3 Weak Decays at Tree Level

3.1 Muon Decay

3.1.1 The Decay Amplitude

We begin our discussion of weak decays with the simplest but a very important decay:

$$\mu^- \to \nu_\mu e^- \bar{\nu}_e, \tag{3.1}$$

for which the Feynman diagram is given in Figure 3.1 We just use the Feynman rules for the vertices and the W propagator collected in Appendix B. Multiplying the result by i to obtain the decay amplitude, we find

$$\mathcal{A}(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = -\frac{1}{8}\frac{g_2^2}{k^2 - M_W^2}[\bar{\nu}_\mu \gamma_\mu (1-\gamma_5)\mu][\bar{e}\gamma^\mu(1-\gamma_5)\nu_e], \tag{3.2}$$

where the Lorentz index $\mu = 0, 1, 2, 3$ in γ_μ should not be confused with μ in the names of the involved particles. The four-momentum squared, k^2, in the W-propagator is due to the smallness of the muon mass negligible when compared with M_W^2 and can be set to zero to an excellent approximation. Comparing then the result with the one from the Fermi theory

$$\mathcal{A}(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = \frac{G_F}{\sqrt{2}}[\bar{\nu}_\mu \gamma_\mu (1-\gamma_5)\mu][\bar{e}\gamma^\mu(1-\gamma_5)\nu_e], \tag{3.3}$$

we obtain an important relation between the Fermi constant G_F and the parameters of the SM

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}.} \tag{3.4}$$

This amplitude has a very simple structure. It is a product of $G_F/\sqrt{2}$, which as seen in (3.4) contains the information about the strength of the interaction and the four-fermion operator composed of charged leptons and corresponding neutrino fields. We will see that such products will also be fundamental for other decays.

There is one important point to be made. As seen in (3.1) there is $\bar{\nu}_e$ in the final state, while both in (3.3) and in the diagram in Figure 3.1 we see ν_e. This is not a misprint. In popular talks in which no arrows are indicated, and even if they are indicated, one would see $\bar{\nu}_e$ in Figure 3.1. But in this book, in which the diagrams will be evaluated, we will

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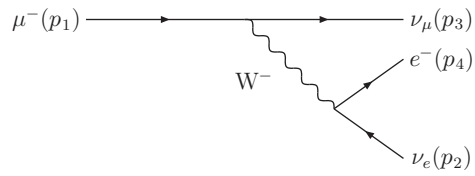


Figure 3.1 Muon decay

proceed as done in this figure: Neutrino moving back in time represents an antineutrino and similar in other diagrams and other particles in our book.

At this stage a few additional remarks should be made:

- Muon decay is strongly suppressed not because of the smallness of g_2, which in fact is $O(1)$, but because of the large $W^\pm$-mass. This is the main reason why G_F is so small.
- The four-fermion operator is a product of two left-handed currents, signaling the maximal parity violation in decays mediated by $W^\pm$. This is one of the crucial properties of electroweak interactions in the SM.
- This operator has energy dimension six.
- As leptons do not carry color, the result in (3.3) is not affected by strong interactions. While the diagrams with additional exchanges of photons $W^\pm$ and Z^0 have to be taken into account for precision test of the SM, they are small and can be calculated in perturbation theory.
- Very importantly, the branching ratio for this decay is more than 0.99, so that the muon lifetime is almost entirely determined by this decay.

3.1.2 The Decay Width, Muon Lifetime, and G_F

Having the decay amplitude in (3.3) at hand, we can calculate the decay width. This calculation can be found in many textbooks, and to save few pages we will only present those details that can be useful for other decays considered by us. A more detailed derivation can be found, e.g., in Langacker's book [10]. In any case, any reader who was exposed to elementary relativistic quantum field theory should be able to perform this calculation without any help from Langacker or the present author. I have chosen Langacker's book because after the routine calculation of the total decay rate in the leading order, he also discusses higher-order QED and electroweak corrections to this decay, which are beyond the scope of our book but relevant for electroweak precision tests. Useful calculations can also be found in other textbooks, in particular in the one by Schwartz [47].

If we are interested only in the total decay rate, we have to average over the decaying muon polarization states and sum over electron polarization states. Denoting the momenta of involved particles as in Figure 3.1 and neglecting the electron mass for the moment gives

$$\overline{|\mathcal{A}(\mu^- \to \nu_\mu e^- \bar{\nu}_e)|^2} = \frac{1}{2}\frac{G_F^2}{2}\mathrm{Tr}[\gamma_\mu(1-\gamma_5)(\not{p}_1 + m_\mu)\gamma_\nu(1-\gamma_5)\ \not{p}_3] \times \mathrm{Tr}[\gamma^\mu(1-\gamma_5)\ \not{p}_2\gamma^\nu(1-\gamma_5)\ \not{p}_4]. \tag{3.5}$$

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With the standard technology for the evaluation of traces, it is straightforward to find

$$\overline{|\mathcal{A}(\mu^- \to \nu_\mu e^- \bar{\nu}_e)|^2} = 64 G_F^2 (p_1 \cdot p_2)(p_3 \cdot p_4). \tag{3.6}$$

Detailed steps are given on page 251 in [10]. The integration over momenta, performed in detail on page 252 in [10], is a bit tedious but the final result is very simple and depends only on G_F and m_μ:

$$\boxed{\Gamma_\mu \equiv \Gamma(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = \frac{G_F^2 m_\mu^5}{192\pi^3}.} \tag{3.7}$$

General formulas for phase space integration can be found in any edition of PDG, but the explicit examples in [10] are certainly very useful.

