J. Appl. Prob. **55,** 862–886 (2018)
*doi:*10.1017/jpr.2018.55
© *Applied Probability Trust* 2018

STOCHASTIC LU FACTORIZATIONS, DARBOUX TRANSFORMATIONS AND URN MODELS

F. ALBERTO GRÜNBAUM,* *University of California, Berkeley*
MANUEL D. DE LA IGLESIA,** *Universidad Nacional Autónoma de México*

Abstract

We consider upper–lower (UL) (and lower–upper (LU)) factorizations of the one-step transition probability matrix of a random walk with the state space of nonnegative integers, with the condition that both upper and lower triangular matrices in the factorization are also stochastic matrices. We provide conditions on the free parameter of the UL factorization in terms of certain continued fractions such that this stochastic factorization is possible. By inverting the order of the factors (also known as a Darboux transformation) we obtain a new family of random walks where it is possible to state the spectral measures in terms of a Geronimus transformation. We repeat this for the LU factorization but without a free parameter. Finally, we apply our results in two examples; the random walk with constant transition probabilities, and the random walk generated by the Jacobi orthogonal polynomials. In both situations we obtain urn models associated with all the random walks in question.

Keywords: Darboux transformation; LU factorization; orthogonal polynomial; random walk; urn model

2010 Mathematics Subject Classification: Primary 60J10
Secondary 33C45; 42C05

1. Introduction

Our main motivation in this paper is to find simple ways to describe certain random walks on the nonnegative integers in terms of urn models. The starting point is Grünbaum [8] in which an urn model associated with Jacobi orthogonal polynomials was investigated. This was a rather contrived model compared to more familiar ones such as those of Ehrenfest and Bernoulli–Laplace, and it raised the question as to whether there was an alternative way of setting a less elaborate urn model.

Our approach in this paper is to divide the urn model associated with the random walk into two different and simpler urn experiments, and combine them to obtain a simpler description of the original urn model. To that end, we propose LU factorizations of the one-step transition probability matrix P of a fixed random walk (which is a tridiagonal matrix; see (2.1)). By LU factorizations we mean either an upper–lower (UL) or a lower–upper (LU) factorization. We discuss below the differences between them. For the UL factorization we aim to factorize P in the form $P = P_{\mathrm{U}}P_{\mathrm{L}}$. Here P_{L} and P_{U} are lower and upper triangular matrices (bidiagonal), respectively, and we want them to be *stochastic*. It is well known that this factorization is not unique and comes with a *free parameter*. The factorization, if it can be achieved in terms of

Received 1 August 2017; revision received 3 August 2018.
* Postal address: Department of Mathematics, University of California, Berkeley, CA 94720, USA.
** Postal address: Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510, Ciudad de México, México. Email address: mdi29@im.unam.mx

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

stochastic factors, will represent a family of factorizations of the original transition matrix P. From a probabilistic point of view, the meaning of this factorization is that of the composition of a family of two random walks (one pure birth and another pure death), first the one associated with P_{U} followed by the one associated with P_{L}. We study the conditions on the free parameter mentioned above under which this stochastic UL factorization is possible and relate it to the theory of continued fractions. We will see that this type of stochastic factorization is not always possible for certain random walks. For instance, the symmetric random walk with constant transition probabilities is one such example. We can do the same if we consider the LU factorization but now the difference is that the stochastic factorization (if possible) is unique and does not come with a free parameter, as was the case with the UL factorization. The stochastic LU factorization is now possible under certain boundary conditions on one of the transition probabilities of the random walk, also related with a continued fraction. This will be the subject of Section 2. We mention that our factorization is in the same spirit as that exhibited in a much more elaborate problem involving matrix-valued orthogonal polynomials; see [9]. For more on the use of this kind of orthogonal polynomials in the study of stochastic processes, we refer the reader to [3], [7], and [12].

UL and LU factorizations of stochastic matrices have been addressed by a number of authors; see, for example, [5], [6], [13], and [19]. The importance of these factorizations is that they lead to interesting methods to compute invariant measures and more generally to study structured Markov chains such as those appearing in queuing systems; see [19]. Heyman [13] obtained a factorization as $I - P = (A - I)(B - S)$, where P is a stochastic matrix, A is strictly upper triangular, B is strictly lower triangular, and S is diagonal. The entries of A (expected values) and the entries of B and S (probabilities) are related with certain Markov chains that are also known as censored Markov chains. Vigon [19] also considered UL and LU factorizations of the form $I - P = (I - L)(I - K)$, where L is upper triangular and K is lower triangular. This means that $P = L + K - LK$. The author explored the relation between this factorization and the Wiener–Hopf factorization for Markov chains, the latter as a particular case of the LU factorizations up to Fourier transforms; see [19] and the references therein for more details. These factorizations are different from those under consideration here, which are of the form $P = P_{\mathrm{U}} P_{\mathrm{L}}$, where the three matrices involved are stochastic. Our approach is to understand how complicated random walks may work in terms of urn models by splitting them into two simpler random walks.

One of the main advantages of considering a factorization of the form $P = P_{\mathrm{U}} P_{\mathrm{L}}$ for tridiagonal stochastic matrices P is that we can make use of the so-called *discrete Darboux transformation,* which consists of inverting the order of the factors. The new matrix $\widetilde{P} = P_{\mathrm{L}} P_{\mathrm{U}}$ will also be tridiagonal and stochastic. Since the factorization comes with a free parameter, we have a family of new random walks different in general from the original family. These Darboux transformations have received attention in the context of the theory of orthogonal polynomials (see, for example, [10], [11], [18], [21], and [22]), in particular in the description of some families of Krall polynomials. The Darboux transformation has played an important role in the study of integrable systems; see [17]. An important property of the transformation is that one knows how to relate the spectral measure associated to P with the spectral measure associated to $\widetilde{P}$ using the so-called *Geronimus transformation*. For the LU factorization, these measures are related through the *Christoffel transformation*; see Section 3. A more general way of performing a discrete Darboux transformation consists of introducing a new parameter $\lambda \in \mathbb{C}$ such that we obtain a factorization of the form $P = AB + \lambda I$, where A is upper triangular and B is lower triangular (or $P = BA + \lambda I$ for the LU factorization). The case we study here is $\lambda = 0$,

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

and A and B are stochastic matrices. Therefore, not only do we provide a new approach to better understand complicated random walks, but with the Darboux transformation we have a new way of constructing infinite different families of random walks and their interpretations in terms of urn models.

Once we have the spectral measure, it is easy to analyze the corresponding random walk in terms of the orthogonal polynomials associated with that measure. This was first addressed in a series of papers by Karlin and McGregor (inspired by the work of Feller and McKean) in the 1950s (see [14]–[16]) where the case of continuous-time birth-and-death processes were first studied and then the case of discrete-time random walks. After that many other authors have been working in this connection, for example, Dette, van Doorn, Ismail, Letessier, and Valent. With knowledge of the spectral measure and the corresponding orthogonal polynomials, it is possible to provide an explicit integral representation of the entries of the n-step transition probability matrix of the random walk (Karlin–McGregor formula) and to study the corresponding invariant measure vector (or distribution if the random walk is ergodic) in terms of the inverse of the norms of the orthogonal polynomials; see Section 3 for more details. It is also possible to study some other probabilistic properties using spectral methods; for example, recurrence, absorbing times, first return times, or limit theorems.

We apply our results to two examples in Sections 4 and 5. The first is the random walk with constant transition probabilities and the second is the random walk generated by the Jacobi orthogonal polynomials. In both cases we show how to choose the free parameter of the UL factorization (or conditions for the LU factorization) such that we obtain a family of stochastic factorizations. In some cases we compute explicitly the coefficients of both stochastic factors. We also state the spectral measure associated with the random walks and explore the corresponding Darboux transformation, as well as other probabilistic properties. Finally, we provide urn models for some particular cases of both examples, exploring the probabilistic meaning of the free parameter in the UL factorization. These urn models can be seen as an application of some rather sophisticated ideas originating in group representation theory in [12] to a much more classical setup. These simplified urn models arising from stochastic factorizations are among the main results in this paper.

2. Stochastic LU and UL factorizations

Let P be the transition probability matrix of an irreducible random walk $\{X_t : t = 0, 1, \ldots\}$ with the space state of nonnegative integers, i.e.

$$P = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \\ c_1 & b_1 & a_1 & 0 & \\ 0 & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.1}$$

As usual, since P is stochastic, all entries are nonnegative and

$$b_0 + a_0 = 1, \qquad c_n + b_n + a_n = 1, \quad n \geq 1.$$

The irreducibility conditions force us to take $0 < a_n,\ c_{n+1} < 1,\ n \geq 0$. In Figure 1 we present a representation of the transitions between the states.

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

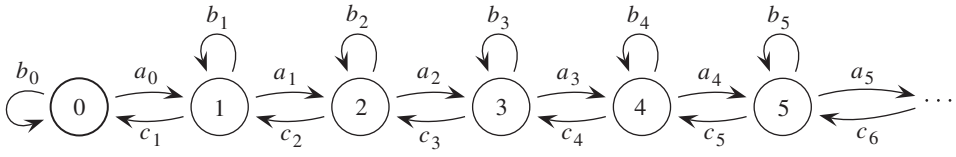


FIGURE 1: Transitions between states for the stochastic matrix P.

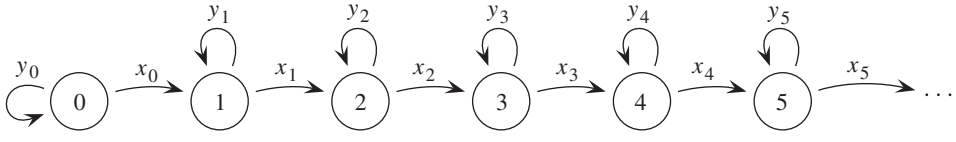


FIGURE 2: Transitions between states for the stochastic matrix P_{U}.

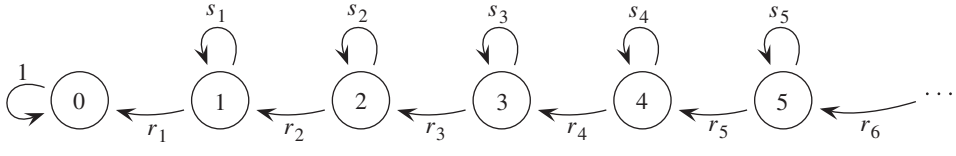


FIGURE 3: Transitions between states for the stochastic matrix P_{L}.

