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Non-computability of *K*-theory for computably presented C*-algebras

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Abstract. We give an example of a unital C*-algebra $\mathbf{A}$ with a computable presentation and for which neither $K_0(\mathbf{A})$ nor $K_1(\mathbf{A})$ has a computable presentation.

1 Introduction

In [EGMM], we initiated the study of the effective content of the construction of the K-theory of a C*-algebra. In particular, we constructed a computable functor that takes as input a c.e. presentation of a C*-algebra $\mathbf{A}$ and outputs a c.e. presentation of the group $K_0(\mathbf{A})$ and another such functor outputting a c.e. presentation of $K_1(\mathbf{A})$. In some cases, such as when $\mathbf{A}$ is an AF algebra, our functor maps computable presentations of $\mathbf{A}$ to computable presentations of $K_0(\mathbf{A})$ (see [EGM, Main Theorem 3]). This naturally raises the question of whether there is a C*-algebra $\mathbf{A}$ such that $\mathbf{A}$ has a computable presentation but $K_0(\mathbf{A})$ does not (see [EGMM, Question 2.17]). In this note, we give an example of a C*-algebra $\mathbf{A}$ such that $\mathbf{A}$ has a computable presentation but neither $K_0(\mathbf{A})$ nor $K_1(\mathbf{A})$ has a computable presentation. We refer the reader to [EGMM] for background on computability theory for C*-algebras and the computable K_0 and K_1 functors.

2 A group with no computable presentation

Fix, for the remainder of the note, a set $R \subseteq \mathbb{N}$ that is c.e. but not computable (e.g., we can take $R = 0'$, the halting problem). Enumerate the prime numbers in order as $(p_n)_{n\in\mathbb{N}}$.

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For each $n \in \mathbb{N}$, set

$$G_n = \begin{cases} \mathbb{Z} & n \in R \\ \mathbb{Z}/p_n\mathbb{Z} & n \notin R. \end{cases}$$

For the rest of the note, set $G = \bigoplus_{n\in\mathbb{N}} G_n$.

Proposition 2.1 *There is no computable presentation of the group G.*

Proof Suppose, toward a contradiction, that $G^\#$ is a computable presentation of G. Let $P = \{n \in \mathbb{N} : G \text{ contains an element of order } p_n\}$. Note that G has an element of order p_n if and only if $G_n = \mathbb{Z}/p_n\mathbb{Z}$ if and only if $n \notin R$, that is, $P = \mathbb{N}\backslash R$.

Since $G^\#$ is computable, we can use it to successively compute the orders of each element of G, whence P is a c.e. set. Since R was chosen to be a c.e. set, this implies that R is a computable set, contradicting our choice of R. ∎

Remark 2.2 It is not too difficult to show that the above group G admits a c.e. presentation. However, using [EGMM, Corollary 2.16], this will be a consequence of the fact that G is isomorphic to $K_0(\mathbf{A})$ for a computably presentable C*-algebra $\mathbf{A}$.

3 Non-computable K_0

For each $n \in \mathbb{N}$, set

$$\mathbf{B}_n = \begin{cases} \mathbb{C} & n \in R \\ \mathcal{O}_{p_n+1} & n \notin R. \end{cases}$$

Here, $\mathcal{O}_k$ is the Cuntz algebra on k generators, that is, $\mathcal{O}_k$ is the universal C*-algebra generated by partial isometries $s_1, \ldots, s_k$ such that each $s_i^* s_i = 1$ and $\sum_{i=1}^k s_i s_i^* = 1$. It is well-known that each $\mathcal{O}_k$ is simple [Cun77, Remarks after Theorem 1.12] and has $K_0(\mathcal{O}_k) \cong \mathbb{Z}/(k-1)\mathbb{Z}$ [Cun81, Theorem 3.7]. We will also need the fact that the standard presentations of the Cuntz algebras are computable, uniformly in k (see [Fox24, Corollary 3.16]).

Lemma 3.1 *The algebras $\mathbf{B}_n$ have computable presentations, uniformly in n.*

Proof We begin by noting that it suffices to show that each $\mathbf{B}_n$ has a c.e. presentation, uniformly in n. Indeed, by a result of Fox [Fox24, Theorem 3.14], any c.e. presentation of a *simple* C*-algebra is computable (and the proof shows that the computability of the presentation is indeed uniform in the c.e. presentation).