Of course, to increase precision electron mass effects, radiative QED corrections and W propagator effects have to be included with the result summarized in (6.51) of [10]. Here we only show the leading contributions of these three corrections:

$$\boxed{\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 - 8\frac{m_e^2}{m_\mu^2}\right)\left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right]\left(1 + \frac{3}{5}\frac{m_\mu^2}{M_W^2}\right).} \tag{3.8}$$

The radiative QED correction is obtained by adding photon exchange between initial muon and the electron in the final state and other QED corrections at this order in α. Note that this is an 0.2% correction to the leading contribution from the diagram in Figure 3.1. But it has to be included along with higher QED corrections to match the precision of the experimental value of τ_μ given in Appendix D, which is used to determine G_F. The other inputs m_e, m_μ, α, and M_W, also given in Appendix D, are very precise as well.

Now the muon lifetime is given by

$$\tau_\mu = \frac{1}{\Gamma_\mu}. \tag{3.9}$$

Consequently (3.8) amended by additional nonleading corrections not explicitly shown here allows us together with (3.9) to determine G_F with high precision:

$$\boxed{G_F = 1.166367(5) \times 10^{-5}\,\text{GeV}^{-2}.} \tag{3.10}$$

G_F is one of the fundamental input parameters in the SM and will appear frequently in this book.

In principle we could next discuss the decays of the τ lepton, but because of its high mass it can also decay to mesons, and it is strategically useful to postpone this discussion and proceed with the leptonic decays of charged mesons.

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3.2 Leptonic Decays of Charged Mesons

3.2.1 Leptonic Decays of π^+ and K^+

Let us next extend our tree-level calculations to the leptonic decays

$$\pi^+ \to \mu^+ \nu_\mu, \qquad K^+ \to \mu^+ \nu_\mu. \tag{3.11}$$

The diagrams for these decays are given in Figure 3.2. The results for the corresponding amplitudes expressed first in terms of quark and lepton fields have a similar structure to (3.3). However, the quarks are confined in π^+ and K^+, and the final amplitudes for the decays in question read

$$\mathcal{A}(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F}{\sqrt{2}} V_{ud}^* (\bar{\nu}_\mu \mu^-)_{V-A} \langle 0 | (\bar{d} u)_{V-A} | \pi^+ \rangle, \tag{3.12}$$

$$\mathcal{A}(K^+ \to \mu^+ \nu_\mu) = \frac{G_F}{\sqrt{2}} V_{us}^* (\bar{\nu}_\mu \mu^-)_{V-A} \langle 0 | (\bar{s} u)_{V-A} | K^+ \rangle, \tag{3.13}$$

where we have introduced the shorthand notation

$$\bar{d}_\alpha \gamma^\mu (1 - \gamma_5) u_\alpha \equiv (\bar{d}_\alpha u_\alpha)_{V-A} \equiv (\bar{d} u)_{V-A} \tag{3.14}$$

and similar for other quark and lepton currents. In what follows, summation over repeated color indices (α in this case) will be understood.

We note the appearance of the CKM factors V_{ud} and V_{us} and the Feynman rules for the quark-W-boson vertices in Figure 3.2 require that the complex conjugates of these factors enter this amplitude. If we considered the corresponding decays of π^- and K^-, no complex conjugation would be required. As V_{ud} and V_{us} are real in our conventions, this difference is immaterial in the decays considered but would be crucial if in particular V_{ub} or V_{td} would be present instead.

In comparison with the muon decay, a complication arises. The last objects on the r.h.s. in (3.12) and (3.13) are matrix elements between the initial mesons and the vacuum. The flavors in the quark currents have been chosen appropriately so that indeed in each case the meson is destroyed by the current, leaving no meson in the final state. But as the initial meson is a bound state of quarks, these matrix elements cannot be evaluated in perturbation theory, and nonperturbative methods, in particular lattice QCD, are required to evaluate them.

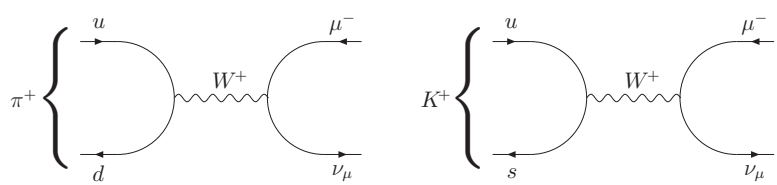


Figure 3.2 $\pi^+ \to \mu^+ \nu_\mu$ and $K^+ \to \mu^+ \nu_\mu$.

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Yet, to make a step forward, we can parametrize these matrix elements as follows

$$\langle 0|(\bar{d}u)_{V-A}|\pi^+\rangle = iF_\pi p_\pi^\mu, \qquad \langle 0|(\bar{s}u)_{V-A}|K^+\rangle = iF_K p_K^\mu, \tag{3.15}$$

where p_π^μ and p_K^μ are the four momenta of the decaying mesons. Recall that a γ^μ is hidden in the matrix element of the current. As this matrix element carries the dimension of the energy squared and we have only the four-momenta p_π^μ and p_K^μ to our disposal, this is a unique parametrization consistent with Lorentz invariance up to an overall numerical factor. In some papers a factor of $\sqrt{2}$ on the r.h.s. of these equations can be found, and this implies a reduction of the numerical values of F_π and F_K by $\sqrt{2}$. When using results found in the literature it is crucial to check which definitions of F_π and F_K are used in a given paper.