We aim to perform a UL factorization of the matrix P in the following way:

$$P = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} y_0 & x_0 & & \\ 0 & y_1 & x_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} s_0 & 0 & & \\ r_1 & s_1 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = P_{\mathrm{U}} P_{\mathrm{L}}, \tag{2.2}$$

with the condition that P_{U} and P_{L} *are also stochastic matrices*. This means that all entries of P_{U} and P_{L} are nonnegative and

$$x_n + y_n = 1, \qquad s_0 = 1, \qquad r_n + s_n = 1, \quad n \geq 1.$$

We use the convention that $x_{-n} = y_{-n} = r_{-n+1} = s_{-n} = 0$ for $n \geq 1$. A direct computation yields

$$a_n = x_n s_{n+1}, \qquad b_n = x_n r_{n+1} + y_n s_n, \qquad c_n = y_n r_n, \quad n \geq 1. \tag{2.3}$$

The irreducibility conditions force us to take $x_n, s_{n+1} > 0$, $n \geq 0$, and $y_n, r_n > 0$, $n \geq 1$. The free parameter y_0 satisfies $0 \leq y_0 < 1$. Observe that P_{U} is a pure-birth random walk on the nonnegative integers; see Figure 2, while P_{L} is a pure-death random walk on the nonnegative integers with absorbing state at 0; see Figure 3.

We could have performed the factorization the other way around, i.e.

$$P = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \tilde{s}_0 & 0 & & \\ \tilde{r}_1 & \tilde{s}_1 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \tilde{y}_0 & \tilde{x}_0 & & \\ 0 & \tilde{y}_1 & \tilde{x}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \widetilde{P}_{\mathrm{L}} \widetilde{P}_{\mathrm{U}}, \tag{2.4}$$

in which case we have an LU factorization with relations

$$a_n = \tilde{s}_n \tilde{x}_n, \qquad b_n = \tilde{r}_n \tilde{x}_{n-1} + \tilde{s}_n \tilde{y}_n, \qquad c_n = \tilde{r}_n \tilde{y}_{n-1}, \quad n \geq 1.$$

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

The important difference between both cases is that in the UL factorization case there is a *free parameter* y_0, while in the LU case the factorization is unique. As before, the irreducibility conditions force us to take $\tilde{x}_n, \tilde{s}_n > 0$, $n \geq 0$ and $\tilde{y}_n, \tilde{r}_{n+1} > 0$, $n \geq 0$. Here we also use the convention that $\tilde{x}_{-n} = \tilde{y}_{-n} = \tilde{r}_{-n+1} = \tilde{s}_{-n} = 0$ for $n \geq 1$.

Lemma 2.1. *Let $P = P_{\mathrm{U}} P_{\mathrm{L}}$ be as in (2.2) and assume that $s_0 = 1$. Then P_{U} is stochastic if and only if P_{L} is stochastic. The same result holds for the LU factorization in (2.4).*

Proof. Assume that P_{U} is stochastic. We will prove that P_{L} is stochastic by induction. If $x_0 + y_0 = 1$ and using the first and second relations in (2.3), we obtain

$$s_1 + r_1 = \frac{a_0}{x_0} + \frac{b_0 - y_0}{x_0} = \frac{1 - y_0}{x_0} = 1.$$

Assume now that $s_n + r_n = 1$. Again using (2.3), wehave

$$\begin{aligned} s_{n+1} + r_{n+1} &= \frac{a_n}{x_n} + \frac{b_n - y_n s_n}{x_n} \\ &= \frac{1}{x_n}(a_n + b_n - y_n(1 - r_n)) \\ &= \frac{1}{x_n}(a_n + b_n + c_n - y_n) \\ &= \frac{1}{x_n}(1 - y_n) \\ &= 1. \end{aligned}$$

Finally, the entries of P_{L} are all nonnegative following (2.3). On the other hand, if P_{L} is stochastic then all entries of P_{U} are nonnegative and adding the three relations in (2.3), we have

$$x_n(s_{n+1} + r_{n+1}) + y_n(s_n + r_n) = a_n + b_n + c_n = 1.$$

The proof for the LU factorization is similar. □

From (2.3), it is clear that the we can generate all the sequences x_n, y_n, s_n, r_n directly from a_n, b_n, c_n and there will be a unique *free parameter* y_0 (something that will not be true for the LU factorization). Indeed, alternatively, we first calculate the sequences s_{n+1}, $n \geq 0$, and y_n, $n \geq 1$, by using the first and third relations in (2.3). The sequences x_n and r_n satisfy $x_n = 1 - y_n$, $n \geq 0$, and $r_n = 1 - s_n$, $n \geq 1$. The best order in which to calculate all sequences is (y_0 is a free parameter and $s_0 = 1$), $s_1, r_1, y_1, x_1, s_2, r_2, y_2, x_2, \ldots$. Another way to generate these sequences is by using certain recurrence relations as we demonstrate in the next lemma.

Lemma 2.2. *Let x_n, y_n, s_n, r_n be the sequences obtained by (2.2) with P_{U} having all its rows summing up to 1 (or P_{L}). Then the sequence y_n satisfies the recurrence relation (y_0 is a free parameter)*

$$y_{n+1} = \frac{c_{n+1}}{1 - a_n/(1 - y_n)}, \qquad n \geq 0,\ y_0 \in \mathbb{R}, \tag{2.5}$$

while the sequence s_n satisfies the recurrence relation

$$s_1 = \frac{a_0}{1 - y_0}, \qquad s_{n+1} = \frac{a_n}{1 - c_n/(1 - s_n)} \qquad n \geq 1. \tag{2.6}$$

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

Proof. From the first and third relations in (2.3), we have

$$y_{n+1} = \frac{c_{n+1}}{r_{n+1}} = \frac{c_{n+1}}{1 - s_{n+1}} = \frac{c_{n+1}}{1 - a_n/x_n} = \frac{c_{n+1}}{1 - a_n/(1 - y_n)},$$
$$s_{n+1} = \frac{a_n}{x_n} = \frac{a_n}{1 - y_n} = \frac{a_n}{1 - c_n/(1 - s_n)}.$$

□

Remark 2.1. A simpler way of writing (2.5) and (2.6) is by calculating both sequences using

$$y_{n+1} = \frac{c_{n+1}}{1 - s_{n+1}}, \qquad s_{n+1} = \frac{a_n}{1 - y_n}, \quad n \geq 0. \tag{2.7}$$

Remark 2.2. For the LU factorization, we have similar recurrence relations, but now for $\tilde{r}_n$ and $\tilde{x}_n$. Indeed,

$$\tilde{r}_{n+1} = \frac{c_{n+1}}{1 - a_n/(1 - \tilde{r}_n)} = \frac{c_{n+1}}{1 - \tilde{x}_n}, \qquad n \geq 0, \tag{2.8}$$

and

$$\tilde{x}_0 = a_0, \qquad \tilde{x}_{n+1} = \frac{a_{n+1}}{1 - c_{n+1}/(1 - \tilde{x}_n)} = \frac{a_{n+1}}{1 - \tilde{r}_{n+1}}, \quad n \geq 0. \tag{2.9}$$

Unlike the UL factorization, there is no free parameter in this case.

In the previous lemma we stated one way to obtain recursively all coefficients x_n, y_n, s_n, r_n in terms of a free parameter y_0. Lemma 2.2 does not provide any information about the positivity of the coefficients x_n, y_n, s_n, r_n. This will be the goal of the next theorem, where we will study how to choose appropriately the parameter y_0 in such a way that P_{U} and P_{L} are both stochastic. Before that, we need to introduce some notation about *continued fractions*. We refer the reader to [20] for an introduction to the subject.

Let H be the continued fraction generated by alternatively choosing a_n and c_n, i.e.

$$H = 1 - \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \cfrac{c_2}{1 - \cdots}}}}, \tag{2.10}$$

Continued fractions can be written in different notation. In this paper, we also use

$$H = 1 - \frac{a_0}{1} \Big| - \frac{c_1}{1} \Big| - \frac{a_1}{1} \Big| - \frac{c_2}{\cdots} \Big|.$$

The sequence $a_0, c_1, a_1, c_2, \ldots$ is called the sequence of *partial numerators* of the continued fraction. Consider the so-called convergents $(h_n)_{n\geq 0}$ of the continued fraction H given by the sequence of truncated continued fractions of H, i.e.

$$h_{2n} = 1 - \frac{a_0}{1} \Big| - \frac{c_1}{1} \Big| - \cdots - \frac{c_n}{1} \Big|, \qquad n \geq 0,$$
$$h_{2n+1} = 1 - \frac{a_0}{1} \Big| - \frac{c_1}{1} \Big| - \cdots - \frac{a_n}{1} \Big|, \qquad n \geq 0.$$

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

It is well known (see [20]) that the convergents of a continued fraction can be written in the form

$$h_n = \frac{A_n}{B_n}, \tag{2.11}$$

where the numbers A_n and B_n can be calculated recursively as

$$\begin{aligned} A_0 &= 1, \qquad A_n = A_{n-1} - \xi_n A_{n-2}, \qquad A_{-1} = 1, \quad n \geq 1, \\ B_0 &= 1, \qquad B_n = B_{n-1} - \xi_n B_{n-2}, \qquad B_{-1} = 0, \quad n \geq 1. \end{aligned}$$

Here ξ_n is the sequence $\xi_{2n} = c_n$ and $\xi_{2n+1} = a_n$ for $n \geq 1$. Therefore, we have

$$A_{2n} = A_{2n-1} - c_n A_{2n-2}, \qquad B_{2n} = B_{2n-1} - c_n B_{2n-2}, \quad n \geq 1,$$

and

$$A_{2n+1} = A_{2n} - a_n A_{2n-1}, \qquad B_{2n+1} = B_{2n} - a_n B_{2n-1}, \quad n \geq 0.$$

Using the relations above it is not difficult to prove that

$$\begin{aligned} A_{2n+2}B_{2n+1} - A_{2n+1}B_{2n+2} &= -a_0c_1a_1c_2\cdots a_{n-1}c_n, \qquad n \geq 0, \\ A_{2n+1}B_{2n} - A_{2n}B_{2n+1} &= -a_0c_1a_1c_2\cdots c_n a_n, \qquad n \geq 0, \end{aligned}$$

or, in other words,

$$h_{2n+2} - h_{2n+1} = \frac{A_{2n+2}}{B_{2n+2}} - \frac{A_{2n+1}}{B_{2n+1}} = -\frac{a_0c_1a_1c_2\cdots a_{n-1}c_n}{B_{2n+1}B_{2n+2}}, \qquad n \geq 0, \tag{2.12}$$

$$h_{2n+1} - h_{2n} = \frac{A_{2n+1}}{B_{2n+1}} - \frac{A_{2n}}{B_{2n}} = -\frac{a_0c_1a_1c_2\cdots c_n a_n}{B_{2n}B_{2n+1}}, \qquad n \geq 0. \tag{2.13}$$

This leads us to the our first result.

