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Exploring a local variant of the Buchsbaum–Eisenbud–Horrocks conjecture

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The Buchsbaum–Eisenbud–Horrocks conjecture has attracted the attention of many researchers working in Commutative Algebra and Algebraic Geometry in the last fifty years. Quite recently, a variant of this conjecture has been formulated by Lima-Pereira, Nuño-Ballesteros, Orefice-Okamoto, and Tomazella in their study of complete intersection singularities. The purpose of this paper is, on the one hand, to exhibit some cases where this conjecture holds. On the other hand, we show that the conjecture fails in general by exhibiting some counterexamples. Finally, we formulate a conjecture that can be regarded as a local variant of Buchsbaum–Eisenbud–Horrocks' one.

Keywords: Buchsbaum–Eisenbud–Horrocks conjecture; Tor modules; Koszul homology; complete intersection ideals; complete intersection singularities; regular local rings; length function in commutative algebra

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1. Introduction

The Buchsbaum–Eisenbud–Horrocks conjecture, formulated at the end of the 80's independently by Buchsbaum and Eisenbud [5, 1.4] and Horrocks [11, Problem 24], can be stated in the following way.

Conjecture (Buchsbaum–Eisenbud–Horrocks). *Let R be a commutative Noetherian ring such that* $\operatorname{Spec}(R)$ *is connected, and let M be a nonzero, finitely generated R-module of finite projective dimension. Then, for any finite projective resolution of M*

$$0 \longrightarrow P_d \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

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we have

$$\operatorname{rank}_R(P_i) \geq \binom{c}{i},$$

where $c = \operatorname{ht}((0 :_R M))$.

The interested reader in the Buchsbaum–Eisenbud–Horrocks conjecture may like to consult [1], a nice survey of what is known and unknown about this problem, especially when R is a polynomial ring in several variables over a field. We also want to mention that in [1, Conjecture 4.8 and Lemma 4.9], it is observed that, in order to prove the Buchsbaum–Eisenbud–Horrocks problem in the case where R is a local ring, it is enough to restrict attention to the class of modules with finite length.

The purpose of this paper is to study the following conjecture, formulated by Lima-Pereira, Nuño-Ballesteros, Orefice-Okamoto, and Tomazella in their study of complete intersection singularities (see [14, Conjecture 4.9]). It may be regarded as a variant of the Buchsbaum–Eisenbud–Horrocks conjecture in the local case.

Conjecture 1 (Lima-Pereira, Nuño-Ballesteros, Orefice-Okamoto, Tomazella). *Let* $(R, \mathfrak{m})$ *be a regular local ring of dimension* n*, let* I *be an ideal generated by an* R*-regular sequence of length* $k \leq n$*, and let* J *be an ideal generated by a regular sequence of length* n*. Assume, in addition, that* $I + J$ *is* $\mathfrak{m}$*-primary. Then, if* ℓ *denotes the length function in the category of* R*-modules, then*

$$\ell\left(\operatorname{Tor}_i^R(R/I, R/J)\right) = \binom{k}{i} \ell\left(\frac{R}{I+J}\right)$$

for any $0 \leq i \leq k$.

We have two main goals in this paper: on the one hand, we give particular settings where Conjecture 1 holds. On the other hand, we show that it is in general not true by exhibiting some counterexamples. At the end, we formulate a more general conjecture, exhibiting some particular examples where it holds.

Now, let us provide a more detailed overview of the contents of this paper for the convenience of the reader. In Section 2, we present some general settings where Conjecture 1 holds. More precisely, we can phrase the main results obtained in Section 2 as follows.

Theorem 1. *Conjecture 1 holds if at least one of the following assumptions holds.*

- *(i) Either* $I \subseteq J$ *or* $J \subseteq I$ *(see Proposition 3).*
- *(ii) If* $k = 2$ *(see Discussion 1).*
- *(iii)* $J = (f_1, \ldots, f_n)$*,* $(f_1, \ldots, f_k) \subseteq I$ *for some* $1 \leq k \leq n-1$*, and* $f_{k+1}, \ldots, f_n$ *is an* (R/I)*-regular sequence (see Proposition 6).*
- *(iv)* $n \geq 4$*,* $J = (f_1, \ldots, f_n)$*,* $(f_1, \ldots, f_{k-1}) \subseteq I$ *for some* $3 \leq k \leq n-1$*, and* $f_{k+1}, \ldots, f_n$ *is an* (R/I)*-regular sequence (see Proposition 7).*

Some remarks are in order here. First, part (i) of Theorem 1 was already proved by Borna [2] in his Ph.D. thesis. Second, part (ii) of Theorem 1 was proved in [14].

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However, in this paper we present alternative proofs of these cases, which allow us to obtain more general statements, as the ones obtained in parts (iii) and (iv) of Theorem 1.

In Section 3, we will show that Conjecture 1 always holds for a certain subclass of complete intersection monomial ideals (see Notation 1). In Section 4, we exhibit explicit counterexamples to Conjecture 1 showing, in particular, that this conjecture cannot hold for any complete intersection binomial ideal.

Section 5 can be independently read from the rest of the paper. In this section, we briefly review the motivation for studying Conjecture 1 from the perspective of Singularity Theory. Finally, in Section 6, we formulate a more general conjecture that we feel is the real generalization of Buchsbaum–Eisenbud–Horrocks' one in this setting. We also raise the corresponding Total Rank Conjecture in this local framework, showing that it cannot be deduced directly from the celebrated papers by VandeBogert and Walker [23, 25] proving the classical Total Rank Conjecture.

Throughout this paper, all the rings are commutative with unity. Moreover, given a commutative ring R, ℓ_R (or simply ℓ if the context is clear) will always denote the length function in the category of R-modules. Finally, by a local ring $(R, \mathfrak{m})$ we mean a commutative Noetherian ring R with a unique maximal ideal $\mathfrak{m}$.

2. Some positive general results

The goal of this section is to provide several situations where Conjecture 1 has a positive answer. We start by reviewing an elementary fact about the behaviour of the length function for the sake of completeness, see [4, page 15, 1.2.25] and [22, Tag 02M1] for a proof.

LEMMA 1. *Let $(R, \mathfrak{m}_R) \xrightarrow{\phi} (S, \mathfrak{m}_S)$ be a homomorphism of local rings, and let N be an S-module which is flat as R-module (N is regarded as R-module by restriction of scalars under ϕ) such that $\ell_S(N/\mathfrak{m}_R N) < \infty$. Then, for any R-module M of finite length, we have*

$$\ell_S(M \otimes_R N) = \ell_R(M) \cdot \ell_S\left(\frac{N}{\mathfrak{m}_R N}\right).$$

The next thing we plan to show is that there is no loss of generality in assuming in Conjecture 1 that R is a complete regular local ring. This is a consequence of the following technical statement.