F_π and F_K are the so-called weak decay constants for pion and kaon, respectively. In the past these constants were usually extracted from the data by combining the leptonic decays in question with the semileptonic ones that we will discuss soon. But the advances in lattice calculations allow us presently to calculate F_π and F_K directly from QCD [48]

$$F_\pi = 129.8(15)\,\text{MeV}, \qquad F_K = 156.1(11)\,\text{MeV}. \tag{3.16}$$

The evaluation of the decay widths for these two decays is easy as the final states contain only two particles and not three as in the muon decay. Let us do it for the pion decay without being confused that the indices on the four-momenta p_μ and p_ν denote *muon* and *neutrino* and are not Lorentz indices.

Inserting (3.15) into (3.12), we find first

$$\mathcal{A}(\pi^+ \to \mu^+\nu_\mu) = i\frac{G_F}{\sqrt{2}}V_{ud}^* F_\pi[\bar{\nu}_\mu \not{p}_\pi(1-\gamma_5)\mu^-]. \tag{3.17}$$

But using Dirac equation, we have

$$\bar{\nu}_\mu \not{p}_\nu = m_\nu \bar{\nu}_\mu, \qquad \not{p}_\nu \mu^- = m_\mu \mu^-. \tag{3.18}$$

Setting neutrino mass to zero, we can use $\not{p}_\nu = 0$ to obtain

$$\not{p}_\pi = \not{p}_\nu + \not{p}_\mu = \not{p}_\mu, \qquad \not{p}_\mu(1-\gamma_5)\mu^- = m_\mu(1+\gamma_5)\mu^-. \tag{3.19}$$

Consequently

$$\mathcal{A}(\pi^+ \to \mu^+\nu_\mu) = i\frac{G_F}{\sqrt{2}}V_{ud}^* F_\pi m_\mu[\bar{\nu}_\mu(1+\gamma_5)\mu^-]. \tag{3.20}$$

Thus denoting this amplitude simply by $\mathcal{A}$, we have

$$|\mathcal{A}|^2 = \frac{G_F^2}{2}|V_{ud}|^2 F_\pi^2 m_\mu^2 |\bar{\nu}_\mu(1+\gamma_5)\mu^-|^2. \tag{3.21}$$

Summing over polarizations of final particles, we find using standard methods (see [10])

$$\overline{|\mathcal{A}|^2} = \frac{G_F^2}{2}|V_{ud}|^2 F_\pi^2 m_\mu^2 [8 p_\nu p_\mu] \tag{3.22}$$

with the last factor in square brackets resulting from

$$\text{Tr}[\not{p}_\nu(1+\gamma_5)(\not{p}_\mu + m_\mu)(1-\gamma_5)] = 2\text{Tr}[\not{p}_\nu \not{p}_\mu(1-\gamma_5)] = 8 p_\nu p_\mu. \tag{3.23}$$

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Denoting by E_ν the neutrino energy in the pion rest frame, we have $p_\nu p_\mu = E_\nu m_\pi$ and

$$\overline{|\mathcal{A}|^2} = 4\, G_F^2 |V_{ud}|^2 F_\pi^2 m_\mu^2 m_\pi E_\nu. \tag{3.24}$$

The decay width is given by

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{\overline{|\mathcal{A}|^2}}{2m_\pi} \left[\frac{E_\nu}{4\pi m_\pi} \right], \tag{3.25}$$

with the last factor in square brackets coming from phase-space volume.

But simple kinematics tells us that

$$E_\nu = \frac{m_\pi}{2} \left(1 - \frac{m_\mu^2}{m_\pi^2} \right). \tag{3.26}$$

Therefore, combining the last three formulas, we reproduce the known result

$$\boxed{\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F^2}{8\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2} \right)^2 F_\pi^2 |V_{ud}|^2.} \tag{3.27}$$

The result for K^+ decay is then obtained from (3.27) in no time by just replacing properly various indices and changing the CKM factor from $|V_{ud}|$ to $|V_{us}|$. We find

$$\boxed{\Gamma(K^+ \to \mu^+ \nu_\mu) = \frac{G_F^2}{8\pi} m_K m_\mu^2 \left(1 - \frac{m_\mu^2}{m_K^2} \right)^2 F_K^2 |V_{us}|^2.} \tag{3.28}$$

These two formulas are very important as they can be used for the leptonic decays of heavy mesons with proper replacement of masses, weak decay constants, and CKM factors.

The important property of these formulas is the helicity suppression factor m_μ^2 that is related to the $V - A$ structure of the quark current. In the rest frame of a meson that has spin-0, the momenta of the muon and of the neutrino must have opposite directions, and their spins must add to zero as shown in Figure 3.3. This means that the projection of the spin onto the momentum $\vec{\sigma}\vec{p}/|\sigma p|$ is the same for both particles. However, because neutrinos are always left-handed (assuming $m_{\nu_\mu} = 0$), only the left-handed part of μ^+ can contribute. If both neutrino and muon were exactly massless, this decay would not be possible because then the W boson only couples to left-handed particles (and right-handed antiparticles). But due to the nonvanishing $\mu^\pm$ mass ($m_\mu \neq 0$) the μ^+ couples to W through its left-handed component. Thus the amplitude must be proportional to m_μ.

With the values of G_F, F_π, and F_K in (3.10) and (3.16), the only unknowns in the decays widths are the CKM elements $|V_{ud}|$ and $|V_{us}|$. They can be determined from semileptonic decays of π^+ and K^+, respectively. Before doing this let us turn our attention to leptonic decays of heavy mesons.