Theorem 2.1. *Let H the continued fraction given by (2.10) and the corresponding convergents $h_n = A_n/B_n$ defined by (2.11). Assume that*

$$0 < A_n < B_n, \qquad n \geq 1. \tag{2.14}$$

Then H is convergent. Moreover, let $P = P_{\mathrm{U}}P_{\mathrm{L}}$ as in (2.2). Then both P_{U} and P_{L} are stochastic matrices if and only if we choose y_0 in the range

$$0 \leq y_0 \leq H. \tag{2.15}$$

Proof. First, following the assumptions in (2.14) and the properties (2.12) and (2.13), we have (recall that $0 < a_n, c_{n+1} < 1$, $n \geq 0$)

$$0 < \cdots < h_{2n+2} < h_{2n+1} < h_{2n} < \cdots < h_1 < h_0 = 1.$$

Therefore, h_n is a bounded strictly decreasing sequence, so it is convergent to H.

Assume first that P_{U} and P_{L} are stochastic. This means that $0 \leq y_0 < 1$, $0 < x_n, s_{n+1} < 1$, $n \geq 0$, and $0 < y_n, r_n < 1$, $n \geq 1$, following the irreducibility conditions for matrix P. Therefore, we already have the lower bound for y_0, i.e. $0 \leq y_0$. In order to prove that $y_0 \leq H$, we will use the definition of y_n and s_n in (2.7). Consider the sequence of numbers α_n, $n \geq 0$, given by $\alpha_{2n} = y_n$, $\alpha_{2n+1} = s_{n+1}$, $n \geq 0$. We will see that the condition $\alpha_n < 1$ implies that $y_0 < h_n$ for every n. Therefore, since h_n is a positive bounded and decreasing sequence, we have (2.15).

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

For $\alpha_0 = y_0$, it is clear that $0 \le y_0 < 1 = h_0$. For $\alpha_1 = s_1$, by definition, we have

$$\alpha_1 = s_1 = \frac{a_0}{1 - y_0} < 1 \quad \Longleftrightarrow \quad y_0 < 1 - a_0 = h_1.$$

For $\alpha_2 = y_1$, using (2.7) and the previous bounds, we have

$$\begin{aligned} \alpha_2 = y_1 = \frac{c_1}{1 - s_1} < 1 \quad &\Longleftrightarrow \quad s_1 < 1 - c_1 \\ &\Longleftrightarrow \quad \frac{a_0}{1 - y_0} < 1 - c_1 \\ &\Longleftrightarrow \quad y_0 < 1 - \frac{a_0}{1 - c_1} = h_2. \end{aligned}$$

For an even index $2n$, using (2.7) and all the previous bounds, we have

$$\begin{aligned} \alpha_{2n} = y_n = \frac{c_n}{1 - s_n} < 1 \quad &\Longleftrightarrow \quad s_n < 1 - c_n \\ &\Longleftrightarrow \quad \frac{a_{n-1}}{1 - y_{n-1}} < 1 - c_n \\ &\Longleftrightarrow \quad y_{n-1} < 1 - \frac{a_{n-1}}{1 - c_n} \\ &\Longleftrightarrow \quad s_{n-1} < 1 - \frac{c_{n-1}}{1 - a_{n-1}/(1 - c_n)} \\ &\quad \vdots \\ &\Longleftrightarrow \quad y_0 < 1 - \cfrac{a_0}{1} - \cfrac{c_1}{1} - \cdots - \cfrac{c_n}{1} = h_{2n}. \end{aligned}$$

Similarly, for an odd index $2n + 1$, we have, using (2.7) and all the previous bounds,

$$\begin{aligned} \alpha_{2n+1} = s_{n+1} = \frac{a_n}{1 - y_n} < 1 \quad &\Longleftrightarrow \quad y_n < 1 - a_n \\ &\Longleftrightarrow \quad \frac{c_n}{1 - s_n} < 1 - a_n \\ &\Longleftrightarrow \quad s_n < 1 - \frac{c_n}{1 - a_n} \\ &\Longleftrightarrow \quad y_{n-1} < 1 - \frac{a_{n-1}}{1 - c_n/(1 - a_n)} \\ &\quad \vdots \\ &\Longleftrightarrow \quad y_0 < 1 - \cfrac{a_0}{1} - \cfrac{c_1}{1} - \cdots - \cfrac{a_n}{1} = h_{2n+1}. \end{aligned}$$

On the contrary, in particular if (2.15) holds, we have $0 \le y_0 \le H < h_n$ for every $n \ge 0$. Following the same steps as before, using an argument of strong induction will lead us to the fact that both P_{U} and P_{L} are stochastic matrices the conditions $0 \le y_0 < 1$, $0 < x_n, s_{n+1} < 1$, $n \ge 0$, and $0 < y_n, r_n < 1$, $n \ge 1$. □

Remark 2.3. For the LU factorization there is no free parameter, so the positivity condition is in terms of an upper bound of the coefficient $\tilde{y}_0 = a_0$. Indeed, we must have

$$0 \le a_0 \le \widetilde{H},$$

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

where

$$\widetilde{H} = 1 - \cfrac{c_1}{1} - \cfrac{a_1}{1} - \cfrac{c_2}{1} - \cfrac{a_2}{\cdots} \tag{2.16}$$

as long as we have $0 < \widetilde{A}_n < \widetilde{B}_n$, $n \geq 1$, where $\tilde{h}_n = \widetilde{A}_n/\widetilde{B}_n$ are the convergents of $\widetilde{H}$. The proof is similar to the proof in the previous proposition, but now using the recurrence relations (2.8) and (2.9).

Remark 2.4. The subject of calculating the exact value of a continued fraction or establishing its convergence is a delicate matter. There are many results allowing one to decide whether a continued fraction is convergent or not. We refer the reader again to [20] for a collection of convergence results for different types of continued fractions. For instance, one of the earliest results is *Worpitzky's theorem*, which states that a continued fraction converges if all partial numerators have moduli less than $\frac{1}{4}$. Another interesting case is when the sequence of partial numerators is a *chain sequence*; see, for example, [2, Theorem 3.1]. Indeed, consider the continued fraction

$$C = 1 - \cfrac{\alpha_1}{1} - \cfrac{\alpha_2}{1} - \cfrac{\alpha_3}{1} - \cfrac{\alpha_4}{\cdots}.$$

Assume that α_n can be written as $\alpha_n = (1 - m_{n-1})m_n$, where $0 \leq m_0 < 1$ and $0 < m_n < 1$ for $n \geq 1$. Then

$$C = m_0 + \frac{1 - m_0}{1 + L},$$

where

$$L = \sum_{n=1}^{\infty} \frac{m_1 m_2 \cdots m_n}{(1 - m_1)(1 - m_2) \cdots (1 - m_n)}. \tag{2.17}$$

Moreover, in this case the convergents satisfy condition (2.14). Indeed, following the approach in [2, Section 3.3], we see that the convergents $C_n = A_n/B_n$ of C satisfy

$$A_n = \prod_{k=1}^{n}(1 - m_k) + m_0\left(\sum_{k=1}^{n-1} m_1 \cdots m_k(1 - m_{k+1}) \cdots (1 - m_n) + \prod_{k=1}^{n} m_k\right),$$

$$B_n = \prod_{k=1}^{n}(1 - m_k) + \sum_{k=1}^{n-1} m_1 \cdots m_k(1 - m_{k+1}) \cdots (1 - m_n) + \prod_{k=1}^{n} m_k.$$

From here, it is clear that $0 < A_n < B_n$, $n \geq 1$. The argument above will be used for the Jacobi polynomials in Section 5.

Remark 2.5. Observe that the $y_0 = 0$ case allows for UL factorization with a *free parameter* s_0 not necessarily equal to 1. Indeed, $x_0 = 1$ and the sequences y_n, x_n, $n \geq 1$, can be obtained by (2.5) and they are independent of s_0. The same holds for the sequences r_n, s_n, $n \geq 1$, which can be obtained from (2.6) and they are independent of s_0. Therefore, we can take any s_0 such that $0 \leq s_0 \leq 1$. The probabilistic meaning of taking $y_0 = 0$ is that the random walk P_{U} has a repelling barrier at state 0. For the random walk P_{L} it is convenient to think of an ignored state (say -1) as an absorbing state from the state 0 with probability $1 - s_0$.

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

3. Stochastic Darboux transformations

With knowledge about the conditions under which a stochastic tridiagonal matrix can be decomposed as a UL (or LU) factorization where both factors are still stochastic matrices, we can perform what is called a *discrete Darboux transformation*. The Darboux transformation has a long history but probably the first reference of a discrete Darboux transformation similar to the one we study here appeared in [17] in connection with the Toda lattice. We now explain the role of a Darboux transformation in our context.

If $P = P_{\mathrm{U}} P_{\mathrm{L}}$ as in (2.2) then by inverting the order of the factors we obtain another tridiagonal matrix of the form

$$\widetilde{P} = P_{\mathrm{L}} P_{\mathrm{U}} = \begin{pmatrix} s_0 & 0 & & \\ r_1 & s_1 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} y_0 & x_0 & & \\ 0 & y_1 & x_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \tilde{b}_0 & \tilde{a}_0 & & \\ \tilde{c}_1 & \tilde{b}_1 & \tilde{a}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.1}$$

Now the new coefficients are

$$\tilde{a}_n = s_n x_n, \qquad \tilde{b}_n = r_n x_{n-1} + s_n y_n, \qquad \tilde{c}_n = r_n y_{n-1}, \quad n \geq 1.$$

The matrix $\widetilde{P}$ is actually stochastic, since the multiplication of two stochastic matrices is again a stochastic matrix. Therefore, it yields a *family* of new random walks $\{\widetilde{X}_t : t = 0, 1, \ldots\}$ with coefficients $(\tilde{a}_n)_n$, $(\tilde{b}_n)_n$, and $(\tilde{c}_n)_n$ depending on a free parameter y_0. In terms of a model driven by urn experiments, the factorization $P = P_{\mathrm{U}} P_{\mathrm{L}}$ may be thought of as two urn experiments; Experiment 1 and Experiment 2, respectively. We first perform Experiment 1 and with the result we immediately perform Experiment 2. The urn model for $\widetilde{P} = P_{\mathrm{L}} P_{\mathrm{U}}$ will proceed in the reverse order, first Experiment 2 and with the result Experiment 1. We will return to this later with some specific examples.