PROPOSITION 1. *Let $(R, \mathfrak{m})$ be a local ring of dimension n, and let $R \xrightarrow{\phi} S$ be a flat local ring homomorphism, with $(S, \mathfrak{m}_S)$ a regular local ring, such that $\dim(S) = \dim(R)$ and that $\ell_S(S/\mathfrak{m}S) < \infty$. Assume that Conjecture 1 holds for S. Then, Conjecture 1 is also valid for R.*

Proof. First, we observe that, since S is regular and flat over R we have, by [15, page 162, Theorem 51], that R is also regular. Now, let I be an ideal of R, generated by a regular sequence of length $k \leq n$, and let J be an ideal of R generated by a regular sequence of length n. Assume in addition that $I + J$ is $\mathfrak{m}$-primary. By [4, Proposition 1.1.2], we know that IS and JS are ideals of S generated by regular

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sequences of lengths k and n, respectively. Moreover, since $I + J$ is $\mathfrak{m}$-primary, we also have, since by assumption ϕ is flat with closed fibre of finite length, that $(I + J)S = IS + JS$ is also $\mathfrak{m}_S$-primary because of Lemma 1. In this way, since Conjecture 1 holds for S we have

$$\ell_S\left(\mathrm{Tor}_i^S(S/IS, S/JS)\right) = \binom{k}{i}\ell_S\left(\frac{S}{(I+J)S}\right).$$

Notice that this equality is equivalent to the following one, because ϕ is flat [20, Corollary 10.61].

$$\ell_S\left(\mathrm{Tor}_i^R(R/I, R/J) \otimes_R S\right) = \binom{k}{i}\ell_S\left(\frac{R}{I+J} \otimes_R S\right). \tag{1}$$

Now, on the one hand, we observe that, by assumption, $\ell_S(S/\mathfrak{m}S) < \infty$. On the other hand, since $I+J$ is $\mathfrak{m}$-primary, and $\mathrm{Tor}_i^R(R/I, R/J)$ is a finitely generated R-module that is annihilated by $I+J$, we have, by [22, Tag 00J0], that $\mathrm{Tor}_i^R(R/I, R/J)$ has finite length. In this way, keeping in mind these facts, we can use Lemma 1 again to rewrite (1) as follows.

$$\ell_R\left(\mathrm{Tor}_i^R(R/I, R/J)\right) \cdot \ell_S(S/\mathfrak{m}S) = \binom{k}{i}\ell_R\left(\frac{R}{I+J}\right) \cdot \ell_S(S/\mathfrak{m}S).$$

And this equality is equivalent, using again our assumption that $\ell_S(S/\mathfrak{m}S) < \infty$, to

$$\ell_R\left(\mathrm{Tor}_i^R(R/I, R/J)\right) = \binom{k}{i}\ell\left(\frac{R}{I+J}\right),$$

which is precisely what we want to prove. □

Corollary 1. *In Conjecture 1 we can assume, without loss of generality, that $(R, \mathfrak{m})$ is a regular local ring, complete with respect to the $\mathfrak{m}$-adic topology. Equivalently, by the Cohen structure theorem [15, pages 217–218], we can assume that R is a formal power series ring with coefficients either on a field or over a Cohen ring.*

We also need a graded variant of Proposition 1, which we formulate now. The following statement will be useful for us in Section 3.

Proposition 2. *Let $\mathbb{K}$ be a field, let $S := \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$, regarded as graded ring with its natural standard $\mathbb{Z}$-grading. Let $I, J \subset S$ be ideals generated by regular sequences of length k and n, respectively, such that both regular sequences are made up of homogeneous elements, and set $\mathfrak{m} := (x_1, \ldots, x_n)$. Moreover, assume that $\sqrt{I+J} = \mathfrak{m}$. Then, Conjecture 1 holds for S, I and J if and only if it holds for $R := S_{\mathfrak{m}}$, IR and JR.*

Proof. Once again, by [4, Proposition 1.1.2], IR and JR are also generated by regular sequences in R of lengths k and n, respectively. Moreover, by [4, Proposition 1.5.15], $- \otimes_S R$ is faithfully flat in the category of graded S-modules. Since both

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$\operatorname{Tor}_i^S(S/I, S/J)$ and $S/(I+J)$ are finite length, graded S-modules supported at $\mathfrak{m}$, and for these modules localization at $\mathfrak{m}$ preserves length, we have that

$$\ell_S\left(\operatorname{Tor}_i^S(S/I, S/J)\right) = \binom{k}{i}\ell\left(\frac{S}{I+J}\right)$$

if and only if

$$\ell_R\left(\operatorname{Tor}_i^R(R/IR, R/JR)\right) = \binom{k}{i}\ell\left(\frac{R}{(I+J)R}\right),$$

just what we want to prove. □

As already pointed out in [14], a particular case of Conjecture 1 was proved in [2, Lemma 3.4.1]; in the next result we reproduce the proof presented there for the convenience of the reader.

PROPOSITION 3. *Conjecture 1 holds if either $I \subseteq J$ or $J \subseteq I$.*

Proof. First, assume that $I \subseteq J$. Let $f_1, \ldots, f_k$ be an R-regular sequence generating I, and let $K_\bullet(f_1, \ldots, f_k; R)$ be the Koszul complex with respect to this sequence of elements. Since $f_1, \ldots, f_k$ is an R-regular sequence, $K_\bullet(f_1, \ldots, f_k; R)$ is a free resolution of R/I of length k. Therefore, one has that

$$\operatorname{Tor}_i^R(R/I, R/J) \cong H_i\left(K_\bullet(f_1, \ldots, f_k; R) \otimes_R R/J\right).$$

However, since by assumption $I \subseteq J$ one has that all the differentials of $K_\bullet(f_1, \ldots, f_k; R) \otimes_R R/J$ are zero, and therefore one ends up, for each $i \geq 0$, with an R/J-isomorphism

$$\operatorname{Tor}_i^R(R/I, R/J) \cong (R/J)^{\oplus\binom{k}{i}}.$$

Since the length function is additive and $I \subseteq J$, one obtains that

$$\ell\left(\operatorname{Tor}_i^R(R/I, R/J)\right) = \binom{k}{i}\ell\left(R/J\right) = \binom{k}{i}\ell\left(R/(I+J)\right),$$

as claimed.