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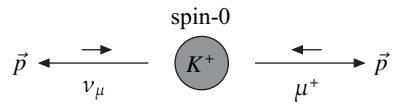


Figure 3.3 Helicity suppression of $K^+ \to \mu^+\nu_\mu$ in the SM.

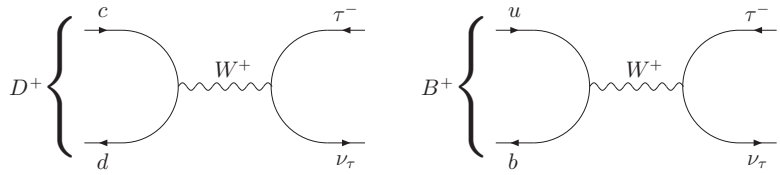


Figure 3.4 $D^+ \to \tau^+\nu_\tau$ and $B^+ \to \tau^+\nu_\tau$.

3.2.2 Leptonic Decays of D^+ and B^+

We will next consider the decays

$$D^+ \to \tau^+\nu_\tau, \qquad B^+ \to \tau^+\nu_\tau \tag{3.29}$$

with $D^+ = (c\bar{d})$ and $B^+ = (u\bar{b})$. Analogous calculations can be done for decays into $\mu^+\nu_\mu$ and $e^+\nu_e$. Only the branching ratios in these cases are much smaller because of the helicity suppression. The Feynman diagrams for these two decays are given in Figure 3.4. There is no need to repeat the calculation of the decay widths as they can be directly obtained from previous formulas by just changing flavor indices. As both decays do not represent the dominant decay channels of D^+ and B^+ it is customary to consider the branching ratios instead of the width for them. They are obtained by dividing the width for these particular decays by the total width or equivalently by multiplying the width for a given decay with the lifetime of the decaying meson. We find then

$$\boxed{\mathcal{B}(D^+ \to \tau^+\nu_\tau)_{\rm SM} = \frac{G_F^2 m_{D^+} m_\tau^2}{8\pi}\left(1 - \frac{m_\tau^2}{m_{D^+}^2}\right)^2 F_{D^+}^2 |V_{cd}|^2 \tau_{D^+},} \tag{3.30}$$

$$\boxed{\mathcal{B}(B^+ \to \tau^+\nu_\tau)_{\rm SM} = \frac{G_F^2 m_{B^+} m_\tau^2}{8\pi}\left(1 - \frac{m_\tau^2}{m_{B^+}^2}\right)^2 F_{B^+}^2 |V_{ub}|^2 \tau_{B^+}.} \tag{3.31}$$

The subscript SM reminds us that these formulas apply only to the SM. As discussed in later chapters of this book, these branching ratios, being strongly suppressed, can be modified in the extensions of this model. While this is in principle also possible for π^+ and K^+ leptonic decays, one expects that new physics effects there are very small because the corresponding branching ratios are already measured very precisely, and the SM describes them very well.

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The relevant weak decay constants are already accurately known from lattice calculations and given as follows [49]

$$F_{D^+} = 212.7(5)\,\text{MeV}, \qquad F_{B^+} = 189.4(13)\,\text{MeV}. \tag{3.32}$$

To predict these branching ratios, we need the values of the CKM parameters. Here semileptonic decays of mesons play an important role, and we will discuss them next.

3.3 Semileptonic Decays of Charged Mesons

3.3.1 Semileptonic Decays of K^+

We next consider the semileptonic decay

$$K^+ \to \pi^0 e^+ \nu_e, \tag{3.33}$$

for which the Feynman diagram is given in Figure 3.5. The part of this diagram that involves the exchange of the $W^\pm$ boson is familiar from previous decays, but in addition there is a quark, this time the u-quark, that belongs to K^+ but does not couple to $W^\pm$. Such a quark is called spectator quark as its role is passive in this decay. In the final state this quark belongs to π^0. With the experience gained so far, we can immediately write down the relevant decay amplitude

$$\mathcal{A}(K^+ \to \pi^0 e^+ \nu_e) = \frac{G_F}{\sqrt{2}} V_{us}^* (\bar{\nu}_e e^-)_{V-A} \langle \pi^0 | (\bar{s}u)_{V-A} | K^+ \rangle. \tag{3.34}$$

This decay can be used to determine $|V_{us}|$ provided the last factor can be determined by nonperturbative methods.

Similar to what we did in (3.15), we parametrize this factor through

$$\langle \pi^0 | (\bar{s}u)_{V-A} | K^+ \rangle = q_+^\mu f_+(q_-^2) + q_-^\mu f(q_-^2), \tag{3.35}$$

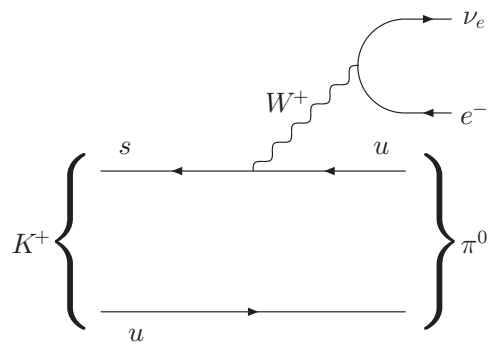


Figure 3.5 Decay $K^+ \to \pi^0 e^+ \nu_e$.