We can do the same for the LU factorization (2.4) of the form $P = \widetilde{P}_{\mathrm{L}} \widetilde{P}_{\mathrm{U}}$ but this does not yield a family of random walks and the transformation is unique. The corresponding Darboux transformation is

$$\widehat{P} = \widetilde{P}_{\mathrm{U}} \widetilde{P}_{\mathrm{L}} = \begin{pmatrix} \tilde{y}_0 & \tilde{x}_0 & & \\ 0 & \tilde{y}_1 & \tilde{x}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \tilde{s}_0 & 0 & & \\ \tilde{r}_1 & \tilde{s}_1 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \hat{b}_0 & \hat{a}_0 & & \\ \hat{c}_1 & \hat{b}_1 & \hat{a}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.2}$$

The new coefficients are

$$\hat{a}_n = \tilde{x}_n \tilde{s}_{n+1}, \qquad \hat{b}_n = \tilde{x}_n \tilde{r}_{n+1} + \tilde{y}_n \tilde{s}_n, \quad n \geq 0, \qquad \hat{c}_n = \tilde{y}_n \tilde{r}_n, \quad n \geq 1.$$

One important property of the Darboux transformation is in transforming the spectral measure associated with a random walk with one-step transition probability P. It is well known (see [16]) that for every tridiagonal stochastic matrix P (also known as a Jacobi matrix), there exists a unique positive measure ω defined on the interval $-1 \leq x \leq 1$. This is just a consequence of the well-known spectral theorem (or Favard's theorem in the context of orthogonal polynomials). The Darboux transformation (3.1) yields a family of random walks $\widetilde{P}$ which is also a Jacobi matrix. Therefore, there exists a unique family of positive measures $\widetilde{\omega}$ associated with $\widetilde{P}$. It is possible to derive (see, for example, [11, Theorem 2]) that the moments of this new $\widetilde{\omega}$ are

$$\widetilde{\mu}_0 = 1, \qquad \widetilde{\mu}_n = y_0 \mu_{n-1}, \quad n \geq 1,$$

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

where μ_n is the nth moment of ω and y_0 is the free parameter from the UL factorization. If the moment $\mu_{-1} = \int_{-1}^{1} x^{-1}\,\mathrm{d}\omega(x)$ is well defined then a candidate for the family of spectral measures is

$$\widetilde{\omega}(x) = y_0 \frac{\omega(x)}{x} + M\delta_0(x), \qquad M = 1 - y_0\mu_{-1}, \tag{3.3}$$

where $\delta_0(x)$ is the Dirac delta function located at $x = 0$. This transformation of the spectral measure ω is also known as a *Geronimus transformation*.

Similarly, for the LU factorization the corresponding Darboux transformation $\widehat{P}$ in (3.2) gives rise to a Jacobi matrix and a spectral measure $\widehat{\omega}$. In this case, we see that this new spectral measure is

$$\widehat{\omega}(x) = x\omega(x), \tag{3.4}$$

or, in other words, a *Christoffel transformation* of ω. For more information about the connection between Darboux transformations and Geronimus or Christoffel transformations, we refer the reader to [10], [11], [18], [21], and [22].

One important consequence of having the spectral measure ω of P (or $\widetilde{P}$ or $\widehat{P}$) is that we can compute the entries of the n-step transition probability matrix P^n (or $\widetilde{P}^n$ or $\widehat{P}^n$). Indeed, using the coefficients of P in (2.1), consider $Q_n(x)$, $n \geq 0$, the polynomials constructed recursively by

$$Q_0(x) = 1, \qquad Q_{-1}(x) = 0, \qquad a_n Q_{n+1}(x) + b_n Q_n(x) + c_n Q_{n-1}(x) = x Q_n(x), \quad n \geq 1.$$

We know from the spectral theorem that $(Q_n)_n$ is a system of orthogonal polynomials with respect to ω. Then we have (see [16]) the so-called *Karlin–McGregor integral representation formula*

$$P_{ij}^n \doteq \mathbb{P}(X_n = j \mid X_0 = i) = \frac{1}{\|Q_j\|_\omega^2} \int_{-1}^{1} x^n Q_i(x) Q_j(x)\,\mathrm{d}\omega(x), \tag{3.5}$$

where $\|Q_j\|_\omega^2$ is the L^2-norm of the polynomial Q_j with respect to ω. Additionally, it is possible to obtain an explicit expression of the components of the invariant measure $\boldsymbol{\pi}$ associated with P; see [7]. Indeed, we have

$$\boldsymbol{\pi} = \left(\frac{1}{\|Q_0\|_\omega^2}, \frac{1}{\|Q_1\|_\omega^2}, \frac{1}{\|Q_2\|_\omega^2}, \cdots \right). \tag{3.6}$$

Finally, we can also study recurrence from knowledge of the spectral measure ω (see [16]). Indeed, the random walk is recurrent if and only if $\int_{-1}^{1} (1-x)^{-1}\,\mathrm{d}\omega(x) = \infty$; otherwise it is transient. Also, the random walk is positive recurrent (or ergodic) if and only if ω has a jump at $x = 1$. It is also possible to study other probabilistic properties using spectral methods such as absorbing times, first return times, or limit theorems; see [1], [3], and [14]–[16].

The same probabilistic properties can be studied for the random walks built from the Darboux transformations $\widetilde{P}$ in (3.1) and $\widehat{P}$ in (3.2) and their spectral measures $\widetilde{\omega}$ in (3.3) and $\widehat{\omega}$ in (3.4).

Remark 3.1. Following Remark 2.5, we now see how the choice of $y_0 = 0$ affects the random walk $\widetilde{P}$ in (3.1) obtained from the Darboux transformation. Indeed, in this case we must have $\tilde{b}_0 = 0$, $\tilde{a}_0 = s_0$, *and* $\tilde{c}_1 = 0$, so $\widetilde{P}$ will be a random walk with a free parameter s_0. The random walk $\widetilde{P}$ can be decomposed as

$$\widetilde{P} = \left(\begin{array}{c|ccc} 0 & s_0 & 0 & \cdots \\ \hline 0 & & & \\ 0 & & \breve{P} & \\ \vdots & & & \end{array} \right),$$

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

where $\breve{P}$ is stochastic, irreducible, and does not depend on s_0. If $s_0 < 1$, there is a probability that the process is absorbed at state -1 if the random walk is at state 0, given by $1 - s_0$. But the random walk $\widetilde{P}$ can never reach the state 0, unless it starts there. In terms of the spectral measure for $\widetilde{P}$ in (3.3), we see that if $y_0 = 0$ then $\widetilde{\omega}(x) = \delta_0(x)$ and we have a *degenerate measure*.

Remark 3.2. In general, given a tridiagonal matrix P as in (2.1), it is difficult to obtain an explicit expression of the spectral measure ω associated with P. One way to obtain the spectral measure is by comparing P with the 0th associated process $\breve{P}$, or the process built from P by deleting the first row and column of P, i.e.

$$P = \left(\begin{array}{c|ccc} b_0 & a_0 & 0 & \cdots \\ \hline c_1 & & & \\ 0 & & \breve{P} & \\ \vdots & & & \end{array}\right), \qquad \breve{P} = \begin{pmatrix} b_1 & a_1 & & \\ c_2 & b_2 & a_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.7}$$

If we call $\breve{\omega}$ the spectral measure associated with $\breve{P}$ then we have (see [1] or [16, Equation (7)])

$$1 + B(z; \omega)[z - b_0 + a_0 c_1 B(z; \breve{\omega})] = 0, \tag{3.8}$$

where $B(z; \psi)$ is the *Stieltjes transform* of some measure ψ supported on $[-1, 1]$, defined by

$$B(z; \psi) \doteq \int_{-1}^{1} \frac{\mathrm{d}\psi(x)}{x - z}.$$

Once we have the Stieltjes transform of ω, we can compute ω by using the well-known *Stieltjes–Perron inversion formula*

$$\int_a^b \mathrm{d}\omega(x) + \frac{1}{2}\omega(\{a\}) + \frac{1}{2}\omega(\{b\}) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_a^b \operatorname{Im} B(x + \mathrm{i}\varepsilon; \omega)\, \mathrm{d}x. \tag{3.9}$$

4. Random walk with constant transition probabilities

In this section we apply the machinery developed in the previous sections to the study of the random walk $\{X_t : t = 0, 1, \ldots\}$ with the one-step transition probability matrix

$$P = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \\ c & b & a & 0 & \\ 0 & c & b & a & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4.1}$$

where $b_0 + a_0 = 1$ and $a + b + c = 1$. First, we will use Theorem 2.1 to find the conditions under which it is possible to have a stochastic UL factorization.

Proposition 4.1. *Assume that $c \le (1 - \sqrt{a})^2$. Then the continued fraction H in (2.10) converges to $H = 1 - a_0/F$, where*

$$F = \tfrac{1}{2}\big(1 + a - c + \sqrt{(1 + a - c)^2 - 4a}\big). \tag{4.2}$$

Moreover, let P be as in (4.1) with $a_0 \le F$. Then $P = P_{\mathrm{U}} P_{\mathrm{L}}$ as in (2.2), where both P_{U} and P_{L} are stochastic matrices if and only if we choose y_0 in the range

$$0 \le y_0 \le 1 - \frac{a_0}{F}.$$

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

Proof. First, we observe that the continued fraction H in (2.10) can be written as $H = 1 - a_0/F$, where

$$F = 1 - \cfrac{c}{1} - \cfrac{a}{1} - \cfrac{c}{1} - \cfrac{a}{\cdots}. \tag{4.3}$$

Then F is a continued fraction of period 2. This means that F, if it converges, will be the solution of a certain quadratic equation. Heuristically, F can be written as

$$F = 1 - \frac{c}{1 - a/F}.$$

Then F satisfies the quadratic equation

$$F^2 - (1 + a - c)F + a = 0,$$

and the solutions are

$$F_{\pm} = \tfrac{1}{2}\big(1 + a - c \pm \sqrt{(1 + a - c)^2 - 4a}\big).$$

We now see that the condition $c \le (1 - \sqrt{a})^2$ implies that F is actually a real number and the value of the continued fraction F is exactly the solution having the larger modulus. A more rigorous proof can be carried out following the approach of [20, Theorem 8.1]. For the second part, the condition $a_0 \le F$ yields that $1 - a_0/F$ is actually a number between 0 and 1 since F is always positive. □

Remark 4.1. We can already see that there are many cases of random walks with constant transition probabilities where *it is not possible* to obtain a stochastic UL factorization. For instance, the case of a symmetric random walk with $c = 1 - a$ (and, therefore, $b = 0$). In this case, the condition $c \le (1 - \sqrt{a})^2$ is equivalent to $a \ge 1$ which is possible only if $a = 1$. But then P would not be irreducible.

