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HARMONIC FUNCTION SPACES ON GROUPS

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Abstract

A class of harmonic function spaces is introduced and studied, namely the spaces $H^p_\sigma(G)$ of σ-harmonic L^p functions on a locally compact group G, for $1 \leqslant p \leqslant \infty$ and a given complex measure σ on G of unit norm. It is shown that there is a contractive projection from $L^p(G)$ onto $H^p_\sigma(G)$, for $1 < p \leqslant \infty$, and structural results for $H^p_\sigma(G)$ are deduced. Given an adapted probability measure σ on G, a uniqueness result is proved, that the space $H^p_\sigma(G)$ contains only constant functions, for $1 \leqslant p < \infty$. For any σ, a result on the dimension of $H^1_\sigma(G)$ is proved.

1. *Introduction*

In this paper, we introduce and study a class of harmonic function spaces which are analogous to the Bergman spaces of holomorphic functions. Let G be a locally compact group with a right-invariant Haar measure λ and let $L^p(G)$ be the complex Lebesgue spaces with respect to λ, for $1 \leqslant p \leqslant \infty$. Let σ be a complex regular Borel measure on G with $\|\sigma\| = 1$. A Borel function $f : G \longrightarrow \mathbb{C}$ is called *σ-harmonic* if the convolution $f * \sigma$ exists locally λ-almost everywhere and $f(x) = f * \sigma(x)$ whenever

$$f * \sigma(x) = \int_G f(xy^{-1})\, d\sigma(y)$$

exists. We recall that a Borel set $S \subset G$ is *locally λ-null* if $\lambda(S \cap K) = 0$ for every compact set $K \subset G$ [**17**, 11.26]. We study the following 'Bergman spaces' of harmonic functions on G:

$$H^p_\sigma(G) = \{f \in L^p(G) : f * \sigma = f\}.$$

The space $H^\infty_\sigma(G)$ has been studied in [**7**] (see also [**5**]). We focus our attention on the case of $1 \leqslant p < \infty$ and the basic structures of $H^p_\sigma(G)$. This work paves the way for a subsequent study of operators on $H^p_\sigma(G)$.

We begin with some background. Let G be a Lie group and let Δ be the Laplace operator on G. A function $f \in C^\infty(G)$ is *harmonic* if $\Delta f = 0$. It is well known that such harmonic functions f can be characterized by a family of convolution equations $f = f * \sigma_t$ $(t > 0)$ for some adapted absolutely continuous probability measures $\{\sigma_t\}_{t>0}$ on G which motivates the above definition of σ-harmonic functions on any locally compact group. Harmonic functions on groups have been widely studied. Recently, the space $H^\infty_\sigma(G)$ of (essentially) bounded σ-harmonic functions has been studied in [**7**]. It has been shown that there is a contractive projection from $L^\infty(G)$ onto $H^\infty_\sigma(G)$ which leads to many useful consequences including the Poisson representation of $H^\infty_\sigma(G)$ and semigroup properties of the Poisson boundaries. Following [**7**], we study the spaces $H^p_\sigma(G)$, for $p \geqslant 1$, and we outline our results below.

We prove, in Section 2, the existence of a contractive projection from $L^p(G)$ onto its subspace $H^p_\sigma(G)$ of σ-harmonic functions, for $\|\sigma\| = 1$ and $p > 1$. The existence of

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a contractive projection is analogous to a fundamental result in Bergman spaces of holomorphic functions [**14**], and as in the L^∞ case, is pivotal in the derivation of structural results for $H^p_\sigma(G)$. For instance, it implies that, for $p > 1$, the space $H^p_\sigma(G)$ is linearly isometric to an L^p space and that, if σ is a probability measure, then each real σ-harmonic L^p function is a difference of two non-negative ones. This property is used to prove a uniqueness result for σ-harmonic L^p functions in Section 3. We also show that $H^p_\sigma(G)$ is the dual space of $H^q_{\widetilde{\sigma}}(G)$ where $1 < p < \infty$, $1/p + 1/q = 1$ and $d\widetilde{\sigma}(x) = d\sigma(x^{-1})$. Such duality does not exist for $H^\infty_\sigma(G)$.

We prove the uniqueness results in Section 3. Given a Riemannian manifold M and the Laplace operator Δ with respect to the Riemannian metric, to ensure a uniqueness property on the equation $\Delta f = 0$ for harmonic functions, f is often restricted to lie in some $L^p(M)$ with respect to the Riemannian measure. In this context, a uniqueness result states that if f is an L^p harmonic function on M, then f is constant. It has been shown by Yau [**27**] (see also [**15**]) that on a complete manifold M, a harmonic function $f \in L^p(M)$ is constant, for $1 < p < \infty$. This result is false for $p = 1, \infty$, but true if in addition, M has non-negative Ricci curvature [**21**, **26**] and in fact, the positive harmonic functions on M are constant in this case, by [**26**]. We prove analogous results for σ-harmonic functions on groups. We first note that, given an adapted probability measure σ on a compact group G, every σ-harmonic L^p function on G must be constant (almost everywhere), where $1 \leqslant p \leqslant \infty$. Using this result and some basic representation theory, we show that, for $1 \leqslant p < \infty$ and for any adapted probability measure σ on any locally compact group G, all σ-harmonic L^p functions are constant (almost everywhere). This result is false for $p = \infty$, and also false if σ is not adapted. For any σ, we prove a result on the dimension of $H^1_\sigma(G)$, and if σ is symmetric, we show that the L^p σ-harmonic functions on a nilpotent group are periodic, with periods in the support of σ. It may be of interest to note that our uniqueness results also apply to Δ-harmonic functions on Lie groups, in view of the earlier remarks.

2. *Harmonic function spaces*

We denote throughout by G a locally compact group with identity e and the right invariant Haar measure λ. Let $C_0(G)$ be the Banach space of complex continuous functions on G vanishing at infinity. Then its dual identifies, via the Riesz representation theorem, with the space $M(G)$ of complex regular Borel measures on G. The norm of $\mu \in M(G)$ is the total variation norm : $\|\mu\| = |\mu|(G)$. We denote by $\mu = \omega \cdot |\mu|$ the polar representation of μ, where $|\omega(x)| = 1$ for all $x \in G$. Let $C_c(G)$ be the subspace of $C_0(G)$ consisting of functions of compact support.

As noted in [**5**], it is more convenient to consider the equation $f = f * \sigma$ instead of $f = \sigma * f$ in the study of matrix-valued harmonic functions. For this reason, we work with a right-invariant Haar measure λ in this paper, in which case the convolutions, when they exist, should be defined as follows:

$$(f * g)(x) = \int_G f(xy^{-1}) g(y)\, d\lambda(y),$$

$$(f * \sigma)(x) = \int_G f(xy^{-1})\, d\sigma(y),$$

$$(\sigma * f)(x) = \int_G f(y^{-1}x) \triangle_G(y^{-1})\, d\sigma(y),$$

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where $\triangle_G$ is the modular function satisfying $d\lambda(xy) = \triangle_G(x)\, d\lambda(y)$ and $d\lambda(x^{-1}) = \triangle_G(x^{-1})\, d\lambda(x)$.

Given $\mu, \sigma \in M(G)$, their convolution is given by

$$\mu * \sigma(f) = \int_G \int_G f(xy)\, d\mu(x)\, d\sigma(y)$$

for $f \in C_0(G)$. Evidently, we have $\mu * \sigma \in M(G)$ and $\|\mu * \sigma\| \leqslant \|\mu\|\|\sigma\|$.

Since we use the right-invariant Haar measure, which is different from most standard references where the left-invariant measure is used, we include the proofs of some basic results for clarity and completeness.

