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Clifford algebra of three dimensional geometry

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SUMMARY

The use of Clifford or geometric algebra for dealing with three dimensional geometry is discussed. One issue is the representation of the rigid body motions of rotations and translations as elements within the algebra. The approach used is to work with projective geometry and choose the square of an additional basis element to be large (infinite). This allows Euclidean points to be represented as vectors in the algebra and transforms on these to be handled using bivectors. This paper looks at the use of Clifford algebra for handling the types of transforms required in robotic applications. A number of example applications are given.

KEYWORDS: Clifford algebra; Geometric algebra; Rigid body motion; Transforms.

1. INTRODUCTION

Many si tuations in robotics require the ability to perform transforms of three dimensional Euclidean space. Such transforms are usually rigid body ones: that is they are combinations of translations and rotations which thus preserve the shape of any body that is transformed. The motions of the individual joints of a robot are governed by such transforms. Similarly the motion of an end-effector through space is obtained by a continuously varying transform. Such a motion often needs to be defined to be smooth and to pass through two or more prescribed points. In the area of robot vision, transforms are regularly required to move between the spaces of the camera, the robot and the global frame of reference.

The conventional computer graphics approach is the use of 4×4 matrices.[1] These allow both rotations and translations of Euclidean space to be defined by working in terms of projective coordinates. The forms of matrix for each type of transform are different but they can be combined using ordinary matrix multiplication. Letting the matrices vary continuously allows smooth motions to be generated.[2]

Quaternions can also be used to represent rotations exactly.[1] These, and variations such as dual quaternions and biquaternions, have also led to work in creation of smooth motions.[3,4] The relation between the quaternion and matrix approaches has also been investigated.[5]

The ideas of quaternions arose in from the work of Hamilton (and others) in the 1800s. The interest was in finding a means for representing exactly the properties of rotations. At the same time, the work of Clifford led to the creation of the geometric algebra which bears his name. These ideas do not seem to have received much attention or application. This is partly because of the successful introduction of matrices for handling transformations. However interest has recently been rekindled[6,7] with application in a number of diverse areas. In particular, the algebra can be applied to define smooth motions.[8] One attraction is that the algebra provides a single framework in which to handle geometry and represents geometric entities as actual elements within itself rather than as equations relating coordinates.

As a geometric algebra automatically includes the quaternions and ordinary Euclidean space, it can certainly be used to deal with rotations. Translations are more problematic. One approach[8] is to treat a translation as a rotation about a distant point. This leads to a level of error which can be controlled by adjusting the distance to this remote point. A geometric algebra is essentially a vector space upon which a multiplication between vectors has been imposed. The products formed by multiplying a basis element of the space with itself are defined to be scalars. There is a certain amount of freedom as to what those scalars should be. By choosing one of the basis elements to have a square which is zero, it is possible to represent both rotations and translations exactly and in the same form.[9] A consequence of this choice it that the representation of Euclidean space that has to be used is not a natural one. Points are represented by elements of the algebra which would more naturally be regarded as planes, and vice versa.

The purpose of this paper is to discuss a form of dual of this approach. Here a geometric algebra is used to handle both translations and rotations in a way which preserves the natural representation of Euclidean space. To deal with three dimensional space, four basis vectors are required. One of these has a square which needs to be treated as infinite. This is handled by defining it to be the reciprocal of an arbitrarily small number which is allowed to tend to zero after the necessary manipulations have been performed. In this approach, all rigid body transforms are represented by even grade elements in the algebra and act upon vectors which are the natural representation of Euclidean points. The additional basis element (and the value of its square) means that geometry is being handled in a projective manner. Use is made of the inner and outer products of the algebra to give concise statements of geometric properties (and the use of the Hodge star operator[9] is avoided).

In Section 2, the general ideas of geometric algebra are reviewed, and, in Section 3, they are applied to three dimensional geometry. Here the definition of the fourth basis element is discussed and some results and examples

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relating to geometry are given for later reference. In Section 4, the use of even grade elements for representing rigid body transforms is investigated. To illustrate the range of possible uses, some application examples are discussed in Section 5. These are essentially the same examples as presented in the previous paper[9] but they are handled with the form of the geometric algebra presented here.

2. GEOMETRIC ALGEBRA

Conventional Euclidean n dimensional space can be regarded as a vector space over the real numbers generated by n basis vectors $e_1, e_2, \ldots, e_n$. Then the typical vector in the space has the form $a_1 e_1 + \ldots + a_n e_n$, where the a_i are real scalars.

What is different for a Clifford or geometric algebra is the introduction of a multiplication between the elements of the space. This is defined firstly for the basis elements. If S is an ordered set of (different) subscripts, then e_S is a new element which is considered to be the product of the basis elements with the individual subscripts in the same order. So, for example, there are additional elements such as the following:

$$e_{12} = e_1 e_2 \qquad e_{123} = e_1 e_2 e_3$$

If S and T are two ordered subsets containing the same subscripts but in possibly different orders, then e_T is taken to be the same as $\pm e_S$, where the positive sign is taken if T is an even permutation of S, and the minus sign if odd. So, for example,

$$e_1 e_2 = e_{12} = -e_{21} = -e_2 e_1 \qquad e_{123} = e_{231} = -e_{213}$$

In the case that S is the empty set, the element e_S is taken to be the real number unity. It remains to define the product of a basis element with itself. For this, e_i^2 is taken to be a scalar λ_i. Assuming that it is non-zero, the value of λ_i does not matter and is here taken to be unity.

$$e_i^2 = 1$$

If this were not the case, then e_i can be replaced by $(e_i/\sqrt{\lambda_i})$, where the scalars may need to be extended to the complex numbers if $\lambda_i < 0$.

In this way, the geometric algebra $\mathcal{G}_n$ can be regarded as being, in particular, a vector space with basis elements e_S where S is any of the possible 2^n subsets of subscripts, and the typical element has the form:

$$x = \sum_S a_S e_S$$

where the a_S are scalars. The algebra contains within it, the field of real scalars and the members of the original vector space. The element x is said to be a *vector* and have *grade* 1, if a_S is non-zero only in cases when $|S| = 1$, and is a *bivector* and has *grade* 2, if a_S is non-zero only when $|S| = 2$. Similarly, the element x is said to have even (odd) grade if a_S is non-zero only for sets S with even (odd) size. Multiplication of general elements is defined as the obvious extension of multiplication of the basis elements. For example:

$$(1 + e_1)(e_2 + e_{13}) = e_2 + e_{13} + e_{12} + e_1 e_{13}$$
$$= e_2 + e_{13} + e_{12} + e_3$$