Toward this end, fix n and fix a computable enumeration $(R_s)_{s\in\mathbb{N}}$ of R, that is, R_s consists of those natural numbers that have been determined to belong to R by stage s. Let q_k denote the kth special point of the standard presentation of $\mathcal{O}_{p_n+1}$. For all $k, s \in \mathbb{N}$ with $s \geq k$, we define $p_{k,s} \in \mathbf{B}_n$ as follows:

- If $n \notin R_s$, set $p_{k,s} := q_k$.
- If $n \in R_s\backslash R_k$, set $p_{k,s} := 0$.
- If $n \in R_k$, set $p_{k,s} := 1$.

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If $s < k$, we declare that $p_{k,s}$ is undefined. Given $k \in \mathbb{N}$, note that $p_{k,s}$ is eventually defined and constant; we denote this eventual constant value by p_k and we set $Q := \{p_k : k \in \mathbb{N}\}$.

Next note that in the case that $n \notin R$, we have $p_k = q_k$ for all k. On the other hand, if $n \in R$ and $s_0 \in \mathbb{N}$ is the least s for which $n \in R_s$, then we have $p_k = 0$ for $k < s_0$ and $p_k = 1$ for $k \geq s_0$. As a result, in either case, Q generates a dense $*$-subalgebra of $\mathbf{B}_n$, whence we may consider p_k as the kth special point of a presentation $\mathbf{B}_n^\#$ of $\mathbf{B}_n$.

It remains to observe that $\mathbf{B}_n^\#$ is a c.e. presentation of $\mathbf{B}_n$, uniformly in n. To see this, consider a rational $*$-polynomial $\rho(x_0, \ldots, x_m)$. Let $(r_s)_{s\in\mathbb{N}}$ be a computable nonincreasing sequence converging to $\|\rho(q_0, \ldots, q_m)\|$ in $\mathcal{O}_{p_{n+1}}$. We define a sequence $(t_s)_{s\geq m}$ as follows. If $n \notin A_s$, set $t_s := r_s$. If $n \in R_s \backslash R_k$, set $t_s := \rho(\vec{0})$. If $n \in R_k$, set $t_s := \rho(\vec{1})$. It follows that $(t_s)_{s\in\mathbb{N}}$ is a computable nonincreasing sequence, uniformly in n, which converges to $\rho(p_0, \ldots, p_m)$, completing the proof. ∎

Set $\mathbf{B} = \bigoplus_{n\in\mathbb{N}} \mathbf{B}_n$.

Proposition 3.2 $\mathbf{B}$ *has a computable presentation.*

Proof Using Lemma 3.1, fix presentations $\mathbf{B}_n^\#$ for the algebras $\mathbf{B}_n$ such that these presentations are computable uniformly in n. For each N, set $\mathbf{B}_N = \bigoplus_{n=1}^N \mathbf{B}_n$. Then each $\mathbf{B}_N$ has a computable presentation $\mathbf{B}_N^\#$, uniformly in N, obtained by taking the direct sum of the presentations $\mathbf{B}_1^\#, \ldots, \mathbf{B}_N^\#$. The inductive limit presentation constructed from the presentations $\mathbf{B}_N^\#$ as described in [Gol24, Section 2] is then a computable presentation of $\lim_N \mathbf{B}_N = \mathbf{B}$, by [Gol24, Lemma 2.7(3)]. ∎

Theorem 3.3 $K_0(\mathbf{B}) \cong G$ *and* $K_1(\mathbf{B}) = 0$.

Proof If $n \in R$, then $\mathbf{B}_n = \mathbb{C}$, so $K_0(\mathbf{B}_n) = \mathbb{Z}$, while if $n \notin R$, then $\mathbf{B}_n = \mathcal{O}_{p_n+1}$, so $K_0(\mathbf{B}_n) = \mathbb{Z}/p_n\mathbb{Z}$. That is, $K_0(\mathbf{B}_n) = G_n$ for all n.

Applying the fact that K_0 commutes with direct sums and direct limits [RLL00, Proposition 4.3.4 and Theorem 6.3.2], we have

$$\begin{aligned} K_0(\mathbf{B}) &= K_0\left(\bigoplus_{n\in\mathbb{N}} \mathbf{B}_n\right) \\ &\cong \bigoplus_{n\in\mathbb{N}} K_0(\mathbf{B}_n) \\ &\cong \bigoplus_{n\in\mathbb{N}} G_n \\ &= G. \end{aligned}$$

∎

Putting together the results so far, we have the following corollary.

