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Robust transitivity of partially hyperbolic diffeomorphisms with interval central leaves

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Abstract. For a boundary-preserving partially hyperbolic diffeomorphism with interval central leaves, we completely characterize the C^k-robust transitivity ($k \geq 2$) by boundary interconnection. As an application, if the boundary Sinai–Ruelle–Bowen (SRB) measures admit negative central Lyapunov exponents, then boundary interconnection also completely characterizes the phenomenon of robustly intermingled basins for boundary SRB measures.

Key words: partially hyperbolic diffeomorphism, robust transitivity, intermingled basins
2020 Mathematics Subject Classification: 37D30 (Primary); 37D05, 37E05 (Secondary)

1. *Introduction*

As one of the most studied topological properties of dynamical systems, transitivity has been a cornerstone of chaotic behavior. The transitivity of a dynamical system implies the indecomposability of the system. Robust transitivity requires the system to remain transitive under small perturbations. Since we can never measure the system with infinite precision in the physical world, robust transitivity produces a stable and observable phenomenon of chaos.

The simplest and most well-known systems with robust transitivity are transitive Anosov diffeomorphisms. They are robustly transitive due to structural stability [**Ano67**]. Although the question remains widely open as to which manifold can support an Anosov diffeomorphism and whether Anosov diffeomorphisms are all transitive [**Sma67**], all of the discovered Anosov diffeomorphisms are transitive and supported on infra-nilmanifolds [**Man74**], which are finitely covered by nilmanifolds.

It then becomes a question whether robust transitivity is completely characterized by hyperbolicity, in a uniform sense or a weaker sense. In 1971, Shub [**Shu71**] constructed

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examples of robustly transitive diffeomorphisms with partial hyperbolicity on $\mathbb{T}^4$. In 1978, Mañé [Mn78] constructed robustly transitive examples on $\mathbb{T}^3$ with partial hyperbolicity. In 1996, Bonatti and Díaz [BD96] introduced the powerful mechanism of a blender to construct robustly transitive systems from perturbations. In 2000, Bonatti and Viana [BV00] produced further examples that are not partially hyperbolic, but admit a dominated splitting. These examples indicate that generally robust transitivity does not imply uniform hyperbolicity, or even partial hyperbolicity.

However, robust transitivity indeed implies some hyperbolic behavior of the dynamical system. In 1982, Mañé [Mn82] proved that C^1-robust transitivity implies uniform hyperbolicity in the case of surface diffeomorphisms. In 1999, Díaz *et al* [DPU99] showed that C^1-robust transitivity implies partial hyperbolicity in dimension three. Finally, in 2003, Bonatti *et al* [BDP03] proved that C^1-robustly transitive diffeomorphisms on closed manifolds must be volume partially hyperbolic.

In this paper, we give a complete characterization of C^k-robust transitivity ($k \geq 2$) for the simplest partially hyperbolic diffeomorphisms, that is, the interval extensions of Anosov diffeomorphisms on nilmanifolds.

Let N be a nilmanifold and $M = N \times [0, 1]$ be the thickened nilmanifold. Note that ∂M is a disjoint union of $M_0 := N \times \{0\}$ and $M_1 := N \times \{1\}$. Let $\mathrm{Diff}^k_\partial(M)$ be the collection of C^k diffeomorphisms on M preserving M_0 and M_1, equipped with the usual C^k topology.

Definition 1.1. We say that $F \in \mathrm{Diff}^k_\partial(M)$ is *C^k-robustly transitive* if there exists an open neighborhood $\mathcal{U} \subseteq \mathrm{Diff}^k_\partial(M)$ of F such that every $G \in \mathcal{U}$ is transitive.

The study of boundary-preserving systems has its own interests. Pacífico [Pac84] studied the structural stability of vector fields on 3-manifolds with a boundary. Ilyashenko [Ily11] also studied the quasi-open set in the space of boundary-preserving diffeomorphisms of $\mathbb{T}^2 \times [0, 1]$, formed by the maps exhibiting the different types of thick attractors, which is almost topologically mixing.

Definition 1.2. We say that $F \in \mathrm{Diff}^k_\partial(M)$ is a *partially hyperbolic diffeomorphism with interval central leaves* if F satisfies the following properties.

- The diffeomorphism F is *partially hyperbolic*, that is, there exists a continuous DF-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that for any $x \in M$,

$$\|DF|_{E^s(x)}\| < 1 < m(DF|_{E^u(x)}),$$
$$\|DF|_{E^s(x)}\| < m(DF|_{E^c(x)}) \leq \|DF|_{E^c(x)}\| < m(DF|_{E^u(x)}).$$

  As a result, there are two F-invariant foliations, the stable foliation $\mathcal{F}^s$ and the unstable foliation $\mathcal{F}^u$, tangent to E^s and E^u, respectively.
- The diffeomorphism F is *dynamically coherent*, that is, there are two F-invariant foliations, the center-stable foliation $\mathcal{F}^{cs}$ and the center-unstable foliation $\mathcal{F}^{cu}$, tangent to $E^{cs} := E^c \oplus E^s$ and $E^{cu} := E^c \oplus E^u$, respectively. As a result, their intersection, the central foliation $\mathcal{F}^c$, is an F-invariant foliation tangent to E^c.
- Each central leaf $\mathcal{F}^c(\cdot)$ is an interval that intersects transversely with M_0 and M_1 exactly at its endpoints.

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The collection of partially hyperbolic diffeomorphisms with interval central leaves in $\mathrm{Diff}^k_\partial(M)$ will be denoted by $\mathrm{PHI}^k(M)$.

Remark 1.3. Since E^c is uniformly transverse to the boundary M_0 and M_1, $\mathrm{PHI}^k(M)$ is open in $\mathrm{Diff}^k_\partial(M)$ [**HPS77**]. For $F \in \mathrm{PHI}^k(M)$ and $x \in M_i$ $(i = 0, 1)$, since M_i is F-invariant and transverse to $E^c(x)$, we have

$$T_xM = T_xM_i \oplus E^c(x) \quad \text{where } T_xM_i = E^s(x) \oplus E^u(x).$$

In particular, $F_i := F|_{M_i} : M_i \to M_i$ is an Anosov diffeomorphism. Let $\pi_i : M \to M_i$ be the natural projection along central leaves, defined by $\pi_i(x) := \mathcal{F}^c(x) \pitchfork M_i$. Then, we have $\pi_i \circ F = F_i \circ \pi_i$. In particular, we have $\pi_0|_{M_1} \circ F_1 = F_0 \circ \pi_0|_{M_1}$, where $\pi_0|_{M_1} : M_1 \to M_0$ is a homeomorphism. Therefore, F_0 and F_1 are topologically conjugated.

Definition 1.4. We say that $F \in \mathrm{PHI}^k(M)$ is *boundary interconnected* if there exist two pairs of periodic points $p_i, q_i \in M_i$ $(i = 0, 1)$ such that

$$\lambda^c(p_0) < 0 < \lambda^c(p_1) \quad \text{and} \quad W^s(p_0) \cap W^u(p_1) \neq \varnothing;$$
$$\lambda^c(q_1) < 0 < \lambda^c(q_0) \quad \text{and} \quad W^s(q_1) \cap W^u(q_0) \neq \varnothing.$$

Here, $\lambda^c(\cdot)$ is the central Lyapunov exponent at the given periodic point that is the logarithm of the absolute value of corresponding eigenvalue of the derivative restricted to E^c, divided by the period; $W^s(\cdot)$ and $W^u(\cdot)$ are the stable and unstable sets, respectively.

Remark 1.5. In fact, $W^s(p_0)$ is an open submanifold of $\mathcal{F}^{cs}(p_0)$ and $W^u(p_1)$ is an open submanifold of $\mathcal{F}^{cu}(p_1)$, given by the hyperbolicity of the periodic points (see Lemma 3.2). Hence, $\dim W^s(p_0) + \dim W^u(p_1) = \dim M + 1$ and their intersection $W^s(p_0) \pitchfork W^u(p_1)$ consists of central intervals. Similarly, $W^s(q_1) \pitchfork W^u(q_0)$ also consists of central intervals.

1.1. *Robust transitivity.* In this paper, we obtain the following characterization of robust transitivity for partially hyperbolic diffeomorphisms with interval central leaves.

THEOREM A. *For $F \in \mathrm{PHI}^k(M)$ $(k \geq 2)$, the following statements are equivalent:*

(1) *F is C^k-robustly transitive;*
(2) *F is boundary interconnected.*

Remark 1.6. Note that the boundary interconnection property is C^1-open in $\mathrm{PHI}^1(M)$ (see Figure 1). We actually prove that for $F \in \mathrm{PHI}^1(M)$, if F is boundary interconnected, then there exists an open neighborhood $\mathcal{U} \subseteq \mathrm{PHI}^1(M)$ of F such that every $G \in \mathcal{U} \cap \mathrm{PHI}^2(M)$ is topologically transitive, see Corollary 3.7. This kind of robust transitivity is exactly similar to the concept of stable ergodicity: It relies on a C^1-robust structure provided by a C^2 partially hyperbolic diffeomorphism, which exactly corresponds to the role of the absolute continuity in the stably ergodic issue with Hopf's argument.

Remark 1.7. The geometric and algebraic structure of M is crucial for Theorem A. In fact, the conclusion does not hold when $F_i : M_i \to M_i$ is a general uniformly hyperbolic

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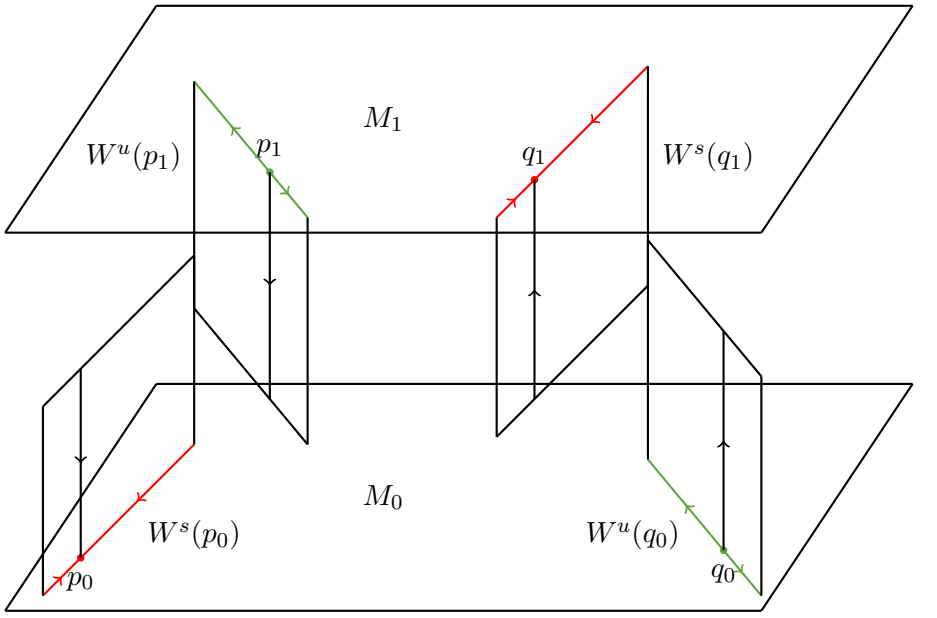


FIGURE 1. Boundary interconnection (color online).

system, rather than an Anosov diffeomorphism on a nilmanifold. For example, let $\sigma : \Sigma \to \Sigma$ be the classical horseshoe, where $\Sigma = \{0, 1\}^{\mathbb{Z}}$ and $\sigma(x)_i = x_{i+1}$. Assume that $\Phi : [0, 1] \to [0, 1]$ is a smooth diffeomorphism with exactly two fixed points, a source at 1 and a sink at 0. Assume further that Φ' is dominated by the hyperbolicity of σ, and consider the partially hyperbolic system $F : \Sigma \times [0, 1] \to \Sigma \times [0, 1]$, $F(x, t) = (\sigma(x), \Phi_x(t))$, where $\Phi_x = \Phi$ when $x_0 = 0$ and $\Phi_x = \Phi^{-1}$ when $x_0 = 1$. Then, F is boundary interconnected but not topologically transitive. In fact, choose two fixed points $p = (0)$, $q = (1)$ of σ, then $p_0 := (p, 0)$, $p_1 := (p, 1)$, $q_0 := (q, 0)$, and $q_1 := (q, 1)$ satisfy the property that we need for boundary interconnection. However, if we take open intervals U, $V \subseteq (0, 1)$ such that $\Phi^n(U) \cap V = \varnothing$ for any $n \in \mathbb{Z}$, then $\Sigma \times \bigcup_{n\in\mathbb{Z}} \Phi^n(U)$ and $\Sigma \times \bigcup_{n\in\mathbb{Z}} \Phi^n(V)$ are disjoint F-invariant open subsets, and hence F is not topologically transitive.