Note that the elements of even grade within $\mathcal{G}_2$ are those of the form $a + be_{12}$. This set is the same as the field of complex numbers. In particular, e_{12} is equivalent to i and its square is -1. Similarly, the elements of even grade in $\mathcal{G}_3$ are the same as the ring of quaternions. In the notation used here, i, j and k are identified as follows:

$$i = e_{32} \qquad j = e_{13} \qquad k = e_{21}$$

One important operation on the algebra is the *reverse*. This is analogous to the idea of the complex conjugate and for that reason it is here denoted by a bar. If e_S is one of the generators, then $\overline{e_S}$ is defined to be e_T where T is the same set as S but taken in the reverse order. In fact, $\overline{e_S}$ is equal to e_S if $|S|$ is congruent to 0 or 1 modulo 4, and is $-e_S$ otherwise. For example:

$$\overline{e_{12}} = e_{21} = -e_{12} \qquad \overline{e_{123}} = e_{321} = -e_{123}$$

The rules familiar for complex conjugation apply: for general elements x and y

$$\overline{x + y} = \overline{x} + \overline{y} \qquad \overline{xy} = \overline{x}\,\overline{y}$$

Inner and outer products are defined on $\mathcal{G}_n$. For general elements x and y, their inner product is given as

$$x \cdot y = \frac{1}{2}(xy + yx)$$

and their outer product is:

$$x \wedge y = \frac{1}{2}(xy - yx)$$

These have all the properties to be expected of products, except that the outer product is anti-commutative, so that $y \wedge x = -(x \wedge y)$. The inner product is commutative so that $y \cdot x = x \cdot y$.

If x, y, z are elements of $\mathcal{G}_4$, which all have the same grade, then it can be shown that:

$$(x \wedge y) \cdot z = (y \wedge z) \cdot x = (z \wedge x) \cdot y$$
$$= \frac{1}{6}(xyz + yzx + zxy - xzy - yxz - zyx)$$

This is valid also in any dimension if x, y, z are all vectors. When the equality applies, the common product is called the *triproduct* of the three elements and is denoted by $[x, y, z]$.

3. THREE DIMENSIONAL GEOMETRY

As previously noted, $\mathcal{G}_n$ contains a copy of Euclidean n-space. The next stage is to extend the algebra so that projective geometry can be handled. Attention is restricted to three dimensions, and $\mathcal{G}_4$ is used. The way this is used to represent three dimensional geometry is now discussed, followed by three results which relate geometric properties

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to expressions involving inner and outer products. Finally in this section an example is given.

The geometric algebra $\mathcal{G}_4$ has four basis elements. Three of these e_1, e_2 and e_3 are taken as before. Their squares are unity.

$$e_1^2 = e_2^2 = e_3^2 = 1$$

The fourth basis element is denoted by e_0. Its square needs, in essence, to be infinite. To handle this, it is assumed that:

$$e_0^2 = \epsilon^{-1}$$

where ϵ is a small, real number. The procedure, in all manipulations, is to carry ϵ through the calculations and then let it tend to zero.

The typical vector in $\mathcal{G}_4$ has the form:

$$We_0 + Xe_1 + Ye_2 + Ze_3$$

This is taken to represent the point $(X/W, Y/W, Z/W)$ in Euclidean space, assuming that W is non-zero. If W is zero, then the vector represents a point at infinity. Thus every finite point has many representations; any (non-zero) scalar multiple of a vector represents the same point. If it is necessary to be specific, the coefficient of e_0 can be taken to be unity, by dividing throughout by W.

If p and q are two vectors in $\mathcal{G}_4$ then $l = p \wedge q$ is a bivector. This represents the line joining the two points represented by p and q. It has the property that $\bar{l}l$ is scalar. More generally any bivector l with this property represents a line. As with points, a scalar multiple of the bivector is regarded representing the same line. It is sometimes convenient, as with points, to normalise a line. For a line l, its *size* is denoted by $\|l\|$ and is defined by:

$$\|l\|^2 = \epsilon\bar{l}l = -\epsilon l^2$$

Note that the size of l does not relate to the length of the line; indeed the line has infinite length. However, if $l = p \wedge q$ and p and q are normalised points so that their coefficients of e_0 are both unity, then $\|l\|$ is the distance between p and q.

Similarly, elements of $\mathcal{G}_4$ of grade three represent planes. In particular, if p, q, r are three vectors, then their triproduct $P = [p, q, r]$ is defined and P represents the plane containing the points represented by the vectors.

In Cartesian geometry, lines and planes are represented by equations involving the coordinates of points. It is important to note that here lines and planes are elements of $\mathcal{G}_4$ in their own right. They are not themselves equations. In Cartesian geometry, the check is made for whether a point lies on a line by putting its coordinates into the appropriate equation. Similar tests are needed in the geometric algebra. These are provided by use of the inner and outer product and are summarised in the following result. For convenience, the distinction between the elements of $\mathcal{G}_4$ and the entities they represent is dropped.

Result 1.
For geometric elements of $\mathcal{G}_4$:

(i) the line joining (distinct) points p, q is $p \wedge q$;
(ii) the plane containing (non-collinear) points p, q, r is $[p, q, r]$;
(iii) the plane containing point p and line l (with p not lying on l) is $p \cdot l$;
(iv) point p lies on line l if and only if $p \cdot l = 0$;
(v) point p lies on plane P if and only if $p \wedge P = 0$;
(vi) line l lies in plane P if and only if $l \wedge P = 0$.

In dealing with intersections of entities, it is useful to denote the element e_{0123} by ω. This commutes with elements of even grade and anti-commutes with those of odd grade. Further, suppose that x and y are elements of $\mathcal{G}_4$. Then, if x has odd grade:

$$x \cdot (y\omega) = (x \wedge y)\,\omega$$
$$x \wedge (y\omega) = (x \cdot y)\,\omega$$

and if x has even grade:

$$x \cdot (y\omega) = (x \cdot y)\,\omega$$
$$x \wedge (y\omega) = (x \wedge y)\,\omega$$

With this notation, the following result deals with intersections geometric entities.

Result 2.
For geometric elements of $\mathcal{G}_4$:

(i) line l intersects plane P in the point $\epsilon(l \wedge P)\ \omega = \epsilon(l \wedge (P\omega))$;
(ii) two planes P, Q intersect in the line $\epsilon(P \wedge Q)\omega$;
(iii) three planes P, Q, R meet in the point $\epsilon[P, Q, R]$.

The inner product has analogies with the scalar product for Cartesian vectors. In the case of the inner product of two lines, the result carries information about the separation of the lines as well the angle between them.

Result 3.
For two lines l and m in $\mathcal{G}_4$, in the limit as ϵ tends to zero:

$$\epsilon(l \cdot m)\,\omega = \|l\|\ \|m\|\ [(\sin\alpha)h - (\cos\alpha)\omega]$$

where α is the angle between the two lines and h is the distance between them.

Example 1.
Consider the four points shown in Figure 1.

