m_H(\partial W) = 0$, $m_{\hat{X}}(C_W) = 1$: that is, $\nu_W : \hat{X} \to \mathcal{M}_W$ is $m_{\hat{X}}$-almost everywhere continuous. It follows that $\{\hat{x} \in \hat{X} : \mathrm{card}((\pi^{\hat{X}}_*)^{-1}\{\hat{x}\}) = 1\}$ has full Haar measure. In particular, each invariant measure on $\mathcal{GM}_W$ that projects to $m_{\hat{X}}$ coincides with $Q_{\mathcal{GM}}$ and, as each such invariant measure projects to the unique invariant measure $m_{\hat{X}}$ of $(\hat{X}, \hat{T})$, the system $(\mathcal{GM}_W, S)$ is indeed uniquely ergodic. As $(\mathcal{M}_W, S)$ and $(\mathcal{M}^G_W, S)$ are continuous factors of $(\mathcal{GM}_W, S)$, the unique ergodicity immediately carries over to these systems. □

Proof of Lemma 4.4. $\Rightarrow$ Assume that $\hat{x} = x + \mathcal{L}$, $\hat{x}' = x' + \mathcal{L} \in \hat{X}$ are such that $\pi^G_* \nu_W(\hat{x}) = \pi^G_* \nu_W(\hat{x}')$.

If one of $\nu_W(\hat{x})$ and $\nu_W(\hat{x}')$ is $\underline{0}$, then also $\pi^G_* \nu_W(\hat{x}) = \pi^G_* \nu_W(\hat{x}') = \underline{0}$ and hence $\nu_W(\hat{x}) = \underline{0} = \nu_W(\hat{x}')$.

So assume from now on that $\nu_W(\hat{x})$, $\nu_W(\hat{x}') \neq \underline{0}$. By Lemma 7.1(d), there is some $\tilde{\ell} \in \mathcal{L}$ such that $x'_G = x_G + \tilde{\ell}_G$. Let $\tilde{x} = x + \tilde{\ell}$. Then $\tilde{x}_G = x'_G$, $\nu_W(x + \mathcal{L}) = \nu_W(\tilde{x} + \mathcal{L})$ and

$$\sum_{\ell\in\mathcal{L}} 1_W(\tilde{x}_H + \ell_H)\, \delta_{\tilde{x}_G + \ell_G} = \pi^G_* \nu_W(\tilde{x} + \mathcal{L}) = \pi^G_* \nu_W(x + \mathcal{L}) = \pi^G_* \nu_W(x' + \mathcal{L})$$
$$= \sum_{\ell\in\mathcal{L}} 1_W(x'_H + \ell_H)\, \delta_{\tilde{x}_G + \ell_G}.$$

Therefore, for each $\ell \in \mathcal{L}$, $(\tilde{x}_H + \ell_H) \in W$ if and only if $(x'_H + \ell_H) \in W$, which can be expressed as $(-\tilde{x}_H + W) \cap \pi^H(\mathcal{L}) = (-x'_H + W) \cap \pi^H(\mathcal{L})$. These sets are non-empty because $\pi^G_* \nu_W(x' + \mathcal{L}) \neq \underline{0}$. As $(W, \mathcal{L})$ is uniquely coding, this implies that $\tilde{x}_H = x'_H$ and hence $\tilde{x} = x'$, so that $x' - x = \tilde{\ell} \in \mathcal{L}$: that is, $\hat{x} = \hat{x}'$ and hence $\nu_W(\hat{x}) = \nu_W(\hat{x}')$.

$\Leftarrow$ Let $h, h' \in H$ be given such that $(-h + W) \cap \pi^H(\mathcal{L}) = (-h' + W) \cap \pi^H(\mathcal{L}) \neq \emptyset$. Then $\pi^G_* \nu_W((0, h) + \mathcal{L}) = \pi^G_* \nu_W((0, h') + \mathcal{L}) \neq \underline{0}$, so, by assumption, also $\nu_W((0, h) + \mathcal{L}) = \nu_W((0, h') + \mathcal{L}) \neq \underline{0}$. Then Lemma 5.4 implies that $(0, h) + \mathcal{L} = (0, h') + \mathcal{L}$, so $(0, h - h') \in \mathcal{L}$. As $\pi^G|_{\mathcal{L}}$ is one-to-one, this means that $h = h'$. Hence $(W, \mathcal{L})$ is uniquely coding. □

Proof of Lemma 4.5. First, suppose that $(W, \mathcal{L})$ is strongly uniquely coding. Because of Lemma 7.1(b) and the compactness of $\mathcal{M}_W$, it only remains to prove that π^G_* is one-to-one. Suppose that $\pi^G_* \nu = \pi^G_* \nu'$ for some ν, $\nu' \in \mathcal{M}_W$. If $\pi^G_* \nu = \pi^G_* \nu' = \underline{0}$, then also $\nu = \underline{0} = \nu'$.

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Otherwise, Lemma 5.4 shows that there exist unique $x, x' \in X$ such that $\nu \leqslant \nu_W(x+\mathcal{L})$ and $\nu' \leqslant \nu_W(x'+\mathcal{L})$. Hence $\pi_*^G\nu \leqslant \pi_*^G\nu_W(x+\mathcal{L})$ and $\pi_*^G\nu' \leqslant \pi_*^G\nu_W(x'+\mathcal{L})$ and, as $0 \neq \pi_*^G\nu = \pi_*^G\nu'$, Lemma 7.1(d) implies that $x'_G = x_G + \tilde{\ell}_G$ for some $\tilde{\ell} \in \mathcal{L}$ and, as in the previous proof, we can replace x by $\tilde{x} = x + \tilde{\ell}$ such that $\nu_W(\tilde{x}+\mathcal{L}) = \nu_W(x+\mathcal{L})$ and $\tilde{x}_G = x'_G$. Hence we can assume, without loss of generality, that $x_G = x'_G$, and it remains to show that $x_H = x'_H$.