Corollary 3.4 $\mathbf{B}$ *is a computably presentable* C*-*algebra and* $K_0(\mathbf{B})$ *has no computable presentation.*

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4 Non-computable K_1

Using the algebra $\mathbf{B}$ from the previous section, let $\mathbf{C}$ be the suspension of $\mathbf{B}$, that is, $\mathbf{C} = S\mathbf{B} := C_0(0,1) \otimes \mathbf{B}$.

Proposition 4.1 *$\mathbf{C}$ has a computable presentation.*

Proof This follows immediately from Proposition 3.2 and [EGMM, Lemma 3.1]. ∎

Proposition 4.2 *$K_0(\mathbf{C}) \cong G$ and $K_1(\mathbf{C}) = 0$.*

Proof Since $\mathbf{C} = S\mathbf{B}$, we have $K_0(\mathbf{C}) \cong K_1(\mathbf{B})$ and $K_1(\mathbf{C}) \cong K_0(\mathbf{B})$ (see [RLL00, Theorem 10.1.3 and Corollary 11.3.1]). ∎

We therefore have the following corollary.

Corollary 4.3 *$\mathbf{C}$ is a computably presentable C^*-algebra and $K_1(\mathbf{C})$ has no computable presentation.*

5 The main examples

Putting together the examples from the previous two sections, we get a single computably presentable C^*-algebra where neither K_0 nor K_1 is computably presented.

Theorem 5.1 *There is a computably presentable C^*-algebra $\mathbf{A}$ such that neither $K_0(\mathbf{A})$ nor $K_1(\mathbf{A})$ are computably presentable.*

Proof Using the algebras $\mathbf{B}$ and $\mathbf{C}$ defined earlier, let $\mathbf{A} = \mathbf{B} \oplus \mathbf{C}$. The algebra $\mathbf{A}$ has a computable presentation because $\mathbf{B}$ and $\mathbf{C}$ have computable presentations.

As mentioned above, K_0 commutes with direct sums, so Propositions 3.3 and 4.2 yield that

$$K_0(\mathbf{A}) \cong K_0(\mathbf{B}) \oplus K_0(\mathbf{C}) \cong G \oplus 0 \cong G.$$

Likewise, K_1 commutes with direct sums [RLL00, Proposition 8.2.6], so we have

$$K_1(\mathbf{A}) \cong K_1(\mathbf{B}) \oplus K_1(\mathbf{C}) \cong 0 \oplus G \cong G.$$

By Proposition 2.1, neither $K_0(\mathbf{A})$ nor $K_1(\mathbf{A})$ has a computable presentation. ∎

Finally, we reach our main result.

Theorem 5.2 *There is a computably presentable, unital, nuclear C^*-algebra $\mathbf{D}$ such that neither $K_0(\mathbf{D})$ nor $K_1(\mathbf{D})$ is computably presentable.*

Proof Let $\mathbf{D}$ be the unitization of the algebra $\mathbf{A}$ used in the proof of Theorem 5.1. Note that since each $\mathbf{B}_n$ is nuclear, $\mathbf{D}$ is nuclear as well. Since $\mathbf{A}$ has a computable

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presentation, $\mathbf{D}$ does as well by [EGMM, Proposition 2.21]. We have $K_1(\mathbf{D}) \cong K_1(\mathbf{A})$ [RLL00, Equation (8.4)], so $K_1(\mathbf{D})$ has no computable presentation.

For K_0, we have $K_0(\mathbf{D}) \cong K_0(\mathbf{A}) \oplus \mathbb{Z}$ [RLL00, Example 4.3.5]. Since $G \oplus \mathbb{Z} \cong G$, $K_0(\mathbf{D}))$ has no computable presentation. ∎

6 A stably finite example

The algebra $\mathbf{D}$ in the previous section is *infinite* in the sense that it has a proper isometry. As mentioned in the introduction, AF algebras have the property that any computable presentation of them yields a computable induced presentation on the K_0 group. Since AF algebras are finite, one might suspect that the distinction at play here is the finite versus infinite one. In this section, we show that one can in fact find stably finite C*-algebras satisfying the conclusion of Theorem 5.2.

For each $n \geq 2$, let $\Gamma_n := \langle a, b \mid bab^{-1} = a^n \rangle$, the so-called Baumslag–Solitar group $BS(1, n)$. The following lemma captures the key facts we need about the group C*-algebras of the Baumslag–Solitar groups.