They are:

$$p_1 = e_0 + e_1 + e_2 + e_3$$
$$p_2 = e_0 + 4e_1 + e_2 + e_3$$
$$p_3 = e_0 + 3e_1 + 3e_2 + e_3$$
$$p_4 = e_0 + 2e_1 + 2e_2 + 3e_3$$

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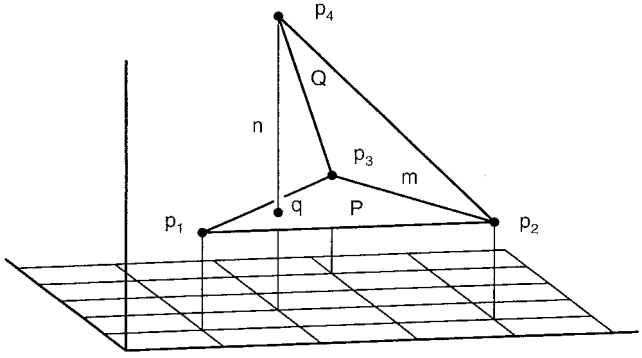


Fig. 1. Geometric example.

The line joining p_2 and p_3 is:

$$m = p_2 \wedge p_3 = -e_{01} + 2e_{02} + 9e_{12} + e_{13} - 2e_{23}$$

and the two planes shown are:

$$P = [p_1, p_2, p_3] = 6e_{012} + 6e_{123}$$
$$Q = [p_2, p_3, p_4] = 3e_{012} - 2e_{013} + 4e_{023} + 21e_{123}$$

Using Result 2(ii), the intersection of the two planes is the line:

$$\epsilon(P \wedge Q)\omega = -12e_{01} + 24e_{02} + 108e_{12} + 12e_{13} - 24e_{23}$$

which is 12 times m and so represents the same line.

The vertical line n through p_4 can be obtained as:

$$n = p_4 \wedge (p_4 + e_3) = e_{03} + 2e_{13} + 2e_{23}$$

By Result 2(i), this cuts plane P in the point:

$$q = \epsilon(n \wedge P)\omega = -6e_0 - 12e_1 - 12e_2 - 6e_3$$

which represents the point with Cartesian coordinates (2, 2,1).

Forming the various products and taking the limit gives:

$$\|m\| = \sqrt{5}$$
$$\|n\| = 1$$
$$\epsilon(m \cdot n)\omega = 3$$

and so, by Result 3, lines m and n are at right angles and are $(3/\sqrt{5})$ units apart.

4. RIGID BODY MOTIONS

The purpose of this section is to show that rigid body motions can be represented in a natural way as elements of even grade within the geometric algebra $\mathcal{G}_4$. Such motions are combinations of rotations and translations, and these are considered separately to begin with. The general idea is that if S is an element representing a motion then its action on the typical vector p of $\mathcal{G}_4$ is to map it to $p' = \bar{S}pS$. It follows that p' has odd grade and is equal to its own reverse; hence, in $\mathcal{G}_4$ at least, p' must also be a vector. Rotations are considered first. These can be represented in terms of quaternions[1] and the approach is essentially to cast those ideas into geometric algebra.

Consider the following element R of $\mathcal{G}_4$.

$$R = c + s_1e_{23} + s_2e_{31} + s_3e_{12}$$

where:

$$c^2 + s_1^2 + s_2^2 + s_3^2 = 1$$

Let b be the bivector:

$$b = q_1e_{23} + q_2e_{31} + q_3e_{12}$$

where:

$$s^2 = s_1^2 + s_2^2 + s_3^3$$

and

$$q_1 = s_1/s, \qquad q_2 = s_2/s, \qquad q_3 = s_3/s$$

then:

$$R = c + sb$$

By multiplication it is seen that:

$$\bar{b}b = q_1^2 + q_2^2 + q_3^2 = 1$$

so that b is a unit bivector and:

$$\bar{R}R = c^2 + s^2 = 1$$

Suppose that c and s are the cosine and sine of angle θ.

$$c = \cos\theta \qquad s = \sin\theta$$

The typical Euclidean point (x, y, z) is represented in $\mathcal{G}_4$ by the vector $p = e_0 + r$ where $r = xe_1 + ye_2 + ze_3$, and this can be written in the form:

$$r = \mathbf{e}^T\mathbf{r}$$

where $\mathbf{e} = [e_1\ e_2\ e_3]^T$ and $\mathbf{r} = [x\ y\ z]^T$, which is the position vector of the point represented by p.

Consider the product $\bar{R}pR$. Note that R is an element not involving e_0 and hence it commutes with e_0.

$$\bar{R}pR = e_0 + (c + s\bar{b})\,r\,(c + sb)$$
$$= e_0 + c^2r + cs(rb + \bar{b}r) + s^2(\bar{b}rb)$$

Multiplication shows that:

$$s(rb + \bar{b}r) = 2\mathbf{e}^T \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} \mathbf{r}$$

and

$$s^2\bar{b}rb = 2\mathbf{e}^T\mathbf{s}\mathbf{s}^T\mathbf{r} - s^2\mathbf{e}^T\mathbf{r}$$

where $\mathbf{s} = [s_1\ s_2\ s_3]^T$.

Hence:

$$\bar{R}pR = e_0 + \mathbf{e}^TA\mathbf{r}$$

where A is the 3×3 matrix:

$$A = c^2I + 2c \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} + 2 \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} [s_1\ s_2\ s_3] - s^2I$$

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If $\mathbf{q} = [q_1\ q_2\ q_3]^T$ is the unit column vector of the $q_i = s_i/s$, then:

$$A = (\cos 2\theta)I + (\sin 2\theta)\begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} + (1 - \cos 2\theta)\,\mathbf{q}\mathbf{q}^T$$

So the map that takes p to $\bar{R}pR$, takes the position vector $\mathbf{r}$ to $A\mathbf{r}$, and the matrix A is Rodrigues formula for a rotation through an angle 2θ about an axis through the origin in the direction of the vector $\mathbf{q}$. Thus the element R of $\mathcal{G}_4$, used as above, defines a rotation. Further, $\bar{R}R = 1$ and R has even grade. The axis of rotation is related to the bivector b; q is a point on the axis and $q = -e_{123}b$. As with complex numbers, the element R can be written in exponential form.

$$R = (\cos\theta) + b(\sin\theta) = \exp(b\theta)$$

To investigate translations, consider now the following element T of $\mathcal{G}_4$:

$$T = 1 + \epsilon e_0 t$$

where $t = t_1 e_1 + t_2 e_2 + t_3 e_3$

The product $\bar{T}T$ can be evaluated by multiplying it out and using the fact that $e_0 t = -te_0$.

$$\bar{T}T = 1 + \epsilon t^2$$

Here $t^2 = (t_1^2 + t_2^2 + t_3^2)$ is a scalar. Thus $\bar{T}T$ tends to unity as ϵ tends to zero.

The action of T in transforming the point $p = e_0 + r$ is now investigated, where $r = xe_1 + ye_2 + ze_3$. It maps it to the following:

$$\begin{aligned}\bar{T}pT &= (1 + \epsilon te_0)(e_0 + r)(1 + \epsilon e_0 t) \\ &= e_0 + r + 2t - \epsilon trt - \epsilon e_0(tt + rt + tr)\end{aligned}$$

Now, as ϵ tends to zero, it is seen that:

$$\bar{T}pT \rightarrow e_0 + r + 2t$$

Thus, in the limit, the transformation using T preserves the e_0 component and takes r to $r + 2t$. In this way, the transformation represents a translation through $2t$.