1.2. *Robustly intermingled basins.* The phenomenon of intermingled basins was originally investigated by Kan [**Kan94**], and summarized by Bonatti *et al* [**BDV05**]. In such a system, there are two Sinai–Ruelle–Bowen (SRB; physical) measures, both of whose basins admit positive volume in every non-empty open subset. Some of these examples are topologically transitive or even topologically mixing [**GS19**, **Xia24**], indicating the significant difference between the topological and measure-theoretical notions of indecomposability. Moreover, Kan's intermingled basins are robust [**KS11**].

This type of construction continues to be a source of interesting research and has been especially studied in the aspect of intermingled basins [**MW05**, **DVY16**, **BP18**, **UV18**]. There is also a family of topologically transitive diffeomorphisms constructed by inserting

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a blender and embedding into a boundaryless manifold [**CGS18**]. More recently, there have been some advances focusing on the number of measures of maximal entropy for systems related to this type of construction [**NnRV21**, **RT22**].

Specifically, we describe the phenomenon of intermingled basins in the setting of partially hyperbolic diffeomorphisms with interval central leaves.

Given $F \in \mathrm{PHI}^k(M)$ $(k \geq 2)$, for $i = 0, 1$, there exists a unique SRB measure μ_i on M_i for F_i, since F_i is an Anosov diffeomorphism on a nilmanifold M_i and therefore topologically transitive. We call them *the boundary SRB measures* of F.

Denote by $\mathrm{PHI}^k_-(M)$ $(k \geq 2)$ the collection of diffeomorphisms $F \in \mathrm{PHI}^k(M)$ that are mostly contracting along the center direction on each boundary, that is, both of the boundary SRB measures μ_i $(i = 0, 1)$ admit negative central Lyapunov exponents, in the sense that

$$\int_{M_i} \ln \|DF|_{E^c}\| \, d\mu_i < 0.$$

Note that $\mathrm{PHI}^k_-(M)$ is open in $\mathrm{PHI}^k(M)$, and for $F \in \mathrm{PHI}^k_-(M)$, μ_i $(i = 0, 1)$ is not only an SRB measure for F_i, but also a physical measure for F.

Definition 1.8. We say that $F \in \mathrm{PHI}^k_-(M)$ admits *intermingled basins* if the basins of its boundary SRB (physical) measures, $B(\mu_0)$ and $B(\mu_1)$, both admit positive volume inside any non-empty open subset of M, and their union has full volume. Further, we say that F admits *C^k-robustly intermingled basins* if there exists an open neighborhood $\mathcal{U} \subseteq \mathrm{PHI}^k_-(M)$ of F such that every $G \in \mathcal{U}$ admits intermingled basins.

An interesting result of Ures and Vásquez [**UV18**] showed that the phenomenon of robustly intermingled basins does not appear on $\mathbb{T}^3$, indicating the particularity of manifolds with a boundary.

In this paper, we also obtain the characterization of robustly intermingled basins for partially hyperbolic diffeomorphisms with interval central leaves that are mostly contracting along the center direction on each boundary.

THEOREM B. *For $F \in \mathrm{PHI}^k_-(M)$ $(k \geq 2)$, the following statements are equivalent:*

(1) *F admits C^k-robustly intermingled basins;*
(2) *F is C^k-robustly transitive;*
(3) *F is boundary interconnected.*

As an application of Theorems A and B, consider the skew-product system

$$A_\varphi(x, t) := (Ax, \ \varphi(x, t)) : \mathbb{T}^d \times [0, 1] \to \mathbb{T}^d \times [0, 1],$$

where $A \in \mathrm{Aut}(\mathbb{T}^d)$ is a hyperbolic toral automorphism with hyperbolic splitting $L^s \oplus L^u$ and $\varphi(x, t) : \mathbb{T}^d \times [0, 1] \to [0, 1]$ is a C^k $(k \geq 2)$ map such that $\varphi(x, \cdot) \in \mathrm{Diff}^k_\partial([0, 1])$ for each $x \in \mathbb{T}^d$. Assume that

$$\|A|_{L^s}\| < \left|\frac{\partial \varphi}{\partial t}\right| < m(A|_{L^u}).$$

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Then, $A_\varphi \in \mathrm{PHI}^k(\mathbb{T}^d \times [0,1])$. For convenience, we denote the iterations of A_φ by

$$A_\varphi^n(x,\ t) = (A^n x,\ \varphi_x^n(t)).$$

Following the definition proposed by Bonatti *et al* [**BDV05**, Ch. 11.1.1], we say that A_φ is a *Kan-type skew-product system* if there exist two periodic points p and q of $A \in \mathrm{Aut}(\mathbb{T}^d)$ with periods $\pi(p)$ and $\pi(q)$, such that $\varphi_p^{\pi(p)}$ has exactly two fixed points, a source at $(p, 1)$ and a sink at $(p, 0)$; $\varphi_q^{\pi(q)}$ also has exactly two fixed points, a source at $(q, 0)$ and a sink at $(q, 1)$. Further, we say that A_φ is mostly contracting if

$$\int_{\mathbb{T}^d} \ln \frac{\partial \varphi}{\partial t}(x, 0)\, dx < 0 \quad \text{and} \quad \int_{\mathbb{T}^d} \ln \frac{\partial \varphi}{\partial t}(x, 1)\, dx < 0.$$

By Theorem A, Kan-type skew-product systems are C^k-robustly transitive. By Theorem B, mostly contracting Kan-type skew product systems admit C^k-robustly intermingled basins for which the boundary SRB measures coincide with the Lebesgue measure restricted to each boundary component.

1.3. *Organization.* This paper is organized as follows. In §2, we introduce some basic definitions and results. In §3, we establish topological transitivity from boundary interconnection (Theorem 3.1), which exactly shows C^2-robust transitivity in Theorem A. Then, we complete the proof of Theorem A in §4, showing that C^∞-robust transitivity implies boundary interconnection. In §5, we prove Theorem B by establishing the equivalence between robustly intermingled basins and boundary interconnection.

2. *Preliminaries*

2.1. *Partially hyperbolic diffeomorphisms with interval central leaves.* As defined in §1, M is a thickened nilmanifold, $\mathrm{Diff}_\partial^k(M)$ is the collection of C^k diffeomorphisms on M preserving each boundary M_0 and M_1, $\mathrm{PHI}^k(M)$ is the collection of partially hyperbolic diffeomorphisms in $\mathrm{Diff}_\partial^k(M)$ with interval central leaves, and $\mathrm{PHI}_-^k(M)$ is the collection of diffeomorphisms in $\mathrm{PHI}^k(M)$ whose boundary SRB measures admit negative central Lyapunov exponents.

For $F \in \mathrm{PHI}^2(M)$, the dynamical foliations $\mathcal{F}^\sigma$ are Hölder continuous with C^{1+} leaves, for $\sigma = s, c, u, cs, cu$. A subset of M is *σ-saturated* if it is a union of σ-leaves. There is a well-known result on the regularity of holonomy maps in [**PSW97**], which we state in our setting as follows. Note that $F \in \mathrm{PHI}^2(M)$ is dynamically coherent and *center bunched*, in the sense that for any $x \in M$,

$$\|DF|_{E^s(x)}\| < \frac{m(DF|_{E^c(x)})}{\|DF|_{E^c(x)}\|} \quad \text{and} \quad m(DF|_{E^u(x)}) > \frac{\|DF|_{E^c(x)}\|}{m(DF|_{E^c(x)})}.$$

LEMMA 2.1. [**PSW97**, Theorem B] *For $F \in \mathrm{PHI}^2(M)$, the holonomy map along the stable foliation (respectively the unstable foliation) restricted to a center-stable leaf (respectively a center-unstable leaf) is a C^1 diffeomorphism. Moreover, the derivative of the holonomy map is uniformly continuous with respect to the base points.*

The following result on the estimate of central twist is useful in our proof.

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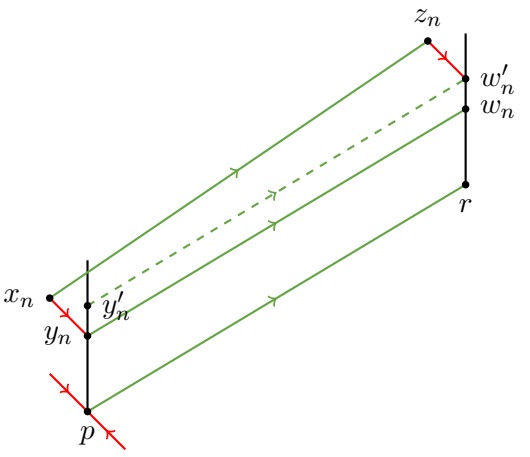


FIGURE 2. Central twist (color online).

LEMMA 2.2. *Assume that* $F \in \mathrm{PHI}^2(M)$ *and* $p \in M_0$ *is a fixed point of* F_0 *with* $\lambda^c(p) < 0$. *Fix* $r \in \mathcal{F}^u(p)$ *and let* $\mathrm{Hol}^r_p : \mathcal{F}^{cs}_{\mathrm{loc}}(p) \to \mathcal{F}^{cs}_{\mathrm{loc}}(r)$ *be the holonomy map along the unstable foliation. For* $x_0 \in \mathcal{F}^{cs}_{\mathrm{loc}}(p)$, *denote* $x_n = F^n(x_0)$ *for every* $n \in \mathbb{N}$ *and define*

$$y_n := \mathcal{F}^s_{\mathrm{loc}}(x_n) \pitchfork \mathcal{F}^c_{\mathrm{loc}}(p), \quad z_n := \mathrm{Hol}^r_p(x_n), \quad w_n := \mathrm{Hol}^r_p(y_n), \quad w'_n := \mathcal{F}^s_{\mathrm{loc}}(z_n) \pitchfork \mathcal{F}^c_{\mathrm{loc}}(r).$$

Then, we have the following estimate:

$$d_c(w'_n,\ w_n) \le C e^{n\lambda_p} \quad \textit{for all } n \in \mathbb{N}.$$

Here, $C \ge 1$ *is a constant depending on* p *and* r, *and*

$$\lambda_p := \frac{\lambda^u(p) - \lambda^c(p)}{\lambda^u(p) - \lambda^s(p)} \lambda^s(p) < \lambda^c(p) < 0,$$

where

$$\ln \|DF|_{E^s(p)}\| =: \lambda^s(p) < \lambda^c(p) < \lambda^u(p) := \ln m(DF|_{E^u(p)}).$$

Proof. Take $y'_n := (\mathrm{Hol}^r_p)^{-1}(w'_n)$, see Figure 2. By Lemma 2.1, the holonomy map Hol^r_p restricted to $\mathcal{F}^c_{\mathrm{loc}}(p)$ is a C^1 diffeomorphism. Hence, it suffices to estimate $d_c(y'_n,\ y_n)$. First, note that

$$d_c(F^{-N}(y'_n),\ F^{-N}(y_n)) \le 2d(F^{-N}(y'_n),\ F^{-N}(y_n))$$

when $n - N$ is sufficiently large. Therefore, there exists a constant $C \ge 1$ such that

$$\begin{aligned} &d_c(F^{-N}(y'_n),\ F^{-N}(y_n)) \\ &\quad \le 2[d_u(F^{-N}(y'_n),\ F^{-N}(w'_n)) + d_s(F^{-N}(w'_n),\ F^{-N}(z_n)) \\ &\qquad + d_u(F^{-N}(z_n),\ F^{-N}(x_n)) + d_s(F^{-N}(x_n),\ F^{-N}(y_n))] \\ &\quad \le C[e^{-N\lambda^u(p)} + e^{(n-N)\lambda^s(p)}]. \end{aligned}$$

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Now, let $-N\lambda^u(p) \approx (n-N)\lambda^s(p)$, then we take

$$N = \left[\frac{-\lambda^s(p)}{\lambda^u(p)-\lambda^s(p)}n\right] + 1.$$

It follows that when n is sufficiently large, we have

$$\begin{aligned} d_c(y_n',\ y_n) &\le Ce^{N\lambda^c(p)}d_c(F^{-N}(y_n'),\ F^{-N}(y_n)) \\ &\le 2C^2e^{N\lambda^c(p)+(n-N)\lambda^s(p)} \\ &\le 4C^2e^{n\lambda_p}, \end{aligned}$$

where

$$\lambda_p = \lim_{n\to+\infty}\left(\frac{N}{n}\lambda^c(p) + \frac{n-N}{n}\lambda^s(p)\right) = \frac{\lambda^u(p)-\lambda^c(p)}{\lambda^u(p)-\lambda^s(p)}\lambda^s(p) < \lambda^c(p) < 0.$$

This completes the proof of Lemma 2.2. □

2.2. *Asymptotically rational independence of Birkhoff sums.* Here, we collect some results related to Birkhoff sums in the uniformly hyperbolic setting.