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where

$$q_+^\mu = p_K^\mu + p_\pi^\mu, \qquad q_-^\mu = p_K^\mu - p_\pi^\mu = p_{\nu_e}^\mu + p_e^\mu. \tag{3.36}$$

We recall that the Lorentz index μ is hidden in $(\bar{s}u)_{V-A}$. But using Dirac equation, we find that

$$(\bar{\nu}_e e^-)_{V-A} q_-^\mu \propto m_e, \tag{3.37}$$

and consequently the second term in (3.35) can be neglected to first approximation so that only the knowledge of the formfactor $f_+(q_-^2)$ is relevant for the determination of $|V_{us}|$. The evaluation of the decay rate requires the knowledge of the q^2 dependence of $f_+(q_-^2)$ and is discussed in detail in section IV of [50], where also QED corrections and hadronic effects affecting the determination of $|V_{us}|$ are discussed in detail.

This presentation provides just an idea what is required to be able to extract the value of $|V_{us}|$ from the experimental branching ratio for this decay. Usually what one finds in the literature is the value of the product $f_+(0)|V_{us}|$ extracted from the data. Calculating then $f_+(0)$ by means of lattice QCD and using some information from chiral perturbation theory, it is possible to find the value of $|V_{us}|$.

The second decay used for the determination of $|V_{us}|$ is $K_L \to \pi^+ e^- \bar{\nu}_e$. The relevant amplitude is easily obtained from (3.34), and a different hadronic matrix element of the current $(\bar{s}u)_{V-A}$ must be known to extract $|V_{us}|$: this time between K_L and π^+. We again refer to [50] and the references given there.

The determination of $|V_{us}|$ is very important for any considerations in flavor physics as it determines the parameter λ in the Wolfenstein parametrization. By now there have been many sophisticated analyses that include corrections to the tree-level formulas given here and use other decays. In the CKM workshops that take place every second year, a separate working group is devoted to this determination and to the determination of $|V_{ud}|$. We will not go into details of this topic as it is adequately covered in [50] and in any proceedings of the CKM workshops. For us what is most important is the value of $|V_{us}|$, which we will use in the rest of this book, and the fact that $|V_{ud}|$, determined independently agrees well with the value obtained from the unitarity of the CKM matrix neglecting the contribution from $|V_{ub}|$. As of 2019 the CKM element $|V_{us}|$ is known very precisely, and we will use

$$\boxed{|V_{us}| = \lambda = 0.2252(9), \qquad |V_{ud}| = 1 - \frac{\lambda^2}{2} = 0.9746} \tag{3.38}$$

in analytic expressions, although in final numerical computations it is common these days to use the standard parametrization of the CKM as given in (2.72), which imposes the exact unitarity.

We can now insert these values of CKM parameters into (3.27) and (3.28) to see whether the relevant branching ratios agree with experiment. Using the input from Table D.1 and the conversion from sec to GeV in (D.1), we find

$$\mathcal{B}(\pi^+ \to \mu^+ \nu_\mu) = \Gamma(\pi^+ \to \mu^+ \nu_\mu)\, \tau(\pi^+) = 1.031 |V_{ud}|^2 = 0.979, \tag{3.39}$$

$$\mathcal{B}(K^+ \to \mu^+ \nu_\mu) = \Gamma(K^+ \to \mu^+ \nu_\mu)\, \tau(K^+) = 12.45 |V_{ud}|^2 = 0.631, \tag{3.40}$$

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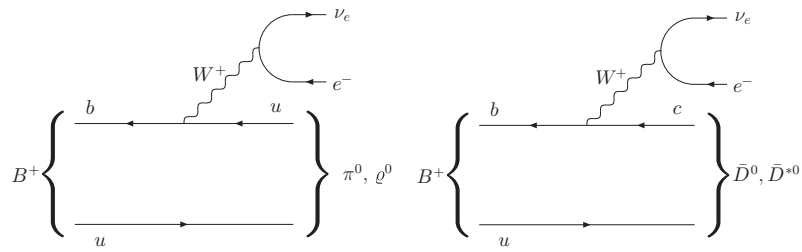


Figure 3.6 $B^+ \to \pi^0(\rho^0)e^+\nu_e$ and $B^+ \to \bar{D}^0(\bar{D}^{*0})e^+\nu_e$.

both in remarkable agreement with the data in Table D.2. Small differences result from QED corrections.

3.3.2 Semileptonic Decays of B^+

We can next consider the decays

$$B^+ \to \pi^0 e^+ \nu_e, \qquad B^+ \to \rho^0 e^+ \nu_e, \tag{3.41}$$

for which the Feynman diagrams are given in Figure 3.6, and the amplitude in the first case reads

$$\mathcal{A}(B^+ \to \pi^0 e^+ \nu_e) = \frac{G_F}{\sqrt{2}} V_{ub}^* (\bar{\nu}_e e^-)_{V-A} \langle \pi^0 | (\bar{b}u)_{V-A} | B^+ \rangle. \tag{3.42}$$

In the second case π^0 in the last factor should be replaced by ρ^0. Evidently these decays allow the determination of $|V_{ub}|$ provided the relevant hadronic matrix elements of the current $(\bar{b}u)_{V-A}$ can be evaluated by lattice QCD.