Thus rotations and translations have been represented by elements of even grade in $\mathcal{G}_4$. These elements only involve terms of grade 0 or 2. In $\mathcal{G}_4$, the only other possible even grade is 4 and this implies the presence of a component of e_{0123}.

Consider the following, more general, element of even grade in $\mathcal{G}_4$:

$$U = 1 + \epsilon e_0(t + \gamma e_{123})$$

where, as before, $t = t_1 e_1 + t_2 e_2 + t_3 e_3$ is a vector, and γ is a scalar. Thus the part in brackets is a typical element of odd grade in the subalgebra generated by e_1, e_2, e_3. The product $\bar{U}pU$ can be found for $p = e_0 + r$ with $r = xe_1 + ye_2 + ze_3$.

$$\begin{aligned}\bar{U}(e_0 + r)U = e_0 + r + 2t \\ - \epsilon[trt + \gamma e_{123}(tr + rt)] \\ - \epsilon e_0[\gamma^2 + \gamma^2 r + tt + rt + tr] \\ + 2\epsilon\gamma e_{0123}[r + t]\end{aligned}$$

Hence, in the limit as ϵ tends to zero, U has the same effect as T, and the presence of the grade 4 term has no effect. The product S of rotation and translation elements also has even grade. It has the property that the coefficients of components involving e_0 are multiples of ϵ. Also, in the limit, the product $\bar{S}S$ is unity. It can be shown that that any element S of $\mathcal{G}_4$ with these properties represents a motion which is a combination of a rotation and a translation.

The condition that $\bar{S}S = 1$ can also be dropped. This ensures that in the image $\bar{S}pS$ of $p = e_0 + r$ the coefficient of e_0 is again unity. If the condition does not hold, then that coefficient may take some other value. As noted previously, all scalar multiples of a vector represent the same point in Euclidean space, and the image vector can be normalised by dividing throughout by the coefficient of e_0. Doing this gives the previous image vector.

Example 2.
In two dimensions, a rigid body transform is made up of a rotation through 60° about the origin followed by a translation of two units in the x direction. Following the above approaches, the rotation element is:

$$R = c + se_{12}$$

where $c = \cos\left(\frac{\pi}{6}\right)$ and $s = \sin\left(\frac{\pi}{6}\right)$, and the translation element is:

$$T = 1 + \epsilon e_{01}$$

The combined transform corresponds to the product of these.

$$S = RT = c + \epsilon ce_{01} - \epsilon se_{02} + se_{12}$$

If $x = e_0 + e_1$ and $y = e_0 + e_2$ are points on the x- and y-axes at unit distance from the origin, then the images of these under the transform are the following:

$$\begin{aligned}\bar{S}xS = (1 - 3\epsilon c^2 + \epsilon s^2)e_0 + (1 + 2c^2 - \epsilon c^2 + \epsilon s^2)e_1 \\ + 2(1 + \epsilon)cse_2\end{aligned}$$

$$\begin{aligned}\bar{S}yS = (1 - \epsilon c^2 + 4\epsilon cs - \epsilon s^2)e_0 + 2(1 - cs + \epsilon cs)e_1 \\ + (1 + \epsilon)(c^2 - s^2)e_2\end{aligned}$$

In the limit, these become:

$$\bar{S}xS = e_0 + (1 + 2c^2)e_1 + 2cse_2 = e_0 + 2.5e_1 + 0.866e_2$$

$$\bar{S}yS = e_0 + 2(1 - cs)e_1 + (c^2 - s^2)e_2 = e_0 + 1.134e_1 + 0.5e_2$$

and it is straightforward to check that these are correct images of the given points.

Example 3.
This example is to create a smooth motion between two given rigid body positions in space. The first position has Cartesian coordinates (5, 0, 0). The second is at (−2, 8, 6) and the object has been rotated through −90° about the z-

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axis and 60° about the radial axis. The corresponding transform elements in $\mathcal{G}_4$ are as follows:

$$S_1 = 1 + 2.5\epsilon e_{01}$$

$$S_2 = 0.612 + 2.639\epsilon e_{01} + 1.742\epsilon e_{02} + 3.864\epsilon e_{03} + 0.354e_{12} + 0.612e_{13} - 0.354e_{23} - 1.035\epsilon e_{0123}$$

The following linear combination of these:

$$S = (1 - t)S_1 + tS_2$$

with $0 \le t \le 1$, is another element of $\mathcal{G}_4$ of the form to represent a rigid body motion. Figure 2 shows the given positions (in bold) and intermediate positions generated by this interpolation for equally spaced values of the parameter t.

5. APPLICATIONS

Three application examples are considered in Selig's paper.[9] The aim here is to consider those same examples, now dealing with them in the version of the geometric algebra discussed earlier.

5.1. Visibility

The first example considers the view of two skew lines l_1 and l_2. These are viewed on an image plane P from a centre of projection at point p_0 (Figure 3). On this plane the images of the lines cross (unless they are parallel) and the aim is to find the intersection point q. The various constructions can be carried out in $\mathcal{G}_4$ as follows. Let Q_1 be the plane that contains l_1 and p_0. Similarly let Q_2 be the plane containing l_2 and p_0. Then, by Result 1(iii):

$$Q_1 = p_0 \cdot l_1$$
$$Q_2 = p_0 \cdot l_2$$

These planes intersect in the line m joining p_0 and q and Result 2(ii) gives:

$$m = \epsilon(Q_1 \wedge Q_2)\omega$$

The point q is where m meets P. So, from Result 2(i):

$$q = \epsilon(m \wedge P)\omega = -\epsilon(Q_1 \wedge Q_2) \cdot P$$

since $\omega^2 = \epsilon^{-1}$.

The viewing ray for the intersection q meets l_1 at the point p_1 where l_1 intersects Q_2. So, again from Result 2(i):

$$p_1 = \epsilon(l_1 \wedge Q_2)\omega$$

Particular values for the geometry are now introduced.[9] The centre of projection is placed at the origin so that:

$$p_0 = e_0$$

The image plane P is a plane parallel to the xy-plane and distance f from it. So P contains the points $e_0 + fe_3$, $e_0 + fe_3 + e_1$ and $e_0 + fe_3 + e_2$ and so P is the triproduct of these as in Result 1 (ii).

$$P = [(e_0 + fe_3), (e_0 + fe_3 + e_1), (e_0 + fe_3 + e_2)]$$
$$= e_{012} + fe_{123}$$

General forms are taken for l_1 and l_2.