LEMMA 2.1. *Let $f \in L^p(G)$ and $\sigma \in M(G)$. Then $f * \sigma \in L^p(G)$ and $\|f * \sigma\|_p \leqslant \|f\|_p \|\sigma\|$ for $1 \leqslant p \leqslant \infty$.*

Proof. For $1 \leqslant p < \infty$, the function f vanishes outside a countable union of compact sets and $f*\sigma$ exists λ-almost everywhere. We define $f*\sigma(x)$ to be 0 if it does not exist. Minkowski's inequality gives

$$\begin{aligned}\left(\int_G |f * \sigma(x)|^p\, d\lambda(x)\right)^{1/p} &= \left(\int_G \left|\int_G f(xy^{-1})\, d\sigma(y)\right|^p d\lambda(x)\right)^{1/p} \\ &\leqslant \left(\int_G \left(\int_G |f(xy^{-1})|\, d|\sigma|(y)\right)^p d\lambda(x)\right)^{1/p} \\ &\leqslant \int_G \left(\int_G |f(xy^{-1})|^p\, d\lambda(x)\right)^{1/p} d|\sigma|(y).\end{aligned}$$

For $p = \infty$, $f*\sigma$ exists outside a locally λ-null set N and we define $f*\sigma = 0$ on N. □

REMARK 2.2. If $\|\sigma\| < 1$ and $f * \sigma = f \in L^p(G)$, Lemma 2.1 implies that $f = 0$.

Throughout the paper, we let $1 \leqslant p, q \leqslant \infty$ be such that $1/p + 1/q = 1$ and we denote the dual pairing between $L^p(G)$ and $L^q(G)$ by

$$\langle \cdot, \cdot \rangle : L^p(G) \times L^q(G) \longrightarrow \mathbb{C}.$$

For $\sigma \in M(G)$, $f \in L^p(G)$ and $g \in L^q(G)$, we have the duality

$$\langle f * \sigma, g \rangle = \langle f, g * \widetilde{\sigma} \rangle,$$

where $d\widetilde{\sigma}(x) = d\sigma(x^{-1})$ and $g * \widetilde{\sigma} \in L^q(G)$ by Lemma 2.1. It follows that, if (f_α) is a net $w(L^p(G), L^q(G))$-converging to f in $L^p(G)$, then the net $(f_\alpha * \sigma)$ is $w(L^p(G), L^q(G))$-convergent to $f * \sigma$, where $w(L^p(G), L^q(G))$ denotes the weak topology on $L^p(G)$ and $w(L^\infty(G), L^1(G))$ is the weak* topology. Using the above duality, we have another description of $H^p_\sigma(G) = \{f \in L^p(G) : f * \sigma = f\}$ as follows. Let

$$J^q_{\widetilde{\sigma}}(G) = \{g * \widetilde{\sigma} - g : g \in L^q(G)\}.$$

Then $H^p_\sigma(G)$ is the annihilator of $J^q_{\widetilde{\sigma}}(G)$:

$$H^p_\sigma(G) = J^q_{\widetilde{\sigma}}(G)^\perp = \left\{f \in L^p_\sigma(G) : \left\langle f, J^q_{\widetilde{\sigma}}(G)\right\rangle = 0\right\}.$$

In particular, $H^p_\sigma(G)$ is weakly closed in $L^p(G)$ for $1 \leqslant p < \infty$, and is weak* closed for $p = \infty$. Also $H^p_\sigma(G) = (L^q(G)/\overline{J^q_{\widetilde{\sigma}}(G)})^*$ for $1 < p \leqslant \infty$.

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In [**14**], Forelli and Rudin have shown that the Bergman spaces of holomorphic functions on balls are ranges of bounded projections on L^p-spaces, for $1 \leqslant p < \infty$. We prove below a similar result for harmonic functions but also note that there is no bounded projection from the space $L^1[0, 2\pi]$ onto the Hardy space H^1 [**23**].

THEOREM 2.3. *Let $\sigma \in M(G)$ with $\|\sigma\| = 1$. For $1 < p \leqslant \infty$, there is a contractive projection $P_\sigma : L^p(G) \longrightarrow L^p(G)$ with range $H^p_\sigma(G)$. Further, for $1 < p < \infty$, P_σ is the dual map of the projection $P_{\tilde{\sigma}} : L^q(G) \longrightarrow L^q(G)$.*

Proof. The case of $p = \infty$ has been proved in [**7**]. Let $1 < p < \infty$. For $m = 1, 2, \ldots$, we define a linear map $\Lambda_m : L^p(G) \longrightarrow L^p(G)$ by

$$\Lambda_m(f) = f * \overbrace{\sigma * \ldots * \sigma}^{m \text{ times}}.$$

By Lemma 2.1, we have $\|\Lambda_m\| \leqslant 1$. Also, Λ_m is weakly continuous when $L^p(G)$ is equipped with the weak topology. Let $\mathcal{K} = \overline{\text{co}}\{\Lambda_m : m = 1, 2, \ldots\}$ be the closed convex hull of $\{\Lambda_m : m = 1, 2, \ldots\}$ with respect to the product topology $\mathcal{T}$ of $L^p(G)^{L^p(G)}$ where $L^p(G)$ is equipped with the weak topology. Then $\mathcal{K}$ is compact. Define $\Phi : \mathcal{K} \longrightarrow \mathcal{K}$ by

$$\Phi(\Lambda)(f) = \Lambda(f) * \sigma \qquad (\Lambda \in \mathcal{K},\ f \in L^p(G)).$$

It is straightforward to verify that Φ is well-defined and $\mathcal{T}$-continuous. Therefore, by the Schauder–Tychonoff fixed-point theorem (cf. [**11**, p. 456]), there exists $P_\sigma \in \mathcal{K}$ such that $\Phi(P_\sigma) = P_\sigma$. We have clearly $P_\sigma(f) = f$ for $f \in H^p_\sigma(G)$ since $\Lambda_m(f) = f$ for all m. Given $f \in L^p(G)$, we have $P_\sigma(f) = \Phi(P_\sigma)(f) = P_\sigma(f) * \sigma$, that is, $P_\sigma(f) \in H^p_\sigma(G)$. This proves that $P_\sigma(L^p(G)) = H^p_\sigma(G)$ and $P_\sigma^2 = P_\sigma$. Since each Λ_m is contractive, so is P_σ.

Next, we show that $P_\sigma = P^*_{\tilde{\sigma}}$. We note that $P_\sigma(f * \sigma) = P_\sigma(f) * \sigma$ for $f \in L^p(G)$ since $\Lambda_m(f * \sigma) = \Lambda_m(f) * \sigma$. For $f \in L^p(G)$, we have

$$P^*_{\tilde{\sigma}} f * \sigma = P^*_{\tilde{\sigma}} f$$

since, for any $g \in L^q(G)$, we have

$$\begin{aligned} \langle g, P^*_{\tilde{\sigma}} f * \sigma \rangle &= \langle g * \tilde{\sigma}, P^*_{\tilde{\sigma}} f \rangle = \langle P_{\tilde{\sigma}}(g * \tilde{\sigma}), f \rangle \\ &= \langle P_{\tilde{\sigma}} g * \tilde{\sigma}, f \rangle = \langle P_{\tilde{\sigma}} g, f \rangle \\ &= \langle g, P^*_{\tilde{\sigma}} f \rangle. \end{aligned}$$

Likewise we have $P^*_\sigma g * \tilde{\sigma} = P^*_\sigma g$ for $g \in L^q(G)$. It follows that $P_\sigma P^*_{\tilde{\sigma}} = P_\sigma$ since

$$\langle g, P_\sigma P^*_{\tilde{\sigma}} f \rangle = \langle P_{\tilde{\sigma}}(P^*_\sigma g), f \rangle = \langle P^*_\sigma g, f \rangle = \langle g, P_\sigma f \rangle$$

for $g \in L^q(G)$ and $f \in L^p(G)$. Hence, for $f \in L^p(G)$, we have $P^*_{\tilde{\sigma}} f = P_\sigma(P^*_{\tilde{\sigma}} f) = P_\sigma f$. Hence $P^*_{\tilde{\sigma}} = P_\sigma$. □

If σ is absolutely continuous (with respect to λ), then so is $\mu * \sigma$ for $\mu \in M(G)$. Although $L^1(G)$ can be identified as the subspace of $M(G)$ consisting of absolutely continuous measures, the above construction cannot be applied to $M(G)$ with the weak* topology $w(M(G), C_0(G))$ to yield a projection $P_\sigma : M(G) \longrightarrow M(G)$ with range $H^1_\sigma(G)$, even assuming that σ is absolutely continuous. The reason is that the map $\Lambda_m(\mu) = \mu * \sigma^m$ need not be weak* continuous.