First, there is a concept of asymptotically rational independence introduced in [**GS19**]. For real numbers $a, b \in \mathbb{R}$, define

$$(a,\ b) := \inf\{|ka+lb| : k, l \in \mathbb{Z},\ ka+lb \neq 0\}.$$

Particularly, this notation $(a,\ b)$ coincides with $\gcd(a, b)$ when $a, b \in \mathbb{Z}$. Recall that two real numbers a, b are *rationally independent* if and only if $(a,\ b) = 0$. Inspired by this fact, the asymptotically rational independence is defined as follows.

Definition 2.3. Let $b \in \mathbb{R}$. A sequence $\{a_m\} \subseteq \mathbb{R}$ is *asymptotically rationally independent* of b if $(a_m,\ b) \to 0$ as $m \to +\infty$.

To understand the asymptotically rational independence, we state the following results.

LEMMA 2.4. *If $a, b \in \mathbb{R}$ and $b \neq 0$, then $(a,\ b) < \varepsilon$ for some $\varepsilon > 0$ if and only if each orbit of $T_a(x) := x + a \pmod{b\mathbb{Z}}$ is ε-dense in $\mathbb{R}/(b\mathbb{Z})$. In particular, if $(a,\ b) > 0$, then $(a,\ b) = |b|/q$ for the reduced fraction $a/b = p/q$ with $p \in \mathbb{Z},\ q \in \mathbb{N},\ (p,\ q) = 1$.*

Proof. The conclusion follows directly from the definition of $(a,\ b)$. □

LEMMA 2.5. *If $a_1, a_2, b \in \mathbb{R}$ and $b \neq 0$, then $0 < |a_1 - a_2| < \varepsilon \pmod{b\mathbb{Z}}$ implies that there exists $a \in \{a_1, a_2\}$ such that $(a,\ b) < \sqrt{|b|\,\varepsilon}$.*

Proof. Given $b \neq 0$, notice that

$$\begin{aligned} &\{a \pmod{b\mathbb{Z}} : (a,\ b) \ge \sqrt{|b|\,\varepsilon}\} \\ &\quad = \left\{\frac{k}{n}b \pmod{b\mathbb{Z}} : n \in \mathbb{Z}^+,\ n \le \sqrt{\frac{|b|}{\varepsilon}},\ k \in \mathbb{Z},\ (k,\ n) = 1\right\} \end{aligned}$$

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is a finite set, in which for any a_1, a_2 with $a_1 \neq a_2 \pmod{b\mathbb{Z}}$, we have $|a_1 - a_2| \geq \varepsilon \pmod{b\mathbb{Z}}$. In fact, taking the reduced fractions

$$\frac{a_i}{b} = \frac{p_i}{q_i} (\mathrm{mod}\ \mathbb{Z}) \quad \text{with } p_i \in \mathbb{Z},\ q_i \in \mathbb{N},\ (p_i,\ q_i) = 1,$$

we have $p_1q_2 - p_2q_1 - kq_1q_2 \neq 0$ for any $k \in \mathbb{Z}$; hence,

$$\left| \frac{p_1}{q_1} - \frac{p_2}{q_2} - k \right| = \frac{|p_1q_2 - p_2q_1 - kq_1q_2|}{q_1q_2} \geq \frac{1}{q_1q_2},$$

and then

$$\begin{aligned} |a_1 - a_2| \pmod{b\mathbb{Z}} &= \inf_k \left\{ \left| \frac{p_1}{q_1} - \frac{p_2}{q_2} - k \right| \cdot |b| \right\} \\ &\geq \frac{|b|}{q_1q_2} = \frac{|b|/q_1 \cdot |b|/q_2}{|b|} = \frac{(a_1,\ b) \cdot (a_2,\ b)}{|b|} \\ &\geq \frac{\sqrt{|b|\ \varepsilon} \cdot \sqrt{|b|\ \varepsilon}}{|b|} = \varepsilon. \end{aligned}$$

Therefore, $0 < |a_1 - a_2| < \varepsilon \pmod{b\mathbb{Z}}$ implies that at least one of them does not belong to the finite set. □

Now, we introduce the Birkhoff sums. Let $A : N \to N$ be an Anosov diffeomorphism on a nilmanifold and $\varphi : N \to \mathbb{R}$ be a Hölder continuous function. Denote by

$$S_\varphi A(p) := \sum_{i=0}^{\pi(p)-1} \varphi(A^i(p))$$

the *Birkhoff sum* of the function φ along a periodic orbit of $p \in \mathrm{Per}(A)$ with period $\pi(p)$.

The following *positive Livšic theorem* (see [**LT03**]) or *Mañé–Conze–Guivarc'h–Bousch lemma* (see [**Bou01**, **Bou11**]) is a classical method to obtain the information of the whole space from that of periodic orbits.

LEMMA 2.6. (Positive Livšic theorem) *Let $A : N \to N$ be an Anosov diffeomorphism on a nilmanifold and $\varphi : N \to \mathbb{R}$ be a Hölder continuous function. If*

$$S_\varphi A(p) \geq 0 \quad \textit{for all } p \in \mathrm{Per}(A),$$

then there exists a Hölder continuous function $\psi : N \to \mathbb{R}$ such that $\varphi \geq \psi - \psi \circ A$.

For $F \in \mathrm{PHI}^1(M)$, we have an Anosov diffeomorphism $F_0 : M_0 \to M_0$. Define

$$\varphi : M_0 \to \mathbb{R} \quad \text{and} \quad \varphi(x) := \ln \|DF|_{E^c(x)}\|.$$

Then, for a periodic point $p \in \mathrm{Per}(F_0)$ with period $\pi(p)$, the corresponding Birkhoff sum $S_\varphi F_0(p)$ is exactly the *total central Lyapunov exponent* $\pi(p)\lambda^c(p)$. Hence, the Birkhoff sum is regarded as an observation function involving the information of central Lyapunov exponents.

In particular, we have the following corollary of Lemma 2.6.

COROLLARY 2.7. *For $F \in \mathrm{PHI}^1(M)$, if $\lambda^c(p) \leq 0$ for every $p \in \mathrm{Per}(F_0)$, then $\lambda^c(x) \leq 0$ whenever $\lambda^c(x)$ exists.*

There is the following rigidity result on the distribution of all Birkhoff sums.

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LEMMA 2.8. [GSX22, Theorem 1.1] *Let $A : N \to N$ be an Anosov diffeomorphism on a nilmanifold and $\varphi : N \to \mathbb{R}$ be a Hölder continuous function. If there are two periodic points $p_0, q_0 \in \mathrm{Per}(A)$ satisfying*

$$S_\varphi A(p_0) < 0 < S_\varphi A(q_0),$$

then the set $\{S_\varphi A(p) : p \in \mathrm{Per}(A)\}$ is dense in $\mathbb{R}$.

2.3. *Analysis in the one-dimensional dynamics.* The following quantitative result is prepared for analyzing the one-dimensional center dynamics connected by a holonomy map. The proof is technical and may be skipped on a first reading.

PROPOSITION 2.9. *Let $f, g \in \mathrm{Diff}^{1+}_\partial([0,1])$ be diffeomorphisms with exactly two fixed points 0, 1, where 0 is a hyperbolic sink; and $h \in \mathrm{Diff}^1_\partial([0,1])$. Then, there exist $\varepsilon_0 > 0$ and an increasing function $L(\varepsilon) : (0, \varepsilon_0) \to (0,1)$ satisfying $L(x) \to 0$ as $x \to 0^+$ and depending on $\beta := g'(0)$, such that the following conclusion holds. If $\alpha := f'(0)$ satisfies*

$$(\ln \alpha,\ \ln \beta) < \varepsilon,$$

then for any two closed intervals $I, J \subseteq (0,1)$ with $|J| > L(\varepsilon)$, there exist two strictly increasing subsequences $\{k_n\}, \{l_n\}$ of $\mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}}\{|k_n \ln \alpha - l_n \ln \beta|\} < +\infty;$$
$$h \circ f^{k_n}(I) \cap g^{l_n}(J) \neq \varnothing \quad \text{for all } n \in \mathbb{N}.$$

Moreover, there exists a constant $c > 0$ such that

$$|h \circ f^{k_n}(I) \cap g^{l_n}(J)| \geq c \cdot \max\{e^{k_n \ln \alpha},\ e^{l_n \ln \beta}\} \quad \text{for all } n \in \mathbb{N}.$$

Proof. First, we have the following expansions:

$$\begin{aligned}
&f(x) = \alpha x + R_f(x), && 0 < \alpha = f'(0) < 1, && R_f(x) = o(x)\ (x \to 0^+);\\
&g(x) = \beta x + R_g(x), && 0 < \beta = g'(0) < 1, && R_g(x) = o(x)\ (x \to 0^+);\\
&h(x) = \gamma x + R_h(x), && \gamma = h'(0) > 0, && R_h(x) = o(x)\ (x \to 0^+).
\end{aligned}$$

For any $k, l \in \mathbb{N}$, we also have

$$g^{-l} \circ h \circ f^k(x) = \beta^{-l}\alpha^k h(x) + R_{k,l}(x), \quad R_{k,l}(x) = o(x)\ (x \to 0^+).$$

Hence, for any $\varepsilon > 0$, we can take $0 < \delta < 1$ sufficiently small such that

$$\max\{|R_f(x)|,\ |R_g(x)|,\ |R_h(x)|\} < \varepsilon x \quad \text{for all } x \in [0,\ \max\{\delta, h(\delta)\}];$$
$$\max\{|R'_f(x)/\alpha|,\ |R'_g(x)/\beta|,\ |R'_h(x)/\gamma|\} < \tfrac{1}{2} \quad \text{for all } x \in [0,\ \max\{\delta, h(\delta)\}].$$

Since $(\ln \alpha,\ \ln \beta) < \varepsilon$, there exist $K^*,\ L^* \in \mathbb{Z}^+$ depending on β such that for any $\eta \in (\beta^2, 1)$, there exist $1 \leq K \leq K^*$, $1 \leq L \leq L^*$ such that

$$K \ln \alpha - L \ln \beta \in (\ln \eta,\ \ln \eta + \varepsilon).$$

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Therefore, by taking δ smaller, we may further require that

$$\max\{|R_{k,l}(x)| : 1 \le k \le K^*,\ 1 \le l \le L^*\} < \varepsilon h(x) \quad \text{for all } x \in [0, \delta].$$

It follows that

$$g^{-L} \circ h \circ f^K(x) \in ((\eta - \varepsilon)h(x),\ (e^\varepsilon \eta + \varepsilon)h(x)).$$

For $x \in [0, \delta]$, define $I_x := (g(h(x)),\ h(x)]$. Then, we have the following conclusion.

CLAIM 2.10. *For any $x \in [0, \delta]$ and any closed interval $J \subseteq I_x$ with*

$$\frac{|J|}{|I_x|} > \frac{3\varepsilon}{1 - \beta - \varepsilon},$$

there exist $1 \le K \le K^*$ and $1 \le L \le L^*$ such that

$$g^{-L} \circ h \circ f^K(x) \in J.$$

Proof of Claim 2.10. Note that $|I_x| > (1 - \beta - \varepsilon)h(x)$ and, hence, $|J| > 3\varepsilon h(x)$. It follows that there exists $\eta \in (\beta^2, e^{-\varepsilon})$ such that

$$((\eta - \varepsilon)h(x),\ (e^\varepsilon \eta + \varepsilon)h(x)) \subseteq J,$$

because the error $((e^\varepsilon - 1)\eta + 2\varepsilon)h(x) < 3\varepsilon h(x)$. It follows that there exist $1 \le K \le K^*$, $1 \le L \le L^*$ such that

$$g^{-L} \circ h \circ f^K(x) \in ((\eta - \varepsilon)h(x),\ (e^\varepsilon \eta + \varepsilon)h(x)) \subseteq J.$$

This completes the proof of Claim 2.10. □

For any point $y \in (0, 1)$ and closed interval $J \subseteq (0, 1)$, define the fundamental domains $\Omega_i := (g^{i+1}(y),\ g^i(y)]$, and denote $J^* := \bigcup_{i \in \mathbb{Z}} g^i(J)$ and $J_i := J^* \cap \Omega_i$ for any $i \in \mathbb{Z}$. Then, we have the following conclusion.