$$l_1 = v_{x1}e_{01} + v_{y1}e_{02} + v_{z1}e_{03} + u_{x1}e_{23} + u_{y1}e_{31} + u_{x1}e_{12}$$

$$l_2 = v_{x2}e_{01} + v_{y2}e_{02} + v_{z2}e_{03} + u_{x2}e_{23} + u_{y2}e_{31} + u_{x2}e_{12}$$

and it follows that:

$$Q_1 = p_0 \cdot l_1 = e_0 \cdot l_1 = u_{x1}e_{023} - u_{y1}e_{013} + u_{z1}e_{102}$$

$$Q_2 = p_0 \cdot l_2 = e_0 \cdot l_2 = u_{x2}e_{203} - u_{y2}e_{013} + u_{z2}e_{102}$$

The outer product of these is:

$$Q_1 \wedge Q_2 = -\epsilon^{-1}(u_{y1}u_{z2} - u_{z1}u_{y2})e_{23} + \epsilon^{-1}(u_{z1}u_{x2} - u_{x1}u_{z2})e_{13} - \epsilon^{-1}(u_{x1}u_{y2} - u_{y1}u_{x2})e_{12}$$

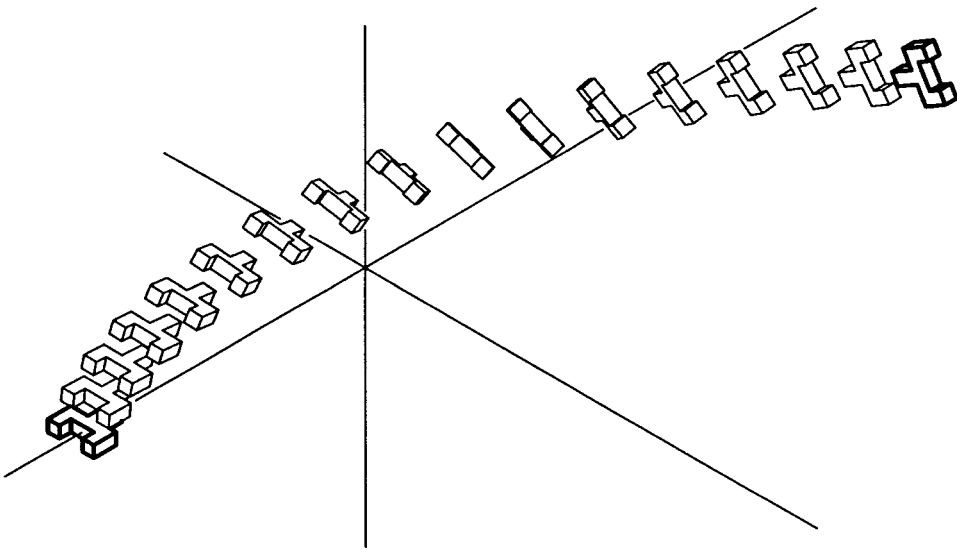


Fig. 2. Linear motion between two given positions.

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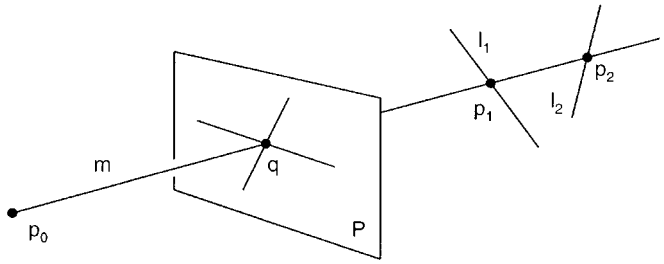


Fig. 3. Apparent intersection of two skew lines.

and so:

$$\begin{aligned} m &= \epsilon(Q_1 \wedge Q_2)\omega \\ &= (u_{y1}u_{z2} - u_{z1}u_{y2})e_{01} + (u_{z1}u_{x2} - u_{x1}u_{z2})e_{02} \\ &+ (u_{x1}u_{y2} - u_{y1}u_{x2})e_{03} \end{aligned}$$

The following product allows q to be found:

$$\begin{aligned} m \wedge P = &-\epsilon^{-1}(u_{x1}u_{y2} - u_{y1}u_{x2})e_{123} \\ &+ f(u_{y1}u_{z2} - u_{z1}u_{y2})e_{023} \\ &- f(u_{z1}u_{x2} - u_{x1}u_{z2})e_{013} \\ &+ f(u_{x1}u_{y2} - u_{y1}u_{x2})e_{012} \end{aligned}$$

and so q is given by:

$$\begin{aligned} q &= \epsilon(m \wedge P)\omega \\ &= -(u_{x1}u_{y2} - u_{y1}u_{x2})e_0 \\ &\quad - f(u_{y1}u_{z2} - u_{z1}u_{y2})e_1 - f(u_{z1}u_{x2} - u_{x1}u_{z2})e_2 \\ &\quad - f(u_{x1}u_{y2} - u_{y1}u_{x2})e_3 \end{aligned}$$

Now q is the point on the image plane where the lines appear to intersect. The Cartesian coordinates in Euclidean space can be found by dividing the coefficients of e_1, e_2, e_3 by the coefficient of e_0. The results are as follows.

$$q_x = f\frac{(u_{y1}u_{z2} - u_{z1}u_{y2})}{(u_{x1}u_{y2} - u_{y1}u_{x2})}$$

$$q_y = f\frac{(u_{z1}u_{x2} - u_{x1}u_{z2})}{(u_{x1}u_{y2} - u_{y1}u_{x2})}$$

$$q_z = f\frac{(u_{x1}u_{y2} - u_{y1}u_{x2})}{(u_{x1}u_{y2} - u_{y1}u_{x2})} = f$$

and these agree with previous formulae.[9]

It is now investigated whether the intersection point is indeed visible. This means checking whether p_1 and p_2 lie on the opposite side of the plane P from the centre of projection p_0. The two points can be checked separately and the following sets up the test for p_1. Since:

$$\begin{aligned} l_1 \wedge Q_2 = &-\epsilon^{-1}v_{x1}u_{x2}e_{123} - \epsilon^{-1}v_{y1}u_{y2}e_{123} - \epsilon^{-1}v_{z1}u_{z2}e_{123} \\ &- u_{x1}u_{y2}e_{012} - u_{x1}u_{z2}e_{013} \\ &+ u_{y1}u_{x2}e_{012} - u_{y1}u_{z2}e_{023} \\ &+ u_{z1}u_{x2}e_{013} - u_{z1}u_{y2}e_{023} \end{aligned}$$

$$\begin{aligned} = &-\epsilon^{-1}(v_{x1}u_{x2} + v_{y1}u_{y2} + v_{z1}u_{z2})e_{123} \\ &- (u_{x1}u_{y2} - u_{y1}u_{x2})e_{012} \\ &- (u_{x1}u_{z2} - u_{z1}u_{x2})e_{013} \\ &- (u_{y1}u_{z2} - u_{z1}u_{y2})e_{023} \end{aligned}$$

point p_1 is determined from Result 2(i) as follows:

$$\begin{aligned} p_1 &= \epsilon(l_1 \wedge Q_2)\omega \\ &= -(v_{x1}u_{x2} + v_{y1}u_{y2} + v_{z1}u_{z2})e_0 \\ &\quad + (u_{y1}u_{z2} - u_{z1}u_{y2})e_1 - (u_{x1}u_{z2} - u_{z1}u_{x2})e_2 \\ &\quad + (u_{x1}u_{y2} - u_{y1}u_{x2})e_3 \end{aligned}$$