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REMARK 2.4. By [**1**], a contractive projection onto a closed subspace of $L^p(G)$, for $1<p<\infty$, is equivalent to a conditional expectation. By the above construction, there is a net (μ_α) in the convex hull of $\{\sigma^n : n=1,2,\ldots\}$ such that

$$P_\sigma(f) = w\text{-}\lim_\alpha f*\mu_\alpha \qquad (f\in L^p(G)).$$

By the Mackey–Arens theorem, the net (μ_α) can be chosen such that the above convergence is uniform on compact sets in $L^q(G)$.

COROLLARY 2.5. *Let $\|\sigma\|=1$. Then for $1<p<\infty$, the space $H^p_\sigma(G)$ is linearly isometric to the dual space $H^q_{\tilde\sigma}(G)^*$, with duality*

$$\langle g,f\rangle = \int_G g(x)f(x)\,d\lambda(x) \qquad \big(g\in H^q_{\tilde\sigma}(G),\ f\in H^p_\sigma(G)\big).$$

Proof. Write L^q for $L^q(G)$. We have $H^q_{\tilde\sigma}(G)^* = P_{\tilde\sigma}(L^q)^* \simeq L^p/P_{\tilde\sigma}(L^q)^\perp \simeq H^p_\sigma(G)$ where the last isometry is given by $f\in H^p_\sigma(G)\longmapsto f+P_{\tilde\sigma}(L^q)^\perp$ as

$$\begin{aligned}\|f\| &\geqslant \inf\{\|f+g\| : g\in P_{\tilde\sigma}(L^q)^\perp\}\\ &\geqslant \inf\{\|P_\sigma(f+g)\| : g\in P_{\tilde\sigma}(L^q)^\perp\}\\ &= \|P_\sigma(f)\| = \|f\|.\end{aligned}$$

□

REMARK 2.6. The above result is false for $p=\infty$. Indeed, if $\sigma=\delta_{2\pi}$ is the unit mass at $2\pi\in\mathbb{R}$, then the $\tilde\sigma$-harmonic functions on $\mathbb{R}$ are the periodic functions with period 2π, and evidently, $H^1_{\tilde\sigma}(\mathbb{R})=\{0\}$ while $H^\infty_\sigma(\mathbb{R})\neq\{0\}$.

COROLLARY 2.7. *Let $\|\sigma\|=1$. Then for $1<p\leqslant\infty$, the space $H^p_\sigma(G)$ is linearly isometric to an $L^p(\Omega)$ space.*

Proof. The case of $p=\infty$ has been proved in [**7**]. The case of $1<p<\infty$ follows from the fact that a 1-complemented subspace of $L^p(G)$ is again an $L^p(\Omega)$ space, as shown in [**1**, **10**] for sigma-finite λ, and in [**25**] for any λ on G. □

For $p<\infty$, we call a real L^p function f on G *non-negative* and write $f\geqslant 0$ if $f(x)\geqslant 0$ for λ-almost every $x\in G$. Given $f\geqslant 0$ and a positive measure $\sigma\in M(G)$, we have $f*\sigma\geqslant 0$ since f is equivalent to an L^p function g which is negative nowhere and $f*\sigma = g*\sigma$ λ-almost everywhere by Lemma 2.1. Likewise a real L^∞ function f is non-negative if $f\geqslant 0$ locally λ-almost everywhere and we have $f*\sigma\geqslant 0$ locally λ-almost everywhere. We note that the cone of non-negative L^p functions is $w(L^p(G),L^q(G))$-closed.

COROLLARY 2.8. *Let σ be a probability measure on G and let $1<p\leqslant\infty$. Then for any real function $f\in H^p_\sigma(G)$, there are non-negative $g,h\in H^p_\sigma(G)$ such that $f=g-h$.*

Proof. Since σ is a probability measure, constant functions are σ-harmonic and every real bounded σ-harmonic function f can be written as $f=M-(M-f)$ for some $M\geqslant f,0$.

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Let $1<p<\infty$. Then the projection $P=P_\sigma$ in Theorem 2.3 is positive, that is, $f\geqslant 0$ implies that $P(f)\geqslant 0$ since $P(f)=w\text{-}\lim_\alpha f*\mu_\alpha$ with $\mu_\alpha\geqslant 0$. Given $f\in H^p_\sigma(G)$ with positive part $f^+\in L^p(G)$, we have $P(f^+)\in H^p_\sigma(G)$ and $f=P(f^+)-(P(f^+)-f)$ where $P(f^+)\geqslant 0$ and $P(f^+)-f=P(f^+-f)$ is σ-harmonic and non-negative. □

REMARK 2.9. If σ is a real measure on G, then every $f\in H^p_\sigma(G)$ is a linear combination of two real functions in $H^p_\sigma(G)$, by considering the real and imaginary parts of f.

EXAMPLE 2.10. Let $\mathcal{T}=\{z\in\mathbb{C}:|z|=1\}$ be the circle group and let $\sigma=\delta_i$ be the unit mass at $i=\sqrt{-1}$. Then a complex function f on G is σ-harmonic if and only if $f(z)=f(-iz)$ for all $z\in G$. For example, the function $f(z)=z^4$ is σ-harmonic, thus $H^p(\mathcal{T})$ contains non-constant functions for $1\leqslant p\leqslant\infty$. In the next section, we show that $H^p_\sigma(G)$ contains only constant functions for any compact group G when σ has 'sufficient' support.

EXAMPLE 2.11. For any absolutely continuous probability measure σ on the additive group $(\mathbb{R}^n,+)$, we have $H^1_\sigma(\mathbb{R}^n)=\{0\}$. Indeed, if there is a non-negative σ-harmonic function $f\in L^1(\mathbb{R}^n)\backslash\{0\}$, then taking Fourier transforms, we have $\widehat{f}\widehat{\sigma}=\widehat{f}$ on $\mathbb{R}^n$ and by continuity, there is a neighbourhood U of $0\in\mathbb{R}^n$ such that $\gamma\in U$ implies that

$$|\widehat{f}(\gamma)-\widehat{f}(0)|<\widehat{f}(0),$$

so $\widehat{f}(\gamma)\neq 0$ and $\widehat{\sigma}(\gamma)=1$, which gives

$$\int_{\mathbb{R}^n}\cos(\gamma\cdot x)\,d\sigma(x)=1.$$

Since $\sigma(\mathbb{R}^n)=1$, we have $\sigma\{x\in\mathbb{R}^n:\gamma\cdot x\in 2\pi\mathbb{Z}\}=1$. Choose $\gamma=(\gamma_1,0,\dots,0)$ and $\eta=(\eta_1,0,\dots,0)$ in U such that γ_1/η_1 is irrational. Then we have

$$\{x\in\mathbb{R}^n:\gamma\cdot x\in 2\pi\mathbb{Z}\}\cap\{x\in\mathbb{R}^n:\eta\cdot x\in 2\pi\mathbb{Z}\}\subset\{0\}\times\mathbb{R}^{n-1},$$

which has zero $\mathbb{R}^n$-Lebesgue measure, and hence zero σ-measure and is impossible. In fact, we also have $H^p_\sigma(\mathbb{R}^n)=\{0\}$ for $1<p<\infty$. This will be shown in Section 3.