Let $G \times H = \bigcup_{j\in\mathbb{N}_0} K_j$, where $(K_j)_j$ is an increasing sequence of compact subsets of $G \times H$. Set $\mathcal{L}_j := \mathcal{L} \cap K_j$ so that $\mathcal{L} = \bigcup_{j\in\mathbb{N}_0} \mathcal{L}_j$ and all $\mathcal{L}_j$ are finite. As ν, ν' are in the closure of $\nu_W(\hat{X})$, there exist $\hat{x}_n = (x_{n,G}, x_{n,H}) + \mathcal{L}$, $\hat{x}'_n = (x'_{n,G}, x'_{n,H}) + \mathcal{L} \in \hat{X}$ such that $x_n, x'_n \in X$, $\hat{x}_n \to x + \mathcal{L}$, $\hat{x}'_n \to x' + \mathcal{L}$ and $\nu_W(\hat{x}_n) \to \nu$, $\nu_W(\hat{x}'_n) \to \nu'$ (see Lemma 5.2). Let $y_n = (x_G, x_{n,H})$ and $y'_n = (x_G, x'_{n,H})$ (recall that $x_G = x'_G$). As $\nu_W(y_n + \mathcal{L}) = \nu_W(\hat{T}_{x_G - x_{n,G}}\hat{x}_n) = S_{x_G - x_{n,G}}\nu_W(\hat{x}_n)$; also $\nu_W(y_n+\mathcal{L}) \to \nu$ and, similarly, $\nu_W(y'_n+\mathcal{L}) \to \nu'$. As this is vague convergence, this means that

$$\text{for all } j \in \mathbb{N}_0, \text{ there exists } n_j \in \mathbb{N}_0, \text{ for all } \ell \in \mathcal{L}_j, \text{ for all } n \geqslant n_j$$
$$\begin{cases} x_{n,H} + \ell_H = (y_n + \ell)_H \in W & \text{if and only if } \nu\{x+\ell\} = 1 \\ \text{and} \\ x'_{n,H} + \ell_H = (y'_n + \ell)_H \in W & \text{if and only if } \nu'\{x'+\ell\} = 1. \end{cases}$$

As $x_G = x'_G$ and as $\pi^G|_{\mathcal{L}}$ is one-to-one,

$$\nu\{x+\ell\} = 1 \quad\Leftrightarrow\quad \pi_*^G\nu\{x_G+\ell_G\} = 1 \quad\Leftrightarrow\quad \pi_*^G\nu'\{x'_G+\ell_G\} = 1 \quad\Leftrightarrow\quad \nu'\{x'+\ell\} = 1.$$

Hence

$$\text{for all } \ell \in \mathcal{L}, \text{ there exists } n_\ell \in \mathbb{N}_0, \text{ for all } n \geqslant n_\ell : \\ 1_W(x_{n,H} + \ell_H) = \nu\{x+\ell\} = \nu'\{x'+\ell\} = 1_W(x'_{n,H} + \ell_H). \tag{12}$$

As $(W, \mathcal{L})$ is assumed to be strongly uniquely coding and as $\nu, \nu' \neq \underline{0}$, it follows that $x_H = x'_H$.

Conversely, suppose now that π_*^G is one-to-one. Let h, h', h_n, h'_n be as in Definition 4.3(b). Let $x = (0,h)$, $x_n = (0,h_n)$, $x' = (0,h')$ and $x'_n = (0,h'_n)$. Then $x_n \to x$, $x'_n \to x'$ and

$$\text{for all } \ell \in \mathcal{L}, \text{ there exists } n_\ell \in \mathbb{N}_0, \text{ for all } n \geqslant n_\ell : 1_{G\times W}(x_n + \ell) = 1_{G\times W}(x'_n + \ell), \tag{13}$$

where this common value is 1 for at least one $\tilde{\ell} \in \mathcal{L}$ and all $n \geqslant n_\ell$. Passing to subsequences, if necessary, we can assume that there exist $\nu, \nu' \in \mathcal{M}_W \backslash \{\underline{0}\}$ such that $\nu_W(x_n+\mathcal{L}) \to \nu$ and $\nu_W(x'_n+\mathcal{L}) \to \nu'$ vaguely. In particular, $\nu \leqslant \nu_W(x+\mathcal{L})$ and $\nu' \leqslant \nu_W(x'+\mathcal{L})$, by Lemma 5.2. Abusing slightly the notation $S_g\nu$, we define $S_y\nu$ by $(S_y\nu)(A) = \nu(-y+A)$ for all $y \in G \times H$. Let $K_1 \subseteq K_2 \subseteq \cdots \subseteq G \times H$ be as in the first part of the proof. Then

$$S_y\nu_W(x+\mathcal{L}) = \sum_{\ell\in\mathcal{L}} 1_{G\times W}(x+\ell)\cdot S_y\delta_{x+\ell} = \sum_{\ell\in\mathcal{L}} 1_{G\times W}(x+\ell)\cdot\delta_{y+x+\ell},$$

so that, in view of (13), there exist $n(j) \in \mathbb{N}_0$ such that, for each K_j and all $n \geqslant n(j)$,

$$(S_{-x_n}\nu_W(x_n+\mathcal{L}))|_{K_j} = (S_{-x'_n}\nu_W(x'_n+\mathcal{L}))|_{K_j}.$$