Let:

$$\alpha = -(v_{x1}u_{x2} + v_{y1}u_{y2} + v_{z1}u_{z2})$$

be the coefficient of e_0 in p_1, and normalise p_1 by redefining it to be:

$$\begin{aligned} p_1 = e_0 &+ \frac{(u_{y1}u_{z2} - u_{z1}u_{y2})}{\alpha}e_1 - \frac{(u_{x1}u_{z2} - u_{z1}u_{x2})}{\alpha}e_2 \\ &+ \frac{(u_{x1}u_{y2} - u_{y1}u_{x2})}{\alpha}e_3 \end{aligned}$$

Then:

$$\begin{aligned} P \wedge p_1 &= (e_{012} + fe_{123}) \wedge p_1 \\ &= \left[\frac{(u_{x1}u_{y2} - u_{y1}u_{x2})}{\alpha} - f\right]\omega \end{aligned}$$

whereas

$$P \wedge e_0 = (e_{012} + fe_{123}) \wedge e_0 = -f\omega$$

These last two expressions represent volumes of tetrahedra whose vertices are the three points in the plane P together with p_1 and e_0 respectively. So for p_1 to lie on the other side of P to the origin (e_0), the signs need to be opposite. Assuming that $f > 0$, the condition is that:

$$\frac{(u_{x1}u_{y2} - u_{y1}u_{x2})}{\alpha} - f > 0$$

or, after rearrangement and adjusting signs,

$$f < \frac{(u_{y1}u_{x2} - u_{x1}u_{y2})}{(v_{x1}u_{x2} + v_{y1}u_{y2} + v_{z1}u_{z2})}$$

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This essentially agrees with Selig's result.[9] (In that paper, the denominator has been assumed to be positive and has been multiplied across.)

As previously noted, the above condition relates to p_1, the intersection of l_1 with Q_2. There is a second condition relating to p_2, the intersection of l_2 with Q_1. This condition can be obtained from the one above by interchanging the the subscripts 1 and 2. This gives:

$$f < \frac{(u_{y2}u_{x1} - u_{x2}u_{y1})}{(v_{x2}u_{x1} + v_{y2}u_{y1} + v_{z2}u_{z1})}$$

Example 4.
The following gives a particular example of the above approach. Consider the following two lines.

$$\begin{aligned} l_1 &= (e_0 + e_1 + \lambda e_3) \wedge (e_0 + 2e_1 + \lambda e_3) = e_{01} - \lambda e_{13} \\ l_2 &= (e_0 + e_1 + \mu e_3) \wedge (e_0 + e_1 + e_2 + \mu e_3) \\ &= e_{02} + e_{12} - \mu e_{23} \end{aligned}$$

These lines are chosen to lie in planes parallel to the image plane. Line l_1 1ies in the plane $z = \lambda$, and l_2 in the plane $z = \mu$.

In the above notation,

$$\begin{aligned} Q_1 &= -\lambda e_{013} \\ Q_2 &= e_{012} - \mu e_{023} \\ Q_1 \wedge Q_2 &= \epsilon^{-1}(-\lambda e_{23} - \lambda\mu e_{12}) \\ (Q_1 \wedge Q_2) \cdot P &= \epsilon^{-1}(\lambda f e_1 + \lambda\mu e_0 + \lambda\mu f e_3) \end{aligned}$$

where, as before, $P = e_{012} + f e_{123}$.

Hence the apparent point of intersection q, after normalisation, is:

$$q = e_0 + \left(\frac{f}{\mu}\right) e_1 + f e_3$$

and this lies in the plane $z = f$ as expected, and the intersection is at $x = (f/\mu)$, $y = 0$ in that plane. (This does not depend upon λ since the line l_1 1ooks like the line $y = 0$ in the image plane.) In particular, if l_2 lies in the image plane, so that $\mu = f$, then the intersection is at $x = 1$, again as would be expected.

Now consider the visibility. Using the previous notation for the coefficients of the two lines, the values for this example are as follows:

$$\begin{array}{llll} v_{x1} = 1 & u_{x1} = 0 & v_{x2} = 0 & u_{x2} = -\mu \\ v_{y1} = 0 & u_{y1} = \lambda & v_{y2} = 1 & u_{y2} = 0 \\ v_{z1} = 0 & u_{z1} = 0 & v_{z2} = 0 & u_{z2} = 1 \end{array}$$

So:

$$\begin{aligned} u_{y1}u_{x2} - u_{x1}u_{y2} &= -\lambda\mu \\ v_{x1}u_{x2} + v_{y1}u_{y2} + v_{z1}u_{z2} &= -\mu \end{aligned}$$

and so the first inequality condition above yields:

$$f < \lambda$$

For the second inequality,

$$\begin{aligned} u_{y2}u_{x1} - u_{x2}u_{y1} &= \lambda\mu \\ v_{x2}u_{x1} + v_{y2}u_{y1} + v_{z2}u_{z1} &= \lambda \end{aligned}$$

leading to:

$$f < \mu$$

As expected, these two inequalities say that the image plane needs to be closer to the origin than the planes of the lines.

5.2. Assembly

The underlying problem[10] is one of inserting a pair of pegs into two holes. Since each peg and hole determines a line in Euclidean space, the abstracted problem is as follows: It is to find a rigid body motion to transform two given lines l_1 and l_2 into two other given lines m_1 and m_2. If the transform element of $\mathcal{G}_4$ is S, then it is required that:

$$\begin{aligned} \bar{S} l_1 S &= m_1 \\ \bar{S} l_2 S &= m_2 \end{aligned}$$

Since, in the limit, $\bar{S}S = 1$, these can be arranged as follows:

$$\begin{aligned} l_1 S - S m_1 &= 0 \\ l_2 S - S m_2 &= 0 \end{aligned}$$

which allow these conditions to be regarded as a number of simultaneous equations involving the components of S. There are redundancies involved. One particular one is investigated as follows. From the first pair of equations,

$$\begin{aligned} m_1 \cdot m_2 &= \frac{1}{2}(m_1 m_2 + m_2 m_1) \\ &= \frac{1}{2}(\bar{S} l_1 S \bar{S} l_2 S + \bar{S} l_2 S \bar{S} l_1 S) \\ &= \frac{1}{2}\bar{S}(l_1 l_2 + l_2 l_1) S \\ &= \bar{S}(l_1 \cdot l_2) S \\ &= \bar{S}S(l_1 \cdot l_2) \\ &= l_1 \cdot l_2 \end{aligned}$$

using here the fact $(l_1 \cdot l_2)$, being a linear combination of 1 and e_{0123}, commutes with all even grade elements. Similarly, taking inner products of elements with themselves, it follows that:

$$m_1 \cdot m_1 = l_1 \cdot l_1 \qquad m_2 \cdot m_2 = l_2 \cdot l_2$$

Now Result 3 applies showing that:

$$\|m_1\| = \|l_1\| \qquad \|m_2\| = \|l_2\|$$

and the distance and angle between m_1 and m_2 are the same as those between l_1 and l_2. These are necessary conditions for the transform S to exist.