EXAMPLE 2.12. Let

$$G=\left\{\begin{pmatrix}x & y\\ 0 & 1\end{pmatrix}:(x,y)\in(0,\infty)\times\mathbb{R}\right\}$$

be the affine group of $\mathbb{R}$. We denote an element in G by (x,y). The right Haar measure of G is given by $d\lambda(x,y)=1/x\,dx\,dy$. Let

$$\sigma=\tfrac{1}{2}\delta_{(2,-1)}+\tfrac{1}{2}\delta_{(2/3,1/3)}.$$

Since

$$f*\sigma(x,y)=\int_{-\infty}^{\infty}\int_0^{\infty}f\left(\frac{x}{u},-\frac{xv}{u}+y\right)d\sigma(u,v),$$

a function f on G is σ-harmonic if and only if

$$f(x,y)=\frac{1}{2}f\left(\frac{x}{2},\frac{x}{2}+y\right)+\frac{1}{2}f\left(\frac{3x}{2},-\frac{x}{2}+y\right).$$

Thus $H^1_\sigma(G)$ contains the non-constant function $f(x,y)=x(x+y)e^{-(x+y)^2}$.

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If we take the unit mass $\sigma = \delta_{(1,-1)}$ on G, then a function f on G is σ-harmonic if and only if $f(x,y) = f(x, x+y)$. Hence $f(x,y) = e^{-x}$ is a non-constant function in $H^\infty_\sigma(G) \backslash H^1_\sigma(G)$.

To end this section, we consider the Hilbert space $H^2_\sigma(G)$. Let $\|\sigma\| = 1$. By Theorem 2.3, the dual of the projection $P_\sigma : L^2(G) \longrightarrow H^2_\sigma(G)$ is $P_{\widetilde{\sigma}}$. Therefore P_σ is self-adjoint if σ is symmetric, that is, if $\widetilde{\sigma} = \sigma$. In this case, P_σ is just the orthogonal projection $Q_\sigma : L^2(G) \longrightarrow H^2_\sigma(G)$.

LEMMA 2.13. *Let $\sigma = \widetilde{\sigma}$ and let $P_\sigma : L^2(G) \longrightarrow H^2_\sigma(G)$ be the above projection. Then $P_\sigma = Q_\sigma$.*

Proof. The projections P_σ and Q_σ have the same kernel. Indeed, we have $Q_\sigma(u) = 0$ if and only if $\langle P_\sigma(f), u\rangle = 0$ for all $f \in L^2(G)$ where $\langle P_\sigma(f), u\rangle = \langle f, P_\sigma(u)\rangle$. Hence for every $u \in L^2(G)$, we have $Q_\sigma(u - P_\sigma(u)) = 0$ and $Q_\sigma(u) = Q_\sigma(P_\sigma(u)) = P_\sigma(u)$. □

By the Riesz representation theorem, for each $x \in G$, there exists a function $K_\sigma(x, \cdot) \in H^2_\sigma(G)$ such that

$$u(x) = \langle u, K_\sigma(x, \cdot)\rangle \qquad \big(u \in H^2_\sigma(G)\big).$$

We call $K_\sigma(x,y)$ the *reproducing kernel* of $H^2_\sigma(G)$. Since

$$(Q_\sigma u)(x) = \int_G u(y) K_\sigma(x,y)\, d\lambda(y),$$

we have, for $\sigma = \widetilde{\sigma}$,

$$(P_\sigma u)(x) = \int_G u(y) K_\sigma(x,y)\, d\lambda(y) \qquad (x \in G)$$

and an L^2 function u on G is σ-harmonic if and only if $u(x) = \int_G u(y) K_\sigma(x,y)\, d\lambda(y)$.

EXAMPLE 2.14. If $\{u_\alpha\}$ is an orthonormal basis of $H^2_\sigma(G)$, then the reproducing kernel is given by, in the L^2-norm of $H^2_\sigma(G)$,

$$K_\sigma(x,y) = \sum_\alpha \overline{u_\alpha(x)} u_\alpha(y).$$

Let $\mathcal{T} = \{e^{i\theta} : \theta \in \mathbb{R}\}$ be the circle group and let $\sigma = \frac{1}{2}\delta_i + \frac{1}{2}\delta_{-i}$ which is symmetric. Then we have $H^2_\sigma(\mathcal{T}) = \{f \in L^2(\mathcal{T}) : f(t) = \frac{1}{2} f(it) + \frac{1}{2} f(-it)\}$. The functions $\{e^{im\theta}\}_{m \in \mathbb{Z}}$ form an orthonormal basis for $L^2(\mathcal{T})$ and $e^{im\theta} \in H^2_\sigma(\mathcal{T})$ if and only if $m = 4n$ for some $n \in \mathbb{Z}$. Hence the reproducing kernel of $H^2_\sigma(\mathcal{T})$ is given by

$$K_\sigma(e^{i\theta}, e^{i\psi}) = \sum_{n=-\infty}^{\infty} e^{4ni(\theta - \psi)}.$$

3. *Uniqueness results*

In this section, we prove the uniqueness results for σ-harmonic functions on groups. On Riemannian manifolds, the L^p harmonic functions are constant, under suitable conditions mentioned in Section 1. On a locally compact group G, we show that the σ-harmonic L^p functions are constant (λ-almost everywhere) if σ is an

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adapted probability measure on G and if $p<\infty$. This result is false if either of these conditions is not satisfied. We have noted before that if f is a harmonic function on a Lie group M with respect to the Laplace operator, then f is σ-harmonic on M for some adapted absolutely continuous probability measure σ on M. Therefore our uniqueness results also apply to Δ-harmonic functions on Lie groups.

Trivially, if σ is the unit mass δ_e at the identity of a group G, then every Borel function on G is σ-harmonic. The uniqueness question for σ-harmonic functions as well as their applications in probability theory are meaningful only when the measure σ has *sufficient* support, in the following sense. A positive measure σ on a locally compact group G is called *adapted* if its support, supp σ, generates a dense subgroup of G. If σ satisfies the stronger condition that

$$G=\overline{\bigcup_{n=1}^{\infty}(\operatorname{supp}\sigma)^n},$$

then we say that σ is *non-degenerate*. Note that we do not assume absolute continuity of σ in the definition of adaptedness and non-degeneracy. However, absolutely continuous measures on connected groups are always adapted. If a locally compact group admits an adapted measure, then it must be a countable union of compact sets [**7**, Lemma 2.1.1].

We begin with compact groups.

LEMMA 3.1. *Let G be a compact group and let σ be an adapted probability measure on G. Then for $1\leqslant p\leqslant\infty$, we have $H^p_\sigma(G)=\mathbb{C}\mathbf{1}$ where $\mathbf{1}$ denotes the constant function on G having value* 1.

