Removing a vertex from a tetrahedron

AIMS Class 2013-14 and A. F. BEARDON

1. *Introduction*

In [1] Nick Lord considered the problem of characterising those triangles that can be obtained by cutting a single vertex from a cube, and from a tetrahedron. Explicitly, with a single plane cut we remove a vertex from the solid, thereby exposing a triangular face, and we would like to characterise these triangular faces up to similarity. The solution in the case of the cube is easy: *each such triangular face is an acute-angled triangle, and any acute-angled triangle can arise in this way*. The characterisation of the triangular faces in the case of even a regular tetrahedron remains open.

The African Institute of Mathematical Sciences (AIMS) is located in the town of Muizenberg on the Cape Peninsula in South Africa. While teaching a course in problem solving there, the second author introduced the class to this problem, and this paper contains an account of the progress made with the AIMS students 2013-14. We discuss the cube first, and then the regular tetrahedron.

2. *Cutting a vertex from a cube*

Consider the situation in which a single vertex V is cut from a cube: this is illustrated in Figure 1 where the three solid lines represent three edges of the cube that meet at V, and the three dotted lines represent the edges of the triangular face ΔABC that is exposed once V has been cut from the cube.

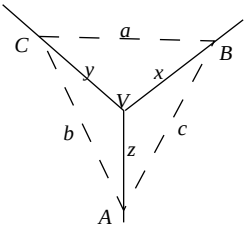


FIGURE 1: The vertex V

As each of the three angles at V are $\frac{\pi}{2}$, Pythagoras' Theorem gives $a^2 = x^2 + y^2$, $b^2 = y^2 + z^2$ and $c^2 = z^2 + x^2$. Thus, for any cut, we have

$$a^2 + b^2 > c^2, \qquad b^2 + c^2 > a^2, \qquad c^2 + a^2 > b^2, \tag{1}$$

and these inequalities together with the cosine rule show that ΔABC is an acute-angled triangle.

Suppose now that an acute-angled triangle is given, say with sides of lengths a, b and c. Then, because the triangle is acute, (1) holds, so there are positive x, y and z such that

$$2x^2 = c^2 + a^2 - b^2, \qquad 2y^2 = a^2 + b^2 - c^2, \qquad 2z^2 = b^2 + c^2 - a^2.$$

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By using these x, y and z to define the cut (as in Figure 1), we find that the exposed triangular face has sides of lengths a, b and c.

In addition to this analytic solution to the problem for the cube, the following two geometric proofs (that any acute-angled triangle can arise in this way) might be of interest to the reader. In each case, we are given an acute-angled triangle T whose vertices are labelled A, B and C.

The first proof

Place the side AB on the (horizontal) floor, with the vertex C above the floor, but not vertically above the side AB. Let D be the vertical projection of C onto the floor, and consider the tetrahedron with vertices A, B, C and D whose base ΔABD is on the floor. Because T is acute, when C is almost above AB, D is almost on the segment AB, and $\angle ADB > \frac{\pi}{2}$. When T is flat on the floor, $C = D$ and $\angle ADB = \angle ACB < \frac{\pi}{2}$. Thus, as we rotate T about the side AB, the angle $\angle ADB$ changes continuously from being greater than $\frac{\pi}{2}$ to less than $\frac{\pi}{2}$. We conclude that for some position of C, say C', we have $\angle ADB = \frac{\pi}{2}$. As the segment $C'D$ is orthogonal to ΔABD, we see that $\angle ADB = \angle ADC' = \angle BDC' = \frac{\pi}{2}$. We may now take D to be the vertex of the cube and the problem is solved.

The second proof

Construct the semicircle S with diameter AB as in Figure 2. Now choose D on the line containing AB such that CD is orthogonal to AB. As T is acute, D will lie between A and B. Now let V be the point where CD meets S; then $\angle AVB = \frac{\pi}{2}$ and, as $\angle ACB < \frac{\pi}{2}$, V will be inside T. Now place T on top of the cube so that V coincides with the vertex to be cut off, and VA and VB are along the edges of the cube. If we now 'rotate' T about the line AB, eventually C will be vertically below V and this solves the problem.