Similarly for the decays

$$B^+ \to \bar{D}^0 e^+ \nu_e, \qquad B^+ \to \bar{D}^{*0} e^+ \nu_e, \tag{3.43}$$

shown also in Figure 3.6, we find

$$\mathcal{A}(B^+ \to \bar{D}^0 e^+ \nu_e) = \frac{G_F}{\sqrt{2}} V_{cb}^* (\bar{\nu}_e e^-)_{V-A} \langle \bar{D}^0 | (\bar{b}c)_{V-A} | B^+ \rangle \tag{3.44}$$

with $\bar{D}^0$ replaced by $\bar{D}^{*0}$ in the second decay in (3.43). These decays allow the determination of $|V_{cb}|$ provided the hadronic matrix elements of the current $(\bar{b}c)_{V-A}$ can be evaluated by lattice QCD.

3.4 The Determination of $|V_{cb}|$ and $|V_{ub}|$

The extraction of $|V_{ub}|$ and $|V_{cb}|$ from the decays listed here, the so-called exclusive decays, in which the final state is specified, developed over the years to a big industry. In the case of $|V_{cb}|$ it is based on the distributions as given, e.g., in [51]

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$$\frac{d\Gamma(B \to D^* \ell \nu)}{d\omega} = \frac{G_F^2 m_{D^*}^3}{48\pi^3}(m_B - m_{D^*})^2 \sqrt{\omega^2 - 1}\chi(\omega)\mathcal{F}^2(\omega)|V_{cb}|^2, \tag{3.45}$$

$$\frac{d\Gamma(B \to D \ell \nu)}{d\omega} = \frac{G_F^2 m_D^3}{48\pi^3}(m_B + m_D)^2(\omega^2 - 1)^{3/2}\mathcal{G}^2(\omega)|V_{cb}|^2, \tag{3.46}$$

with $\mathcal{F}(\omega)$ and $\mathcal{G}(\omega)$ being formfactors related to the matrix element in (3.44) for D^* and D, respectively. Their expressions are given in (19) of [51]. Moreover,

$$\chi(\omega) = 1 + \frac{4\omega}{\omega + 1}\frac{1 - 2\omega r_{D^*} + r_{D^*}^2}{(1 - r_{D^*})^2}, \tag{3.47}$$

with

$$r_{D^*} = \frac{m_{D^*}}{m_B}, \qquad \omega = \frac{p_B \cdot p_{D^{(*)}}}{m_B m_{D^{(*)}}} = \frac{m_B^2 + m_{D^{(*)}}^2 - q^2}{2 m_B m_{D^{(*)}}}. \tag{3.48}$$

But there is another route to $|V_{ub}|$ and $|V_{cb}|$. It uses the so-called inclusive decays in which one sums over all final states that are reached through the $b \to u$ or $b \to c$ transitions driven by $|V_{ub}|$ and $|V_{cb}|$, respectively. Here very much developed methods of *heavy quark expansions* are essential for the determination of $|V_{ub}|$ and $|V_{cb}|$. We will describe them briefly in Section 7.5.

Although the extraction of these two CKM matrix elements using these decays is in principle the optimal strategy, being basically independent of new physics contributions, it has been weakened by disagreements between the exclusive and inclusive determinations of $|V_{ub}|$ and to a lesser extent in the case of $|V_{cb}|$. This is clearly seen in a number of reviews and discussions, which make efforts to understand this disagreement. A subset of relevant papers can be found in [52–60].

As an example for this disagreement we quote the *exclusive* determinations from lattice QCD form factors [61–64]

$$|V_{ub}|_{\text{excl}} = (3.72 \pm 0.14) \times 10^{-3}, \qquad |V_{cb}|_{\text{excl}} = (39.36 \pm 0.75) \times 10^{-3}, \tag{3.49}$$

and the *inclusive* ones given by [64, 65]

$$|V_{ub}|_{\text{incl}} = (4.40 \pm 0.25) \times 10^{-3}, \qquad |V_{cb}|_{\text{incl}} = (42.21 \pm 0.78) \times 10^{-3}, \tag{3.50}$$

typical values that one can encountered in the literature.

The problem with $|V_{ub}|$ will hopefully be solved in the BELLE II era. On the other hand, a breakthrough in the case of $|V_{cb}|$ has been made in [55, 57, 58] by realizing that the so-called CLN parametrization [66] of $\mathcal{F}(\omega)$ and $\mathcal{G}(\omega)$ formfactors on which the exclusive determinations of $|V_{cb}|$ were based, being model-dependent, is not optimal, and the model-independent BGL parametrization [67] should be favored instead. As an example, one finds [55]

$$|V_{cb}|_{\text{excl}}^{\text{CLN}} = (37.4 \pm 1.3) \times 10^{-3}, \qquad |V_{cb}|_{\text{excl}}^{\text{BGL}} = (41.9 \pm 1.9) \times 10^{-3}, \tag{3.51}$$

with the second value very close to the inclusive determination in (3.50).

The story of both $|V_{ub}|$ and $|V_{cb}|$ determinations is not over. In particular, there is still a significant difference between the exclusive and inclusive determination of $|V_{ub}|$.

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Interested readers should have a look at the talks presented at the CKM 2018 workshop, in particular those of Gambino and Schwanda, from various lattice QCD analyses, the summary of the 2018 CKM working group II [68] and [69].

For this book we will take as nominal values

$$\boxed{|V_{ub}|^{\rm nom} = 3.7 \times 10^{-3}, \qquad |V_{cb}|^{\rm nom} = 42.0 \times 10^{-3},} \tag{3.52}$$

with the one for $|V_{ub}|$ close to exclusive determinations and the one for $|V_{cb}|$ closer to the inclusive ones. However, on various places, we will stress which SM predictions are particularly sensitive to the values of $|V_{ub}|$ and $|V_{cb}|$ so that their precise determinations are very important. The choice of exclusive value of $|V_{ub}|$ is the author's prejudice that with improved lattice QCD calculations its exclusive value in (3.49) will be favored.