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In the limit $n \to \infty$ this yields $S_{-x}\nu = S_{-x'}\nu'$. As $\pi^G(x) = \pi^G(x') = 0$, this implies that $\pi^G_*\nu = \pi^G_*(S_{-x}\nu) = \pi^G_*(S_{-x'}\nu') = \pi^G_*\nu'$ and, as π^G_* is one-to-one, it follows that $\nu = \nu' \in \mathcal{M}_W\backslash\{\underline{0}\}$. Hence $\nu_W((0, h) + \mathcal{L}) = \nu_W((0, h') + \mathcal{L}) \neq \underline{0}$, and $h = h'$ follows in the same way as at the end of the proof of Lemma 4.4. □

8. *Configurations with maximal density and their orbit closures*

Proof of Theorem 4. (Following the arguments in [**43**, Proof of Theorem 1]) If P is the one-point mass concentrated on $\underline{0} \in \mathcal{M}$, then the limit D_P is clearly zero. Otherwise, P is an ergodic S-invariant probability measure on $\mathcal{M}_W\backslash\{\underline{0}\}$, so that $P \circ \hat{\pi}^{-1}$ is a $\hat{T}$-invariant probability measure on $\hat{X}$, where $\hat{\pi} : \mathcal{M}_W\backslash\{\underline{0}\} \to \hat{X}\backslash Z_W$ is as defined in Definition 5.5. It follows that $P \circ \hat{\pi}^{-1} = m_{\hat{X}}$. Hence it suffices to prove (4) for $m_{\hat{X}}$-almost every $\hat{x}$ and all $\nu \leqslant \nu_W(\hat{x})$, that is, for all $\nu \in \hat{\pi}^{-1}\{\hat{x}\}$.

First, observe that

$$\limsup_{n\to\infty} \frac{\nu(A_n \times H)}{m_G(A_n)} \leqslant \limsup_{n\to\infty} \frac{\nu_W(\hat{x})(A_n \times H)}{m_G(A_n)} \leqslant \operatorname{dens}(\mathcal{L}) \cdot m_H(W) \quad \text{for all } \hat{x} \in \hat{X} \text{ and all } \nu \in \hat{\pi}^{-1}\{\hat{x}\}, \quad (14)$$

by Theorem 3. Next we investigate the convergence of the left-hand side of (14) and identify its limit D_P†. To this end, fix a compact subset $H_0 \subseteq H$ such that $\nu(G \times (H\backslash H_0)) = 0$ for all $\nu \in \mathcal{M}_W$. Then fix a compact zero neighbourhood $B \subseteq G$ such that $(\ell_G + B) \cap (\ell'_G + B) = \emptyset$ for all $\ell \neq \ell'$ in $\mathcal{L} \cap (G \times H_0)$. With $\hat{x} = x + \mathcal{L}$, this assumption has the following consequence.

$$\begin{aligned}
\nu((-\partial^B A_n) \times H_0) \leqslant \nu_W(\hat{x})((-\partial^B A_n) \times H_0) &= \sum_{y \in (x+\mathcal{L}) \cap (G \times W)} 1_{-\partial^B A_n}(y_G) \\
&\leqslant \sum_{\ell \in \mathcal{L} \cap (G \times (-x_H + H_0))} 1_{-\partial^B A_n}(x_G + \ell_G) \\
&= \frac{1}{m_G(B)} \sum_{\ell \in \mathcal{L} \cap ((-x_G - \partial^B A_n) \times (-x_H + H_0))} \int 1_{x_G + \ell_G + B}(g)\, dm_G(g) \\
&\leqslant \frac{1}{m_G(B)}\, m_G\Bigg(\bigcup_{\ell \in \mathcal{L} \cap ((-x_G - \partial^B A_n) \times (-x_H + H_0))} (x_G + \ell_G + B) \Bigg) \\
&\leqslant \frac{1}{m_G(B)}\, m_G(-\partial^B A_n + B) \\
&= \frac{1}{m_G(B)}\, m_G(-B + \partial^B A_n) \leqslant \frac{1}{m_G(B)}\, m_G(\partial^{-B+B} A_n),
\end{aligned}$$

so that

$$\begin{aligned}
&\left| \int_{G \times H} (1_{-A_n} * 1_B)(x_G)\, d\nu(x) - m_G(B)\, \nu((-A_n) \times H) \right| \\
&\quad \leqslant m_G(B) \int_{G \times H} \left| \left(1_{-A_n} * \frac{1_B}{m_G(B)} \right)(x_G) - 1_{-A_n}(x_G) \right| d\nu(x) \\
&\quad \leqslant m_G(B)\, \nu((-\partial^B A_n) \times H_0) \leqslant m_G(\partial^{-B+B} A_n).
\end{aligned}$$

† The following argument also applies to non-abelian but unimodular G.