CLAIM 2.11. *There exists a constant $0 < \rho < 1$ such that*

$$\rho \frac{|J_i|}{|\Omega_i|} \le \frac{|J_j|}{|\Omega_j|} \le \rho^{-1} \frac{|J_i|}{|\Omega_i|} \quad \textit{whenever } \Omega_i, \Omega_j \subseteq [0, h(\delta)].$$

In particular, there exists a constant $c > 0$ such that $|g^i(J)| \ge c\beta^i$ for all $i \in \mathbb{N}$.

Proof of Claim 2.11. Note that J_i does not change when we replace J by J_0. Since J_0 has at most two components, by considering each component, we may assume at first that $J = [a, b] \subseteq \overline{\Omega}_0$. Without loss of generality, assume that $j > i$. Then, we have

$$\begin{aligned}
\frac{|J_j|}{|\Omega_j|} &= \frac{g^j(b) - g^j(a)}{g^j(y) - g^{j+1}(y)} \\
&= \frac{g^i(b) - g^i(a)}{g^i(y) - g^{i+1}(y)} \cdot \frac{(g^{j-i})'(\xi_1)}{(g^{j-i})'(\xi_2)} \\
&= \frac{|J_i|}{|\Omega_i|} \cdot \exp\left[\sum_{k=0}^{j-i-1} (\ln g'(g^k(\xi_1)) - \ln g'(g^k(\xi_2))) \right],
\end{aligned}$$

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where $\xi_1 \in J_i$ and $\xi_2 \in \Omega_i$, and hence,

$$\begin{aligned}
&\Big| \sum_{k=0}^{j-i-1} (\ln g'(g^k(\xi_1)) - \ln g'(g^k(\xi_2))) \Big| \\
&\quad \le \sum_{k=0}^{j-i-1} |\ln g'(g^k(\xi_1)) - \ln g'(g^k(\xi_2))| \\
&\quad \le \sum_{k=0}^{j-i-1} C|g^k(\xi_1) - g^k(\xi_2)|^\theta \\
&\quad \le \sum_{k=0}^{j-i-1} C^2\beta^{k\theta}|\xi_1 - \xi_2|^\theta \le \frac{C^2}{1-\beta^\theta}.
\end{aligned}$$

Here, $0 < \theta < 1$ is the Hölder exponent of $\ln g'$ in $[0, h(\delta)]$ and $C \ge 1$ is a constant depending on g. Therefore, we can take

$$\rho = \exp\left[-\frac{C^2}{1-\beta^\theta}\right].$$

This completes the proof of Claim 2.11. □

Consider the family of fundamental domains

$$\mathcal{D} := \{I_t := (g(t), t] : t \in D_0 := (g(h(\delta)), h(\delta)]\}, \quad D := \sup_{t\in D_0} |I_t|.$$

CLAIM 2.12. *There exist $\varepsilon_0 > 0$ and an increasing function $G : (0, \varepsilon_0) \to (0, 1)$ satisfying $G(x) \to 0$ as $x \to 0^+$ and $r = r(s, t) : D_0 \times D_0 \to \{-1,\ 0,\ 1\}$ such that*

$$|g^r(J_s(\varepsilon)) \cap I_t| \ge \varepsilon \quad \textit{for all } s, t \in D_0,$$

where $J_s(\varepsilon) := [s - G(\varepsilon),\ s]$.

Proof of Claim 2.12. Fix any $s \in D_0$. Considering the relative position of s and I_t, we need the following sufficient conditions:

$$\begin{aligned}
t - g^{-1}(s - G(\varepsilon)) &\ge \varepsilon \quad \text{when } g(t) < s < g(t) + \varepsilon; \\
G(\varepsilon) &\ge \varepsilon \quad \text{when } g(t) + \varepsilon \le s \le t; \\
t - (s - G(\varepsilon)) &\ge \varepsilon \quad \text{when } t < s < g^{-1}(g(t) + \varepsilon); \\
g(s) - g(s - G(\varepsilon)) &\ge \varepsilon \quad \text{when } s \ge g^{-1}(g(t) + \varepsilon).
\end{aligned}$$

Finally, we take

$$G(\varepsilon) = \sup_{u\in D_0} \{g(u) + \varepsilon - g(u - \varepsilon),\ \varepsilon,\ g^{-1}(g(u) + \varepsilon) - u + \varepsilon,\ u - g^{-1}(g(u) - \varepsilon)\}.$$

One can check that $G(\cdot)$ is increasing and $G(x) \to 0$ as $x \to 0^+$. Note that we take $\varepsilon_0 > 0$ sufficiently small so that when $0 < \varepsilon < \varepsilon_0$, the above discussions make sense. □

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Now, we prove Proposition 2.9. Define

$$L(\varepsilon) := \sup\left\{\left|g^{-l}\left[J_s\left(\rho^{-1}\frac{3\varepsilon}{1-\beta-\varepsilon}D\right)\right]\right| : s \in D_0,\ l \in \mathbb{Z}\right\}.$$

Note that we take $\varepsilon_0 > 0$ sufficiently small so that when $0 < \varepsilon < \varepsilon_0$, the function $L(\cdot)$ is well defined. One can check that $L(\cdot)$ is increasing and $L(x) \to 0$ as $x \to 0^+$.

We claim that $|J| > 3L(\varepsilon)$ will be sufficient.

In fact, for any point $x \in (0, 1)$ and closed interval $J \subseteq (0, 1)$ with $|J| > L(\varepsilon)$, there exist $k^* \in \mathbb{N}$ and $l^* \in \mathbb{Z}$ such that $f^{k^*}(x) \in (0, \delta)$ and $g^{l^*}(J) = [s - |g^{l^*}(J)|,\ s]$ for some $s \in D_0$. Moreover,

$$|g^{l^*}(J)| > \left|J_s\left(\rho^{-1}\frac{3\varepsilon}{1-\beta-\varepsilon}D\right)\right| = G\left(\rho^{-1}\frac{3\varepsilon}{1-\beta-\varepsilon}D\right).$$

By Claim 2.12, we have

$$\frac{|g^{l^*+r}(J) \cap I_t|}{|I_t|} > \rho^{-1}\frac{3\varepsilon}{1-\beta-\varepsilon} \quad \text{for all } t \in D_0,$$

where $r \in \{-1,\ 0,\ 1\}$ depends on J and $t \in D_0$. In particular, for each $n \in \mathbb{N}$ large enough, consider the fundamental domain

$$I_{f^{k^*+n}(x)} = (g(h(f^{k^*+n}(x))),\ h(f^{k^*+n}(x))]$$

and let $\widehat{l}_n \in \mathbb{N}$ be the unique integer such that

$$g^{-\widehat{l}_n}(I_{f^{k^*+n}(x)}) = I_t \quad \text{there exists } t \in D_0.$$

Then, we have

$$\frac{|g^{l^*+r_n}(J) \cap g^{-\widehat{l}_n}(I_{f^{k^*+n}(x)})|}{|g^{-\widehat{l}_n}(I_{f^{k^*+n}(x)})|} > \rho^{-1}\frac{3\varepsilon}{1-\beta-\varepsilon},$$

where $r_n \in \{-1,\ 0,\ 1\}$ depends on x and J.

By Claim 2.11, we have

$$\frac{|g^{l^*+r_n+\widehat{l}_n}(J) \cap I_{f^{k^*+n}(x)}|}{|I_{f^{k^*+n}(x)}|} > \frac{3\varepsilon}{1-\beta-\varepsilon}.$$

By Claim 2.10, there exist $1 \le K_n \le K^*$ and $1 \le L_n \le L^*$ such that

$$g^{-L_n} \circ h \circ f^{K_n}(f^{k^*+n}(x)) \in g^{l^*+r_n+\widehat{l}_n}(J).$$

Finally, we take $k_n = n + k^* + K_n$ and $l_n = \widehat{l}_n + l^* + r_n + L_n$, and then we have

$$g^{-l_n} \circ h \circ f^{k_n}(x) \in J.$$

Since $k_n \to +\infty$ and $l_n \to +\infty$ as $n \to +\infty$, by taking a subsequence if necessary, we can require that the sequences $\{k_n\}$ and $\{l_n\}$ are positive and strictly increasing.

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It follows that for any two closed intervals $I, J \subseteq (0,1)$ with $|J| > 3L(\varepsilon)$, there exist strictly increasing subsequences $\{k_n\}$, $\{l_n\}$ of $\mathbb{N}$ such that

$$h \circ f^{k_n}(I) \cap g^{l_n}(J') \neq \varnothing,$$

where J' is the middle third of J. Then, we have the following two cases.

Case 1. $h \circ f^{k_n}(I)$ is totally contained in $g^{l_n}(J)$. Applying Claim 2.11 to f, we have

$$|h \circ f^{k_n}(I) \cap g^{l_n}(J)| = |h \circ f^{k_n}(I)| \geq c\alpha^{k_n}$$

for some constant $c > 0$.

Case 2. $h \circ f^{k_n}(I)$ is not totally contained in $g^{l_n}(J)$. Assume that $J = [a, b]$ and $J' = [a', b']$, then $h \circ f^{k_n}(I)$ contains $g^{l_n}([a, a'])$ or $g^{l_n}([b', b])$. Applying Claim 2.11, we have

$$|h \circ f^{k_n}(I) \cap g^{l_n}(J)| \geq \min\{|g^{l_n}([a, a'])|,\ |g^{l_n}([b', b])|\} \geq c\beta^{l_n}$$

for some constant $c > 0$.

Recall that by the choice of $\widehat{l_n}$, we have

$$h(f^{k^*+n}(x)) \in (g^{\widehat{l_n}+1}(h(\delta)),\ g^{\widehat{l_n}}(h(\delta))).$$

Hence, there exists a constant $C \geq 1$ such that

$$C\gamma\alpha^{k^*+n}x \geq C^{-1}\beta^{\widehat{l_n}+1}h(\delta);$$
$$C^{-1}\gamma\alpha^{k^*+n}x \leq C\beta^{\widehat{l_n}}h(\delta).$$

We conclude that

$$\sup_{n\in\mathbb{N}}\{|k_n \ln\alpha - l_n \ln\beta|\} < +\infty \quad \text{and} \quad \lim_{n\to+\infty}\frac{l_n}{k_n} = \frac{\ln\alpha}{\ln\beta}.$$

As a result, there exists a constant $c > 0$ such that

$$|h \circ f^{k_n}(I) \cap g^{l_n}(J)| \geq c \cdot \max\{e^{k_n \ln\alpha},\ e^{l_n \ln\beta}\} \quad \text{for all } n \in \mathbb{N}.$$

This completes the proof of Proposition 2.9. □

3. *Boundary interconnection implies topological transitivity*

In this section, we prove that boundary interconnection implies topological transitivity.

THEOREM 3.1. *Assume that $F \in \mathrm{PHI}^2(M)$. If F is boundary interconnected, then F is topologically transitive.*

First, we characterize the stable set of a periodic point with a uniformly contracting center.

LEMMA 3.2. *Let $p \in \mathrm{Per}(F_0)$ be a periodic point with period $\pi(p)$ and $\lambda^c(p) < 0$. Assume that $\gamma_p : [0,1] \to \mathcal{F}^c(p)$ is a C^{1+} parameterization of the central leaf of p with $\gamma_p(0) = p$. Then, there exists $0 < s_p \leq 1$ satisfying*

$$I_p^c := W^s(p) \cap \mathcal{F}^c(p) = \gamma_p([0, s_p)) \quad \textit{and} \quad W^s(p) = \bigcup_{x\in I_p^c}\mathcal{F}^s(x).$$

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In particular, define $\ell_p : [0, 1] \to [0, s_p]$ *and* $\ell_p(t) := s_p t$, *then* $\gamma_p^s := \gamma_p \circ \ell_p : [0, 1] \to \bar{I}_p^c$ *is a* C^{1+} *parameterization, and*

$$(\gamma_p^s)^{-1} \circ F^{\pi(p)} \circ \gamma_p^s \in \mathrm{Diff}_\partial^{1+}([0, 1])$$

is a diffeomorphism with exactly two fixed points 0, 1, *where* 0 *is a hyperbolic sink.*

Proof of Lemma 3.2. Notice that $f_p := (\gamma_p)^{-1} \circ F^{\pi(p)} \circ \gamma_p \in \mathrm{Diff}_\partial^{1+}([0, 1])$. Since $\lambda^c(p) < 0$, we have $0 < f_p'(0) < 1$ and, therefore, f_p has a smallest positive fixed point $0 < s_p \le 1$. It follows that $\ell_p{}^{-1} \circ f_p \circ \ell_p \in \mathrm{Diff}_\partial^{1+}([0, 1])$ is a diffeomorphism with exactly two fixed points 0, 1, where 0 is a hyperbolic sink.