It is possible to calculate the coefficients x_n, y_n, s_n, r_n of the UL factorization using Lemma 2.2. These coefficients will depend on a_0 and y_0. In general, the formulae are quite cumbersome, but in some special cases (see the next subsection) these formulae simplify considerably. For instance, the sequence y_n has the following form ($a_0 \le F$):

$$y_n = \frac{\beta_1 a_0 + \xi_n^1(1 - y_0 - a_0/F)}{\beta_2 a_0 + \xi_n^2(1 - y_0 - a_0/F)}, \qquad n \ge 1,$$

for certain constants β_i, $i = 1, 2$, and sequences ξ_n^i, $i = 1, 2$, with $\beta_1 < \beta_2$ and $\xi_n^1 < \xi_n^2$, $n \ge 1$.

Remark 4.2. Following Remark 2.3, for the LU factorization we must have $0 \le a_0 \le \widetilde{H} = F$, where F is the continued fraction (4.3). As long as we are in the conditions of Proposition 4.1, we have a convergent continued fraction F with a value given by (4.2). The coefficients $\tilde{x}_n, \tilde{y}_n, \tilde{s}_n, \tilde{r}_n$ in (2.4) can be calculated in the same way as before. For instance, for the case of $a = c = \frac{1}{4}$ (and, therefore, $b = \frac{1}{2}$), we have $F = \frac{1}{2}$ and then a_0 must be taken in the range $0 \le a_0 \le \frac{1}{2}$. Then a simple computation yields

$$\tilde{x}_n = \frac{a_0 + n(1 - 2a_0)}{2a_0 + 2n(1 - 2a_0)}, \qquad \tilde{y}_n = 1 - \tilde{x}_n, \quad n \ge 0, \tag{4.4}$$

$$\tilde{r}_n = \frac{2a_0 + (2n - 1)(1 - 2a_0)}{4a_0 + 4n(1 - 2a_0)}, \qquad \tilde{s}_0 = 1 \qquad \tilde{s}_n = 1 - \tilde{r}_n, \quad n \ge 1.$$

We now focus on the spectral measure associated with P in (4.1).

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

Proposition 4.2. *Let P be the one-step transition probability matrix of the random walk given by (4.1). Then the spectral measure ω associated with P is*

$$\omega(x) = \omega_c(x) + \omega(\{1\})\delta_1(x) + \omega(\{\gamma\})\delta_\gamma(x), \tag{4.5}$$

where

$$\gamma = \frac{a_0 - a + aa_0 - a_0c - a_0^2}{a_0 - a}, \tag{4.6}$$

and $\delta_\alpha(x)$ is the Dirac delta function located at $x = \alpha$. The continuous part of the measure is

$$\omega_c(x) = \frac{a_0\sqrt{(x - \sigma_-)(\sigma_+ - x)}}{2\pi(a_0 - a)(1 - x)(x - \gamma)}, \qquad x \in [\sigma_-, \sigma_+] \subseteq [-1, 1], \tag{4.7}$$

where $\sigma_\pm = 1 - (\sqrt{a} \mp \sqrt{c})^2$, while the discrete masses are

$$\omega(\{1\}) = \frac{c - a}{a_0 + c - a}\,\mathbf{1}_{\{c>a\}}, \qquad \omega(\{\gamma\}) = \frac{(a_0 - a)^2 - ac}{(a_0 - a)^2 - ac + a_0c}\,\mathbf{1}_{\{(a_0-a)^2>ac\}},$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

Proof. We compute the spectral measure ω using the formulae from Remark 3.2. First, we observe that $\breve{P}$ in (3.7) is

$$\breve{P} = \begin{pmatrix} b & a & & \\ c & b & a & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

The spectral measure $\breve{\omega}$ associated with $\breve{P}$ can be easily calculated since the 0th associated process of $\breve{P}$ (or the process built from $\breve{P}$ by deleting the first row and column of $\breve{P}$) is exactly the same as $\breve{P}$. Therefore, using (3.8), the Stieltjes transform of the spectral measure $\breve{\omega}$ is (with $b_0 = 1 - a_0$ and $b = 1 - a - c$)

$$B(z; \breve{\omega}) = \frac{b - z \pm \sqrt{(z - \sigma_+)(z - \sigma_-)}}{2ac}, \qquad \sigma_\pm = 1 - (\sqrt{a} \mp \sqrt{c})^2.$$

Now we compute the spectral measure ω of the process P. Substituting $B(z; \breve{\omega})$ into (3.8) and following straightforward computations, it follows that the Stieltjes transform of the spectral measure ω is

$$B(z; \omega) = \frac{2a - aa_0 - a_0 + a_0c + (a_0 - 2a)z + a_0\sqrt{(z - \sigma_+)(z - \sigma_-)}}{2(1 - z)((a_0 - a)z + a - aa_0 + a_0c + a_0^2 - a_0)}.$$

Observe that the Stieltjes transform has two real poles (or one if $a_0 = a$). One at $z = 1$ and the other at $z = \gamma$, where γ is given by (4.6). This means that the measure ω will consist of a continuous density plus possibly some delta masses located at $x = 1$ and/or $x = \gamma$ with certain weights, i.e. as in (4.5). Using the Stieltjes–Perron inversion formula (3.9), we see that the continuous part of the measure is given by (4.7). The discrete masses of ω in (4.5) come from the residues at the simple poles of $B(z; \omega)$. □

Once we have the spectral measure ω in (4.5) associated with P, we can compute the spectral measure $\widetilde{\omega}$ associated with the Darboux transformation $\widetilde{P}$ in (3.1), which is the

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

Geronimus transformation of ω. This measure $\widetilde{\omega}$ is well defined if and only if the moment $\mu_{-1} = \int x^{-1}\, d\omega(x)$ is well defined. The delta at $x = 1$ does not add any additional problem to this moment. The delta at $x = \gamma$ in (4.6) may produce a problem if $\gamma = 0$, which is possible if

$$c = \frac{(1-a_0)(a_0-a)}{a_0}, \qquad ac < (a_0-a)^2.$$

Substituting the value of c into the inequality yields $a_0 > \sqrt{a}$. With this assumption, we have $F = a_0$. According to Proposition 4.1, the only possible choice is for $y_0 = 0$, which we already know yields a degenerate measure. If $ac \geq (a_0-a)^2$ then there is no delta at $x = \gamma$. It is possible to see that the location of γ is outside of $[\sigma_-, \sigma_+]$ and that with the condition $(a_0-a)^2 > ac$, we must have $\gamma \in [-1, 1] \setminus [\sigma_-, \sigma_+]$. Then there is no problem with the continuous part ω_c of ω. As for the continuous part ω_c in (4.7), since we are dividing by x in the Geronimus transformation, there may be a problem of integrability if $0 \in (\sigma_-, \sigma_+)$. On the one hand, it is easy to see that $\sigma_+ > 0$. On the other hand, we have $\sigma_- < 0$ if and only if $\sqrt{a} + \sqrt{c} > 1$. In other words, $ac > \frac{1}{4}b^2$. But in this case we should have $c > (1-\sqrt{a})^2$ and this is a contradiction with the assumptions of Proposition 4.1. Therefore, under the conditions of Proposition 4.1, the Geronimus transformation of ω is always well defined.

We can do the same for the spectral measure $\widehat{\omega}$ of the Darboux transformation $\widehat{P}$ in (3.2), which is the Christoffel transformation of ω. In this case, multiplying ω by x does not bring any integrability problems to ω, so $\widehat{\omega}$ is always well defined.

Remark 4.3. As we mentioned in (3.5), it is possible to compute the entries of the n-step transition probability matrices P^n, $\widetilde{P}^n$, and $\widehat{P}^n$. For instance, for P in (4.1), the polynomials $Q_n(x)$, $n \geq 0$, can be constructed recursively by

$$a_0 Q_1(x) + b_0 Q_0(x) = x\, Q_0(x), \qquad Q_0(x) = 1,$$

and

$$a\, Q_{n+1}(x) + b\, Q_n(x) + c\, Q_{n-1}(x) = x\, Q_n(x), \qquad n \geq 1.$$

In this case, these polynomials are basically affine transformations of the Chebychev polynomials of the first and second kind. Once we have these polynomials, we can compute the n-step transition probability matrices P^n through the Karlin–McGregor formula (3.5) and the corresponding invariant measure using (3.6).

4.1. A factorization and its associated urn models

In this section we assume that $a = c = \frac{1}{4}$ (and, therefore, $b = \frac{1}{2}$). Then according to Proposition 4.1, we have $F = \frac{1}{2}$ and then $0 \leq y_0 \leq 1 - 2a_0$ as long as $a_0 \leq \frac{1}{2}$. If we take $y_0 = 1 - ka_0$, $k \geq 2$, with $a_0 \leq 1/k$, then the coefficients x_n, y_n, r_n, s_n are independent of a_0 and are

$$\begin{aligned} y_n &= \frac{2 + (2n-1)(k-2)}{4 + 4n(k-2)}, \qquad x_n = 1 - y_n, \quad n \geq 1, \\ s_n &= \frac{1 + (n-1)(k-2)}{2 + (2n-1)(k-2)}, \qquad r_n = 1 - s_n, \quad n \geq 1, \qquad s_0 = 1. \end{aligned} \tag{4.8}$$

We state a probabilistic implementation of the UL factorization in terms of a family of urn models in this case. We assume that $k \geq 2$ is a positive integer (certain number of balls).

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

The stochastic matrix P is then

$$P = \begin{pmatrix} 1-a_0 & a_0 & 0 & 0 & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

This is a very simple random walk and one could argue that analyzing it any further is unnecessary. It is provided here as an illustration of what will be carried out in the more complicated case in the next section.

From (4.8), we see that the factors P_{U} and P_{L} in (2.2) are

$$P_{\mathrm{U}} = \begin{pmatrix} 1-ka_0 & ka_0 & 0 & 0 & \\ 0 & \frac{k}{4k-4} & \frac{3k-4}{4k-4} & 0 & \\ 0 & 0 & \frac{3k-4}{8k-12} & \frac{5k-8}{8k-12} & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$P_{\mathrm{L}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \\ \frac{k-1}{k} & \frac{1}{k} & 0 & 0 & \\ 0 & \frac{2k-3}{3k-4} & \frac{k-1}{3k-4} & 0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Each of the matrices P_{U} and P_{L} represents an experiment in terms of an urn model, which we call Experiment 1 and Experiment 2, respectively. Observe that $k \geq 2$ is a free positive integer, so we have a *family* of urn experiments which combined gives the *same* urn model for P. There is a simple case for $k = 2$ and then $x_n = y_n = s_n = r_n = \frac{1}{2}$, $n \geq 1$, so we will assume that $k \geq 3$.

Experiment 1 (for P_{U}) consists of a discrete-time pure-birth random walk on the nonnegative integers (see Figure 2) where each state represents the number of blue balls in the urn. We call this random walk $\{X_t^{(1)} : t = 0, 1, \ldots\}$. We assume that this urn sits in an environment, or bath for short, consisting of an infinite number of blue and red balls. If the state of the system is n blue balls ($n \geq 1$), take $k + n(2k-5)$ blue balls and $2 + (2n-1)(k-2)$ red balls from the bath and add them to the urn. Draw one ball from the urn at random with the uniform distribution. We have two possibilities.