The next part of this example is to investigate whether more than one transform can take the one pair of lines onto

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the other. Before looking at this, consider the problem of transforming a single line l to another m. If a point is chosen on each line, then it is a simple matter to achieve a translation to map one point onto the other. So this leaves the need for a rotation (about the common point) to take one line onto the other. Further, because it is also straightforward to translate the common point to and from the origin, it is assumed that the two lines intersect at the origin (e_0). So take

$$l = e_0 \wedge b \qquad m = e_0 \wedge c$$

Further assume that:

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3$$
$$c = c_1 e_1 + c_2 e_2 + c_3 e_3$$

since any e_0 component disappears in the outer product with e_0. Then b and c anticommute with e_0, and:

$$l = e_0 b \qquad m = e_0 c$$

Assume also that b and c have the same length so that

$$b^2 = b_1^2 + b_2^2 + b_3^2 = c^2 = c_1^2 + c_2^2 + c_3^2 = \lambda^2$$

Now transforming l by b has no effect since multiplication shows that:

$$\bar{b} l b = -\lambda^2 l$$

Set

$$v = b + c$$

and then multiplying out gives

$$\bar{v} l v = -\mu^2 m$$

where μ is the scalar $(b + c)^2$. The following element

$$S = \frac{1}{\lambda\mu} b (b + c)$$

is an element of even grade and

$$\bar{S} l S = m$$

So it represents a transform taking l to m. It is clear that S does not involve e_0 and hence S is a rotation.

Returning to the main example, consider under what conditions more than one transform can map the first pair of given lines onto the second pair. If S_1 and S_2 are two such, then $S = S_1 \overline{S_2}$ fixes both l_1 and l_2.

$$\bar{S} l_1 S = l_1 \qquad \bar{S} l_2 S = l_2$$

Without loss of generality, it can be assumed that l_1 is the z-axis and that $l_1 = e_{03}$. So it is being assumed

$$e_{03} S = S e_{03} \qquad l_2 S = S l_2$$

Thus S commutes with e_{03} and l_2. Examination of the multiplication table for $\mathcal{G}_4$ shows that the only basis elements commuting with e_{03} are: 1, e_{03}, e_{12} and ω. (These are also the only basis elements commuting with e_{12}.) Hence S is a linear combination of these.

If S does not contain either e_{03} or e_{12} then it has the form $\alpha + \beta\omega$ where α and β are scalars. This commutes with all even grade elements and hence with all lines and so corresponds to the identity transform on lines.

If S does contain a component of e_{03} or e_{12}, then the same argument says that l_2 is a linear combination of 1, e_{03}, e_{12}, ω. As l_2 is a bivector, it must have the form $l_2 = \alpha e_{03} + \beta e_{12}$. As $\overline{l_2} l_2$ is scalar, either α or β is zero. If $\alpha = 0$, then l_2 is a line at infinity. If $\beta = 0$, then l_2 is another line along the z-axis. Hence, if S represents a non-trivial transform which fixes both l_1 and l_2, then l_2 is collinear with l_1. Again this agrees with the previous conclusion.[9]

5.3. Stereo vision

The final example is concerned with stereo views of an object. Here two cameras are used to form images of the same scene. The situation is shown in Figure 4. Point p_1 is the right "eye point" or focus looking at an object. The line shown from p_1 is towards a point on the object. Let p_3 be the point where this line cuts the right image plane, Q_R. Then p_3 is the point on the image plane that represents the viewed point. However from the view, it is only known that the object point lies somewhere on the view line joining p_1 and p_3.

The left view can be regarded as looking at this line. Its image m on the left image plane Q_L, is shown together with the left focus denoted as p_2. The image of the line in the left image plane is called the *epipolar line*.

An expression for m can be found as follows. The three points p_1, p_2, p_3 form a plane Q, say, given, as in Result 1(ii), by:

$$Q = [p_1, p_2, p_3]$$

Where this plane cuts the left image plane Q_L is the epipolar line. So, from Result 2(ii),

$$m = \epsilon(Q_L \wedge Q)\omega = \epsilon(Q_L \wedge [p_1, p_2, p_3])\omega$$

A particular case is considered[9] in which the two camera systems are identical. The optical axis of a camera is the line from the focus, normal to the image plane. The optical axes are taken to be coplanar. They meet at a point distance d from both foci and they are separated by an angle θ as shown in Figure 5.

Interest is in finding the epipolar line in the left image plane, but basing this upon the coordinates of the image point of p_3. There is a need to move between coordinate systems for the two images planes. For convenience it is supposed that focus point p_2 is at the origin:

$$p_2 = e_0$$

and its optical axis lies along the z-axis. Figure 5 shows the direction of the x- and z-axes; the y-axis is out of the plane of the figure. (Note that, compared to reference [9] the z-axis has been reversed to obtain right-handed sets of axes.)

Firstly, find the transform S that maps from the position of the left camera to that of the right. This is a combination

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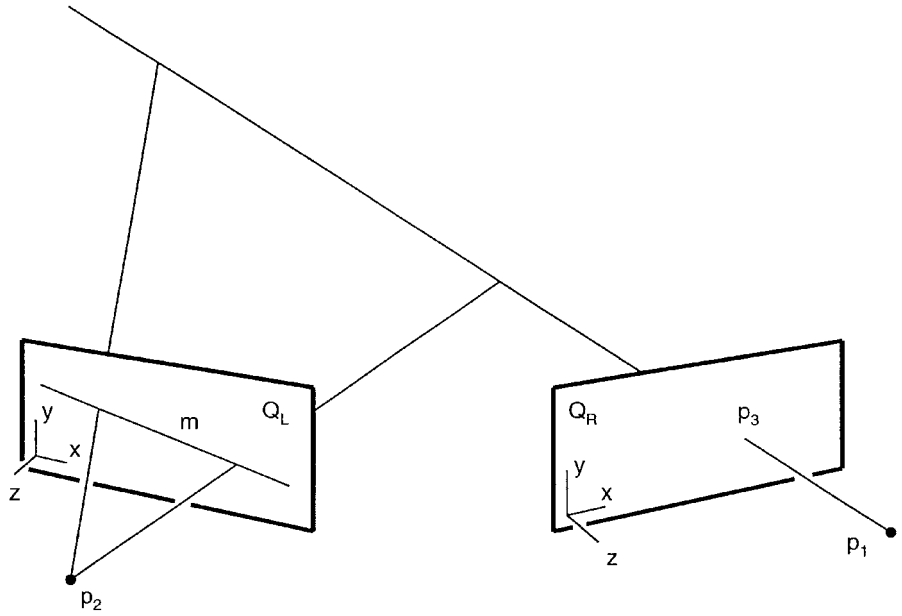


Fig. 4. Epipolar line.

of a translation through $-d$ in the z-direction, a rotation of θ about the y-axis and a translation of $+d$ in the new z-direction.

$$S = (1 + \frac{1}{2}\epsilon d e_{03})((\cos\frac{1}{2}\theta) + (\sin\frac{1}{2}\theta)e_{31})(1 - \frac{1}{2}\epsilon d e_{03})$$

$$= (\cos\frac{1}{2}\theta) - (\sin\frac{1}{2}\theta)e_{13} + \epsilon d(\sin\frac{1}{2}\theta)e_{01}$$

where, in the last line, terms in epsilon are ignored except those which also involve e_0. Then,

$$p_1 = \bar{S}p_2S$$

$$= e_0 + (d\sin\theta)e_1 - d(1 - \cos\theta)e_3$$

where, at the end, ϵ has been allowed to tend to zero.