On the other hand, assume that $J \subseteq I$. Since J is generated by a regular sequence of length n, we have that I is $\mathfrak{m}$-primary, hence $k = n$. Now, let $g_1, \ldots, g_n$ be an R-regular sequence generating J, and let $K_\bullet(g_1, \ldots, g_n; R)$ be the Koszul complex with respect to this sequence of elements, which is a free resolution of R/J of length n. In this way, we have $\operatorname{Tor}_i^R(R/I, R/J) \cong H_i\left(K_\bullet(g_1, \ldots, g_n; R) \otimes_R R/I\right)$. Since by assumption $J \subseteq I$ one has that all the differentials of $K_\bullet(g_1, \ldots, g_n; R) \otimes_R R/I$ are

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zero, and therefore one ends up, for each $i \geq 0$, with an R/I-isomorphism

$$\mathrm{Tor}_i^R(R/I, R/J) \cong (R/I)^{\oplus \binom{n}{i}}.$$

Since the length function is additive and $J \subseteq I$, one finally obtains that

$$\ell\left(\mathrm{Tor}_i^R(R/I, R/J)\right) = \binom{n}{i}\ell\left(R/I\right) = \binom{n}{i}\ell\left(R/(I+J)\right).$$

This concludes the proof. □

Proposition 3 can also be proved using the following result obtained in [19, Lemma 2.2].

LEMMA 2. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, let $I \subseteq R$ be an ideal, and let $F_\bullet \longrightarrow M$ be a minimal free resolution of a finitely generated R-module M. Moreover, set*

$$\beta_j(M) := \dim_{R/\mathfrak{m}}\left(\mathrm{Tor}_j^R(R/\mathfrak{m}, M)\right),$$

the j-th Betti number of M. Then, the following statements are equivalent:

(i) Every differential d_j of $F_\bullet$ satisfies $d_j(F_j) \subseteq I \cdot F_{j-1}$.
(ii) For any $j \geq 0$, $\mathrm{Tor}_j^R(M, R/I)$ is a free R/I-module of rank $\beta_j(M)$.

The motivation for considering Lemma 2 stems from the following notion, which was introduced in [19, Definition 2.1], see also [6, Definition 2.4] for a closely related notion.

DEFINITION 1. *Let R be a commutative Noetherian local ring, and let I be an ideal of R. We say that I is a **weak complete intersection ideal** if, for any integer $j \geq 0$, $\mathrm{Tor}_j^R(R/I, R/I)$ is a free R/I-module of rank $\beta_j(R/I)$.*

Actually, thanks to Lemma 2, we can extend Proposition 3 in the following way:

PROPOSITION 4. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, let I be an ideal of R with finite projective dimension, and let $J \subseteq R$ be an ideal. Assume that J is $\mathfrak{m}$-primary, that $I \subseteq J$, and that I is a weak complete intersection ideal. Then, one has that*

$$\ell(\mathrm{Tor}_j^R(R/I, R/J)) = \beta_j(R/I)\ell\left(\frac{R}{J}\right).$$

Proof. Let $F_\bullet$ be the minimal free resolution of R/I as R-module; since by assumption I is a weak complete intersection, Lemma 2 guarantees that all the differentials of $F_\bullet$ can be represented by matrices whose entries are in I. Since $I \subseteq J$, one has

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that $F_\bullet \otimes_R R/J$ is a complex where all its differentials are zero, which implies that

$$H_j(F_\bullet \otimes_R R/J) \cong \left(\frac{R}{J}\right)^{\oplus \beta_j(R/I).}$$

This isomorphism implies, by the additivity of the length, that

$$\ell(\mathrm{Tor}_j^R(R/I, R/J)) = \ell(H_j(F_\bullet \otimes_R R/J)) = \beta_j(R/I)\ell\left(\frac{R}{J}\right),$$

just what we wanted to show. □

The following result is a direct consequence of [16, Theorem 2.2].

PROPOSITION 5. *Preserving the notations of Conjecture 1, for any integer $i \geq 0$, there is an R/J-isomorphism*

$$\mathrm{Hom}_{R/J}(\mathrm{Tor}_i^R(R/I, R/J), R/J) \cong \mathrm{Tor}_{k-i}^R(R/I, R/J).$$

In particular, we get the equality $\ell(\mathrm{Tor}_i^R(R/I, R/J)) = \ell(\mathrm{Tor}_{k-i}^R(R/I, R/J))$.

As an immediate consequence of Proposition 5, one obtains:

COROLLARY 2. *Conjecture 1 holds if and only if, for any $1 \leq i \leq \lfloor k/2 \rfloor$, we have*

$$\ell\left(\mathrm{Tor}_i^R(R/I, R/J)\right) = \binom{k}{i}\ell\left(\frac{R}{I+J}\right).$$

In particular, Corollary 2 leads to a different proof of Conjecture 1 for the case $k = 2$ with respect to the one given in [14].

DISCUSSION 1. *Assume that $I = (f_1, f_2)$, where f_1, f_2 is an R-regular sequence. Set $A := R/J$; in this particular case, the complex $K_\bullet(f_1, f_2; R) \otimes_R A$ is given by*

$$0 \longrightarrow A \xrightarrow{d_2} A^2 \xrightarrow{d_1} A \longrightarrow 0.$$

This complex can be split into the following short exact sequences:

$$0 \longrightarrow \ker(d_2) = \mathrm{Tor}_2^R(R/I, R/J) \longrightarrow A \longrightarrow \mathrm{Im}(d_2) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}(d_2) \longrightarrow \ker(d_1) \longrightarrow \mathrm{Tor}_1^R(R/I, R/J) \longrightarrow 0$$

$$0 \longrightarrow \ker(d_1) \longrightarrow A^2 \longrightarrow \tfrac{I+J}{J} \longrightarrow 0$$

$$0 \longrightarrow \tfrac{I+J}{J} \longrightarrow A \longrightarrow \tfrac{R}{I+J} \longrightarrow 0.$$

We know, thanks to Corollary 2, that

$$\ell(\mathrm{Tor}_2^R(R/I, R/J)) = \ell\left(\frac{R}{I+J}\right).$$

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Keeping in mind this fact, and using the additivity of the length with short exact sequences one has that

$$\ell(\mathrm{Tor}_1^R(R/I,R/J)) = \ell(\ker(d_1)) - \ell(\mathrm{Im}(d_2)) = 2\ell(A) - \ell\left(\frac{I+J}{J}\right) - \ell(\mathrm{Im}(d_2))$$
$$= 2\ell(A) + \ell\left(\frac{R}{I+J}\right) - \ell(A) - \ell(\mathrm{Im}(d_2)) = \ell(A) + \ell\left(\frac{R}{I+J}\right) - \ell(\mathrm{Im}(d_2))$$
$$= \ell\left(\frac{R}{I+J}\right) + \ell(A) - \ell(\mathrm{Im}(d_2)) = \ell\left(\frac{R}{I+J}\right) + \ell(\mathrm{Tor}_2^R(R/I,R/J)) = 2\ell\left(\frac{R}{I+J}\right),$$

just what we wanted to show.