- If we draw a red ball then we remove all red balls and $k + n(2k-5)$ blue balls from the urn (i.e. all balls we introduced at the beginning) and start over. Therefore,

$$\mathbb{P}(X_1^{(1)} = n \mid X_0^{(1)} = n) = \frac{2 + (2n-1)(k-2)}{4 + 4n(k-2)}.$$

  Observe that this probability is given by the coefficient y_n in (4.8).

- If we draw a blue ball then we remove all red balls and $k - 1 + n(2k-5)$ blue balls from the urn (so that there are $n+1$ blue balls in the urn) and start over. Therefore,

$$\mathbb{P}(X_1^{(1)} = n+1 \mid X_0^{(1)} = n) = \frac{2 + (2n+1)(k-2)}{4 + 4n(k-2)}.$$

  Observe that this probability is given by the coefficient x_n in (4.8).

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

We follow this strategy except when the initial number of blue balls is 0, in which case there is an initial test to determine if in one step of time we add one blue ball to the urn. This initial test may be thought of as tossing a (probably biased) coin with probability of heads equal to ka_0. If we have heads then we add 1 blue ball to the urn, otherwise we repeat the coin tossing until we get heads.

Experiment 2 (for P_{L}) is very similar but without an initial test. In this case, we have a discrete-time pure-death random walk on the nonnegative integers (see Figure 3) where each state again represents the number of blue balls in the urn. We call this random walk $\{X_t^{(2)} : t = 0, 1, \ldots\}$. If the state of the system is n blue balls ($n \geq 1$), take $1 + n(k-3)$ blue balls and $1 + (n-1)(k-2)$ red balls from the bath and add them to the urn. Draw again one ball from the urn at random. Again, we have two possibilities.

- If we draw a red ball then we remove all red balls and $1 + n(k-3)$ blue balls from the urn (i.e. all balls we introduce at the beginning) and start over. Therefore,

$$\mathbb{P}(X_1^{(2)} = n \mid X_0^{(2)} = n) = \frac{1 + (n-1)(k-2)}{2 + (2n-1)(k-2)}.$$

  Observe that this probability is given by the coefficient s_n in (4.8).

- If we draw a blue ball then we remove all red balls and $2 + (n-1)(k-2)$ blue balls from the urn (so that there are $n-1$ blue balls in the urn) and start over. Therefore,

$$\mathbb{P}(X_1^{(2)} = n-1 \mid X_0^{(2)} = n) = \frac{1 + n(k-2)}{2 + (2n-1)(k-2)}.$$

  Observe that this probability is given by the coefficient r_n in (4.8).

The urn model for P will be the composition of both experiments, first Experiment 1 and then Experiment 2, while the urn model for the Darboux transformation $\widetilde{P}$ (3.1) is also the composition of both experiments but proceeds in the reverse order. There are four possible results, $n+1$, n (twice), or $n-1$ blue balls in the urn. A representation for this urn model is similar to Figure 4.

Since $a_0 \leq 1/k \leq \frac{1}{2}$, $k \geq 2$, it follows that the spectral measure associated with P is given by only the continuous part of (4.7), i.e.

$$\omega(x) = \frac{2a_0}{\pi[(4a_0-1)x + (1-2a_0)^2]}\sqrt{\frac{x}{1-x}}, \qquad x \in [0, 1].$$

In order to calculate the spectral measure $\widetilde{\omega}$ for the Darboux transformation $\widetilde{P}$, we need to compute the Geronimus transformation of ω (see (3.3)). A straightforward computation yields

$$\mu_{-1} = \int_0^1 \frac{\omega(x)}{x}\,\mathrm{d}x = \frac{1}{1-2a_0}.$$

We exclude the $a_0 = \frac{1}{2}$ case when $k = 2$ in which case we have $y_0 = 0$ and $\widetilde{\omega}$ is a degenerate measure. Therefore,

$$\widetilde{\omega}(x) = \frac{2a_0(1-ka_0)}{\pi[(4a_0-1)x + (1-2a_0)^2]\sqrt{x(1-x)}} + \frac{a_0(k-2)}{1-2a_0}\delta_0(x), \qquad x \in [0, 1].$$

From the probabilistic point of view, and since we have explicit expressions of the spectral measures ω and $\widetilde{\omega}$, we use the Karlin–McGregor formula (3.5) to compute P^n and $\widetilde{P}^n$. From the

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

inverse of the norms of the corresponding orthogonal polynomials, we see that the invariant measure in (3.6) (not a distribution) is

$$\boldsymbol{\pi} = (1, 4a_0, 4a_0, \ldots).$$

Finally, in this case, we can see that $\int_0^1 \omega/(1-x)\,\mathrm{d}x = \infty$ and $\int_0^1 \widetilde{\omega}/(1-x)\,\mathrm{d}x = \infty$. Therefore, both random walks are always *recurrent*. A similar urn model and probabilistic considerations can be derived from the LU factorization (4.4).

5. Random walk generated by the Jacobi polynomials

In this section we study in detail the case of the Jacobi orthogonal polynomials. For $\alpha, \beta > -1$, define the coefficients

$$\begin{aligned} a_n &= \frac{(n+\beta+1)(n+1+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+2+\alpha+\beta)}, \qquad n \geq 0, \\ b_n &= \frac{(n+\beta+1)(n+1)}{(2n+\alpha+\beta+1)(2n+2+\alpha+\beta)} \\ &\quad + \frac{(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}, \qquad n \geq 0, \\ c_n &= \frac{n(n+\alpha)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}, \qquad n \geq 1. \end{aligned} \tag{5.1}$$

Observe that all the coefficients are nonnegative, $a_0 + b_0 = 1$, and $a_n + b_n + c_n = 1$, $n \geq 1$, so they are the coefficients of a discrete-time random walk $\{X_t : t = 0, 1, \ldots\}$ on the nonnegative integers and depend on the state of the system. Also, it is easy to see that the random walk is irreducible for the values $\alpha, \beta > -1$.

The family of polynomials generated by the three-term recursion relation

$$x\,Q_n^{(\alpha,\beta)}(x) = a_n\,Q_{n+1}^{(\alpha,\beta)}(x) + b_n\,Q_n^{(\alpha,\beta)}(x) + c_n\,Q_{n-1}^{(\alpha,\beta)}(x), \qquad n \geq 0,$$

where $Q_{-1}^{(\alpha,\beta)}(x) = 0$ and $Q_0^{(\alpha,\beta)}(x) = 1$ is the well-known family of *Jacobi polynomials*, which are orthogonal with respect to the (normalized) weight

$$w(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha}(1-x)^{\beta}, \qquad x \in [0, 1]. \tag{5.2}$$

Note that the Jacobi polynomials satisfy the condition

$$Q_n^{(\alpha,\beta)}(1) = 1.$$

The normalization in terms of a_n, b_n, c_{n+1}, $n \geq 0$, is natural when one thinks of these polynomials (at least for some values of α, β) as the spherical functions for some appropriate symmetric space, and insist that these functions take the value 1 at the North pole of the corresponding sphere. The simplest case is the one with $\alpha = \beta = 0$ in which case we have the Legendre polynomials and the usual two-dimensional sphere sitting in $\mathbb{R}^3$.

In the following proposition we will use Theorem 2.1 to find the conditions under which it is possible to have a stochastic UL factorization.

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

Proposition 5.1. *Let H be the continued fraction (2.10) with the coefficients a_n, c_{n+1}, $n \geq 0$, in (5.1) of the Jacobi polynomials. Then H converges to $(1+L)^{-1}$, where $L = (\beta+1)/\alpha$. Moreover, $P = P_{\mathrm{U}} P_{\mathrm{L}}$ as in (2.2), where both P_{U} and P_{L} are stochastic matrices if and only if we choose y_0 in the range*

$$0 \leq y_0 \leq \frac{\alpha}{\alpha+\beta+1}. \tag{5.3}$$

Additionally, it is always possible to perform a stochastic LU factorization $P = \widetilde{P}_{\mathrm{L}} \widetilde{P}_{\mathrm{U}}$ as in (2.4).

Proof. The main observation in order to prove this proposition is that the corresponding sequence of alternating numbers $a_0, c_1, a_1, c_2, \ldots$ is a chain sequence. Following the notation of Remark 2.4, we call α_n, $n \geq 1$, the sequence of partial numerators $a_0, c_1, a_1, c_2, \ldots$. Therefore, $\alpha_n = (1 - m_{n-1}) m_n$, where

$$m_{2n} = \frac{n}{2n+\alpha+\beta+1}, \qquad m_{2n+1} = \frac{n+\beta+1}{2n+\alpha+\beta+2}, \quad n \geq 0.$$

Since $m_0 = 0$, we see that H converges to $(1+L)^{-1}$, where L is given by (2.17), which in this case is a hypergeometric series. It is possible to show that $L = (\beta+1)/\alpha$. Since we have a chain sequence, the conditions of Theorem 2.1 hold, so we see that stochastic UL factorization is always possible if we choose the free parameter y_0 in the range (5.3).

For the LU factorization, there is no free parameter and we need

$$a_0 = \frac{\beta+1}{\alpha+\beta+2} \leq \widetilde{H},$$

where $\widetilde{H}$ is the continued fraction given by (2.16). In this case, if we call α_n, $n \geq 1$, the sequence of partial numerators $c_1, a_1, c_2, a_2, \ldots$, then α_n is again a chain sequence with coefficients

$$m_{2n} = \frac{n+\beta+1}{2n+\alpha+\beta+2}, \qquad m_{2n+1} = \frac{n+1}{2n+\alpha+\beta+3}, \quad n \geq 0.$$

Observe now that $m_0 \neq 0$. Therefore, we have

$$\widetilde{H} = m_0 + \frac{1-m_0}{1+L},$$

where L is given by (2.17). As before, it is possible to calculate L, which in this case is given by $L = 1/\alpha$. Therefore,

$$\widetilde{H} = \frac{\beta+1}{\alpha+\beta+2} + \frac{(\alpha+1)/(\alpha+\beta+2)}{1+1/\alpha} = \frac{\alpha+\beta+1}{\alpha+\beta+2}.$$

Clearly, we have $a_0 \leq \widetilde{H}$ and, therefore, we can always perform a stochastic LU factorization. The proof is complete. □

In the next result we state an explicit expression for the coefficients x_n, r_n, s_n, r_n in the UL factorization (2.2) and the coefficients $\tilde{x}_n, \tilde{r}_n, \tilde{s}_n, \tilde{r}_n$ in the LU factorization (2.4).