By the above argument, $I_p^c := W^s(p) \cap \mathcal{F}^c(p) = \gamma_p([0, s_p))$. Therefore, we have

$$W^s(p) \supseteq \bigcup_{x \in I_p^c} \mathcal{F}^s(x).$$

Now, take $y \in W^s(p)$. Then, we have $\pi_0(y) \in \mathcal{F}^s(\pi_0(p))$. Recall that $\pi_0 : M \to M_0$ is the natural projection along central leaves. It follows that $y \in \mathcal{F}^{cs}(p)$. Assume that $y \in \mathcal{F}^s(z)$ for some $z \in \mathcal{F}^c(p)$. Then, $z \in W^s(p)$ and, hence, $z \in I_p^c$ since $W^s(p)$ is s-saturated. This completes the proof of Lemma 3.2. □

Note that similar conclusions hold for periodic points of both F_0 and F_1: The stable set (respectively unstable set) is characterized by the uniformly contracting center (respectively uniformly expanding center) when the central Lyapunov exponent is negative (respectively positive). For simplicity, we omit the statement for the other case.

Now, we prove Theorem 3.1.

Proof of Theorem 3.1. It suffices to show that for any two non-empty open subsets $U, V \subseteq M$, there exists $N \in \mathbb{N}$ such that $F^N(U) \cap V \neq \varnothing$. Equivalently, we only need to show that $U^* \cap V^* \neq \varnothing$, where

$$U^* := \bigcup_{n \in \mathbb{N}} F^n(U) \quad \text{and} \quad V^* := \bigcup_{n \in \mathbb{N}} F^{-n}(V).$$

Note that $F(U^*) \subseteq U^*$, hence, U^* is u-saturated; $F^{-1}(V^*) \subseteq V^*$; and V^* is s-saturated.

Since F is boundary interconnected, there exist two pairs of periodic points $p_i, q_i \in M_i$ such that

$$\lambda^c(p_0) < 0 < \lambda^c(p_1) \quad \text{and} \quad W^s(p_0) \pitchfork W^u(p_1) \neq \varnothing;$$
$$\lambda^c(q_1) < 0 < \lambda^c(q_0) \quad \text{and} \quad W^s(q_1) \pitchfork W^u(q_0) \neq \varnothing.$$

Without loss of generality, we assume that the periodic points p_i and q_i are fixed points. Define $\varphi : M \to \mathbb{R}$ by

$$\varphi(x) := \ln \|DF|_{E^c(x)}\|.$$

Then, φ is Hölder continuous, and there exist constants $C_\varphi \ge 1$ and $0 < \theta < 1$ such that

$$|\varphi(x) - \varphi(y)| \le C_\varphi d(x, y)^\theta \quad \text{for all } x, y \in M.$$

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Recall that we denote by

$$S_\varphi F_0(p) := \sum_{i=0}^{\pi(p)-1} \varphi(F_0^i(p)) = \pi(p)\lambda^c(p)$$

the Birkhoff sum of the function φ for $p \in \mathrm{Per}(F_0)$ with period $\pi(p)$. As a result,

$$S_\varphi F_0(p_0) = \varphi(p_0) = \lambda^c(p_0) < 0 < \lambda^c(q_0) = \varphi(q_0) = S_\varphi F_0(q_0).$$

Generate new periodic points for asymptotically rational independence. Now, compared with the results in [**Xia24**], we do not know whether $S_\varphi F_0(p_0)$ and $S_\varphi F_0(q_0)$ are rationally independent. Therefore, we need the following asymptotic version.

CLAIM 3.3. *There exists a sequence of periodic points* $\{p_{0,m}\} \subseteq \mathrm{Per}(F_0)$ *such that*

$$(S_\varphi F_0(p_{0,m}),\ S_\varphi F_0(q_0)) \to 0 \quad \textit{and} \quad \lambda^c(p_{0,m}) \to \lambda^c(p_0) \quad \textit{as } m \to +\infty.$$

Proof of Claim 3.3. If $S_\varphi F_0(p_0)$ and $S_\varphi F_0(q_0)$ are indeed rationally independent, then we just take $p_{0,m} = p_0$ for any $m \in \mathbb{N}$. Now, we assume that $S_\varphi F_0(p_0)$ and $S_\varphi F_0(q_0)$ are rationally dependent. By Lemma 2.8, the set $\{S_\varphi F_0(p) : p \in \mathrm{Per}(F_0)\}$ is dense in $\mathbb{R}$. Therefore, there exists a sequence of periodic points $\{\bar{p}_{0,m}\} \subseteq \mathrm{Per}(F_0)$ such that

$$2^{-m} < |S_\varphi F_0(\bar{p}_{0,m}) - S_\varphi F_0(p_0)| < 2^{-m+1}. \tag{3.1}$$

For fixed $m \in \mathbb{N}$, we construct $p_{0,m}$ from $\bar{p}_{0,m}$ by the shadowing property.

An Anosov diffeomorphism on a nilmanifold M_0, F_0 is topologically transitive. Hence, $\bar{p}_{0,m}$ and p_0 are homoclinically related, and there are points x_0, $y_0 \in M_0$ with

$$x_0 \in \mathcal{F}^u(p_0) \pitchfork \mathcal{F}^s(\bar{p}_{0,m}) \quad \text{and} \quad y_0 \in \mathcal{F}^u(\bar{p}_{0,m}) \pitchfork \mathcal{F}^s(p_0),$$

see Figure 3. For simplicity, we omit the picture of the other points in $\mathrm{Orb}(\bar{p}_{0,m})$.

We then consider the following pseudo-orbit segment:

$$\mathcal{P}'_m := (F_0^{-K_m}(y_0), \dots, y_0, \dots, F_0^{L_m-1}(y_0), F_0^{-L_m}(x_0), \dots, x_0, \dots, F_0^{K_m-1}(x_0)).$$

Here, K_m, $L_m \in \mathbb{N}$ are large multiples of $\pi(\bar{p}_{0,m})$ to be decided. The sum of φ over $\mathcal{P}'_m$ is

$$S_\varphi(\mathcal{P}'_m) := \sum_{x \in \mathcal{P}'_m} \varphi(x) = \sum_{i=-K_m}^{L_m-1} \varphi(F_0^i(y_0)) + \sum_{i=-L_m}^{K_m-1} \varphi(F_0^i(x_0)).$$

Since F_0 is uniformly contracting (respectively expanding) in stable (respectively unstable) manifolds and φ is Hölder continuous, we have the following convergent series:

$$\begin{aligned} P = &\sum_{i=-\infty}^{-1} [\varphi(F_0^i(y_0)) - \varphi(F_0^i(\bar{p}_{0,m}))] + \sum_{i=0}^{+\infty} [\varphi(F_0^i(y_0)) - \varphi(p_0)] \\ &+ \sum_{i=-\infty}^{-1} [\varphi(F_0^i(x_0)) - \varphi(p_0)] + \sum_{i=0}^{+\infty} [\varphi(F_0^i(x_0)) - \varphi(F_0^i(\bar{p}_{0,m}))]. \end{aligned}$$

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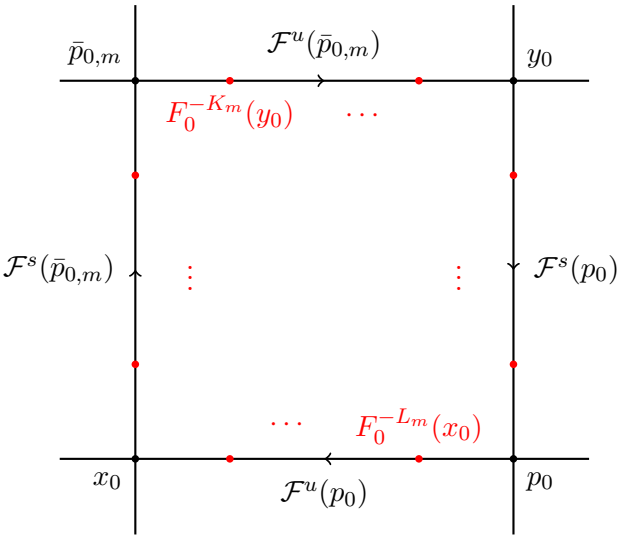


FIGURE 3. Periodic pseudo-orbit (color online).

As a result, given a sequence of positive numbers $\bar{\varepsilon}_m \to 0$, there exist $K_m^0, L_m^0 \in \mathbb{N}$ being multiples of $\pi(\bar{p}_{0,m})$ such that when $K_m \geq K_m^0$ and $L_m \geq L_m^0$, we have

$$|S_\varphi(\mathcal{P}'_m) - (P + 2K_m\lambda^c(\bar{p}_{0,m}) + 2L_m\lambda^c(p_0))| < \bar{\varepsilon}_m. \tag{3.2}$$

However, by the classical *exponential Anosov shadowing lemma*, there are constants $0 < \delta_0 < 1$, $C_0 \geq 1$, and $0 < \mu_0 < 1$ such that for every $0 < \delta < \delta_0$, every δ-pseudo-orbit $\{x_i : i_A \leq i \leq i_B\}$ $(i_A \leq 0 \leq i_B)$ is exponentially shadowed by a periodic point p in the sense that

$$d(F_0^i(p),\ x_i) < C_0\mu_0^{|i|}\delta \quad \text{for all } i_A \leq i \leq i_B.$$

As a result, there exist $K_m^1, L_m^1 \in \mathbb{N}$ such that when $K_m \geq K_m^1$ and $L_m \geq L_m^1$, $\mathcal{P}'_m$ is a $\delta_0\bar{\varepsilon}_m$-pseudo-orbit, exponentially shadowed by a periodic point $p'_{0,m} \in \mathrm{Per}(F_0)$. In fact, it suffices to take K_m^1 and L_m^1 to be multiples of $\pi(\bar{p}_{0,m})$ and so large that

$$\max\{d(F_0^{L_m^1}(y_0),\ p_0),\ d(F_0^{-L_m^1}(x_0),\ p_0)\} < \tfrac{1}{2}\delta_0\bar{\varepsilon}_m;$$
$$\max\{d(F_0^{K_m^1}(x_0),\ \bar{p}_{0,m}),\ d(F_0^{-K_m^1}(y_0),\ \bar{p}_{0,m})\} < \tfrac{1}{2}\delta_0\bar{\varepsilon}_m.$$

Now, we consider $S_\varphi F_0(p'_{0,m})$, which is the sum of φ over the orbit of $p'_{0,m}$. Since φ is Hölder continuous, the difference between $S_\varphi F_0(p'_{0,m})$ and $S_\varphi(\mathcal{P}'_m)$ is controlled by

$$|S_\varphi F_0(p'_{0,m}) - S_\varphi(\mathcal{P}'_m)| < C_\varphi C_0^\theta \frac{2\delta_0^\theta}{1-\mu_0^\theta} \cdot \bar{\varepsilon}_m^\theta. \tag{3.3}$$

Combining (3.2) and (3.3), there exists a constant $C \geq 1$ such that

$$|S_\varphi F_0(p'_{0,m}) - (P + 2K_m\lambda^c(\bar{p}_{0,m}) + 2L_m\lambda^c(p_0))| < C\bar{\varepsilon}_m^\theta. \tag{3.4}$$

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Here, we choose $K_m = \max\{K_m^0, K_m^1, m\pi(\bar{p}_{0,m})\}$ and $L_m = \max\{L_m^0, L_m^1, m(K_m + |P|)\}$ so that the central Lyapunov exponent is also controlled:

$$\begin{aligned}|\lambda^c(p'_{0,m}) - \lambda^c(p_0)| &= \left|\frac{S_\varphi F_0(p'_{0,m})}{\pi(p'_{0,m})} - \lambda^c(p_0)\right| \\ &\le \frac{|P| + 2K_m|\lambda^c(\bar{p}_{0,m}) - \lambda^c(p_0)| + C\bar{\varepsilon}_m^\theta}{2K_m + 2L_m} \\ &\to 0 \quad (\text{as } m \to +\infty).\end{aligned}$$