Proof. Since G is compact, we have $L^\infty(G)\subset L^p(G)\subset L^1(G)$ for $1\leqslant p\leqslant\infty$. Therefore, it suffices to prove the assertion for $H^1_\sigma(G)$. Let $\mathcal{U}$ be a neighbourhood base at the identity $e\in G$. For each $U\in\mathcal{U}$, we can choose a positive bounded function ψ_U with compact support in U and $\int_G\psi_U\,d\lambda=1$. Then for every $f\in L^1(G)$, we have $\|\psi_U*f-f\|_1\to 0$ as $U\to\{e\}$ (cf. [**13**, p. 53]). Given $f\in H^1_\sigma(G)$, the function ψ_U*f is continuous (cf. Lemma 3.2) and σ-harmonic on G, and is therefore constant (see [**5**, Proposition 21] or [**20**]). It follows that f is constant λ-almost everywhere, which completes the proof. □

We note that the assertion $H^p_\sigma(G)=\mathbb{C}\mathbf{1}$ means that every σ-harmonic L^p function on G is constant λ-almost everywhere although we often omit the phrase 'λ-almost everywhere' if no confusion is likely. Actually, Lemma 3.1 does not imply that all σ-harmonic L^p functions are constant everywhere. Consider the circle group $\mathcal{T}=\{\exp(2\pi i\theta):\theta\in[0,1)\}$. One can define a probability measure σ on $\mathcal{T}$ by

$$\sigma=\sum_{n=1}^{\infty}\sum_{m=1}^{2^{n-1}}\frac{1}{2^{2n-1}}\delta_{\exp(2\pi i\theta_m)},\qquad\text{where }\theta_m=\frac{2m-1}{2^n}.$$

Then we have supp $\sigma=\mathcal{T}$. Let

$$S=\{\exp(2\pi i\theta):\theta=\frac{m}{2^n},\ 1\leqslant m<2^n;\ m,n\in\mathbb{N}\}$$

and let χ_S be the characteristic function of S. Since $SS^{-1}\subset S$ and $SS\subset S$, the function $1+\chi_S$ is σ-harmonic on $\mathcal{T}$, but not constant everywhere.

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If G is non-compact, then there is no non-zero constant function in $L^p(G)$ for $p<\infty$, in other words, the only constant function in $H^p_\sigma(G)$ is the zero function. Hence a uniqueness result will be expressed as $H^p_\sigma(G)\subset\mathbb{C}\mathbf{1}$ for arbitrary groups G. In [**6**, **8**], various Liouville-type theorems were proved for right uniformly continuous bounded σ-harmonic functions where harmonicity was defined in terms of the convolution equation $f=\sigma*f$. We first improve these results by removing the uniform continuity condition. In the present context where the equation $f=f*\sigma$ is used for harmonic functions, we should consider left uniform continuity and as we work with the right Haar measure, we include the proof of the following lemma for completeness.

Recall that a function $f:G\longrightarrow\mathbb{R}$ is *left uniformly continuous* if for every $\varepsilon>0$, there is a neighbourhood V of the identity such that $x^{-1}y\in V$ implies that $|f(x)-f(y)|<\varepsilon$. Given a function f on G and $x\in G$, we denote by f_x the right-translation $f_x(\cdot)=f(\cdot x)$ of f by x.

LEMMA 3.2. *Let $\psi\in L^q(G)$ and $f\in L^p(G)$ where $1\leqslant p,q\leqslant\infty$. Then $\widetilde{\psi}*f$ is bounded and left uniformly continuous, where $\widetilde{\psi}(x)=\psi(x^{-1})$.*

Proof. We first consider $p<\infty$. By Hölder's inequality, we have

$$\begin{aligned}|(\widetilde{\psi}*f)(x)-(\widetilde{\psi}*f)(y)|&\leqslant\int_G|\widetilde{\psi}(z^{-1})(f(zx)-f(zy))|\,d\lambda(z)\\&\leqslant\|\psi\|_q\left(\int_G|f(zx)-f(zy)|^p\,d\lambda(z)\right)^{1/p}\\&=\|\psi\|_q\|f_x-f_y\|_p.\end{aligned}$$

Thus $\widetilde{\psi}*f$ is bounded.

Let $\varepsilon>0$. Pick $\varphi\in C_c(G)$ such that $\|\varphi-f\|_p<\varepsilon/4\|\psi\|_q$. Let K be the compact support of φ. Let W be a compact neighbourhood of the identity $e\in G$. Since φ is uniformly continuous, we can choose a compact neighbourhood $V\subset W$ of e such that $x^{-1}y\in V$ implies that

$$|\varphi(x)-\varphi(y)|<\frac{\varepsilon}{2\|\psi\|_q\lambda(KW)^{1/p}}.$$

Then we have

$$\begin{aligned}\|\varphi_x-\varphi_y\|_p&=\left(\int_G|\varphi(zx)-\varphi(zy)|^p\,d\lambda(z)\right)^{1/p}\\&=\left(\int_{KW}|\varphi(z)-\varphi(zx^{-1}y)|^p\,d\lambda(z)\right)^{1/p}\\&<\frac{\varepsilon}{2\|\psi\|_q\lambda(KW)^{1/p}}\lambda(KW)^{1/p}=\frac{\varepsilon}{2\|\psi\|_q}.\end{aligned}$$

Hence, for $x^{-1}y\in V$, we have

$$\begin{aligned}|(\widetilde{\psi}*f)(x)-(\widetilde{\psi}*f)(y)|&\leqslant\|\psi\|_q\|f_x-f_y\|_p\\&\leqslant\|\psi\|_q(\|f_x-\varphi_x\|_p+\|\varphi_x-\varphi_y\|_p+\|\varphi_y-f_y\|_p)\\&=\|\psi\|_q(2\|f-\varphi\|_p+\|\varphi_x-\varphi_y\|_p)<\varepsilon,\end{aligned}$$

which proves the left uniform continuity of $\widetilde{\psi}*f$.

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Finally, for $p=\infty$, we conclude the proof by applying similar arguments to

$$|(\widetilde{\psi} * f)(x) - (\widetilde{\psi} * f)(y)| \leqslant \int_G |(\widetilde{\psi}(xz^{-1}) - \widetilde{\psi}(yz^{-1}))f(z)|\, d\lambda(z)$$
$$\leqslant \|\psi_{x^{-1}} - \psi_{y^{-1}}\|_1 \|f\|_\infty.$$

□

LEMMA 3.3. *Let σ be a complex measure on a locally compact group G with $\|\sigma\|=1$. The following conditions are equivalent.*

(i) *Every σ-harmonic L^∞ function on G is constant locally λ-almost everywhere.*

(ii) *Every left uniformly continuous bounded σ-harmonic function on G is constant.*

Further, if (ii) *holds, then every σ-harmonic L^p function on G is constant λ-almost everywhere, for $1 \leqslant p < \infty$.*

Proof. (ii) $\Longrightarrow$(i). Let $\{\psi_\alpha\}$ be a bounded approximate identity in $L^1(G)$. Then the net $(\widetilde{\psi_\alpha} * f)$ weak* converges to f for every $f \in L^\infty(G)$ since

$$\langle \psi, f\rangle = \lim_\alpha \langle \psi_\alpha * \psi, f\rangle = \lim_\alpha \langle \psi, \widetilde{\psi_\alpha} * f\rangle \qquad (\psi \in L^1(G)).$$

By Lemma 3.2, the function $\widetilde{\psi_\alpha} * f$ is left uniformly continuous and bounded on G. If f is σ-harmonic, then each $\psi_\alpha * f$ is also σ-harmonic and hence constant. Hence f is constant in $L^\infty(G)$.