Another particular instance where Conjecture 1 holds is given in the following statement. We present two proofs of this result for the reader's benefit. In the next result, given a sequence of elements $\mathbf{g} = g_1,\dots,g_t$ in a commutative ring A, $H_i(\mathbf{g};A)$ will denote the ith homology group of the Koszul complex of A with respect to the sequence $\mathbf{g}$.

PROPOSITION 6. *Suppose that I can be generated by a regular sequence of length k, and let $J = (f_1,\dots,f_n)$, where $\mathbf{f} = f_1,\dots,f_n$ is an R-regular sequence. Assume that $(f_1,\dots,f_k) \subseteq I$ and that $f_{k+1},\dots,f_n$ is an R/I-regular sequence. Then, Conjecture 1 holds.*

Proof. Recall that, given B an arbitrary commutative Noetherian ring, and given $\mathbf{b} = b_1,\dots,b_t$ a sequence of elements of B, by [12, Remark 1.4] one has, for any integer $i \geq 0$, that

$$H_i(\mathbf{b},0;B) \cong H_i(\mathbf{b};B) \oplus H_{i-1}(\mathbf{b};B). \tag{2}$$

Now, we come back to our setting; our assumption $(f_1,\dots,f_k) \subseteq I$ implies that

$$H_i(\mathbf{f};R/I) \cong H_i(f_{k+1},\dots,f_n,0,\dots,0;R/I).$$

Therefore, using (2) k times one ends up with the following isomorphism:

$$H_i(\mathbf{f};R/I) \cong \bigoplus_{j=0}^{i} H_{i-j}(f_{k+1},\dots,f_n;R/I)^{\oplus\binom{k}{j}}.$$

However, since by assumption $f_{k+1},\dots,f_n$ is an R/I-regular sequence, the last isomorphism boils down to the one below:

$$H_i(\mathbf{f};R/I) \cong H_0(f_{k+1},\dots,f_n;R/I)^{\oplus\binom{k}{i}} \cong \left(\frac{R}{I+J}\right)^{\oplus\binom{k}{i}}.$$

Summing up, since $\mathrm{Tor}_i^R(R/I,R/J) \cong H_i(\mathbf{f};R/I)$, one concludes, combining these last two isomorphisms, that

$$\ell(\mathrm{Tor}_i^R(R/I,R/J)) = \binom{k}{i}\ell\left(\frac{R}{I+J}\right),$$

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as claimed. □

Alternative proof of Proposition 6. Since, by assumption, $f_{k+1}, \dots, f_n$ is an R/I-regular sequence and that

$$\frac{R}{I + (f_{k+1}, \dots, f_n)} = \frac{R}{I + J},$$

one has, by [24, Corollary 5.16], that

$$H_i(\mathbf{f}; R/I) \cong H_i(f_1, \dots, f_k; R/(I + J)).$$

Finally, since $(f_1, \dots, f_k) \subseteq I + J$ one has that all the differentials of $K_\bullet(f_1, \dots, f_k; R/(I + J))$ are zero, and therefore one finally ends up with an isomorphism of $R/(I + J)$-modules

$$H_i(f_1, \dots, f_k; R/(I + J)) \cong \left(\frac{R}{I + J}\right)^{\oplus \binom{k}{i}}.$$

Summing up, we obtain

$$\ell(\mathrm{Tor}_i^R(R/I, R/J)) = \ell(H_i(\mathbf{f}; R/I)) = \ell(H_i(f_1, \dots, f_k; R/(I+J))) = \binom{k}{i} \ell\left(\frac{R}{I + J}\right),$$

as claimed. □

We can extend the conclusion obtained in Proposition 6 in the following way.

PROPOSITION 7. *Let $k \geq 3$, suppose that I can be generated by a regular sequence of length k, and let $J = (f_1, \dots, f_n)$, where $\mathbf{f} = f_1, \dots, f_n$ is an R-regular sequence. Assume that $(f_1, \dots, f_{k-1}) \subseteq I$, and that $f_{k+1}, \dots, f_n$ is an R/I-regular sequence. Then, Conjecture 1 holds.*

Proof. Let $g_1, \dots, g_k$ be a regular sequence generating I. Moreover, set

$$A := \frac{R}{(g_1, \dots, g_k, f_{k+1}, \dots, f_n)},$$

which by our assumption is a complete intersection ring, in particular Gorenstein. Fix an integer $j \geq 1$, and denote by $\mathbf{0}_{k-1}$ the sequence of $(k-1)$ consecutive zeroes. Now, using [24, Corollary 5.16] one has that

$$H_j(K_\bullet(f_1, \dots, f_n; R/I)) = H_j(K_\bullet(f_k, \mathbf{0}_{k-1}; A)).$$

Now, using (2) $k - 1$ times one ends up with the following isomorphism:

$$H_j(K_\bullet(f_1, \dots, f_n; R/I)) = \bigoplus_{s=0}^{j} H_{j-s}(K_\bullet(f_k; A))^{\oplus \binom{k-1}{s}}. \tag{3}$$

However, since the homology of the Koszul complex with respect to one element is concentrated always in degrees zero and one, the natural isomorphism (3) boils

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down to

$$H_j(K_\bullet(f_1,\dots,f_n;R/I)) = H_1(K_\bullet(f_k;A))^{\oplus\binom{k-1}{j-1}} \oplus H_0(K_\bullet(f_k;A))^{\oplus\binom{k-1}{j}}.$$

Moreover, since A is Gorenstein we have, using [16, Theorem 2.2], that

$$\mathrm{Hom}_A(\mathrm{Hom}_A(H_1(K_\bullet(f_k;A),A),A) \cong \mathrm{Hom}_A(H_0(K_\bullet(f_k;A),A)$$

and therefore, since the length of the linear dual of a module of finite length agrees with the length of the module itself [7, page 350], we get that

$$\ell\left(H_1(K_\bullet(f_k;A))\right) = \ell\left(H_0(K_\bullet(f_k;A))\right).$$

In this way, combining these two last equalities with the additivity of the length and the Pascal identity

$$\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j}$$

we obtain, taking into account

$$\ell\left(H_0(K_\bullet(f_k;A))\right) = \ell\left(\frac{R}{I+J}\right),$$

that

$$\ell(\mathrm{Tor}_j^R(R/I,R/J)) = \binom{k}{j}\ell\left(\frac{R}{I+J}\right).$$

This concludes the proof. □

The same strategy used in Proposition 7 can be used to prove the following generalization of it.