In the same manner as above, we can make a slight modification on the choice of the pseudo-orbit segment. Precisely, we add the periodic orbit of $\bar{p}_{0,m}$ and consider the following pseudo-orbit segment:

$$\mathcal{P}''_m := (\bar{p}_{0,m},\ F_0(\bar{p}_{0,m}),\ \ldots,\ F_0^{\pi(\bar{p}_{0,m})-1}(\bar{p}_{0,m})) \cup \mathcal{P}'_m.$$

Then, we have a periodic point $p''_{0,m} \in \mathrm{Per}(F_0)$ shadowing $\mathcal{P}''_m$ and satisfying a similar estimate as (3.4):

$$|S_\varphi F_0(p''_{0,m}) - (P + 2K_m\lambda^c(\bar{p}_{0,m}) + 2L_m\lambda^c(p_0) + S_\varphi F_0(\bar{p}_{0,m}))| < C\bar{\varepsilon}_m^\theta. \tag{3.5}$$

Moreover, we also have that

$$|\lambda^c(p''_{0,m}) - \lambda^c(p_0)| \to 0 \quad \text{as } m \to +\infty.$$

Finally, we choose $p_{0,m} \in \{p'_{0,m},\ p''_{0,m}\}$ so that $(S_\varphi F_0(p_{0,m}),\ S_\varphi F_0(p_0)) \to 0$ as $m \to +\infty$. In fact, combining (3.4) and (3.5), we have

$$|S_\varphi F_0(p''_{0,m}) - S_\varphi F_0(p'_{0,m}) - S_\varphi F_0(\bar{p}_{0,m})| < 2C\bar{\varepsilon}_m^\theta. \tag{3.6}$$

Now, taking $\bar{\varepsilon}_m$ so small that $2C\bar{\varepsilon}_m^\theta < 2^{-m}$, then we have the following estimate by combining (3.1) and (3.6):

$$0 < |S_\varphi F_0(p''_{0,m}) - S_\varphi F_0(p'_{0,m}) - S_\varphi F_0(p_0)| < 3 \cdot 2^{-m}. \tag{3.7}$$

Taking further $3 \cdot 2^{-m} < |S_\varphi F_0(p_0)|$, then by Lemma 2.5, there exists $p_{0,m} \in \{p'_{0,m}, p''_{0,m}\}$ such that

$$(S_\varphi F_0(p_{0,m}),\ S_\varphi F_0(p_0)) < \sqrt{3|S_\varphi F_0(p_0)|} \cdot 2^{-m/2}.$$

Since $S_\varphi F_0(p_0)$ and $S_\varphi F_0(q_0)$ are rationally dependent, there exists a sequence of positive numbers $\varepsilon_m \to 0^+$ such that $(S_\varphi F_0(p_{0,m}),\ S_\varphi F_0(q_0)) < \varepsilon_m$.

This completes the proof of Claim 3.3. □

In fact, we can modify the choice of $\{p_{0,m}\}$ to obtain more information about the sequence.

CLAIM 3.4. *By replacing the periodic point $p_{0,m}$ by some points in* $\mathrm{Orb}(p_{0,m})$ *and taking a subsequence, we may assume further that the sequence $\{p_{0,m}\}$ constructed in Claim 3.3 satisfies the following properties.*

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- *Better Birkhoff average:*

$$\frac{1}{k}\sum_{i=0}^{k-1}\ln\|DF|_{E^c(F_0^i(p_{0,m}))}\| \le \frac{\lambda^c(p_0)}{3} \quad \text{for all } k\in\mathbb{N},\ \text{for all } m\in\mathbb{N}.$$

- *Uniform size of contracting center: there exists $\eta_0 > 0$ such that*

$$|I^c_{p_{0,m}}| \ge \eta_0 \quad \text{for all } m\in\mathbb{N}.$$

- *Convergence: there exists $r_0 \in M_0$ such that $p_{0,m} \to r_0$ as $m \to +\infty$.*

Proof of Claim 3.4. Since $\lambda^c(p_{0,m}) \to \lambda^c(p_0) < 0$, by taking a subsequence, we assume that

$$\lambda^c(p_{0,m}) < \frac{\lambda^c(p_0)}{2} \quad \text{for all } m\in\mathbb{N}.$$

By the classical *Pliss lemma*, for each $m \in \mathbb{N}$, there exists a hyperbolic time n_m such that

$$\frac{1}{k}\sum_{i=0}^{k-1}\ln\|DF|_{E^c(F_0^{n_m+i}(p_{0,m}))}\| \le \frac{\lambda^c(p_0)}{3} \quad \text{for all } k\in\mathbb{N}.$$

Hence, we can replace $p_{0,m}$ by $F_0^{n_m}(p_{0,m}) \in \mathrm{Orb}(p_{0,m})$. Since $p_{0,m}$ is a periodic point, this replacement does not influence the properties that we need in Claim 3.3.

Now, we show that the length of the contracting center $I^c_{p_{0,m}}$ has a positive lower bound with respect to $m \in \mathbb{N}$. Since $p_{0,m}$ now admits a better Birkhoff average, there exists a maximal closed interval I_m containing $p_{0,m}$ as an endpoint, lying in $I^c_{p_{0,m}}$, satisfying

$$\bar\varphi_k(x) \le \frac{\lambda^c(p_0)}{6} \quad \text{for all } 1\le k\le \pi(p_{0,m}),\ \text{for all } x\in I_m,$$

where

$$\bar\varphi_k(x) := \frac{1}{k}\ln\|DF^k|_{E^c(x)}\| = \frac{1}{k}\sum_{i=0}^{k-1}\varphi(F^i(x)).$$

Then, for $x, y \in I_m$, we have

$$\begin{aligned}
|\bar\varphi_k(x)-\bar\varphi_k(y)| &\le \frac{1}{k}\sum_{i=0}^{k-1}|\varphi(F^i(x))-\varphi(F^i(y))| \le \frac{1}{k}\sum_{i=0}^{k-1} C_\varphi d(F^i(x),F^i(y))^\theta\\
&\le \frac{1}{k}\sum_{i=0}^{k-1} C_\varphi d_c(F^i(x),F^i(y))^\theta \le \frac{1}{k}\sum_{i=0}^{k-1} C_\varphi \max_{z\in I_m}\|DF^i|_{E^c(z)}\|^\theta d_c(x,y)^\theta\\
&\le \frac{1}{k}\sum_{i=0}^{k-1} C_\varphi e^{(1/6)\lambda^c(p_0)i\theta} d_c(x,y)^\theta \le \frac{1}{k}\cdot\frac{C_\varphi}{1-e^{(1/6)\lambda^c(p_0)\theta}} d_c(x,y)^\theta.
\end{aligned}$$

Let $\widetilde{p}_{0,m}$ be the other endpoint of I_m. Since I_m is maximal, there exists $1 \le k_m \le \pi(p_{0,m})$ such that

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$$\begin{aligned}\frac{\lambda^c(p_0)}{6} = \bar{\varphi}_{k_m}(\widetilde{p}_{0,m}) &\le \bar{\varphi}_{k_m}(p_{0,m}) + |\bar{\varphi}_{k_m}(\widetilde{p}_{0,m}) - \bar{\varphi}_{k_m}(p_{0,m})| \\ &\le \frac{\lambda^c(p_0)}{3} + \frac{1}{k_m}\frac{C_\varphi}{1 - e^{(1/6)\lambda^c(p_0)\theta}}|I_m|^\theta \\ &\le \frac{\lambda^c(p_0)}{3} + \frac{C_\varphi}{1 - e^{(1/6)\lambda^c(p_0)\theta}}|I_m|^\theta.\end{aligned}$$

As a result,

$$|I_m| \ge \left[-\frac{1}{6}\lambda^c(p_0) \cdot \frac{1}{C_\varphi}(1 - e^{(1/6)\lambda^c(p_0)\theta}) \right]^{1/\theta} > 0.$$

It follows that $p_{0,m}$ has a uniform size of contracting center.

Finally, since M_0 is compact, we can take a subsequence to ensure that $p_{0,m} \to r_0$ for some $r_0 \in M_0$. □

Show the density of the stable and unstable sets. Next, we show that the intersection of the stable and unstable sets implies their density.

CLAIM 3.5. *The stable and unstable sets $W^s(p_0)$, $W^u(q_0)$, $W^u(p_1)$, $W^s(q_1)$ are dense in M.*

Proof of Claim 3.5. For convenience, we define $\widehat{p}_0 := \pi_1(p_0)$ and $\widehat{p}_1 := \pi_0(p_1)$. Recall that $W^s(p_0) \pitchfork W^u(p_1)$ contains a central interval. Assume that the interval lies in a central leaf with endpoints $z_0 \in M_0$ and $z_1 \in M_1$, then there exists an open central interval $J \subseteq \mathcal{F}^c(z_0)$ with an endpoint z_0, lying in $W^s(p_0)$ and containing an interval in $W^u(p_1)$, see Figure 4.

Now, consider $F^{-N}(J)$. One of its endpoints is $F_0^{-N}(z_0)$, which tends to $\widehat{p}_1$; the other endpoint tends to p_1, since it lies in $W^u(p_1)$. Moreover, $F^{-N}(J)$ is a central interval, and hence, $F^{-N}(J)$ tends to $\mathcal{F}^c(p_1)$, in the sense that

$$\begin{aligned}&\text{for all } x \in \mathcal{F}^c(p_1), \text{ for all } \varepsilon > 0, \text{ there exists } N \in \mathbb{N}, \text{ there exists } x' \in F^{-N}(J), \\ &\quad \text{such that } d(x', x) < \varepsilon.\end{aligned}$$

Since F_0 is an Anosov diffeomorphism on a nilmanifold, the stable foliation restricted to M_0 is minimal, in the sense that each stable leaf is dense in M_0. It follows that $\mathcal{F}^{cs}$ is minimal, because $\mathcal{F}^{cs}(\cdot) = \pi_0^{-1}(\mathcal{F}^s(\pi_0(\cdot)))$.

Now, $\mathcal{F}^{cs}(p_1)$ is dense in M, for any non-empty open subset $U \subseteq M$, there exists $x \in \mathcal{F}^c(p_1)$ such that $\mathcal{F}^s(x) \cap U \neq \varnothing$. Note that

$$\mathcal{F}^s(x) = \bigcup_{k \in \mathbb{N}} F^{-k}(\mathcal{F}^s_\varepsilon(F^k(x))) \quad \text{for all } \varepsilon > 0.$$

Therefore, there exists $K = K(\varepsilon) \in \mathbb{N}$ such that $\mathcal{F}^s_\varepsilon(F^K(x)) \cap F^K(U) \neq \varnothing$. Recall that $F^{-N}(J)$ tends to $\mathcal{F}^c(p_1)$, hence, there exist $N \in \mathbb{N}$ and $x' \in F^{-N}(J)$ such that $\mathcal{F}^s_\varepsilon(F^K(x')) \cap F^K(U) \neq \varnothing$. Now, we have that $\mathcal{F}^s(F^N(F^K(x'))) \cap F^{K+N}(U) \neq \varnothing$. Since $F^N(F^K(x')) \in F^K(J) \subseteq W^s(p_0)$, we conclude that

$$W^s(p_0) \cap U = F^{-(K+N)}(W^s(p_0)) \cap U \neq \varnothing.$$

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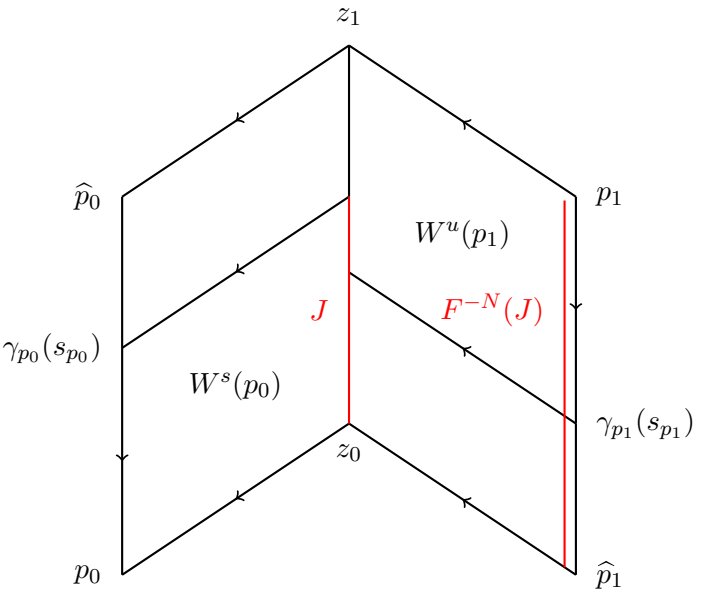


FIGURE 4. Density of $W^s(p_0)$ (color online).