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As $(A_n)_n$ is a van Hove sequence, this implies that

$$\begin{aligned}
&\lim_{n\to\infty} m_G(B)\frac{\nu((-A_n)\times H)}{m_G(A_n)}\\
&\quad=\lim_{n\to\infty}\frac{1}{m_G(A_n)}\int_{G\times H}(1_{-A_n}*1_B)(x_G)\,d\nu(x)\\
&\quad=\lim_{n\to\infty}\frac{1}{m_G(A_n)}\int_{G\times H}\left(\int_{-A_n}1_B(-g+x_G)\,dm_G(g)\right)d\nu(x)\\
&\quad=\lim_{n\to\infty}\frac{1}{m_G(A_n)}\int_{G\times H}\left(\int_{A_n}1_{B\times H}((g,0)+x)\,dm_G(g)\right)d\nu(x)\\
&\quad=\lim_{n\to\infty}\frac{1}{m_G(A_n)}\int_{A_n}\left(\int_{G\times H}1_{B\times H}\circ T_g\,d\nu\right)dm_G(g)\\
&\quad=\lim_{n\to\infty}\frac{1}{m_G(A_n)}\int_{A_n}(S_g\nu)(B\times H)\,dm_G(g)\qquad(15)
\end{aligned}$$

whenever $\hat{x}\in\hat{X}$, $\nu\in\hat{\pi}^{-1}\{\hat{x}\}$ and any of these limits exists. For later reference, we note that the same holds for limits along subsequences.

Next observe that the evaluation $\nu\mapsto\nu(B\times H)=\nu(B\times H_0)$ is upper semicontinuous and, in particular, measurable so that, as in [**43**], the generalized Birkhoff ergodic theorem [**37**] applies to the last line of (15): that is, for P-almost every ν

$$\begin{aligned}
\int_{\mathcal{M}_W}\nu'(B\times H)\,dP(\nu')&=\lim_{n\to\infty}\frac{1}{m_G(A_n)}\int_{A_n}(S_g\nu)(B\times H)\,dm_G(g)\\
&=\lim_{n\to\infty}m_G(B)\frac{\nu((-A_n)\times H)}{m_G(A_n)}\\
&=\lim_{n\to\infty}m_G(B)\frac{\nu(A_n\times H)}{m_G(A_n)},\qquad(16)
\end{aligned}$$

where we note, for the last equation, that $(-A_n)_n$ is a van Hove sequence as well if G is abelian.

We have just shown that $D_P=(1/m_G(B))\int\nu(B\times H)\,dP(\nu)$. It remains to prove that $D_P\leqslant\operatorname{dens}(\mathcal{L})\cdot m_H(W)$ with equality if and only if $P=Q_{\mathcal{M}}=m_{\hat{X}}\circ\nu_W^{-1}$ (see Definition 3.7). First, observe that $D_{Q_{\mathcal{M}}}=\operatorname{dens}(\mathcal{L})\cdot m_H(W)$, by Moody's Theorem 3. Next observe that $\nu\leqslant\nu_W(\hat{x})$, where $\hat{x}=\hat{\pi}(\nu)$. Recalling that $P\circ\hat{\pi}^{-1}=m_{\hat{X}}$, we see that

$$\begin{aligned}
D_P&=\frac{1}{m_G(B)}\int_{\mathcal{M}_W}\nu(B\times H)\,dP(\nu)\leqslant\frac{1}{m_G(B)}\int_{\mathcal{M}_W}\nu_W(\hat{\pi}(\nu))(B\times H)\,dP(\nu)\\
&=\frac{1}{m_G(B)}\int_{\hat{X}}\nu_W(\hat{x})(B\times H)\,dm_{\hat{X}}(\hat{x})=\frac{1}{m_G(B)}\int_{\mathcal{M}_W}\nu(B\times H)\,dQ_{\mathcal{M}}(\nu)=D_{Q_{\mathcal{M}}},
\end{aligned}$$

with equality if and only if $\nu=\nu_W(\hat{\pi}(\nu))$ for P-almost every ν, that is, if $P=Q_{\mathcal{M}}$. □

Proof of Corollary 4. Let P be an ergodic S-invariant probability measure on the Borel-σ-algebra $\mathcal{B}(\mathcal{M}_W^G)$ of $\mathcal{M}_W^G$. It can be transferred to an ergodic S-invariant probability P_0 on the sub-σ-algebra $(\pi_*^G)^{-1}(\mathcal{B}(\mathcal{M}_W^G))\subseteq\mathcal{B}(\mathcal{M}_W)$. As $\mathcal{M}_W^G$ is a Polish space, $\mathcal{B}(\mathcal{M}_W^G)$

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and hence also $(\pi_*^G)^{-1}(\mathcal{B}(\mathcal{M}_W^G))$ are countably generated. Hence P_0 can be extended to a measure P' on $\mathcal{B}(\mathcal{M}_W)$ [**13**, Theorem 9.8.2].

Let $\tilde{P}$ be a weak limit of a subsequence P'_{n_i}, where

$$P'_n := \frac{1}{m_G(A_n)} \int_{A_n} P' \circ S_{-g}\, dm_G(g),$$

and observe that, for each continuous $\varphi : \mathcal{M}_W^G \to \mathbb{R}$,

$$\begin{aligned}\int_{\mathcal{M}_W} \varphi \circ \pi_*^G\, dP'_n &= \frac{1}{m_G(A_n)} \int_{A_n} \left(\int_{\mathcal{M}_W^G} \varphi \circ S_g\, d(P' \circ (\pi_*^G)^{-1}) \right) dm_G(g) \\ &= \frac{1}{m_G(A_n)} \int_{A_n} \left(\int_{\mathcal{M}_W^G} \varphi \circ S_g\, dP \right) dm_G(g) = \int_{\mathcal{M}_W^G} \varphi\, dP,\end{aligned}$$

as $P' \circ (\pi_*^G)^{-1} = P_0 \circ (\pi_*^G)^{-1} = P$ and $P \circ S_{-g} = P$.