Proposition 5.2. *The coefficients x_n, y_n, s_n, r_n in the UL factorization (2.2) are*

$$x_n = \frac{\gamma_n + \delta_n y_0}{(\varepsilon_n + \nu_n y_0)(2n+\alpha+\beta+1)}, \qquad y_n = 1 - x_n, \quad n \geq 0,$$

$$s_n = \frac{\varepsilon_n + \nu_n y_0}{(\gamma_{n-1} + \delta_{n-1} y_0)(2n+\alpha+\beta)}, \qquad r_n = 1 - s_n, \quad n \geq 1, \qquad s_0 = 1,$$

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

where

$$\gamma_n = (\alpha+1)_n(\alpha+\beta+2)_n, \qquad \alpha\delta_n = n!\,(\beta+1)_{n+1} - (\alpha+\beta+1)\gamma_n,$$
$$\varepsilon_n = (\alpha+1)_n(\alpha+\beta+2)_{n-1}, \qquad \alpha\nu_n = n!\,(\beta+1)_n - (\alpha+\beta+1)\varepsilon_n.$$

Here, $(a)_j$ *denotes the Pochhammer symbol defined by* $(a)_0 = 1$ *and* $(a)_j = a(a+1)\cdots(a+j-1)$, $j \geq 1$, $a \in \mathbb{C}$. *The coefficients* $\tilde{x}_n, \tilde{y}_n, \tilde{s}_n, \tilde{r}_n$ *in the LU factorization (2.4) are*

$$\begin{aligned} \tilde{x}_n &= \frac{n+\beta+1}{2n+\alpha+\beta+2}, \qquad \tilde{y}_n = \frac{n+\alpha+1}{2n+\alpha+\beta+2}, \quad n \geq 0, \\ \tilde{s}_n &= \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}, \qquad \tilde{r}_n = \frac{n}{2n+\alpha+\beta+1}, \quad n \geq 1, \qquad \tilde{s}_0 = 1. \end{aligned} \tag{5.4}$$

Proof. Following a direct and straightforward computation, we see that (2.3) holds using the recurrence formulae

$$\varepsilon_n = (2n+\alpha+\beta)\gamma_{n-1} - (n+\beta)(n+\alpha+\beta)\varepsilon_{n-1},$$
$$\nu_n = (2n+\alpha+\beta)\delta_{n-1} - (n+\beta)(n+\alpha+\beta)\nu_{n-1}.$$

We can do the same for the LU factorization. □

Remark 5.1. There is a more convenient way of writing the coefficients x_n, y_n, s_n, r_n in the previous proposition if we make the substitution $y_0 = (\alpha/(\alpha+\beta+1))h_0$ (in which case $0 \leq h_0 \leq 1$). Indeed,

$$\begin{aligned} x_n &= \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} \\ &\quad \times \frac{h_0 n!\,(\beta+1)_n(n+\beta+1) + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_{n+1}}{h_0 n!\,(\beta+1)_n(n+\alpha+\beta+1) + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_{n+1}}, \\ y_n &= \frac{n+\alpha}{2n+\alpha+\beta+1} \\ &\quad \times \frac{h_0 n!\,(\beta+1)_n(n+\alpha) + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_n n}{h_0 n!\,(\beta+1)_n(n+\alpha) + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_n(n+\alpha)}, \\ s_n &= \frac{n+\alpha+\beta}{2n+\alpha+\beta} \\ &\quad \times \frac{h_0(n-1)!\,(\beta+1)_n(n+\alpha+\beta) + (1-h_0)(\alpha+1)_{n-1}(\alpha+\beta+1)_n(n+\beta)}{h_0(n-1)!\,(\beta+1)_n(n+\alpha+\beta) + (1-h_0)(\alpha+1)_{n-1}(\alpha+\beta+1)_n(n+\alpha+\beta)}, \\ r_n &= \frac{n+\alpha}{2n+\alpha+\beta} \frac{h_0 n!\,(\beta+1)_n + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_n}{h_0(n-1)!\,(n+\alpha)(\beta+1)_n + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_n}. \end{aligned} \tag{5.5}$$

From these formulae, we clearly see that each coefficient x_n, y_n, s_n, r_n is the multiplication of two positive numbers less than 1. We will state the probabilistic implementations of these coefficients in the next subsection.

There are two cases where all coefficients x_n, y_n, s_n, r_n in Proposition 5.2 simplify considerably; namely, when y_0 coincides with one of the endpoints of the range (5.3).

Case 1: $y_0 = 0$. The coefficients x_n, y_n, s_n, r_n are

$$\begin{aligned} x_n &= \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}, \qquad y_n = \frac{n}{2n+\alpha+\beta+1}, \quad n \geq 0, \\ s_n &= \frac{n+\beta}{2n+\alpha+\beta}, \qquad r_n = \frac{n+\alpha}{2n+\alpha+\beta}, \quad n \geq 1, \qquad s_0 = 1. \end{aligned}$$

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

Case 2: $y_0 = \alpha/(\alpha+\beta+1)$. The coefficients x_n, y_n, s_n, r_n are given by

$$\begin{aligned} x_n &= \frac{n+\beta+1}{2n+\alpha+\beta+1}, \qquad y_n = \frac{n+\alpha}{2n+\alpha+\beta+1}, \quad n \geq 0, \\ s_n &= \frac{n+\alpha+\beta}{2n+\alpha+\beta}, \qquad r_n = \frac{n}{2n+\alpha+\beta}, \quad n \geq 1, \qquad s_0 = 1. \end{aligned} \tag{5.6}$$

We will provide a detailed urn model associated with these coefficients in the next subsection. These coefficients are scalar counterparts of the matrix-valued coefficients given in [12], where the authors considered a special case of the UL block factorization.

We now explore the spectral measure associated with the Darboux transformation $\widetilde{P}$ in (3.1). To do so, we need to calculate the Geronimus transformation (3.3) of $w(x)$ in (5.2). In this case, it is easy to see that

$$\mu_{-1} = \int_0^1 \frac{w(x)}{x}\,\mathrm{d}x = \frac{\alpha+\beta+1}{\alpha}.$$

Therefore, following (3.3), we have

$$\widetilde{w}(x) = y_0 \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha-1}(1-x)^{\beta} + \left(1 - y_0 \frac{\alpha+\beta+1}{\alpha}\right)\delta_0(x), \qquad x \in [0,1].$$

This measure is integrable as long as $\alpha > 0$ and $\beta > -1$. We see that if y_0 is in the range (5.3) then the mass at 0 is always nonnegative, and vanishes if $y_0 = \alpha/(\alpha+\beta+1)$. The $y_0 = 0$ case was treated in Remark 3.1 and in this case the spectral measure is just the Dirac delta function $\delta_0(x)$ (degenerate).

For the LU factorization, the associated (normalized) weight is given by the Christoffel transform of w in (3.4), i.e.

$$\widehat{w}(x) = \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+1)} x^{\alpha+1}(1-x)^{\beta}, \qquad x \in [0,1].$$

As in the previous example, it is always possible to compute the entries of the n-step transition probability matrices P^n, $\widetilde{P}^n$, and $\widehat{P}^n$ using the Karlin–McGregor formula (3.5) and the corresponding orthogonal polynomials. For instance, for P we also have an explicit expression for the norms of the Jacobi polynomials, i.e.

$$\|Q_n^{(\alpha,\beta)}\|_w^2 = \frac{n!\,(\alpha+1)_n}{(\beta+1)_n(\alpha+\beta+2)_{n-1}(2n+\alpha+\beta+1)}.$$

This yields the explicit expression of the components of the invariant measure (3.6), i.e.

$$\boldsymbol{\pi} = \left(1, \frac{(\beta+1)(\alpha+\beta+3)}{\alpha+1}, \frac{(\beta+1)_2(\alpha+\beta+2)(\alpha+\beta+5)}{2(\alpha+2)_2}, \dots\right).$$

Finally, from the behavior of the integral $\int_0^1 (1-x)^{-1} w(x)\,\mathrm{d}x$, we can study recurrence. This behavior only depends on the parameter β. If $-1 < \beta \leq 0$ then the integral is ∞, so the process is always recurrent. Otherwise, for $\beta > 0$, the process is transient. The process can never be positive recurrent because the measure w does not have a jump at $x = 1$. The same result can be deduced for the random walks generated by $\widetilde{P}$ and $\widehat{P}$.

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

5.1. An urn model for the Jacobi polynomials

We now provide an urn model associated with the Jacobi polynomials. We will first use the simpler UL factorization with coefficients from (5.6) and then make a few comments about the general case (5.5). In the latter case, as in the previous section, we have a family of urn model experiments, which combined yield the same model for the Jacobi polynomials. A similar urn model can be derived from the LU factorization (5.4), but now without a free parameter.

To the best of the authors' knowledge, Grünbaum [8] is the first instance of the urn model being associated with Jacobi polynomials. This is a rather contrived model when compared to the more familiar ones such as those of Ehrenfest and Bernoulli–Laplace. More information on these models can be found in [4]. It turns out that these celebrated models are related to the Krawtchouk and dual Hahn orthogonal polynomials, in the same way that our models are related to the Jacobi polynomials. We propose a new way of understanding the urn model behind the Jacobi polynomials, as the composition of two easier urn experiments given by the UL factorization (5.6). Our approach can be used to analyze other complicated random walks which have yet to be addressed.

From here, we will assume that the parameters α and β are nonnegative integers. Consider the discrete-time random walk $\{X_t : t = 0, 1, \ldots\}$ on the nonnegative integers whose one-step transition probability matrix P coincides with the one for the three-term recursion relation in (5.1). Consider the UL factorization $P = P_{\mathrm{U}} P_{\mathrm{L}}$ in (2.2) with coefficients x_n, y_n, s_n, r_n from (5.6). Each of the matrices P_{U} and P_{L} represents an experiment in terms of an urn model, which we call Experiment 1 and Experiment 2, respectively. At times $t = 0, 1, 2, \ldots$ an urn contains n blue balls and this determines the state of our random walk at that time. Each urn for both experiments sits in an environment, or bath for short, consisting of an infinite number of blue and red balls.

Experiment 1 (for P_{U}) consists of a discrete-time pure-birth random walk on the nonnegative integers; see Figure 2. We call this random walk $\{X_t^{(1)} : t = 0, 1, \ldots\}$. If the state of the system is n blue balls ($n \geq 0$), take $\beta + 1$ blue balls and $n + \alpha$ red balls from the bath and add them to the urn. Draw one ball from the urn at random with the uniform distribution. We have two possibilities.