This shows the density of $W^s(p_0)$. By the same argument, $W^u(q_0)$, $W^u(p_1)$, and $W^s(q_1)$ are also dense in M. □

Move from p_0 to $p_{0,m}$. We first show that the points in U^* appear near the contracting central interval $I^c_{p_{0,m}}$.

Since $\lambda^c(p_0) < 0$, we have

$$W^s(p_0) = \bigcup_{k\in\mathbb{N}} F^{-k}(\mathcal{F}^{cs}_{\mathrm{loc}}(p_0)).$$

Therefore, by the density of $W^s(p_0)$, we have

$$U^* \cap \mathcal{F}^{cs}_{\mathrm{loc}}(p_0) \neq \varnothing,$$

and hence, we can take

$$\widetilde{p}_0 \in \pi_0(U^*) \cap \mathcal{F}^{s}_{\mathrm{loc}}(p_0) \subseteq M_0.$$

Moreover, $U^* \cap \mathcal{F}^c(\widetilde{p}_0)$ contains a closed central interval $J^c_{\widetilde{p}_0}$, satisfying $\mathcal{P} := \mathcal{F}^u_{\mathrm{loc}}(J^c_{\widetilde{p}_0}) \subseteq U^*$, see Figure 5.

Similarly, since $\lambda^c(q_0) > 0$ and $W^u(q_0)$ is dense in M, we have

$$V^* \cap \mathcal{F}^{cu}_{\mathrm{loc}}(q_0) \neq \varnothing,$$

and, hence, we can take

$$\widetilde{q}_0 \in \pi_0(V^*) \cap \mathcal{F}^{u}_{\mathrm{loc}}(q_0) \subseteq M_0.$$

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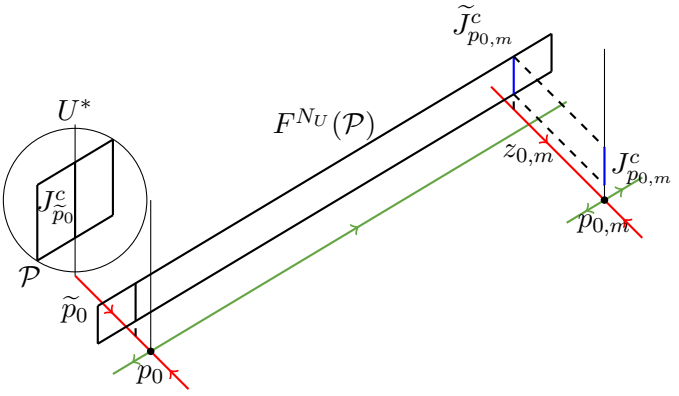


FIGURE 5. Holonomy map from p_0 to $p_{0,m}$ (color online).

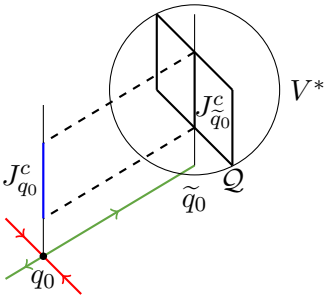


FIGURE 6. Holonomy map near q_0 (color online).

We can further require that $V^* \cap \mathcal{F}^c(\widetilde{q}_0)$ contains a closed central interval $J^c_{\widetilde{q}_0}$ satisfying $\mathcal{Q} := \mathcal{F}^s_{\mathrm{loc}}(J^c_{\widetilde{q}_0}) \subseteq V^*$, see Figure 6.

Note that $\mathcal{F}^u(p_0)$ is dense in M_0. Hence, we can choose $z_{0,m} \in \mathcal{F}^u(p_0) \pitchfork \mathcal{F}^s_{\mathrm{loc}}(p_{0,m})$. Since $p_{0,m}$ converges, we may further assume that $z_{0,m}$ also converges.

In fact, by taking a subsequence, we assume that $p_{0,m}$ lies in the local su-box of r_0, then it suffices to ensure that $\mathcal{F}^u(p_0) \pitchfork \mathcal{F}^s_{\mathrm{loc}}(r_0) \neq \varnothing$.

It follows that there exists $N_U \in \mathbb{N}$ independent of $m \in \mathbb{N}$ such that

$$F^{N_U}(\mathcal{P}) \pitchfork \mathcal{F}^c(\mathcal{F}^s_{\mathrm{loc}}(p_{0,m})) \neq \varnothing.$$

In particular, $F^{N_U}(\mathcal{P}) \pitchfork \mathcal{F}^c(\mathcal{F}^s_{\mathrm{loc}}(p_{0,m}))$ contains a central interval $\widetilde{J}^c_{p_{0,m}}$, whose length has a positive lower bound with respect to $m \in \mathbb{N}$.

Let $J^c_{p_{0,m}} \subseteq \mathcal{F}^c(p_{0,m})$ be the image of $\widetilde{J}^c_{p_{0,m}}$ under the holonomy map along the stable leaves, restricted to $\mathcal{F}^c(\mathcal{F}^s_{\mathrm{loc}}(p_{0,m}))$. By Lemma 2.1, the length of $J^c_{p_{0,m}}$ also has a positive lower bound with respect to $m \in \mathbb{N}$. Moreover, since $p_{0,m}$ has a uniform size of contracting center and $J^c_{\widetilde{p}_0}$ lies in $W^s(p_0)$, by taking a larger N_U, we can assume further that $J^c_{p_{0,m}} \subseteq I^c_{p_{0,m}}$.

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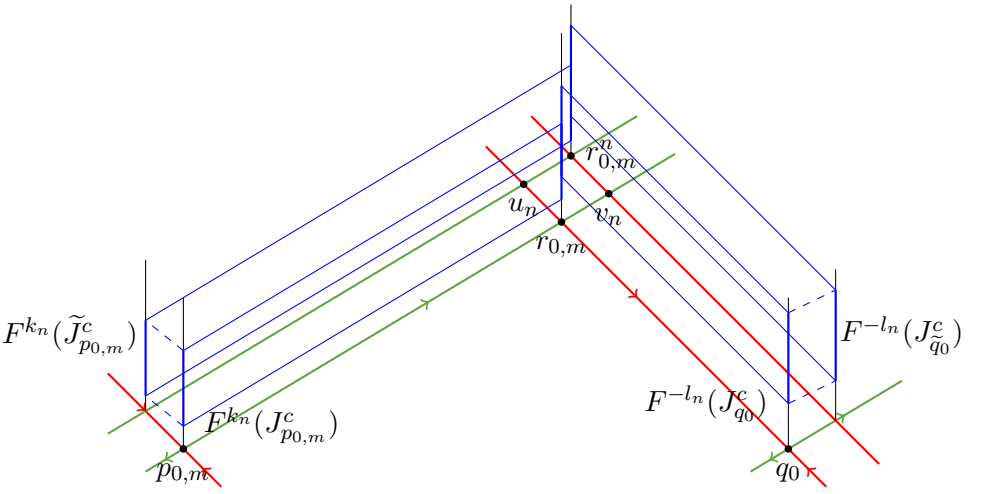


FIGURE 7. Central intersections (color online).

Recall that in Lemma 3.2, we have a C^{1+} parameterization $\gamma^s_{p_{0,m}} : [0,1] \to \bar{I}^c_{p_{0,m}}$ for each $m \in \mathbb{N}$. Since $I^c_{p_{0,m}}$ has a uniform size, we can further require that there exists $0 < \sigma_0 < 1$ such that

$$\sigma_0 < |(\gamma^s_{p_{0,m}})'(t)| < \sigma_0^{-1} \quad \text{for all } t \in [0,1], \text{ for all } m \in \mathbb{N}.$$

However, let $J^c_{q_0} \subseteq \mathcal{F}^c_{\mathrm{loc}}(q_0)$ be the image of $J^c_{\widetilde{q}_0}$ under the holonomy map along the unstable leaves, restricted to $\mathcal{F}^{cu}_{\mathrm{loc}}(q_0)$. Since $\lambda^c(q_0) > 0$, we may assume further that $J^c_{q_0} \subseteq I^c_{q_0}$. Similarly, we have a C^{1+} parameterization $\gamma^s_{q_0} : [0,1] \to \bar{I}^c_{q_0}$ satisfying

$$\sigma_0 < |(\gamma^s_{q_0})'(t)| < \sigma_0^{-1} \quad \text{for all } t \in [0,1].$$

Exhibit central intersection. From now on, we fix $m \in \mathbb{N}$ so large that

$$L(\varepsilon_m) < \min\{\inf_{m\in\mathbb{N}} |(\gamma^s_{p_{0,m}})^{-1}(J^c_{p_{0,m}})|,\ |(\gamma^s_{q_0})^{-1}(J^c_{q_0})|\}.$$

Here, $L(\cdot)$ is given by Proposition 2.9 and ε_m is defined in Claim 3.3 satisfying

$$(S_\varphi F_0(p_{0,m}),\ S_\varphi F_0(q_0)) < \varepsilon_m.$$

Without loss of generality, we assume that $p_{0,m}$ is a fixed point. Fix $r_{0,m} \in \mathcal{F}^u(p_{0,m}) \pitchfork \mathcal{F}^s(q_0)$. For simplicity, we denote by Hol^y_x the holonomy map from x to y along stable or unstable manifolds defined on cu-leaves or cs-leaves, see Figure 7.

CLAIM 3.6. *There exist two strictly increasing subsequences $\{k_n\}$, $\{l_n\}$ of $\mathbb{N}$ such that*

$$H \circ F^{k_n}(J^c_{p_{0,m}}) \cap F^{-l_n}(J^c_{q_0}) \neq \varnothing,$$

where

$$H := \mathrm{Hol}^{q_0}_{r_{0,m}} \circ \mathrm{Hol}^{r_{0,m}}_{p_{0,m}}.$$

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Moreover, there exists a constant $c > 0$ such that

$$|H \circ F^{k_n}(J^c_{p_{0,m}}) \cap F^{-l_n}(J^c_{q_0})| \geq c \cdot \max\{e^{k_n \lambda^c(p_{0,m})},\ e^{-l_n \lambda^c(q_0)}\} \quad \textit{for all } n \in \mathbb{N}.$$

Proof of Claim 3.6. Note that

$$(S_\varphi F_0(p_{0,m}),\ S_\varphi F_0(q_0)) < \varepsilon_m,$$

which means that

$$(\ln \|DF|_{E^c(p_{0,m})}\|,\ \ln \|DF|_{E^c(q_0)}\|) < \varepsilon_m,$$

since $p_{0,m}$ and q_0 are assumed to be fixed points. Equivalently, we have

$$(\ln \|DF|_{E^c(p_{0,m})}\|,\ \ln \|DF^{-1}|_{E^c(q_0)}\|) < \varepsilon_m.$$

The conclusion then follows from Proposition 2.9, by taking

$$\begin{aligned} f &= (\gamma^s_{p_{0,m}})^{-1} \circ F \circ \gamma^s_{p_{0,m}}, \\ g &= (\gamma^s_{q_0})^{-1} \circ F^{-1} \circ \gamma^s_{q_0}, \\ h &= (\gamma^s_{q_0})^{-1} \circ H \circ \gamma^s_{p_{0,m}}. \end{aligned}$$

The initial intervals are $(\gamma^s_{p_{0,m}})^{-1}(J^c_{p_{0,m}})$ and $(\gamma^s_{q_0})^{-1}(J^c_{q_0})$. □

As a result, there exists a constant $c > 0$ such that

$$|J^n_{p_{0,m}} \cap J^n_{q_0}| \geq c \cdot \max\{e^{k_n \lambda^c(p_{0,m})},\ e^{-l_n \lambda^c(q_0)}\} \quad \text{for all } \ n \in \mathbb{N},$$

where

$$\begin{aligned} J^n_{p_{0,m}} &:= \mathrm{Hol}^{r_{0,m}}_{p_{0,m}} \circ F^{k_n}(J^c_{p_{0,m}}); \\ J^n_{q_0} &:= (\mathrm{Hol}^{q_0}_{r_{0,m}})^{-1} \circ F^{-l_n}(J^c_{q_0}). \end{aligned}$$

Now, consider the intersection between $F^{k_n}(\mathcal{F}^u_{\mathrm{loc}}(\widetilde{J}^c_{p_{0,m}}))$ and $F^{-l_n}(\mathcal{F}^s_{\mathrm{loc}}(J^c_{q_0}))$. Since F_0 is an Anosov diffeomorphism on a nilmanifold, for n sufficiently large, their projections on M_0, $\pi_0(F^{k_n}(\mathcal{F}^u_{\mathrm{loc}}(\widetilde{J}^c_{p_{0,m}})))$, and $\pi_0(F^{-l_n}(\mathcal{F}^s_{\mathrm{loc}}(J^c_{q_0})))$, intersect at $r^n_{0,m} \in M_0$, which converges to $r_{0,m}$ as $n \to +\infty$. Moreover, we have

$$\begin{aligned} u_n &\in \mathcal{F}^u_{\mathrm{loc}}(r^n_{0,m}) \pitchfork \mathcal{F}^s_{\mathrm{loc}}(r_{0,m}), \\ v_n &\in \mathcal{F}^s_{\mathrm{loc}}(r^n_{0,m}) \pitchfork \mathcal{F}^u_{\mathrm{loc}}(r_{0,m}). \end{aligned}$$

For simplicity, we denote the intersections by

$$\begin{aligned} \widehat{J}^n_{p_{0,m}} &:= F^{k_n}(\mathcal{F}^u_{\mathrm{loc}}(\widetilde{J}^c_{p_{0,m}})) \cap \mathcal{F}^c(r^n_{0,m}); \\ \widehat{J}^n_{q_0} &:= F^{-l_n}(\mathcal{F}^s_{\mathrm{loc}}(J^c_{q_0})) \cap \mathcal{F}^c(r^n_{0,m}). \end{aligned}$$

Our aim is to show that $\widehat{J}^n_{p_{0,m}} \cap \widehat{J}^n_{q_0} \neq \varnothing$, see Figure 8.