Then $\tilde{P}$ is S-invariant and, for each continuous $\varphi : \mathcal{M}_W^G \to \mathbb{R}$,

$$\int_{\mathcal{M}_W} \varphi \circ \pi_*^G\, d\tilde{P} = \lim_{i\to\infty} \int_{\mathcal{M}_W} \varphi \circ \pi_*^G\, d\bar{P}_{n_i} = \int_{\mathcal{M}_W^G} \varphi\, dP,$$

that is, $P = \tilde{P} \circ (\pi_*^G)^{-1}$. If $\tilde{P}$ is not ergodic, we can decompose it into its ergodic components, $\tilde{P} = \int_{\mathcal{M}_W} \tilde{P}_\nu\, d\tilde{P}(\nu)$. Then $P = \int_{\mathcal{M}_W} \tilde{P}_\nu \circ (\pi_*^G)^{-1}\, d\tilde{P}(\nu)$, and all $\tilde{P}_\nu \circ (\pi_*^G)^{-1}$ are again ergodic. As P itself is ergodic, it follows that $P = \tilde{P}_\nu \circ (\pi_*^G)^{-1}$ for $\tilde{P}$-almost every ν. □

Proof of Theorem 5. (a) First, observe that $m_{\hat{X}}(\hat{X}_{\max}) = 1$, by Theorem 4. The $\hat{T}$-invariance of $\hat{X}_{\max}$ follows from

$$\begin{aligned}\nu_W(\hat{T}_{-g}\hat{x})(A_n \times H) &= S_{-g}\nu_W(\hat{x})(A_n \times H) = \nu_W(\hat{x})(T_g(A_n \times H)) \\ &= \nu_W(\hat{x})((g + A_n) \times H)\end{aligned}$$

and the fact that $(g + A_n)_n$ is a van Hove sequence.

(b) Let $\hat{x} \in \hat{X}_{\max} = \{\hat{x} \in \hat{X} : \lim_{n\to\infty} \nu_W(\hat{x})(A_n \times H)/m_G(A_n) = D_{Q_{\mathcal{M}}}\}$ and suppose that $Q_{n_i,\hat{x}}$ converges weakly to some probability measure P on $\mathcal{M}_W$. Fix a compact zero neighbourhood $B \subseteq G$, as in the proof of Theorem 4, and recall that $\hat{x} \in \hat{X}_{\max}$. Then equation (15) implies that

$$\begin{aligned}m_G(B) \cdot D_{Q_{\mathcal{M}}} &= \lim_{i\to\infty} m_G(B) \frac{\nu_W(\hat{x})((-A_{n_i}) \times H)}{m_G(A_{n_i})} \\ &= \lim_{i\to\infty} \frac{1}{m_G(A_{n_i})} \int_{A_{n_i}} (S_g \nu_W(\hat{x}))(B \times H)\, dm_G(g) \\ &= \lim_{i\to\infty} \frac{1}{m_G(A_{n_i})} \int_{A_{n_i}} \int_{\mathcal{M}_W} \nu(B \times H)\, d\delta_{S_g\nu_W(\hat{x})}(\nu)\, dm_G(g) \\ &= \lim_{i\to\infty} \int_{\mathcal{M}_W} \nu(B \times H)\, dQ_{n_i,\hat{x}}(\nu) \\ &\leqslant \int_{\mathcal{M}_W} \nu(B \times H)\, dP(\nu) = m_G(B) \cdot D_P,\end{aligned}$$

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where the inequality holds due to the portemanteau theorem, as $\{\nu|_{B\times H} : \nu \in \mathcal{M}_W\}$ is closed in $\mathcal{M}_W$. Hence

$$D_{Q_{\mathcal{M}}} \leqslant D_P \leqslant \mathrm{dens}(\mathcal{L}) \cdot m_H(W) = D_{Q_{\mathcal{M}}}.$$

This shows that $D_P = D_{Q_{\mathcal{M}}}$ and hence $P = Q_{\mathcal{M}}$, in view of Theorem 4.

(c) As $Q_{n,\hat{x}}(\mathcal{M}_W(\hat{x})) = 1$ for all $n \in \mathbb{N}_0$ and as $Q_{n,\hat{x}}$ converges weakly to $Q_{\mathcal{M}}$, $1 \geqslant Q_M(\mathcal{M}_W(\hat{x})) \geqslant \lim_{n\to\infty} Q_{n,\hat{x}}(\mathcal{M}_W(\hat{x})) = 1$ for each $\hat{x} \in \hat{X}_{\max}$.

(d) For $\hat{x} \in \nu_W^{-1}(\mathrm{supp}(Q_{\mathcal{M}}))$ also $\mathcal{M}_W(\hat{x}) \subseteq \mathrm{supp}(Q_{\mathcal{M}})$, by translation invariance of $m_{\hat{X}}$ and closedness of $\mathrm{supp}(Q_{\mathcal{M}})$. Hence $\mathcal{M}_W(\hat{x}) = \mathrm{supp}(Q_{\mathcal{M}})$ for all $\hat{x} \in \hat{X}_{\max} \cap \nu_W^{-1}(\mathrm{supp}(Q_{\mathcal{M}}))$ and $m_{\hat{X}}(\hat{X}_{\max} \cap \nu_W^{-1}(\mathrm{supp}(Q_{\mathcal{M}}))) = 1$, because $m_{\hat{X}}(\hat{X}_{\max}) = 1$ and $m_{\hat{X}} \circ \nu_W^{-1} = Q_{\mathcal{M}}$.