Suppose now that the image point p_3 in the right image plane has the form

$$p_3 = \bar{S}(e_0 + Xe_1 + Ye_2 + Ze_3)S$$

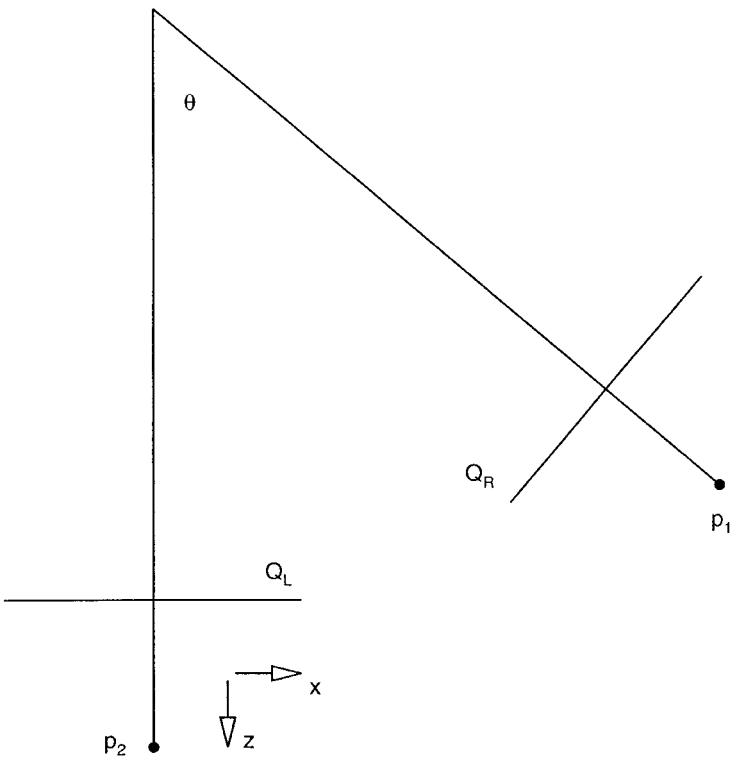


Fig. 5. Example of a stereo configuration.

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where X, Y, Z are the coordinates relative to the frame of reference of the right image. Due to the coordinate system used, Z is $-f$, where f is the distance from the focus to the image plane. Image point p_3 is found by multiplying out the above expression.

$$p_3 = e_0 + [X\cos\theta + (Z+d)\sin\theta]e_1 + Ye_2 - [X\sin\theta - Z\cos\theta + d(1-\cos\theta)]e_3$$

where, at the end, ϵ has been allowed to go to zero.

The plane Q containing p_1, p_2, p_3 is now found.

$$\begin{aligned} Q &= (p_1 \wedge p_2)\cdot p_3 \\ &= -d[Y(1-\cos\theta)e_{023} - (X(1-\cos\theta) - Z\sin\theta)e_{013} \\ &\quad + Y(\sin\theta)e_{012}] \end{aligned}$$

The following three points lie on Q_L,

$$(e_0 + Ze_3) \qquad (e_0 + e_1 + Ze_3) \qquad (e_0 + e_2 + Ze_3)$$

and so Q_L is the triproduct of these, giving

$$Q_L = e_{012} + Ze_{123}$$

Now form the outer product with Q.

$$\begin{aligned} Q_L \wedge Q = -d[&\epsilon^{-1}Y(1-\cos\theta)e_{13} + \epsilon^{-1}(X(1-\cos\theta) \\ &- Z\sin\theta)e_{23} + ZY(1-\cos\theta)e_{01} \\ &+ Z(X(1-\cos\theta) - Z\sin\theta)e_{02} + ZY\sin\theta e_{03}] \end{aligned}$$

and then the epipolar line

$$\begin{aligned} m &= \epsilon(Q_L \wedge Q)\omega \\ &= d[(X(1-\cos\theta) - Z\sin\theta)e_{01} - Y(1-\cos\theta)e_{02} \\ &\quad + ZY(1-\cos\theta)e_{23} - Z(X(1-\cos\theta) - Z\sin\theta)e_{13} \\ &\quad + ZY\sin\theta e_{12}] \end{aligned}$$

Since multiplicative factors do not change the line, the factor of d can be ignored. Hence m can be taken to be:

$$m = Be_{01} - Ae_{02} + AZe_{23} - BZe_{13} + CZe_{12}$$

where

$$\begin{aligned} A &= Y(1-\cos\theta) \\ B &= X(1-\cos\theta) - Z\sin\theta \\ C &= Y\sin\theta \end{aligned}$$

To find the equation of the line, suppose that q is a typical point.

$$q = e_0 + xe_1 + ye_2 + ze_3$$

From Result 1 (iv), it lies on m if

$$\begin{aligned} 0 &= q\cdot m \\ &= (CZ + Ax + By)e_{012} + (-BZ + Bz)e_{013} \\ &\quad + (AZ - Az)e_{023} + (AZx + BZy + CZz)e_{123} \end{aligned}$$

and equating coefficients to zero gives

$$\begin{aligned} Ax + By &= -CZ \\ z &= Z \end{aligned}$$

This is the equation for the epipolar line in the left image plane.[9]

6. CONCLUSIONS

Clifford or geometric algebra is a means for dealing with representation of geometry. It has been seen that it is possible to represent three dimensional geometry by working with a geometric algebra with four basis vectors. One of these is chosen to have a square which is (infinitely) large. This allows a projective representation of three dimensional Euclidean space. In this, there is the natural correspondence of vectors in the algebra and points, bivectors and lines, and trivectors (elements of grade 3) and planes.

Geometric entities are represented by elements in the algebra rather then equations between coordinates. The inner and outer products can be used to provide tests for whether one entity lies within another and to obtain the results of intersections between entities.

It has been seen that any rigid body transform can be represented by an even grade element S in the algebra. The map that sends a vector p to $\bar{S}pS$ corresponds to the transform. In this way, the element S represents a position for a rigid body in space. A smooth motion between such positions can then be achieved by interpolating between the elements of the algebra. The use of the approach to other geometric applications has also been illustrated.

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