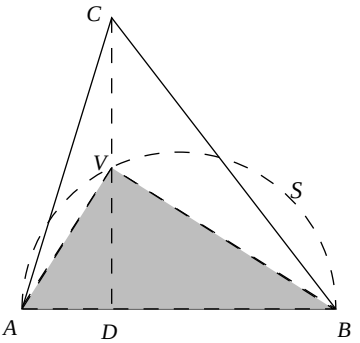


FIGURE 2: The semicircle construction

A graphical representation

We shall now represent our result on the cube graphically by using the familiar theorem of the 17th century mathematician Vincenzo Viviani,

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namely that *the sum of the distances from a point inside an equilateral triangle to the three sides of the triangle is constant, and equal to the height of the triangle* (see [2] for more information). Let $\mathscr{E}$ be the equilateral triangle which has vertices at $U = (0, 0)$, $V = \left(2\pi/\sqrt{3}, 0\right)$ and $W = \left(\pi/\sqrt{3}, \pi\right)$. Then, for any point P inside $\mathscr{E}$, we have $d_U + d_V + d_W = \pi$, where, for example, d_U is the distance of P from the side VW.

Now take any *labelled triangle* T (by this we mean a triangle whose vertices are labelled as, say, A, B and C), and let the angles at A, B and C be θ_1, θ_2 and θ_3, respectively. Then T can be represented by the point P in $\mathscr{E}$, where $\left(d_U, d_V, d_W\right) = \left(\theta_1, \theta_2, \theta_3\right)$. As the set of triangles that can be obtained by removing a vertex from a cube is represented by the inequalities $0 < \theta_j < \frac{\pi}{2}$, $j = 1,2,3$, we see that the set of triangular faces obtained by removing a vertex from a cube is represented by the shaded region in Figure 3.

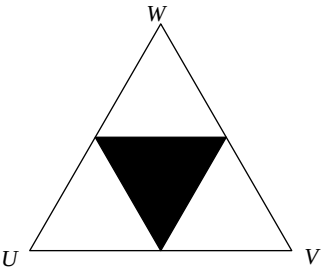


FIGURE 3: Triangular faces from a cube

4. *The problem for a tetrahedron*

Now consider the problem of removing a vertex from a tetrahedron. There is a significant difference between the cube and the tetrahedron, for each tetrahedron has *two* significant angles, namely the angle θ of each face at V, and the dihedral angle Θ between any two faces that contain V. For the cube, $\theta = \Theta$, but for the general tetrahedron this is not so.

As we are only interested in ΔABC up to similarity, and parallel cuts give exposed faces that are similar triangles, we need only consider cuts that lie in some small neighbourhood of V. Thus the part of the tetrahedron that lies outside this neighbourhood is irrelevant. With this observation, we can rephrase the problem as follows. Consider a tetrahedron $\mathscr{T}$ placed on a horizontal surface, and let F be the face of $\mathscr{T}$ that is in contact with this surface. We shall assume that F is an equilateral triangle, and that the vertex V of $\mathscr{T}$ which is above F lies directly above the centroid of F. Then the tetrahedron has rotational symmetry of order three about the vertical line through V and the centroid, and the three faces of $\mathscr{T}$ that meet at V have the same angle, say θ, at V. The problem of the cube corresponds to the case $\theta = \frac{\pi}{2}$, and the case of a regular tetrahedron corresponds to $\theta = \frac{\pi}{3}$. In all cases, $0 < \theta < \frac{2\pi}{3}$, and the tetrahedron is 'flat' when $\theta = \frac{2\pi}{3}$. In [1], Nick

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Lord considered whether or not we can characterise, up to similarity, the set of triangular faces that are exposed after cutting V from $\mathcal{T}$, and proved the following result (which contains the result for a cube as a special case).

Theorem 4.1: Let T be the exposed triangular face formed by cutting the vertex V from $\mathcal{T}$. Then each angle of T is at most $\max\{\theta, \pi - \theta\}$, and this is best possible.