(e) Let $\hat{x} \in C_W$. Suppose, for a contradiction, that $\nu_W(\hat{x}) \notin \mathrm{supp}(Q_{\mathcal{M}})$. Then there is an open neighbourhood $O \subseteq \mathcal{M}$ of $\nu_W(\hat{x})$ such that $Q_{\mathcal{M}}(O) = m_{\hat{X}}(\nu_W^{-1}(O)) = 0$. As $\hat{x} \in C_W$, there is an open neighbourhood $U \subseteq \hat{X}$ of $\hat{x}$ such that $\nu_W(U) \subseteq O$, which implies that $m_{\hat{X}}(U) = 0$, which is a contradiction. □

Proof of Remark 3.11. (a) As C_W is a dense G_δ-set, we only have to show that $\nu_W(C_W) \subseteq \overline{\nu_W(R)}$ for each dense G_δ-set $R \subseteq \hat{X}$. But this is obvious, because R is dense and C_W is the set of continuity points of ν_W.

(b) If $F \subseteq \hat{X}$ has full Haar measure, then $Q_{\mathcal{M}}(\nu_W(F)) = m_{\hat{X}}(F) = 1$, so $\mathrm{supp}(Q_{\mathcal{M}}) \subseteq \overline{\nu_W(F)}$. Hence $\mathrm{supp}(Q_{\mathcal{M}}) \subseteq \bigcap_{F\in\mathcal{F}} \overline{\nu_W(F)}$, where the intersections range over the family $\mathcal{F}$ of all full measure subsets of $\hat{X}$. For the converse inclusion, consider $F = \hat{X}_{\max} \cap \nu_W^{-1}(\mathrm{supp}(Q_{\mathcal{M}}))$. Then $F \in \mathcal{F}$, by Theorem 5(d), and $\nu_W(F) \subseteq \mathrm{supp}(Q_{\mathcal{M}})$, so that also $\overline{\nu_W(F)} \subseteq \mathrm{supp}(Q_{\mathcal{M}})$.

(c) Suppose now that $m_H(\partial W) = 0$. Then $C_W \subseteq \hat{X}$ has full Haar measure. Hence assertion (b) implies that

$$\overline{\nu_W(C_W)} \subseteq \mathrm{supp}(Q_{\mathcal{M}}) = \bigcap_{F\in\mathcal{F}} \overline{\nu_W(F)} \subseteq \overline{\nu_W(C_W)}.$$

The same proofs apply to ν_W^G and $Q_{\mathcal{M}^G}$. □

Proof of Remark 3.12 and Proposition 3.15. We assume that $W \neq \emptyset$ without loss of generality. For both proofs, we need the following construction. Let $U = U_G \times H$ with $U_G \subseteq G$ being an open, relatively compact zero neighbourhood that is small enough such that $\mathcal{L} \cap ((-\overline{U_G} + \overline{U_G}) \times (-W + W)) = \{0\}$. Then the quotient map $G \times H \to \hat{X}$, when restricted to $U \subseteq G \times H$, is one-to-one. Given two finite disjoint sets $\mathcal{L}_1, \mathcal{L}_0 \subseteq \mathcal{L}$ that are not both empty, define $O = O(\mathcal{L}_1, \mathcal{L}_0, U) \subseteq \mathcal{M}_W$ by

$$O = \{\nu \in \mathcal{M}_W : \mathcal{L}_1 \subseteq \mathrm{supp}(\nu) - U,\ \mathcal{L}_0 \cap \mathrm{supp}(\nu) - U = \emptyset\}. \tag{17}$$

Then O is open in the vague topology. Let $\hat{U} := \{\hat{x} \in \hat{X} : \exists u \in U \text{ with } \pi^{\hat{X}}(u) = \hat{x}\}$ and note that if such an element $u \in U$ exists, then it is unique. This allows us to define

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$\xi : \hat{U} \to U$ by $\xi(\hat{x}) = u \in (\pi^{\hat{X}})^{-1}\{u\}$. Then

$$\begin{aligned}
\nu_W^{-1}(O) &= \{\hat{x} = x + \mathcal{L} \in \hat{X} : x \in X,\ \forall \ell_1 \in \mathcal{L}_1\ \exists \ell \in \mathcal{L}\ \exists u \in U : \ell_1 = x + \ell - u \\
&\quad \text{and } \pi^H(x + \ell) \in W \text{ and } \forall \ell_0 \in \mathcal{L}_0\ \forall \ell \in \mathcal{L}\ \forall u \in U, \\
&\quad \ell_0 = x + \ell - u \ \Rightarrow\ \pi^H(x + \ell) \notin W\} \\
&= \{\hat{x} = x + \mathcal{L} \in \hat{U} : \forall \ell_1 \in \mathcal{L}_1 : \pi^H(\xi(\hat{x}) + \ell_1) \in W \\
&\quad \text{and } \forall \ell_0 \in \mathcal{L}_0 : \pi^H(\xi(\hat{x}) + \ell_0) \notin W\} \\
&= \{\hat{x} \in \hat{U} : \forall \ell_1 \in \mathcal{L}_1 : \pi^H(\xi(\hat{x}) + \ell_1) \in W \text{ and } \forall \ell_0 \in \mathcal{L}_0 : \pi^H(\xi(\hat{x}) + \ell_0) \notin W\} \\
&= \left\{\hat{x} \in \hat{U} : (\xi(\hat{x}))_H \in \bigcap_{\ell_1 \in \mathcal{L}_1} (W - \ell_{1,H}) \cap \bigcap_{\ell_0 \in \mathcal{L}_0} (H \backslash (W - \ell_{0,H}))\right\}.
\end{aligned}$$