- If we draw a red ball then we remove all red balls and $\beta + 1$ blue balls from the urn (i.e. all balls we introduce in this first step) and start over. Therefore,

$$\mathbb{P}(X_1^{(1)} = n \mid X_0^{(1)} = n) = \frac{n + \alpha}{2n + \alpha + \beta + 1}.$$

  Observe that this probability is given by the coefficient y_n in (5.6).

- If we draw a blue ball then we remove all red balls and β blue balls from the urn (so that there are $n + 1$ blue balls in the urn) and start over. Therefore,

$$\mathbb{P}(X_1^{(1)} = n + 1 \mid X_0^{(1)} = n) = \frac{n + \beta + 1}{2n + \alpha + \beta + 1}.$$

  Observe that this probability is given by the coefficient x_n in (5.6).

Experiment 2 (for P_{L}) consists of a discrete-time pure-death random walk on the nonnegative integers; see Figure 3. We call this random walk $\{X_t^{(2)} : t = 0, 1, \ldots\}$. If the state of the system is n blue balls ($n \geq 0$), take $n + \alpha + \beta$ red balls from the bath and add them to the urn. Draw again one ball from the urn at random. Again, we have two possibilities.

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

- If we draw a red ball then we remove all red balls in the urn and start over. Therefore,

$$\mathbb{P}(X_1^{(2)} = n \mid X_0^{(2)} = n) = \frac{n+\alpha+\beta}{2n+\alpha+\beta}.$$

  Observe that this probability is given by the coefficient s_n in (5.6).

- If we draw a blue ball then we remove that blue ball and all red balls from the urn (so that there are $n-1$ blue balls in the urn) and start over. Therefore,

$$\mathbb{P}(X_1^{(2)} = n-1 \mid X_0^{(2)} = n) = \frac{n}{2n+\alpha+\beta}.$$

  Observe that this probability is given by the coefficient r_n in (5.6).

If the urn is empty we stop the experiment. Observe that if we have 0 blue balls in the urn, the experiment will not change from that moment on.

As in the previous section, the urn model for P will be the composition of Experiment 1 and then Experiment 2, while the urn model for the Darboux transformation $\widetilde{P}$ (3.1) proceeds in the reverse order. If we first perform Experiment 1 we will end up with an urn with either n (if we draw a red ball) or $n+1$ (if we draw a blue ball) blue balls. Now we perform Experiment 2 with n or $n+1$ blue balls, in which case we may have either $n-1$ (if we draw a blue ball) or n (if we draw a red ball) blue balls, while for the $n+1$ case we may have either n (if we draw a blue ball) or $n+1$ (if we draw a red ball) blue balls. The combination of probabilities of these four cases yields the coefficients of the three-term recurrence relation (5.1) (see also (2.3)) for P for the Jacobi polynomials; see Figure 4.

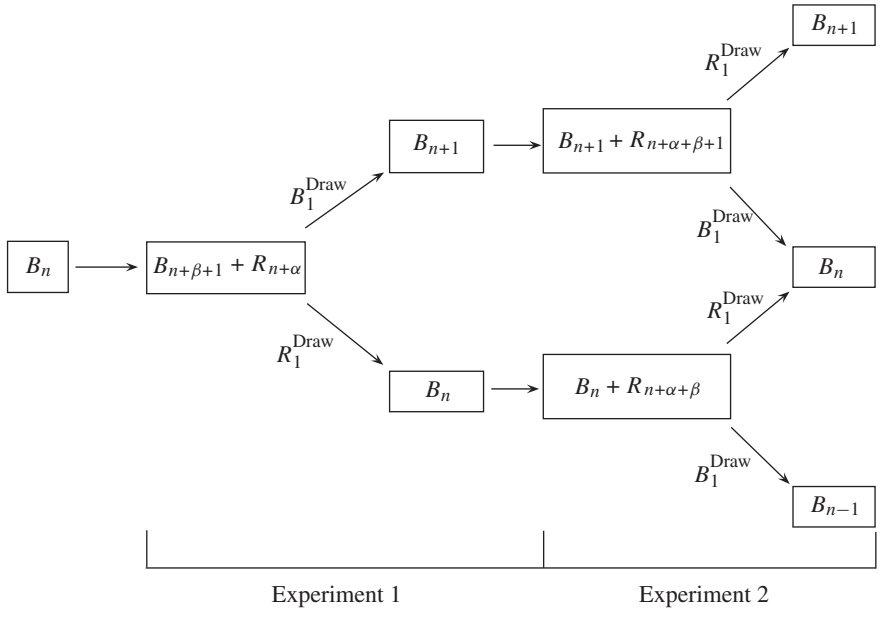


FIGURE 4: A schematic of Experiments 1 and 2. The boxed regions represent the state or urn with $B_\cdot$ and $R_\cdot$ indicating the number of blue or red balls, respectively, contained within the urn. When a ball is drawn from the urn, this is shown as $B_\cdot^{\text{Draw}}$ or $R_\cdot^{\text{Draw}}$. The initial state is B_n.

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

For the general coefficients x_n, y_n, s_n, r_n in (5.5), we obtain a family of Experiments 1 and 2, depending on $0 \leq h_0 \leq 1$. Now both experiments are more complicated, but we observe that each probability x_n, y_n, s_n, r_n is the multiplication of two positive numbers less than 1. This means that for each experiment we will have to perform another experiment (different for each experiment) which combined will yield the probabilities x_n, y_n, s_n, r_n. All these experiments can be seen as urn models with blue and red balls very similar to the ones we have already shown. There are certain choices of the parameters where all coefficients simplify considerably. For instance, for $\alpha = \beta = 0$, we have

$$x_n = \frac{n+1}{2n+1}, \qquad y_n = \frac{n}{2n+1}, \quad n \geq 0, \qquad s_0 = 1 \qquad s_n = r_n = \frac{1}{2}, \quad n \geq 1.$$

Another (more complicated) example is for $\alpha = 1$, $\beta = 0$, and $h_0 = \frac{1}{2}$, i.e.

$$x_n = \frac{1 + (n+1)(n+2)}{2[1 + (n+1)^2]}, \qquad y_n = \frac{1 + n(n+1)}{2[1 + (n+1)^2]}, \quad n \geq 0,$$

$$s_n = \frac{(n+1)(n^2+1)}{(2n+1)(1+n(n+1))}, \qquad r_n = \frac{n(1+(1+n)^2)}{(2n+1)(1+n(n+1))}, \quad n \geq 1, \qquad s_0 = 1.$$

Alternative (and more complicated) urn models for matrix-valued generalizations of Jacobi polynomials, as well as other probabilistic models in terms of Young diagrams, can be found in [12].

Acknowledgements

The work of both authors was supported by UC MEXUS-CONACYT grant CN-16-84. The work of the second author was also supported by PAPIIT-DGAPA-UNAM grant IA102617 (Mexico), MTM2015-65888-C4-1-P (Ministerio de Economía y Competitividad, Spain) and FQM-262, FQM-7276 (Junta de Andalucía).

References

[1] Castro, M. M. and Grünbaum, F. A. (2013). On a seminal paper by Karlin and McGregor. *SIGMA* **9,** 020.
[2] Chihara, T. S. (1978). *An Introduction to Orthogonal Polynomials*. Gordon and Breach, New York.
[3] Dette, H., Reuther, B., Studden, W. J. and Zygmunt, M. (2006). Matrix measures and random walks with a block tridiagonal transition matrix. *SIAM J. Matrix Anal. Appl.* **29,** 117–142.
[4] Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd edn. John Wiley, New York.
[5] Grassmann, W. K. (1987). Means and variances of time averages in Markovian environments. *Europ. J. Operat. Res.* **31,** 132–139.
[6] Grassmann, W. K. (1993). Means and variances in Markov reward systems. In *Linear Algebra, Markov Chains and Queueing Models*. Springer, New York, pp. 193–204.
[7] Grünbaum, F. A. (2008). Random walks and orthogonal polynomials: some challenges. In *Probability, Geometry and Integrable Systems* (Math. Sci. Res. Inst. Publ. **55**), Cambridge University Press, pp. 241–260.
[8] Grünbaum, F. A. (2010). An urn model associated with Jacobi polynomials. *Comm. App. Math. Comput. Sci.* **5**, 55–63.
[9] Grünbaum, F. A. (2011). The Darboux process and a noncommutative bispectral problem: some explorations and challenges. In *Geometric Aspects of Analysis and Mechanics* (Progr. Math. **292**), Birkhäuser, New York, pp. 161–177.
[10] Grünbaum, F. A. and Haine, L. (1996). Orthogonal polynomials satisfying differential equations: the role of the Darboux transformation. In *Symmetries and Integrability of Difference Equations*. American Mathematical Society, Providence, RI, pp. 143–154.
[11] Grünbaum, F. A., Haine, L. and Horozov, E. (1999). Some functions that generalize the Krall-Laguerre polynomials. *J. Comput. Appl. Math.* **106,** 271–297.

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press

[12] Grünbaum, F. A., Pacharoni, I. and Tirao, J. (2013). Two stochastic models of a random walk in the U(n)-spherical duals of U($n+1$). *Ann. Mat. Pura Appl. (4)* **192,** 447–473.
[13] Heyman, D. P. (1995). A decomposition theorem for infinite stochastic matrices. *J. Appl. Prob.* **32,** 893–901.
[14] Karlin, S. and McGregor, J. L. (1957). The differential equations of birth-and-death processes, and the Stieltjes moment problem. *Trans. Amer. Math. Soc.* **85,** 489–546.
[15] Karlin, S. and McGregor, J. (1957). The classification of birth and death processes. *Trans. Amer. Math. Soc.* **86,** 366–400.
[16] Karlin, S. and McGregor, J. (1959). Random walks. *Illinois J. Math.* **3,** 66–81.
[17] Matveev, V. B. and Salle, M. A. (1979). Differential-difference evolution equations II. Darboux transformation for the Toda lattice. *Lett. Math. Phys.* **3,** 425–429.
[18] Spiridonov, V. and Zhedanov, A. (1999). Self-similarity, spectral transformations and orthogonal and biorthogonal polynomials. In *Self-Similar Systems*, Joint Institute of Nuclear Research, Dubna, pp. 349–361.
[19] Vigon, V. (2013). LU-factorization versus Wiener-Hopf factorization for Markov chains. *Acta Appl. Math.* **128,** 1–37.
[20] Wall, H. S. (1948). *Analytic Theory of Continued Fractions*. Van Nostrand, New York.
[21] Yoon, G. J. (2002). Darboux transforms and orthogonal polynomials. *Bull. Korean Math. Soc.* **39,** 359–376.
[22] Zhedanov, A. (1997). Rational spectral transformations and orthogonal polynomials. *J. Comput. Appl. Math.* **85,** 67–86.

https://doi.org/10.1017/jpr.2018.55 Published online by Cambridge University Press