3.5 Leptonic and Semileptonic Decays of Neutral Mesons

A quick look at Table D.2 reveals the following facts about these decays:

- Purely leptonic decays are absent in this table because they are highly suppressed. Best-known examples are $K_{L,S} \to \mu^+\mu^-$ and $B^0_{s,d} \to \mu^+\mu^-$, which will appear in this book but they can only proceed through loop diagrams in the SM.
- Semileptonic decays of K_L dominate the decays of this meson with a significant branching ratio coming from K_L decaying into three pions. This pattern originates in the fact that the K_L meson, as we will discuss in Chapter 8, is dominantly a CP= $-$ eigenstate and can easily decay into an odd number of pions. The decay into two pions can only proceed when CP symmetry is violated.
- As K_S is dominantly a CP= $+$ eigenstate, semileptonic decays in this case are very strongly suppressed, and weak decays of K_S to two pions dominate.
- We list there two branching ratios for B^0_d and B^+ decays. Other can be found in PDG.

We do not show the leading diagrams for semileptonic decays, as they can be obtained from the ones for charged meson decays by changing the *spectator* quark that does not couple to $W^\pm$ from the up-quark u to down-quark d. We only warn the reader that when doing calculations with K_L and K_S mesons factors of $\sqrt{2}$ appear in their definitions and have to be taken properly into account in the evaluation of decay rates.

3.6 Nonleptonic Decays of Mesons

3.6.1 Preliminaries

Finally let us consider nonleptonic decays of mesons like

$$K^+ \to \pi^+\pi^0, \qquad D^+ \to \pi^0 K^+, \qquad B^+ \to D^0 K^+. \tag{3.53}$$

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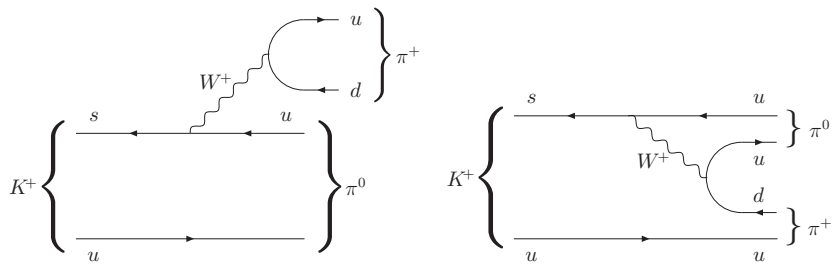


Figure 3.7 Decay $K^+ \to \pi^+\pi^0$.

The flavor structure of the first decay is represented by the left diagram in Figure 3.7. Proceeding like in previous cases, we can immediately write down the corresponding decay amplitude

$$\mathcal{A}(K^+ \to \pi^+\pi^0) = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \langle \pi^+\pi^0 | (\bar{u}d)_{V-A} (\bar{s}u)_{V-A} | K^+ \rangle. \tag{3.54}$$

The new feature relative to the decays considered so far is the appearance of the product of two quark currents, the so-called four-quark operator

$$Q_2 = (\bar{u}d)_{V-A} (\bar{s}u)_{V-A} = (\bar{u}_\alpha d_\alpha)_{V-A} (\bar{s}_\beta u_\beta)_{V-A}. \tag{3.55}$$

The index "2" and not "1" is kept to conform with the literature. Moreover, we exposed this time color indices α and β as they will be relevant soon.

We know already from previous decays that the hadronic matrix element of Q_2 cannot be calculated in perturbation theory, and we will need lattice QCD or some other nonperturbative method to obtain its value. But it is tempting to factorize this matrix element as follows:

$$\langle \pi^+\pi^0 | (\bar{u}d)_{V-A} (\bar{s}u)_{V-A} | K^+ \rangle = \langle \pi^+ | (\bar{u}d)_{V-A} | 0 \rangle \langle \pi^0 | (\bar{s}u)_{V-A} | K^+ \rangle \tag{3.56}$$

so that each factor could be extracted from previous decays. Yet as seen in Figure 3.7, this can only be at best an approximation as the gluon exchanges between the quarks on the two ends of the $W^\pm$ propagator spoil this factorization. Moreover, the appearance of new types of diagrams like the second one in Figure 3.7, not encountered in leptonic and semileptonic decays, clearly shows that (3.56) cannot be true. We conclude therefore that the study of nonleptonic decays is much harder than of leptonic and semileptonic decays because of the breakdown of factorization of a matrix element of a four-quark operator into the product of matrix elements of two quark currents.

For the remaining two decays in (3.53) we find

$$\mathcal{A}(D^+ \to \pi^0 K^+) = \frac{G_F}{\sqrt{2}} V_{cd}^* V_{us} \langle \pi^0 K^+ | (\bar{u}s)_{V-A} (\bar{d}c)_{V-A} | D^+ \rangle, \tag{3.57}$$

$$\mathcal{A}(B^+ \to D^0 K^+) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} \langle D^0 K^+ | (\bar{u}s)_{V-A} (\bar{b}c)_{V-A} | B^+ \rangle. \tag{3.58}$$

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Also in these cases factorization of the matrix element of a four-quark operator into a product of matrix elements of currents is broken. The additional complication arises because in this case lattice QCD alone, as of 2020, is of little help, and other methods like the ones based on flavor symmetries and those using the fact that now heavy mesons are involved have to be used. One of these methods is the so-called QCD factorization, which goes beyond the *naive* factorization in (3.56). We will describe this method briefly in Section 7.4. But let us first look a bit closer at QCD effects in nonleptonic meson decays to be able to make progress in this book.