PROPOSITION 8. *Let $k \geq 3$, suppose that I can be generated by a regular sequence of length k, and let $J = (f_1,\dots,f_n)$, where $\mathbf{f} = f_1,\dots,f_n$ is an R-regular sequence. Assume that $(f_1,\dots,f_{k-2}) \subseteq I$, and that $f_{k+1},\dots,f_n$ is an R/I-regular sequence. Then, Conjecture 1 holds.*

Proof. As before, let $g_1,\dots,g_k$ be a regular sequence generating I, and set

$$A := \frac{R}{(g_1,\dots,g_k,f_{k+1},\dots,f_n)}.$$

Fix $j \geq 1$ be an integer, and denote by $\mathbf{0}_{k-2}$ the sequence of $(k-2)$ consecutive zeroes. Now, using [24, Corollary 5.16] one has that

$$H_j(K_\bullet(f_1,\dots,f_n;R/I)) = H_j(K_\bullet(f_{k-1},f_k,\mathbf{0}_{k-2};A)).$$

Now, using (2) $k-2$ times one ends up with the following isomorphism:

$$H_j(K_\bullet(f_1,\dots,f_n;R/I)) = \bigoplus_{s=0}^{j} H_{j-s}(K_\bullet(f_{k-1},f_k;A))^{\oplus\binom{k-2}{s}}. \tag{4}$$

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However, since the homology of the Koszul complex with respect to two elements is concentrated always in degrees zero, one, and two, the natural isomorphism (4) boils down to

$$H_j(K_\bullet(f_1,\dots,f_n;R/I)) = H_2(K_\bullet(f_{k-1},f_k;A))^{\oplus\binom{k-2}{j-2}} \oplus H_1(K_\bullet(f_{k-1},f_k;A))^{\oplus\binom{k-2}{j-1}} \oplus H_0(K_\bullet(f_{k-1},f_k;A))^{\oplus\binom{k-2}{j}}.$$

Moreover, since A is Gorenstein we have, using [16, Theorem 2.2], that

$$\mathrm{Hom}_A(\mathrm{Hom}_A(H_2(K_\bullet(f_{k-1},f_k;A),A),A) \cong \mathrm{Hom}_A(H_0(K_\bullet(f_{k-1},f_k;A),A),$$

and therefore, using again that the length of the linear dual of a module of finite length agrees with the length of the module itself, we have that

$$\ell\left(H_2(K_\bullet(f_{k-1},f_k;A))\right) = \ell\left(H_0(K_\bullet(f_{k-1},f_k;A))\right).$$

On the other hand, using again that A is Gorenstein, we can repeat the calculation done for proving Conjecture 1 in the case of two elements to obtain the following equality:

$$\ell\left(H_1(K_\bullet(f_{k-1},f_k;A))\right) = 2\ell\left(H_0(K_\bullet(f_{k-1},f_k;A))\right)$$

In this way, combining all these equalities with the additivity of the length and the identity

$$\begin{aligned}&\binom{k-2}{j-2} + 2\binom{k-2}{j-1} + \binom{k-2}{j} = \binom{k-2}{j-2} + \binom{k-2}{j-1} + \binom{k-2}{j-1} + \binom{k-2}{j}\\ &= \binom{k-1}{j-1} + \binom{k-1}{j} = \binom{k}{j}\end{aligned}$$

we end up with

$$\ell(\mathrm{Tor}_j^R(R/I,R/J)) = \binom{k}{j}\ell\left(\frac{R}{I+J}\right),$$

once again keeping in mind that

$$\ell\left(H_0(K_\bullet(f_{k-1},f_k;A))\right) = \ell\left(\frac{R}{I+J}\right).$$

The proof is therefore completed. □

Our next goal is to provide a concrete example where one can effectively apply Proposition 8. We will be back to the monomial case in Section 3.

Example 1. Let $n \geq 4$ be an integer, let $3 \leq k \leq n-1$ be an integer, let $(a_1,\dots,a_k) \in \mathbb{N}^k$, let $(b_1,\dots,b_n) \in \mathbb{N}^n$ such that $a_j \leq b_j$ for any $1 \leq j \leq k-2$, let $S = \mathbb{K}[x_1,\dots,x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$, let $\mathfrak{m} = (x_1,\dots,x_n)$, and set

$$R := S_{\mathfrak{m}},\ I := (x_1^{a_1},\dots,x_k^{a_k})R,\ J := (x_1^{b_1},\dots,x_n^{b_n})R.$$

Notice that the following statements hold.

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(i) I is generated by an R-regular sequence of length k.
(ii) J is generated by an R-regular sequence of length n.
(iii) Since, for any $1 \leq j \leq k-2$, $a_j \leq b_j$, one has that $(x_1^{b_1}, \ldots, x_{k-2}^{b_{k-2}})R \subseteq I$.
(iv) $x_{k+1}^{b_{k+1}}, \ldots, x_n^{b_n}$ is an R/I-regular sequence of length $n-k \geq 1$.

Therefore, by the foregoing statements, we can apply Proposition 8 to conclude that, in this case,

$$\ell(\mathrm{Tor}_i^R(R/I, R/J)) = \binom{k}{i} \ell\left(\frac{R}{I+J}\right).$$

The same strategy used in all our previous results can also be applied to prove the following:

LEMMA 3. *Let $k \geq 3$, suppose that I can be generated by a regular sequence of length k, and let $J = (f_1, \ldots, f_n)$, where $\mathbf{f} = f_1, \ldots, f_n$ is an R-regular sequence. Assume that $(f_1, \ldots, f_s) \subseteq I$ for some $1 \leq s \leq k-2$ and that $f_{k+1}, \ldots, f_n$ is an R/I-regular sequence. Then, one has that*

$$\ell\left(H_j(\mathbf{f}; R/I)\right) = \sum_{i=0}^{k-s} \binom{s}{j-i} \ell\left(H_i(f_{s+1}, \ldots, f_k; R/(I + (f_{k+1}, \ldots, f_n)))\right).$$

REMARK. Notice that Lemma 3 is a key technical fact for reducing the conjecture to the case where our ideal J is generated by less elements. The price we have to pay is that we have to calculate Koszul homologies not over a regular local ring, but over the complete intersection ring R/I. Moreover, in this case J could not be generated by an R/I-regular sequence.