Lemma 6.1 *For every $n \geq 2$, $C^*(\Gamma_n)$ is a stably finite residually finite-dimensional* C*-*algebra and has $K_0(C^*(\Gamma_n)) \cong \mathbb{Z}$ and $K_1(C^*(\Gamma_n)) \cong \mathbb{Z} \oplus \mathbb{Z}/(n-1)\mathbb{Z}$.*

Proof We may view $\Gamma_n = \langle a, b \mid bab^{-1} = a^n \rangle$ as a semi-direct product $\mathbb{Z}[1/n] \rtimes \mathbb{Z}$, where the $\mathbb{Z}$ factor corresponds to the subgroup $\langle b \rangle$, which acts by taking nth powers. Thus, Γ_n is an extension of $\mathbb{Z}[1/n]$ by $\mathbb{Z}$, and hence is an (elementary) amenable group. Therefore, $C^*(\Gamma_n) \cong C_r^*(\Gamma_n)$ (see [BO08, Theorem 2.6.8]). It thus suffices to show that $C_r^*(\Gamma_n)$ is residually finite dimensional and stably finite. The latter property is shared by all reduced group C*-algebras, because they have faithful tracial states (see [BO08, Proposition 2.5.3]), so we only need to show that $C_r^*(\Gamma_n)$ is residually finite dimensional.

Each Γ_n is a linear group, as it is isomorphic to the subgroup

$$\left\{ \begin{bmatrix} n^k & s \\ 0 & 1 \end{bmatrix} : k \in \mathbb{Z}, s \in \mathbb{Z}[1/n] \right\}$$

of $\mathrm{GL}_2(\mathbb{C})$. Thus, by [BL00, Theorem 4.3], each $C_r^*(\Gamma_n)$ is residually finite dimensional. Again, we have $C^*(\Gamma_n) \cong C_r^*(\Gamma_n)$, so the former is also residually finite dimensional. The statements about the K-theory of $C^*(\Gamma_n)$ are proved in [PV18, Theorem 1]. ∎

Now define $\mathbf{B}_n$ as in Section 3, except replace $\mathcal{O}_{p_n+1}$ by $C^*(\Gamma_{p_n+1})$ when $n \notin R$. Equip $C^*(\Gamma_{p_n+1})$ with its standard presentation. Now one observes that Lemma 3.1 continues to hold in this setting. Indeed, the standard presentation of $C^*(\Gamma_{p_n+1})$ is clearly c.e. uniformly in n. Moreover, this presentation is in fact computable uniformly in n by [Fox24, Theorem 3.5], since $C^*(\Gamma_{p_n+1})$ is residually finite dimensional, thereby establishing Lemma 3.1 with the redefined $\mathbf{B}_n$. As a result, setting $\mathbf{B} := \bigoplus_n \mathbf{B}_n$, we have that $K_1(\mathbf{B}) \cong G$, where G is the group admitting no computable presentation defined in Section 2, and $K_0(\mathbf{B}) = \bigoplus_n \mathbb{Z}$. The algebra $\mathbf{B}$ is stably finite because each $\mathbf{B}_n$ is stably

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finite. We also note that since each Γ_n is amenable each $\mathbf{B}_n$ is nuclear, and hence $\mathbf{B}$ is nuclear as well.

We have thus found a stably finite C*-algebra $\mathbf{B}$ that is computably presentable (the validity of Proposition 3.2 depended only on the conclusion of Lemma 3.1) for which $K_1(\mathbf{B})$ is not computably presentable. Setting $\mathbf{A} := \widetilde{S\mathbf{B}}$, the unitization of the suspension of $\mathbf{B}$, we have that $K_0(\mathbf{A}) \cong K_1(\mathbf{B}) \oplus \mathbb{Z} \cong G \oplus \mathbb{Z} \cong G$. Note also that $\mathbf{A}$ is computably presentable, stably finite and that $K_1(\mathbf{A}) \cong K_0(\mathbf{B}) = \bigoplus_n \mathbb{Z}$. Finally, setting $\mathbf{E} := \mathbf{A} \oplus \mathbf{B}$ yields a computably presentable, stably finite C*-algebra with $K_0(\mathbf{E}) \cong K_0(\mathbf{A}) \oplus K_0(\mathbf{B}) \cong G$ and $K_1(\mathbf{E}) \cong K_1(\mathbf{A}) \oplus K_1(\mathbf{B}) \cong G$, neither of which are computably presentable. In summary, we have shown the following theorem.

Theorem 6.2 *There is a computably presentable, stably finite, unital, nuclear* C*-*algebra* $\mathbf{E}$ *such that neither* $K_0(\mathbf{E})$ *nor* $K_1(\mathbf{E})$ *has a computable presentation.*

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