3.6.2 QCD Effects

As we have seen, in leptonic and semileptonic decays of mesons the decay amplitude involves the product of the matrix element of a quark current between meson states and of the leptonic current. Even if such matrix elements cannot be calculated in perturbation theory, they compactly summarize all QCD effects in these decays and are sufficiently simple to be calculated these days by lattice methods with respectable precision.

However, this is no longer possible in the case of nonleptonic decays, and one has to face a nonperturbative evaluation of hadronic matrix element of the four quark operator involving three external meson states. But this is not the end of the story. Once the diagrams with both $W^{\pm}$ and gluon exchanges are considered, we find that the operator Q_2 in (3.55) is not the only operator responsible for the decay $K^+ \to \pi^+\pi^0$. Generally there are several additional operators that are generated by QCD effects. One of them is

$$Q_1 = (\bar{u}_\alpha d_\beta)_{V-A}(\bar{s}_\beta u_\alpha)_{V-A}, \tag{3.59}$$

where again summation over color indices is understood. Note that the flavor structure of Q_1 is the same as the one of Q_2, but the color structure differs. We will demonstrate the existence of Q_1 in Section 5.1.

If the four-quark flavors in Q_1 and Q_2 were all different from each other, then indeed these operators would be the only four-quark operators of dimension six we would have to consider in nonleptonic K decays and analogous operators in B and D nonleptonic decays. In fact, there are meson decays, to be discussed later, which involve only such *current-current* operators. But Q_1 and Q_2 contain two up-quarks and in such a case new operators, the so-called penguin operators, are generated through gluon and photon exchanges, and they also have to be included in the analysis. We demonstrate their existence in Chapter 6.

Thus in general several four-quark operators have to be taken into account. Denoting these operators by Q_i we find that the amplitudes for $K \to \pi\pi$ decays with K and π being charged or neutral have the general structure

$$\boxed{\mathcal{A}(K \to \pi\pi) = \frac{G_F}{\sqrt{2}} \sum_i V^i_{\rm CKM} C_i \langle \pi\pi | Q_i | K \rangle,} \tag{3.60}$$

with V^i_{CKM} being appropriate CKM factors that can be different for different operators. Analogous expressions for other nonleptonic decays exist and will be discussed later.

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The coefficients C_i, the so-called Wilson Coefficients, describe the strength with which a given operator contributes to the decay amplitude, although this strength depends also on the size of the matrix elements $\langle\pi\pi|Q_i|K\rangle$. The coefficients C_i could at first sight be evaluated in the ordinary perturbation theory in α_s but due to the fact that the mass of $W^{\pm}$ is vastly different from the low-energy scales involved in the decays of mesons, large logarithms of the ratio of M_W and these low-energy scales multiply α_s. Therefore, even in the presence of asymptotic freedom in QCD, in nonleptonic decays not only factorization of matrix elements into products of matrix elements of currents but also the ordinary perturbation theory breaks down.

Yet, in the last forty years powerful methods like effective theories, operator product expansion, and renormalization group among others have been developed, and to continue our presentation of the gauge theory of weak decays, we have to learn what all these methods are. We must also develop efficient technology to apply these methods not only to make predictions for weak decays within the SM but also for its extensions. This is the prime goal of the next three chapters.

3.7 Summary and Motivation

We have seen that tree-level decays play an important role in the determination of the elements of the CKM matrix. In this chapter we have briefly described how

$$|V_{us}|, \qquad |V_{ub}|, \qquad |V_{cb}|, \tag{3.61}$$

can be determined from leptonic and semileptonic decays of mesons. This determination is important as it is believed to be not polluted by contributions from unknown dynamics at very short distance scales, which still has to be discovered. We have also discussed the muon decay, which allows to extract the Fermi constant, the crucial parameter in the phenomenology of weak decays in the SM.

We have also stressed that the study of nonleptonic decays is much harder. Yet, such a study is indispensable for many reasons. Here are few of them

- We have not determined the KM phase in the CKM matrix or equivalently the angle γ in the unitarity triangle and also other angles in this triangle. As we will see in Section 8.5, nonleptonic decays of B mesons play a very important role in these determinations.
- The ratio of the real parts of the isospin amplitudes A_0 and A_2 in $K \to \pi\pi$ decays is measured to be 22.4, while a simple application of factorization of hadronic matrix elements gives $\sqrt{2}$ for this ratio. As we will see in Sections 7.2 and 7.3, nonfactorizable QCD affects both at short-distance scales and in particular long distances are responsible dominantly for this rule.
- CP-violation was discovered in 1964 in $K_L \to \pi\pi$ decays, and its size summarized by the value of the parameter ε_K is an important constraint for the SM and NP beyond it. The same applies to the ratio ε'/ε measured also in these decays. It describes the size of another type of CP violation, to be defined in Section 8.1.

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- Nonleptonic transitions between particles and antiparticles known under the name of mixing play crucial role in the tests of the SM and of its extensions. These are $K^0-\bar{K}^0$, $B_s^0-\bar{B}_s^0$, $B_d^0-\bar{B}_d^0$, and $D^0-\bar{D}^0$ mixings.

But before we can begin to study all these phenomena we have to collect various tools that will allow us to go beyond the tree diagrams and also properly include strong interaction effects.

We are starting collecting tools for our expedition!

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