Finally, assume that (ii) holds and let $f \in L^p(G)$ be σ-harmonic for $p<\infty$. As before, choose $\psi_U \in C_c(G)$ such that $\|\psi_U * f - f\|_p \to 0$ as $U \to \{e\}$. By Lemma 3.2, the function $\psi_U * f$ is left uniformly continuous, bounded and σ-harmonic. Thus $\psi_U * f$ is constant and hence f is constant in $L^p(G)$. □

Given an adapted probability measure σ on G, we have $H^\infty_\sigma(G) = \mathbb{C}\mathbf{1}$ for abelian groups, discrete or class 2 nilpotent groups G [**3**, **4**, **12**, **18**]. If σ is non-degenerate and absolutely continuous, then it is known that $H^\infty_\sigma(G) = \mathbb{C}\mathbf{1}$ for locally compact nilpotent groups (cf. [**2**, **6**, **16**, **22**]). The assumption of absolute continuity can be removed.

PROPOSITION 3.4. *Let σ be a non-degenerate probability measure on a nilpotent group G. Then $H^\infty_\sigma(G) = \mathbb{C}\mathbf{1}$.*

Proof. By [**6**, Theorem 1.3], every left uniformly continuous function in $H^\infty_\sigma(G)$ is constant. Hence the result follows from Lemma 3.3. □

We note that it has been shown in [**8**] that if G is an almost connected [IN]-group and if σ is a probability measure on G, non-singular with its translates, but not necessarily absolutely continuous, then $H^\infty_\sigma(G) = \mathbb{C}\mathbf{1}$. For arbitrary groups, we have the following extension of Choquet and Deny's result for abelian groups [**4**].

PROPOSITION 3.5. *Let σ be a non-degenerate probability measure on a locally compact group G with centre Z and let $f \in H^\infty_\sigma(G)$. Then for every $z \in Z$, we have $f(xz) = f(x)$ for locally λ-almost every $x \in G$.*

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Proof. Let $f \in H_\sigma^\infty(G)$. Then for any $\psi \in L^1(G)$, the convolution $\widetilde{\psi} * f$ is left uniformly continuous, bounded and σ-harmonic on G. We note that G is a countable union of compact sets since it supports a non-degenerate probability measure (cf. [**7**, Lemma 2.1.1]). Hence, as in [**6**, Lemma 1.1], $\widetilde{\psi} * f$ is constant on the centre Z. We have, for $z \in Z$,

$$\langle \psi, f_z - f \rangle = \widetilde{\psi} * f(z) - \widetilde{\psi} * f(e) = 0.$$

Therefore $f_z = f$ in $L^\infty(G)$. □

COROLLARY 3.6. *Let σ be a symmetric probability measure on a nilpotent group G and let $f \in H_\sigma^p(G)$ where $1 \leqslant p \leqslant \infty$. Then for every $a \in \operatorname{supp}\sigma$, we have $f_{a^{-1}} = f$ in $L^p(G)$.*

Proof. Let G_σ be the closed subgroup of G generated by $\operatorname{supp}\sigma$. Then G_σ is a countable union of compact sets (cf. [**7**, Lemma 2.1.1]). Since σ is symmetric, it is non-degenerate on G_σ.

We first prove the result for a bounded continuous σ-harmonic function f. By restricting f and σ to G_σ, we regard f as a bounded continuous σ-harmonic function on G_σ. We use induction on the nilpotent class of G_σ. The abelian case is just Choquet and Deny's result (cf. Proposition 3.5). Suppose the result holds for nilpotent class r. Let G_σ be of nilpotent class $r+1$ with class series of closed normal subgroups:

$$G_\sigma = G_0 \supset G_1 \supset \ldots \supset G_r \supset G_{r+1} = \{e\},$$

where G_σ/G_r is of nilpotent class r and G_r is contained in the centre of G_σ. Let $q : G_\sigma \longrightarrow G_\sigma/G_r$ be the quotient map. By Proposition 3.5, we have $f(xy) = f(x)$ for all $y \in G_r$ and $x \in G_\sigma$. Therefore we can define a bounded continuous function $F : G_\sigma/G_r \longrightarrow \mathbb{C}$ by $F(q(x)) = f(x)$ for $x \in G_\sigma$. Let $\widetilde{\sigma} = \sigma \circ q^{-1}$ be the induced measure on G_σ/G_r. Then F is $\widetilde{\sigma}$-harmonic on G_σ/G_r. Let $a \in \operatorname{supp}\sigma$. Then $q(a) \in \operatorname{supp}\widetilde{\sigma}$ and hence we have

$$f(xa^{-1}) = F(q(x)q(a)^{-1}) = F(q(x)) = f(x) \qquad (x \in G_\sigma).$$

The above actually holds for all $x \in G$, by the application of the result to the left translate $f(x\cdot)$ at $e \in G_\sigma$.

Now take any $f \in H_\sigma^p(G)$. For every $\psi \in L^q(G)$, the function $\widetilde{\psi} * f$ is bounded continuous and σ-harmonic. Therefore the above argument gives, for $a \in \operatorname{supp}\sigma$,

$$\langle f - f_{a^{-1}}, \psi \rangle = (\widetilde{\psi} * f)(e) - (\widetilde{\psi} * f)(a^{-1}) = 0.$$

Hence $f - f_{a^{-1}} = 0$ in $L^p(G)$. □

REMARK 3.7. If G is abelian, the symmetric condition on σ in Corollary 3.6, and Corollary 3.8 below, can be dropped by Choquet and Deny's result for $H_\sigma^\infty(G)$ [**4**]. We note that, given a function $f \in H_\sigma^p(G)$, the restriction of f to G_σ need not lie in $L^p(G_\sigma)$ as G and G_σ may have different Haar measures in general.

COROLLARY 3.8. *Let σ be a symmetric probability measure on a nilpotent group G. Then $H_\sigma^\infty(G)$ is a subalgebra of $L^\infty(G)$.*

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Proof. Let $f, g \in H^\infty_\sigma(G)$. Then their pointwise product fg satisfies $(fg)_{a^{-1}} = fg$ for all $a \in \operatorname{supp}\sigma$ and is therefore σ-harmonic. □

We now consider $p < \infty$. The uniqueness result can be extended to non-negative subharmonic functions as the result below shows. Let σ be a probability measure on a locally compact group G. A real Borel function f on G is called *σ-subharmonic* if $f * \sigma$ exists and $f \leqslant f * \sigma$ λ-almost everywhere. We do not assume adaptedness of σ in the following result.

PROPOSITION 3.9. *Let σ be a probability measure on a locally compact group G and let $1 \leqslant p < \infty$. The following conditions are equivalent.*

(i) *Every σ-harmonic L^p function on G is constant λ-almost everywhere.*

(ii) *Every non-negative σ-subharmonic L^p function on G is constant λ-almost everywhere.*

Proof. (ii) $\Longrightarrow$ (i). This follows from the fact that if f is σ-harmonic, then $|f|$ is σ-subharmonic. Also, if f is a real σ-harmonic L^p function on a compact group G, then $f = |f| - (|f| - f)$ is constant λ-almost everywhere.

(i) $\Longrightarrow$ (ii). Let $f \in L^p(G)$ be non-negative and σ-subharmonic. Then we have

$$0 \leqslant f \leqslant f * \sigma \leqslant \ldots \leqslant f * \sigma^n \leqslant \ldots$$

and $\|f * \sigma^n\|_p \leqslant \|f\|_p$. By choosing a subsequence, we may assume that the weak limit $g = \lim_{n\to\infty} f * \sigma^n$ exists in $L^p(G)$ for $1 < p < \infty$. For $p = 1$, the sequence $(f * \sigma^n)$ is weakly Cauchy because the limit

$$\lim_{n\to\infty} \int_E (f * \sigma^n)\, d\lambda$$

exists for every Borel set E in G as the integrals are increasing and bounded above by $\|f\|_1$ (cf. [**11**, p. 342]). Therefore the weak limit $g = \lim_{n\to\infty} f * \sigma^n$ also exists in $L^1(G)$. Evidently g is σ-harmonic and therefore constant.