Proof: This proof is shorter than that given in [1]. We use Figure 1 again, but we now imagine that this represents the vertex of the given tetrahedron where each of the faces has an angle θ at V. Let $\lambda = 2\cos\theta$. As $0 < \theta < \frac{2\pi}{3}$, we see that $-1 < \lambda < 2$; the cube corresponds to $\lambda = 0$, the regular tetrahedron to $\lambda = 1$, and the 'flat' tetrahedron to $\lambda = -1$. Now apply the cosine rule to the triangles in Figure 1; this gives

$$a^2 = x^2 + y^2 - \lambda xy, \quad b^2 = y^2 + z^2 - \lambda yz, \quad c^2 = z^2 + x^2 - \lambda xz.$$

Thus

$$\begin{aligned}
\cos\angle BAC &= \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{\lambda(x - z)(y - z) + (2 - \lambda)z^2}{2bc} \\
&\geqslant \frac{\lambda(x - z)(y - z)}{2bc} \\
&\geqslant \frac{-|\lambda|}{2} \\
&= \begin{cases} -\frac{\lambda}{2} & \text{if } 0 \leqslant \lambda < 2 \\ \frac{\lambda}{2} & \text{if } -1 < \lambda \leqslant 0 \end{cases} \\
&= \begin{cases} \cos(\pi - \theta) & \text{if } 0 \leqslant \lambda < 2 \\ \cos\theta & \text{if } -1 < \lambda \leqslant 0 \end{cases}
\end{aligned} \tag{2}$$

since, from the triangle inequality, $b \geqslant |y - z|$ and $c \geqslant |x - z|$. This proves that $\angle ABC \leqslant \max\{\theta, \pi - \theta\}$, and, by a similar argument, the angles $\angle ABC$ and $\angle ACB$ also satisfy this inequality.

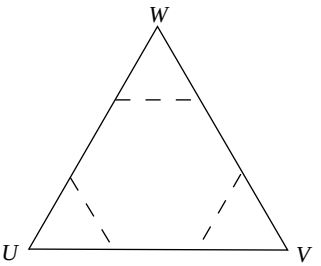


FIGURE 4: A bound on the triangular faces from a regular tetrahedron

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Theorem 4.1 implies that, in the case of a regular tetrahedron, no angle of T exceeds $\frac{2\pi}{3}$, so that if we use the graphical representation of triangles described above, we see that the set of triangular faces obtained by cutting a vertex from a regular tetrahedron is represented by some subset of the regular hexagon illustrated in Figure 4. Now it is stated in [1, p. 314] that 'it seems likely that this is the only restriction on the shape of such a triangle'. However, if this is so, then the set of possible triangular faces would be represented by the entire hexagon. We shall now show that this is incorrect by showing that some points inside this hexagon correspond to triangles that *cannot* be obtained by cutting the vertex from a regular tetrahedron. Explicitly, we shall prove the following result.

Theorem 4.2: Suppose that the exposed triangular face T formed by cutting the vertex V from a regular tetrahedron $\mathcal{T}$ is an isosceles triangle. Then T is an acute-angled triangle.

Proof: We may assume (in the notation of Figure 1) that $b = c$, and that $\angle ABC = \angle ACB = \alpha$ and $\angle BAC = \beta$. We need only show that $\beta \leqslant \frac{\pi}{2}$. First, there are positive x, y and z with

$$x^2 + y^2 - xy = a^2,$$
$$y^2 + z^2 - yz = b^2, \tag{3}$$
$$z^2 + x^2 - xz = c^2. \tag{4}$$

By assumption, $b = c$, and this with (3) and (4) shows that $x = y$ or $z = x + y$. Now from (2) (with $\lambda = 1$) we have

$$\cos\beta = \cos\angle BAC = \frac{(z - x)(z - y) + z^2}{2bc}.$$

As $(z - x)(z - y) + z^2 > 0$ when $x = y$, and when $z = x + y$, we see that $\cos\beta > 0$. As $0 < \beta < \pi$, this implies that $0 < \beta < \frac{\pi}{2}$ as required.