As the restriction of the quotient map to U is one-to-one, by the extended Weil formula [**46**, Theorem 3.4.6],

$$\begin{aligned}
Q_{\mathcal{M}}(O) &= m_{\hat{X}}(\nu_W^{-1}(O)) \\
&= \operatorname{dens}(\mathcal{L}) \cdot m_{G \times H}\left\{u \in U :\ u_H \in \bigcap_{\ell_1 \in \mathcal{L}_1} (W - \ell_{1,H}) \cap \bigcap_{\ell_0 \in \mathcal{L}_0} (H \backslash (W - \ell_{0,H}))\right\} \\
&= \operatorname{dens}(\mathcal{L}) \cdot m_G(U_G) \cdot m_H\left(\bigcap_{\ell_1 \in \mathcal{L}_1} (W - \ell_{1,H}) \cap \bigcap_{\ell_0 \in \mathcal{L}_0} (H \backslash (W - \ell_{0,H}))\right). \quad (18)
\end{aligned}$$

Proof of Remark 3.12. Let $\mathcal{L}_1 \subseteq \mathcal{L}$ be non-empty and finite and let $\mathcal{L}_0 = \emptyset$. We can choose U_G such that $m_G(\partial U_G) = 0$ (compare [**26**, Lemma 3.6]). Then $O = O(\mathcal{L}_1, \emptyset, U)$ is a continuity set for $Q_{\mathcal{M}}$, as is seen from the above calculation with $\partial O = \{\nu \in \mathcal{M}_W : \mathcal{L}_1 \subseteq \operatorname{supp}(\nu) - \partial U\}$. Next, note that, asymptotically as $n \to \infty$,

$$Q_{n,\hat{x}}(O) = f_n(\mathcal{L}_1, \nu_W(\hat{x})) \cdot m_G(U_G) + o(1)$$

on any van Hove sequence. Hence weak convergence in Theorem 5(b) yields, for any $\hat{x} \in \hat{X}_{\max}$, that $\lim_{n\to\infty} Q_{n,\hat{x}}(O) = Q_{\mathcal{M}}(O)$ on the associated tempered van Hove sequence. We thus obtain

$$f(\mathcal{L}_1, \nu_W(\hat{x})) = \frac{Q_{\mathcal{M}}(O)}{m_G(U_G)} = \operatorname{dens}(\mathcal{L}) \cdot m_H\left(\bigcap_{\ell_1 \in \mathcal{L}_1} (W - \ell_{1,H})\right). \qquad \square$$

Proof of Proposition 3.15. For any given point $\hat{x} = x + \mathcal{L} \in \hat{X}$ and any non-empty finite set $\mathcal{L}' \subseteq \mathcal{L}$, let $\mathcal{L}_1 = \{\ell \in \mathcal{L}' : \ell_H \in -x_H + W\}$ and $\mathcal{L}_0 = \mathcal{L}' \backslash \mathcal{L}_1$. Then, for any $h \in H$,

$$(-h + W) \cap \pi^H(\mathcal{L}') = \pi^H(\mathcal{L}_1) \quad \text{if and only if } h \in \bigcap_{\ell \in \mathcal{L}_1} (W - \ell_H) \cap \bigcap_{\ell \in \mathcal{L}_0} (H \backslash (W - \ell_H)).$$

As $\pi^H(\mathcal{L}_1) = (-x_H + W) \cap \pi^H(\mathcal{L}')$,

$$\begin{aligned}
&\{h \in H : (-x_H + W) \cap \pi^H(\mathcal{L}') = (-h + W) \cap \pi^H(\mathcal{L}')\} \\
&\quad = \bigcap_{\ell \in \mathcal{L}_1} (W - \ell_H) \cap \bigcap_{\ell \in \mathcal{L}_0} (H \backslash (W - \ell_H)). \quad (19)
\end{aligned}$$

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$\Longrightarrow$ Suppose that $\nu_W(\hat{x}) \in \mathrm{supp}(Q_{\mathcal{M}})$. Define $O = O(\mathcal{L}_1, \mathcal{L}_0, U)$ as in (17). As $O \subseteq \mathcal{M}_W$ is open and as $\nu_W(\hat{x}) \in O$, by definition of this set, $Q_{\mathcal{M}}(O) > 0$, which implies, in view of (18) and (19), that

$$m_H\{h \in H : (-x_H + W) \cap \pi^H(\mathcal{L}') = (-h + W) \cap \pi^H(\mathcal{L}')\} > 0.$$

$\Longleftarrow$ Suppose that $\nu_W(\hat{x}) \notin \mathrm{supp}(Q_{\mathcal{M}})$. Then there is an open neighbourhood $O_0 \subseteq \mathcal{M}$ of $\nu_W(\hat{x})$ such that $Q_{\mathcal{M}}(O_0) = m_{\hat{X}}(\nu_W^{-1}(O_0)) = 0$. There are finite disjoint sets $\mathcal{L}_1, \mathcal{L}_0 \subseteq \mathcal{L}$, which are not both empty, such that the open set $O = O(\mathcal{L}_1, \mathcal{L}_0, U)$ is contained in O_0, with $U = U'_G \times H$ and $U'_G \subseteq U_G$ being a sufficiently small open zero neighbourhood (compare (17)). Hence, in view of (18) and (19),

$$m_H\{h \in H : (-x_H + W) \cap \pi^H(\mathcal{L}') = (-h + W) \cap \pi^H(\mathcal{L}')\} = 0. \qquad \square$$