Some sequence of convex combinations of $(f * \sigma^n)$ converges in norm to g. Such a sequence is of the form $(f * \mu_k)$, where $\mu_k \in \operatorname{co}\{\sigma^n : n = 1, 2, \ldots\}$, which has a subsequence $(f * \mu_j)$ converging to g pointwise λ-almost everywhere. Since $f \leqslant f * \mu_j$, we have $f \leqslant g$ λ-almost everywhere.

If G is non-compact, then $g = 0$ and hence $f = 0$ λ-almost everywhere.

Let G be compact and $\lambda(G) = 1$. Then $\mathbf{1} \in L^q(G)$ and

$$\begin{aligned} \int_G f d\lambda &= \langle f, \mathbf{1} \rangle \\ &= \lim_{n\to\infty} \langle f, \mathbf{1} * \widetilde{\sigma}^n \rangle \\ &= \lim_{n\to\infty} \langle f * \sigma^n, \mathbf{1} \rangle \\ &= \langle g, \mathbf{1} \rangle = \int_G g d\lambda = c, \end{aligned}$$

where $g = c$ λ-almost everywhere. Since $f \leqslant c$, we have $f = c$ λ-almost everywhere. □

The following example shows that the above result is false for $p = \infty$.

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EXAMPLE 3.10. Consider the multiplicative group $\mathbb{R}^+ = (0, \infty)$ with the Haar measure $d\lambda(x) = dx/x$. Define an adapted (and absolutely continuous) probability measure σ on $\mathbb{R}^+$ by

$$d\sigma(x) = \frac{1}{x^2} \exp\left(-\frac{1}{x}\right) dx.$$

The function $f(x) = \exp(-x)$ is a positive function in $L^\infty(\mathbb{R}^+) \backslash L^1(\mathbb{R}^+)$. We have

$$\begin{aligned}(f * \sigma)(x) &= \int_0^\infty e^{-x/y} \frac{1}{y^2} e^{-1/y}\, dy \\ &= \frac{1}{1+x} \geqslant f(x),\end{aligned}$$

so f is a non-constant σ-subharmonic function on $\mathbb{R}^+$, but $H^\infty_\sigma(\mathbb{R}^+) = \mathbb{C}\mathbf{1}$.

To prove our main uniqueness result, we first recall some definitions in representation theory. Let G be a locally compact group with dual space $\widehat{G}$, consisting of (equivalence classes of) continuous unitary irreducible representations of G. We denote a representation of G by $\pi : G \longrightarrow \mathcal{B}(H_\pi)$ where $\mathcal{B}(H_\pi)$ is the von Neumann algebra of bounded linear operators on a Hilbert space H_π. The *dimension* of π is defined by $\dim \pi = \dim H_\pi$. Let $\iota \in \widehat{G}$ be the one-dimensional representation $\iota(\cdot) = \mathrm{Id} \in \mathcal{B}(\mathbb{C})$.

The irreducible continuous unitary representations of a locally compact group G are in one-to-one correspondence with the irreducible *-representations of the group C^*-algebra $C^*(G)$ which is the completion of $L^1(G)$ with respect to the C^*-norm

$$\|f\|' = \sup_\pi \{\|\pi(f)\|\},$$

where the supremum is taken over all *-representations of $L^1(G)$ (cf. [**9**, 13.9]). Given a continuous unitary representation π of G which determines a *-representation of $C^*(G)$, still denoted by π, we have

$$\pi(f) = \int_G f(x)\pi(x)\, d\lambda(x)$$

for $f \in L^1(G)$. We denote the kernel of π in $C^*(G)$ by

$$\ker \pi = \{a \in C^*(G) : \pi(a) = 0\}.$$

Given a measure $\sigma \in M(G)$, we define

$$\pi(\sigma) = \int_G \pi(x)\, d\sigma(x).$$

Then we have $\pi(f * \sigma) = \pi(f)\pi(\sigma)$ for $f \in L^1(G)$ and $\sigma \in M(G)$.

The following lemma strengthens the result in [**7**, 2.1.4].

LEMMA 3.11. *Let σ be an adapted probability measure on a locally compact group G. Then the operator $I_\pi - \pi(\sigma)$ is invertible in $\mathcal{B}(H_\pi)$ for every representation $\pi \in \widehat{G} \backslash \{\iota\}$.*

Proof. Let $\pi \in \widehat{G} \backslash \{\iota\}$. We first show that $I_\pi - \pi(\sigma)$ has a left inverse in $\mathcal{B}(H_\pi)$. For this, it suffices to show that $\mathcal{B}(H_\pi)\,(I_\pi - \pi(\sigma)) = \mathcal{B}(H_\pi)$. Suppose otherwise, in which case $\mathcal{B}(H_\pi)(I_\pi - \pi(\sigma))$ is a weakly closed left ideal properly contained

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in $\mathcal{B}(H_\pi)$ and there is a projection $p \neq I_\pi$ such that $\mathcal{B}(H_\pi)(I_\pi - \pi(\sigma)) = \mathcal{B}(H_\pi)p$. Then $I_\pi - \pi(\sigma) = (I_\pi - \pi(\sigma))p$ gives $\pi(\sigma)(I_\pi - p) = I_\pi - p$ and we can choose a unit vector ξ in $(I_\pi - p)(H_\pi)$. We have $\pi(\sigma)\xi = \xi$ and therefore

$$1 = \langle \pi(\sigma)\xi, \xi\rangle = \int_G \langle \pi(x)\xi, \xi\rangle \, d\sigma(x) = \int_{\operatorname{supp}\sigma} \operatorname{Re}\langle \pi(x)\xi, \xi\rangle \, d\sigma(x).$$

If $\operatorname{Re}\langle \pi(x)\xi, \xi\rangle < 1$ for some $x \in \operatorname{supp}\sigma$, then the inequality holds in some open neighbourhood V of x by continuity and hence, as $\sigma(V) > 0$,

$$\begin{aligned} 1 &= \int_V \operatorname{Re}\langle \pi(x)\xi, \xi\rangle \, d\sigma(x) + \int_{G\setminus V} \operatorname{Re}\langle \pi(x)\xi, \xi\rangle \, d\sigma(x) \\ &< \sigma(V) + \sigma(G\setminus V) = 1, \end{aligned}$$

which is impossible. Hence for all $x \in \operatorname{supp}\sigma$, we have $\langle \pi(x)\xi, \xi\rangle = 1$ which gives $\pi(x)\xi = \xi$. It follows from the adaptedness of σ that $\pi(x)\xi = \xi$ for all $x \in G$ contradicting $\pi \neq \iota$. Hence $I_\pi - \pi(\sigma)$ has a left inverse.

Since σ is a real measure, we have $\pi(\sigma)^* = \int_G \pi(x)^* \, d\sigma(x) = \int_G \pi(x^{-1}) \, d\sigma(x)$ and we can apply similar arguments to show that $I_\pi - \pi(\sigma)^*$ has a left inverse, that is, $I_\pi - \pi(\sigma)$ has a right inverse and is therefore invertible. □

We denote by

$$L^1_0(G) = \left\{ f \in L^1(G) : \int_G f \, d\lambda = 0 \right\}$$

the augmentation ideal in $L^1(G)$ which has co-dimension 1. Note that $\iota(f) = \int_G f \, d\lambda$ for $f \in L^1(G)$. Now we are ready to prove our main uniqueness result.