3. The monomial case

Here, we present another instance where Conjecture 1 holds; namely, in the monomial setting. More precisely, the aim of this section is to show that Conjecture 1 holds in case both ideals I and J are monomial ideals generated by pure powers of the variables in a polynomial ring.

NOTATION 1. Hereafter in this section, let $\mathbb{K}$ be a field, let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in $n \geq 1$ variables over $\mathbb{K}$, let $(a_1, \ldots, a_k) \in \mathbb{N}^k$ (for some $1 \leq k \leq n$) and let $(b_1, \ldots, b_n) \in \mathbb{N}^n$. Assume also that $a_j \neq 0$ for all $1 \leq j \leq k$ and that $b_l \neq 0$ for all $1 \leq l \leq n$. Finally, set $I := (x_1^{a_1}, \ldots, x_k^{a_k})$, $J := (x_1^{b_1}, \ldots, x_n^{b_n})$.

Keeping in mind the previous notation, the main result of this section is the following:

THEOREM 2. *Preserving Notation 1, one has that*

$$\ell_R(\mathrm{Tor}_i^R(R/I, R/J)) = \binom{k}{i} \ell_R\left(\frac{R}{I+J}\right).$$

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Proof. First, we set the following notation.

$$\begin{aligned} S_j &:= \mathbb{K}[x_j],\ 1 \leq j \leq n, \\ M_j &:= \begin{cases} \dfrac{S_j}{(x_j^{a_j})}, \text{if } 1 \leq j \leq k, \\ S_j, \text{if } k+1 \leq j \leq n. \end{cases} \\ N_j &:= \frac{S_j}{(x_j^{b_j})},\ 1 \leq j \leq n. \end{aligned}$$

Then, we have the following $\mathbb{K}$-vector space isomorphisms:

$$\frac{R}{I} \cong \bigotimes_{j=1}^{n} M_j, \quad \frac{R}{J} \cong \bigotimes_{j=1}^{n} N_j.$$

Now, fix $1 \leq j \leq k$. We have the following short exact sequence of S_j-modules:

$$0 \longrightarrow S_j \xrightarrow{\cdot x_j^{a_j}} S_j \longrightarrow M_j \longrightarrow 0.$$

Tensoring this short exact sequence with N_j one obtains, for any integer $1 \leq j \leq k$, the complex

$$(C_j)_\bullet : \quad 0 \longrightarrow N_j \xrightarrow{\cdot x_j^{a_j}} N_j \longrightarrow 0.$$

On the other hand, for $k+1 \leq j \leq n$, we define $(C_j)_\bullet$ as the complex

$$0 \longrightarrow S_j \xrightarrow{x_j^{b_j}} S_j \longrightarrow 0.$$

Now, fix an integer $1 \leq j \leq k$. The complex $(C_j)_\bullet$ has non-zero homology at degrees 0 and 1. Indeed, on the one hand, the zeroth homology of this complex has length

$$\ell\left(\frac{N_j}{x_j^{a_j} N_j}\right) = \min(a_j, b_j).$$

On the other hand, its first homology has the same length, because $(0 :_{N_j} x_j^{a_j})$ is spanned by the final $\min(a_j, b_j)$ monomials modulo $x_j^{b_j}$. Moreover, for $j > k$, the complex $(C_j)_\bullet$ has only homology at degree zero, with length b_j. Recall also that the full Koszul complex is the tensor product over $\mathbb{K}$ of these one-variable complexes [4, end of page 46].

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Now, we define the following complexes

$$A_\bullet := \bigotimes_{j=1}^{k} (C_j)_\bullet, \quad B_\bullet := \bigotimes_{j=k+1}^{n} (C_j)_\bullet.$$

By the previous remarks, it is clear that, for any $0 \leq i \leq n$,

$$\mathrm{Tor}_i^R\left(\frac{R}{I}, \frac{R}{J}\right) = H_i\left(A_\bullet \otimes B_\bullet\right).$$

Now, by the Kunneth formula in the form displayed in [20, Corollary 10.84], we have a $\mathbb{K}$-vector space isomorphism

$$\mathrm{Tor}_i^R\left(\frac{R}{I}, \frac{R}{J}\right) \cong \bigoplus_{p+q=i} \left(H_p\left(A_\bullet\right) \otimes H_q(B_\bullet)\right). \tag{5}$$

Now, let us first focus on the homology of the complex $B_\bullet$. By combining the Kunneth formula with the definition of $B_\bullet$, we have that

$$H_q(B_\bullet) \cong \bigoplus_{s_{k+1}+\ldots+s_n=q} \left(\bigotimes_{j=k+1}^{n} H_{s_j}((C_j)_\bullet)\right) = \begin{cases} 0, \text{if } q \neq 0, \\ \bigotimes_{j=k+1}^{n} H_0\left((C_j)_\bullet\right), \text{if } q = 0. \end{cases}$$

Taking into account this isomorphism, we can rewrite (5) as follows.

$$\mathrm{Tor}_i^R\left(\frac{R}{I}, \frac{R}{J}\right) \cong H_i\left(A_\bullet\right) \otimes \left(\bigotimes_{j=k+1}^{n} H_0\left((C_j)_\bullet\right)\right). \tag{6}$$

Now, we focus on the homology of $A_\bullet$. Since each complex $(C_j)_\bullet$ has only nonzero homologies at degrees zero and one, and both have the same length, we have the following isomorphism of $\mathbb{K}$-vector spaces, where by convention $\binom{k}{i} = 0$ for $k > i$:

$$H_i\left(A_\bullet\right) \cong \left(\bigotimes_{j=1}^{k} H_0((C_j)_\bullet)\right)^{\oplus\binom{k}{i}}.$$

In this way, by combining this last isomorphism with (6) and the foregoing calculations of lengths, we obtain that

$$\ell\left(\mathrm{Tor}_i^R\left(\frac{R}{I}, \frac{R}{J}\right)\right) = \ell(H_i(A_\bullet)) \cdot \ell(H_0(B_\bullet)) = \binom{k}{i} \prod_{j=1}^{k} \min(a_j, b_j) \prod_{j=k+1}^{n} b_j.$$

On the other hand, since

$$I + J = (x_1^{\min(a_1,b_1)}, \ldots, x_k^{\min(a_k,b_k)}, x_{k+1}^{b_{k+1}}, \ldots, x_n^{b_n}),$$

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hence

$$\ell\left(\frac{R}{I+J}\right)\prod_{j=1}^{k}\min(a_j,b_j)\prod_{j=k+1}^{n}b_j.$$

This concludes the proof. □

4. Some counterexamples

The goal of this section is to exhibit two counterexamples to Conjecture 1 showing that, in general, this conjecture does not always hold.