9. *Outlook*

9.1. *The topological theory.* Theorem 1 shows that the notion of a maximal equicontinuous factor is of no help for studying the system $(\mathcal{M}_W, S)$ when $\mathrm{int}(W)$, the interior of the window, is empty, because then this factor is trivial. This holds, *a fortiori*, for the system $(\mathcal{M}_W^G, S)$ which is a factor of $(\mathcal{M}_W, S)$. On the other hand, in view of Proposition 3.5, the system $(\mathcal{GM}_W, S)$ has always the maximal equicontinuous factor $(\hat{X}, \hat{T})$ and, in view of Lemma 5.3, the same is true for the system $(\mathcal{GM}_W^G, S)$.

This means that if we want to study more general transitive subsystems (M, S) of $(\mathcal{M}^G, S)$, where $\mathcal{M}^G$ denotes the space of all locally finite measures on G endowed with the topology of vague convergence, we should look for a way to extend (M, S) (which generalizes $(\mathcal{M}_W^G, S)$) in a natural way to a system that plays the role of $(\mathcal{GM}_W^G, S)$ (and hence has a maximal equicontinuous factor). A hint for how to accomplish this is given in the proof of Theorem 1(b). It is shown there that $\mathcal{GM}_W^G \subseteq \hat{X} \times \mathcal{M}_W^G$ is a non-trivial joining between these two systems, except if $\mathrm{card}(\mathcal{M}_W^G) = 1$.

So one might attempt to look for maximal equicontinuous systems that can be joined in a non-trivial and non-redundant way to a general transitive system (M, S). By *non-redundant* we mean that the joining is such that, for a dense G_δ-set of points in $\mu \in M$, the μ-section of the joining consists of just one point and that the joining itself is the closure of the set of all these points. Observe that the joining $\mathcal{GM}_W \subseteq \hat{X} \times \mathcal{M}_W$ does have this property because the zero measure, if it belongs to $\mathcal{M}_W$, is the only element in $\mathcal{M}_W$ that is joined to more than one point in $\hat{X}$ and $\mathcal{GM}_W \backslash (\hat{X} \times \{\underline{0}\})$ is dense in $\mathcal{GM}_W$.

An equivalent, but more appealing, way to approach this problem would be to investigate existence and properties of a *generic maximal equicontinuous factor* of (M, S) (see [**25**] for the notion of generic factors). Systems (M, S) generated by relatively compact but non-compact windows might be first candidates to test such ideas.

9.2. *Ergodic theory.* If P is an S-invariant probability measure on $\mathcal{GM}_W$, then the system $(\mathcal{GM}_W, P, S)$ has the structure of a fibred random dynamical system over the base $(\hat{X}, m_{\hat{X}}, \hat{T})$ with fibres that are subsets of copies of $\{0, 1\}^{\mathcal{L}}$. This sets the scene to study entropy properties of such systems along the lines developed, for example, in [**62**], and [**14**] can be the starting point for a thermodynamic formalism—at least when $G = \mathbb{Z}$. Recent results on $\mathcal{B}$-free systems, such as [**11**, **34**], might provide a good testing ground for such ideas.

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9.3. *Non-abelian model sets.* Our analysis carries over to the case, where G and H need not be commutative (see [**63**, Ch. 7] for the basic group theoretic facts). Concerning topological results, note that the compact minimal left coset dynamical system $(\hat{X}, \hat{T})$ might no longer be equicontinuous. Assuming it remains distal, we can then still consider the graph dynamical system $(\mathcal{GM}_W, S)$, and the results of §§4.2, 4.3 and 3.1 remain true; the latter with 'equicontinuous' replaced by 'distal'. For measure theoretic results, note that the left invariant quotient measure $m_{\hat{X}}$ on $\hat{X}$ remains unique up to normalization [**63**, §3.4]. Hence $\hat{X}$, together with its natural left action, is strictly ergodic. (This is true without cocompactness of the discrete group $\mathcal{L}$.) Thus Theorem 2 still holds—up to the spectral statement in part (a). Also, Theorems 3–5 remain true with minor adjustments (compare the proof of Theorem 4). This implies that the entropy estimate for weak model sets [**26**], together with its proof, continues to hold in that case. Note, however, that the latter results rely on the existence of a van Hove sequence (see [**44**] for a discussion of this strong form of amenability in the non-abelian setting).

In the recent work [**12**], an alternative to Schlottmann's approach to the torus parametrization [**54**] is systematically developed for non-abelian regular model sets, with a measure theoretic focus. Then diffraction theory is developed in a very general setting, which does not assume amenability nor cocompactness. In the cocompact case, a pure point diffraction result is obtained for groups G admitting a Gelfand pair. It might be interesting to further explore the relationship between suitable types of dynamical and diffraction spectrum for model sets, thereby extending the pure point result of the abelian case, as in [**6**]. A natural starting point for such an analysis seems diffraction of lattices (compare the abelian result [**47**]). Also, the recent spectral analysis of odometer actions [**18**] might provide further insight.

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