THEOREM 3.12. *Let σ be an adapted probability measure on a locally compact group G. Then we have $H^1_\sigma(G) \subset \mathbb{C}\mathbf{1}$.*

Proof. We first show that

$$H^1_\sigma(G) \subset \bigcap_{\pi \in \widehat{G}\setminus\{\iota\}} \ker \pi.$$

Let $f \in H^1_\sigma(G)$. Then we have $\pi(f) = \pi(f * \sigma) = \pi(f)\pi(\sigma)$ which gives $\pi(f)(I_\pi - \pi(\sigma)) = 0$ and hence $\pi(f) = 0$ for $\pi \in \widehat{G}\setminus\{\iota\}$, by Lemma 3.11. For any $a \in C^*(G)$, we have

$$\|a\|' = \sup\{\|\pi(a)\| : \pi \text{ is an irreducible representation of } C^*(G)\}$$

(cf. [**9**, 2.7.3]). Therefore $H^1_\sigma(G) \cap L^1_0(G) = \{0\}$ since $f \in H^1_\sigma(G) \cap L^1_0(G)$ implies that $\|f\|' = 0$. It follows that $\dim H^1_\sigma(G) \leqslant 1$ as $L^1_0(G)$ has co-dimension one in $L^1(G)$.

Suppose that $H^1_\sigma(G) \neq \{0\}$. By Remark 2.9, we can choose a non-zero real continuous function $u \in H^1_\sigma(G)$ such that $H^1_\sigma(G) = \mathbb{C}u$. We complete the proof by showing that u is constant. For any $x \in G$, the left-translate ${}_x u(\cdot) = u(x\cdot)$ is in $H^1_\sigma(G)$ and is therefore a constant multiple of u which gives $u(xy) = \alpha(x)u(y)$ for some $\alpha(x) \in \mathbb{C}$ and for all $y \in G$. Thus $u(e) \neq 0$ and we may assume that $u(e) = 1$ by normalizing u. It follows that $u(xy) = u(x)u(y)$ for all $x, y \in G$. Hence, by the

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right invariance of λ, we have

$$\int_G |u(x)|\,d\lambda(x) = \int_G |u(xy)|\,d\lambda(x) = \left(\int_G |u(x)|\,d\lambda(x)\right)|u(y)|,$$

which gives $|u(y)| = 1$ for all $y \in G$. As $u \in L^1(G)$, the Haar measure λ must be finite, that is, G is compact and hence u is constant by Lemma 3.1. □

REMARK 3.13. In view of the above proof, we remark that $\bigcap_{\pi \in \widehat{G}\setminus\{\iota\}} \ker \pi$ need not be $\{0\}$ in general. For instance, if f is any non-zero constant function on the circle group $\mathcal{T}$, then for any $\pi \in \widehat{\mathcal{T}}\setminus\{\iota\} \simeq \mathbb{Z}\setminus\{0\}$, we have $\pi(f) = (1/2\pi)\int_0^{2\pi} f(e^{ix})e^{inx}\,dx = 0$ for some $n \in \mathbb{Z}\setminus\{0\}$.

COROLLARY 3.14. *Let σ be an adapted probability measure on a locally compact group G. Then we have $H^p_\sigma(G) \subset \mathbb{C}\mathbf{1}$ for $1 < p < \infty$.*

Proof. Let $1 < p < \infty$. By Corollary 2.8 and Remark 2.9, it suffices to show that the non-negative functions in $H^p_\sigma(G)$ are constant. Let $0 \leqslant f \in H^p_\sigma(G)$. Then by Jensen's inequality, f^p is σ-subharmonic and $f^p \in L^1(G)$. Since $H^1_\sigma(G) \subset \mathbb{C}\mathbf{1}$, Proposition 3.9 implies that f^p, and hence f, is constant λ-almost everywhere. □

The following result also follows directly from Example 2.11 and arguments similar to the above proof.

COROLLARY 3.15. *Let σ be an absolutely continuous probability measure on $\mathbb{R}^n$. Then $H^p_\sigma(\mathbb{R}^n) = \{0\}$ for $1 \leqslant p < \infty$.*

We note that Corollary 3.14 is false for $p = \infty$. Indeed, it is known that if there is a probability measure σ on G such that every σ-harmonic $L^\infty(G)$-function is constant, then G is amenable (cf. [**19**, **24**]). Therefore Corollary 3.14 does not hold for $p = \infty$, for non-amenable groups, for example, $\mathrm{SL}(2, \mathbb{R})$ and the free groups on more than two generators.

We have given examples in the previous section to show that $H^p_\sigma(G)$ can contain non-constant functions if σ is not adapted. We conclude this section with a result on the dimension of $H^1_\sigma(G)$, relating it to the irreducible representations of G, by refining the arguments in Lemma 3.11.

Given a probability measure σ on a locally compact group G, we denote, as before, by G_σ the closed subgroup of G generated by the support of σ. We denote by $\mathrm{Sp}(\pi(\sigma))$ the spectrum of $\pi(\sigma) \in \mathcal{B}(H_\pi)$, and by $\sharp S$ the cardinality of a set S.

The following result extends Theorem 3.12.

THEOREM 3.16. *Let σ be a probability measure on a locally compact group G such that G_σ has finite index $[G : G_\sigma]$ in G. Then we have*

$$\dim H^1_\sigma(G) \leqslant [G : G_\sigma]^2(\sharp\{\pi \in \widehat{G} : \dim \pi \leqslant [G : G_\sigma]\}).$$

Proof. Let $G/G_\sigma = \{a_1 G_\sigma, \ldots, a_m G_\sigma\}$. Let $\pi \in \widehat{G}\setminus\{\iota\}$ have $\dim \pi > [G : G_\sigma]$. Then $I - \pi(\sigma)$ is invertible in $\mathcal{B}(H_\pi)$, for otherwise, as in the proof of Lemma 3.11, there is a unit vector ξ in $\mathcal{B}(H_\pi)$ such that

$$\pi(x)\xi = \xi \qquad (x \in G_\sigma).$$

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It follows that $\pi(G)\xi = \{\pi(a_1)\xi, \ldots, \pi(a_m)\xi\}$ and $\dim H_\pi \leqslant m = [G : G_\sigma]$ by irreducibility of π. Therefore $1 \in \mathrm{Sp}(\pi(\sigma))$ implies that $\dim \pi \leqslant m$.

Let $f \in H^1_\sigma(G)$ and $\pi \in \widehat{G}$. If $I - \pi(\sigma)$ is invertible in $B(H_\pi)$, then $\pi(f) = 0$ as in the proof of Theorem 3.12. Therefore we have

$$H^1_\sigma(G) \subset \bigcap_{\pi \in \widehat{G}, 1 \notin \mathrm{Sp}(\pi(\sigma))} \ker \pi \subset \bigcap_{\pi \in \widehat{G}, \dim \pi > m} \ker \pi,$$

which implies that

$$H^1_\sigma(G) \cap \bigcap_{\pi \in \widehat{G}, \dim \pi \leqslant m} \ker \pi = \{0\},$$

giving

$$\dim H^1_\sigma(G) \leqslant \dim \left(\bigoplus_{\pi \in \widehat{G}, \dim \pi \leqslant m} C^*(G)/\ker \pi \right),$$

where each finite-dimensional representation $\pi : C^*(G) \longrightarrow \mathcal{B}(H_\pi)$ induces a monomorphism $C^*(G)/\ker \pi \longrightarrow \mathcal{B}(H_\pi)$ and $\mathcal{B}(H_\pi)$ has dimension $(\dim H_\pi)^2$. This completes the proof. □

REMARK 3.17. The above proof actually gives the sharper estimate

$$\dim H^1_\sigma(G) \leqslant \sum_{\pi \in \widehat{G}, 1 \in \mathrm{Sp}(\pi(\sigma))} (\dim \pi)^2 \leqslant \sum_{\pi \in \widehat{G}, \dim \pi \leqslant m} (\dim \pi)^2.$$

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