We end this section with a general comment about the graphical representation of the triangles. Given a triangle, say with angles θ_1, θ_2 and θ_3, we wish to locate the corresponding point P, say with coordinates (X, Y), inside $\mathcal{E}$. Thus, in order to plot the points P that correspond to certain triangles, we need an explicit formula for the function $(\theta_1, \theta_2, \theta_3) \to (X, Y)$, where $\theta_1 + \theta_2 + \theta_3 = \pi$. If we use the triangle $\mathcal{E}$, then the point (X, Y) is the intersection of the lines L_1 and L_2, where L_1 is parallel to UW and at a distance θ_2 from it, and L_2 is parallel to UV and at a distance θ_3 from it. Thus L_1 and L_2 have equations $y = \sqrt{3}x - 2\theta_2$ and $y = \theta_3$, respectively, and we find that

$$(X, Y) = \left(\frac{2\theta_2 + \theta_3}{\sqrt{3}}, \theta_3\right).$$

Of course, this general result is not needed for the special case that arises from cutting the cube as in this case the graphical representation is obvious.

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5. *Further remarks*

The fact that not all points inside the regular hexagon represent an exposed triangular face prompted a discussion of exactly which triangles can be obtained from a regular tetrahedron, and the AIMS class of 2013-14 performed numerical experiments which provided experimental data. Individuals selected values of x, y and z, calculated the angles of T, and then plotted their results as a subset of the equilateral triangle referred to above. This experimental data led to a plot of points which suggested that the set of triangular faces obtained in this way might be represented by a set which looks something like the shaded region in Figure 5.

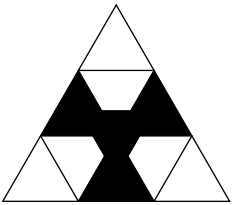


FIGURE 5: The conjectured set of triangles

Our thinking at this point was that the shaded region in Figure 5 might possibly be the union of four shaded triangles, namely a small black triangle corresponding to the condition $\max\{\theta_1, \theta_2, \theta_3\} \leqslant \Theta$, together with three more black triangles, each bounded by two of the three lines $\theta_j = \frac{\pi}{3}$ (which pass through the centroid of $\mathscr{E}$, and which are parallel to the sides of $\mathscr{E}$). These last three triangles might correspond to the condition that at least two of the θ_j are not less that $\frac{\pi}{3}$. However, a more detailed numerical examination suggests that this conjecture may not be true, and the exact shape of the set that represents all exposed faces is not at all clear.

6. *Intersecting cylinders*

Finally, we consider the problem from a different perspective. We begin with the cube, and use the same notation as before. The equation $a^2 = x^2 + y^2$ represents a circular cylinder in $\mathbb{R}^3$ whose axis is the z-coordinate axis. A similar statement holds for the equations $b^2 = y^2 + z^2$ and $c^2 = z^2 + x^2$; thus if a, b and c arise as the lengths of the sides of an exposed face, with distances x, y and z determining the cut, then the three circular cylinders whose radii are a, b and c, and whose axes are the coordinates axes, must intersect at the point (x, y, z) in $\mathbb{R}^3$. If we project the cylinders given by $y^2 + z^2 = b^2$ and $z^2 + x^2 = c^2$ onto the plane $z = 0$, we obtain the two regions $|y| \leqslant b$ and $|x| \leqslant c$. Given that (x, y, z) represents some cut (that is x, y and z are positive), the point (x, y, z) lies on all three cylinders, so that we must have $a^2 < b^2 + c^2$, and so on.

Let us now consider the problem of removing a vertex from a tetrahedron. In a similar way, the equation $x^2 + y^2 - xy = a^2$, which can

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be rewritten as $(x + y)^2 + 3(x - y)^2 = 4a^2$, represents a vertical cylinder in $\mathbb{R}^3$, with an elliptical cross section which, in the plane $z = 0$, is an ellipse with major and minor axes along the lines $x + y = 0$ and $x - y = 0$. If we interpret the other two equations in the same way, the problem becomes that of finding the intersection of three elliptical cylinders whose axes are mutually orthogonal. This suggests that a simple, elegant solution to the tetrahedron problem might not exist.

References

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2. M. de Villiers, 3D Generalisations of Viviani's Theorem, *Math. Gaz.*, **97** (November 2013) pp. 441-445.

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A. F. BEARDON

Department of Pure Mathematics and Mathematical Statistics (DPMMS), University of Cambridge CB3 0WB

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