Our first example also shows that the conjecture is not true, in general, when either I or J is generated by a regular sequence given by binomials. The calculations in the following examples were done with Macaulay2 [8].

EXAMPLE 2. Let $S=\mathbb{Q}[x,y,z]$ be a polynomial ring in three variables over the rational numbers, set $\mathfrak{m}=(x,y,z)$, and set $R:=S_{\mathfrak{m}}$, which is a regular local ring. Consider the ideals of R

$$I=(x^2,y^2,z^2),\ J=(xy+x^2,xz+z^2,yz+y^2).$$

In this case, one has that

$$\ell\left(\operatorname{Tor}_1^R(R/I,R/J)\right)=14\neq 12=\binom{3}{1}\ell\left(\frac{R}{I+J}\right).$$

Our second example is the following one. In this example, there is defect not only in one Tor length, but in two.

EXAMPLE 3. Let $S=\mathbb{Q}[x,y,z,w]$ be a polynomial ring in four variables over the rational numbers, set $\mathfrak{m}=(x,y,z,w)$, and set $R:=S_{\mathfrak{m}}$, which is a regular local ring. Consider the ideals of R

$$I=(x^2,y^2,z^2,w^2),\ J=\left(\frac{\partial h_3}{\partial x},\frac{\partial h_3}{\partial y},\frac{\partial h_3}{\partial z},\frac{\partial h_3}{\partial w}\right),$$

where h_3 denotes the complete symmetric polynomial of degree 3. We know, thanks to [13, Lemma 3.1], that J is generated by an R-regular sequence. However, one has that

$$\ell\left(\operatorname{Tor}_1^R(R/I,R/J)\right)=30\neq 28=\binom{4}{1}\ell\left(\frac{R}{I+J}\right),$$
$$\ell\left(\operatorname{Tor}_2^R(R/I,R/J)\right)=46\neq 42=\binom{4}{2}\ell\left(\frac{R}{I+J}\right).$$

5. Application to singularity theory

As we have already explained in the Introduction of this manuscript, this section can be read independently of the rest of the paper. Our motivation here is to explain

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why Conjecture 1 has some interest from the perspective of Singularity Theory. In this section, $\mathbb{C}\{x_1,\dots,x_n\}$ will denote the ring of convergent power series with coefficients in the field of complex numbers.

Let $(X,0)\subset(\mathbb{C}^n,0)$ be a reduced analytic variety, and let f be a function germ. The Bruce–Roberts number of f with respect to $(X,0)$, denoted $\mu_{\mathrm{BR}}(f,X)$, was defined in [3] as the length of the quotient $\mathbb{C}\{x_1,\dots,x_n\}/tf(\Theta_X)$, where

$$\Theta_X := \{\xi\in\Theta_n \mid \xi(h)\in I_X\}$$

is the set of vector fields that are tangent to $(X,0)$, and Θ_n denotes the $\mathbb{C}\{x_1,\dots,x_n\}$-module of germs of vector fields on $(\mathbb{C}^n,0)$. The notation Θ_X is also commonly written as $\mathrm{Der}\,(-\log)(X)$. This number is a generalization of the Milnor number of a function germ. If $X=\mathbb{C}^n$, then $\Theta_X=\Theta_n$, and in this case, $\mu_{\mathrm{BR}}(f,X)=\mu(f)$. In general, the Bruce–Roberts number of f with respect to $(X,0)$ is greater than or equal to the Milnor number of f.

The first relation between these numbers was presented in [17, Theorem 3.1], where $(X,0)$ is a weighted homogeneous hypersurface with an isolated singularity. In this case,

$$\mu_{\mathrm{BR}}(f,X)=\mu(f)+\mu(X\cap f^{-1}(0)), \tag{7}$$

where $\mu(X\cap f^{-1}(0))$ is the Milnor number of the ICIS $(X\cap f^{-1}(0),0)$, as defined in [9, 10].

In [18, Corollary 4.1], (7) was extended to any isolated hypersurface singularity. In this case,

$$\mu_{\mathrm{BR}}(f,X)=\mu(f)+\mu(X\cap f^{-1}(0))+\mu(X,0)-\tau(X,0), \tag{8}$$

where μ and τ denote, respectively, the Milnor and Tjurina numbers of $(X,0)$. (8) can be regarded as a natural extension of (7). In fact, an isolated hypersurface singularity is weighted homogeneous if and only if the equality $\mu(X,0)=\tau(X,0)$ holds (see [21]).

One class of varieties that extends the concept of an isolated hypersurface singularity is the class of isolated complete intersection singularity (ICIS). Considering any ICIS, it was raised the following conjecture (see [14]):

Conjecture 2. *Let $(X,0)\subset(\mathbb{C}^n,0)$ be an ICIS of codimension $0<k<n$, and let f be a function germ such that $\mu_{\mathrm{BR}}(f,X)<\infty$. Then,*

$$\mu_{\mathrm{BR}}(f,X)=\mu(f)+\mu(X\cap f^{-1}(0),0)+\mu(X,0)-\tau(X,0)+(k-1)\dim_{\mathbb{C}} \times\left(\frac{\mathbb{C}\{x_1,\dots,x_n\}}{J(f)+I_X}\right).$$

Conjecture 2 is an extension of the relation that satisfies an isolated hypersurface singularity. In [14, Theorem 4.7], it was proved that

$$\mu_{\mathrm{BR}}(f,X)=\mu(f)+\mu(X\cap f^{-1}(0),0)+\mu(X,0)-\tau(X,0)-\dim_{\mathbb{C}}\left(\frac{\mathbb{C}\{x_1,\dots,x_n\}}{J(f)+I_X}\right) +\dim_{\mathbb{C}}\mathrm{Tor}_1\left(\frac{\mathbb{C}\{x_1,\dots,x_n\}}{I_X},\frac{\mathbb{C}\{x_1,\dots,x_n\}}{J(f)}\right),$$

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for any ICIS $(X,0)$ of codimension $0<k<n$, and any analytic function germ f such that $\mu_{\mathrm{BR}}(f,X)<\infty$.

To conclude the proof of Conjecture 2, it is necessary to prove the equality

$$\dim_{\mathbb{C}} \operatorname{Tor}_1\left(\frac{\mathbb{C}\{x_1,\ldots,x_n\}}{I_X},\frac{\mathbb{C}\{x_1,\ldots,x_n\}}{J(f)}\right)=k\dim_{\mathbb{C}}\left(\frac{\mathbb{C}\{x_1,\ldots,x_n\}}{J(f)+I_X}\right).$$

With this goal in mind, Conjecture 1 was formulated in [14]. In Section 4, we have presented counterexamples to Conjecture 1. Since in all of them the ideal I is generated by a regular sequence of length n, we can now present a more restrictive conjecture that is still relevant in the context described in this section.

CONJECTURE 3. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension n, let I be an ideal generated by an R-regular sequence of length $k\leq n-1$, and let J be an ideal generated by a regular sequence of length n. Then, for any $0\leq i\leq k$, we have that*

$$\ell\left(\operatorname{Tor}_i^R(R/I,R/J)\right)=\binom{k}{i}\ell\left(\frac{R}{I+J}\right).$$

REMARK. Keeping in mind the content of our new restricted Conjecture 3, we want to single out the following facts.

(i) As noted in Section 2, the case where the ideal I is generated by a regular sequence of length $k=2$ has already been proven.

(ii) The case $k=1$ can be proved as follows. Set $A:=R/J$, which is an Artinian complete intersection ring, and let $f\in R$ be a generator of I. In this case, we have the following exact sequence:

$$0\longrightarrow(0:_A f)\longrightarrow A\xrightarrow{\cdot f}A\longrightarrow\frac{A}{fA}\cong\frac{R}{I+J}\longrightarrow 0.$$

From this short exact sequence we deduce that

$$\ell(\operatorname{Tor}_1^R(R/I,R/J))=\ell((0:_A f))=\ell\left(\frac{A}{fA}\right)=\ell\left(\frac{R}{I+J}\right),$$

which is what we want to show.

(iii) As we have already seen throughout the paper (see Section 2), by imposing certain conditions on the ideal I we have established Conjecture 3 in some particular cases.

6. Raising more general conjectures

In this paper, we see that Conjecture 1 is in general false. However, in all our counterexamples we always have the following lower inequality:

$$\ell\left(\operatorname{Tor}_i^R(R/I,R/J)\right)\geq\binom{k}{i}\ell\left(\frac{R}{I+J}\right).$$

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This motivates us to formulate the following conjecture, which might be regarded as a variant of the Buchsbaum–Eisenbud–Horrocks problem in the local case.

QUESTION 1. Let $(R, \mathfrak{m})$ be a regular local ring of dimension n, let I be an ideal of R that contains a regular sequence of length $k \leq n$, and let J be an ideal of R that contains a regular sequence of length n such that $I + J$ is $\mathfrak{m}$-primary. Let $0 \leq i \leq k$. Is it true that

$$\ell\left(\operatorname{Tor}_i^R(R/I, R/J)\right) \geq \binom{k}{i} \ell\left(\frac{R}{I+J}\right)?$$

We want to observe that, in Question 1, we cannot weaken the regular sequence containment assumption on J, as the next example shows.

EXAMPLE 4. Let $S = \mathbb{Q}[x, y, z, w]$, let $\mathfrak{m} = (x, y, z, w)$, set $R := S_{\mathfrak{m}}$, and

$$I := (x^3, y^3, z^3, w^3)R,\ J := (x(x+y+z), y(x+y+w), z(x+z+w), w(y+z+w))R.$$

One can check that $\sqrt{I+J} = (x, y, z, w)R$, that I is a complete intersection, and that J is an almost complete intersection (it has grade 3 and 4 minimal generators). In this case,

$$\ell(\operatorname{Tor}_1(R/I, R/J)) = 42 < 48 = \binom{4}{1} \cdot 12 = \binom{4}{1} \cdot \ell\left(\frac{R}{I+J}\right),$$
$$\ell(\operatorname{Tor}_2(R/I, R/J)) = 51 < 72 = \binom{4}{2} \cdot 12 = \binom{4}{2} \cdot \ell\left(\frac{R}{I+J}\right).$$

The next example may be regarded as a piece of evidence for Question 1.

EXAMPLE 5. Coming back to Example 4, setting now

$$J := (x^3, y^3, z^3, w^3)R,\ I := (x(x+y+z), y(x+y+w), z(x+z+w), w(y+z+w))R.$$

and keeping in mind Question 1, we have

$$\ell(\operatorname{Tor}_1(R/I, R/J))) = 42 > 36 = \binom{3}{1} \cdot 12 = \binom{3}{1} \cdot \ell\left(\frac{R}{I+J}\right),$$
$$\ell(\operatorname{Tor}_2(R/I, R/J))) = 51 > 36 = \binom{3}{2} \cdot 12 = \binom{3}{2} \cdot \ell\left(\frac{R}{I+J}\right),$$
$$\ell(\operatorname{Tor}_3(R/I, R/J))) = 25 > 12 = \binom{3}{3} \cdot 12 = \binom{3}{3} \cdot \ell\left(\frac{R}{I+J}\right).$$

We can also formulate a local variant of the so-called Total Rank Conjecture.

QUESTION 2. Let $(R, \mathfrak{m})$ be a regular local ring of dimension n, let I be an ideal of R that contains a regular sequence of length $k \leq n$, and let J be an ideal of R

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that contains a regular sequence of length n such that $I+J$ is $\mathfrak{m}$-primary. Is it true that

$$\sum_{i=0}^{n} \ell\left(\operatorname{Tor}_i^R(R/I, R/J)\right) \geq 2^k \ell\left(\frac{R}{I+J}\right)?$$

Notice that Question 2 is not covered by the results obtained by VandeBogert and Walker in [23] because the tensor product of the Koszul complexes of two complete intersection ideals is not a priori concentrated in degrees $[0, \dim(R)+2]$.

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