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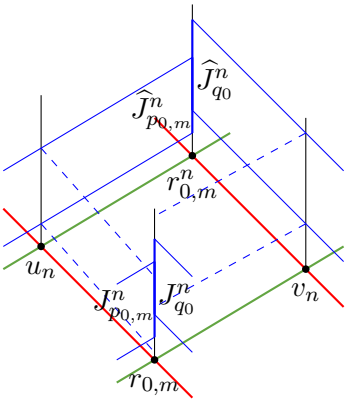


FIGURE 8. Dynamics near $r_{0,m}$ (color online).

By Lemma 2.1 (the C^1 property of holonomy maps) and Lemma 2.2 (the estimate of central twists), there exists a constant $C \geq 1$ such that

$$\begin{aligned}
&\mathrm{Hol}^{r_{0,m}}_{u_n} \circ \mathrm{Hol}^{u_n}_{r^n_{0,m}}(\widehat{J}^n_{p_{0,m}}) \text{ is a } Ce^{k_n\lambda_{p,m}}\text{-perturbation of } J^n_{p_{0,m}};\\
&\mathrm{Hol}^{r_{0,m}}_{v_n} \circ \mathrm{Hol}^{v_n}_{r^n_{0,m}}(\widehat{J}^n_{q_0}) \text{ is a } Ce^{-l_n\lambda_q}\text{-perturbation of } J^n_{q_0},
\end{aligned}$$

as intervals in $\mathcal{F}^c(r_{0,m})$, where

$$\begin{aligned}
\lambda_{p,m} &:= \frac{\lambda^u(p_{0,m}) - \lambda^c(p_{0,m})}{\lambda^u(p_{0,m}) - \lambda^s(p_{0,m})}\lambda^s(p_{0,m}) < \lambda^c(p_{0,m}) < 0,\\
\lambda_q &:= \frac{-\lambda^s(q_0) + \lambda^c(q_0)}{-\lambda^s(q_0) + \lambda^u(q_0)}\lambda^u(q_0) > \lambda^c(q_0) > 0.
\end{aligned}$$

Again, by the argument of central twist in Lemma 2.2, for n sufficiently large and $x \in \widehat{J}^n_q$, we have

$$\begin{aligned}
&d_c(\mathrm{Hol}^{r_{0,m}}_{v_n} \circ \mathrm{Hol}^{v_n}_{r^n_{0,m}}(x),\ \mathrm{Hol}^{r_{0,m}}_{u_n} \circ \mathrm{Hol}^{u_n}_{r^n_{0,m}}(x))\\
&\quad\leq C[d_u(r^n_{0,m},\ u_n) + d_s(u_n,\ r_{0,m}) + d_s(r^n_{0,m},\ v_n) + d_u(v_n,\ r_{0,m})]\\
&\quad\leq C^2[e^{k_n\lambda^s(p_{0,m})} + e^{-l_n\lambda^u(q_0)}].
\end{aligned}$$

Compared with the length of the intersection which is controlled by

$$|J^n_{p_{0,m}} \cap J^n_{q_0}| \geq c \cdot \max\{e^{k_n\lambda^c(p_{0,m})},\ e^{-l_n\lambda^c(q_0)}\} \triangleq \rho_n,$$

and the preceding perturbation errors, which are $o(\rho_n)$, implied by the strict inequalities among the corresponding Lyapunov exponents, we conclude together with the injectivity of the maps $\mathrm{Hol}^{r_{0,m}}_{u_n} \circ \mathrm{Hol}^{u_n}_{r^n_{0,m}}$ and $\mathrm{Hol}^{r_{0,m}}_{v_n} \circ \mathrm{Hol}^{v_n}_{r^n_{0,m}}$ that there exists $n_0 \in \mathbb{N}$ such that $\widehat{J}^n_{p_{0,m}} \cap \widehat{J}^n_{q_0} \neq \varnothing$ for any $n \geq n_0$. Since $\widehat{J}^n_{p_{0,m}} \subseteq U^*$ and $\widehat{J}^n_{q_0} \subseteq V^*$, we have $U^* \cap V^* \neq \varnothing$.

This ends the proof of Theorem 3.1. □

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COROLLARY 3.7. *If $F \in \mathrm{PHI}^1(M)$ is boundary interconnected, then there is an open neighborhood $\mathcal{U} \subseteq \mathrm{PHI}^1(M)$ of F such that every $G \in \mathcal{U} \cap \mathrm{PHI}^2(M)$ is topologically transitive.*

Proof. Notice that the boundary interconnection property is C^1-open in $\mathrm{PHI}^1(M)$. The conclusion then follows directly from Theorem 3.1. □

4. *Robust transitivity implies boundary interconnection*

In this section, we complete the proof of Theorem A by showing that robust transitivity implies boundary interconnection.

THEOREM 4.1. *If $F \in \mathrm{PHI}^k(M)$ $(k \geq 2)$ is C^k-robustly transitive, then F is boundary interconnected.*

As a preparation of the proof, we need the following method to construct an attractor by perturbation.

LEMMA 4.2. *Let $F \in \mathrm{PHI}^k(M)$ $(k \geq 2)$. If $\lambda^c(p) \leq 0$ for every $p \in \mathrm{Per}(F_0)$, then for any neighborhood $\mathcal{U} \subseteq \mathrm{PHI}^k(M)$ of F, there exists $G \in \mathcal{U}$ such that M_0 is an attractor of G.*

Proof. By Corollary 2.7, we have $\lambda^c(x) \leq 0$ whenever $\lambda^c(x)$ exists. To show that M_0 is an attractor of G, it suffices (see [**Cao03**, Lemma 2]) to check that there exists a constant $\varepsilon > 0$ such that

$$\{x \in M_0 : \lambda^c_G(x) \leq -\varepsilon\}$$

has full measure with respect to any ergodic measure of G_0.

Let $X(s) := s(s-1)$ be a C^∞ vector field on $[0, 1]$ and $\Phi_t(s)$ be the C^∞ flow generated by X. Then, by abuse of notation, we define

$$\Phi_t(z, s) := (z, \Phi_t(s)) : N \times [0, 1] \to N \times [0, 1].$$

It follows that Φ_t converges to identity as $t \to 0^+$, in any C^k topology.

Now, consider $G := \Phi_t \circ F$. Since G acts in the same way as F on M_0 and M_1, we have $G \in \mathrm{Diff}^k_\partial(M)$ and, hence, when $t > 0$ is sufficiently small, $G \in \mathrm{PHI}^k(M)$.

Note that the partially hyperbolic splitting of G restricted to M_0 has the same stable and unstable subbundles as F. We denote it by

$$T_x M = E^s(x) \oplus E^c_G(x) \oplus E^u(x) \quad \text{for all } x \in M_0.$$

Define $\alpha_t := \Phi_t'(0) = \partial_s \Phi_t(0) \in (0, 1)$ and $\beta(z) := f_z'(0) = \partial_s f_z(0) > 0$, where we denote $F(z, s) = (F_s(z), f_z(s))$. Here, $\alpha_t = e^{-t}$, which can be solved by $(d/dt)\alpha_t = -\alpha_t$ and $\alpha_0 = 1$, and hence, $\alpha_t \in (0, 1)$ when $t > 0$. Then, we have

$$D\Phi_t(z, 0) = \begin{pmatrix} I & 0 \\ 0 & \alpha_t \end{pmatrix};$$

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$$DF(z,0)=\begin{pmatrix} DF_0(z) & * \\ 0 & \beta(z)\end{pmatrix};$$
$$DG(z,0)=\begin{pmatrix} DF_0(z) & * \\ 0 & \alpha_t\beta(z)\end{pmatrix}.$$

Let $v^c=(*,1)^T$ and $v^c_G=(*,1)^T$ be vectors of $E^c(z,0)$ and $E^c_G(z,0)$, respectively. Then,

$$DF(z,0)(v^c)=(*,\ \beta(z))^T \quad \text{and} \quad DG(z,0)(v^c_G)=(*,\ \alpha_t\beta(z))^T.$$

Let $\|\cdot\|$ be the original induced norm in E^c and E^c_G, and define

$$\|(Z,S)^T\|_F := |S| \quad \text{for all } (Z,S)^T\in E^c;$$
$$\|(Z,S)^T\|_G := |S| \quad \text{for all } (Z,S)^T\in E^c_G.$$

Since E^c and E^c_G are both transverse to M_0, which is compact, the norms $\|\cdot\|_F$ and $\|\cdot\|_G$ are both equivalent with $\|\cdot\|$. That is, there exists a constant $C\geq 1$ such that

$$C^{-1}\|\cdot\|\leq\|\cdot\|_F\leq C\|\cdot\|;$$
$$C^{-1}\|\cdot\|\leq\|\cdot\|_G\leq C\|\cdot\|.$$

Moreover, by the above discussion, we have

$$\|DF(z,0)(v^c)\|_F=\beta(z)\|v^c\|_F \quad \text{and} \quad \|DG(z,0)(v^c_G)\|_G=\alpha_t\beta(z)\|v^c_G\|_G.$$

As a result, for any $x\in M_0$,

$$\|DG^n|_{E^c_G(x)}\|_G=\alpha_t^n\|DF^n|_{E^c(x)}\|_F$$

and, hence,

$$C^{-4}\alpha_t^n\|DF^n|_{E^c(x)}\|\leq\|DG^n|_{E^c_G(x)}\|\leq C^4\alpha_t^n\|DF^n|_{E^c(x)}\|.$$

By the classical *Oseledets theorem*, for each ergodic measure μ of $F_0=G_0: M_0\to M_0$, there exists a μ-full measure subset $B_\mu\subseteq M_0$ such that for any $x\in B_\mu$, the Lyapunov exponents of F and G at x both exist. Consider

$$B_0 := \bigcup_{\mu \text{ ergodic}} B_\mu,$$

then $\mu(B_0)=1$ for any ergodic measure μ.

Fix $x\in B_0$. We have

$$\begin{aligned}&\frac{1}{n}\ln\|DG^n|_{E^c_G(x)}\|\\ &\quad=\frac{1}{n}(\ln\|DG^n|_{E^c_G(x)}\|-\ln\|DF^n|_{E^c(x)}\|)+\frac{1}{n}\ln\|DF^n|_{E^c(x)}\|\\ &\quad\in\left[\frac{1}{n}\ln\|DF^n|_{E^c(x)}\|+\ln\alpha_t-\frac{4}{n}\ln C,\ \frac{1}{n}\ln\|DF^n|_{E^c(x)}\|+\ln\alpha_t+\frac{4}{n}\ln C\right].\end{aligned}$$

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Let $n \to +\infty$ and we have

$$\lambda^c_G(x) = \lambda^c(x) + \ln \alpha_t \le \ln \alpha_t < 0 \quad \text{for all } x \in B_0.$$

This completes the proof of Lemma 4.2. □

Proof of Theorem 4.1. Since F is C^k-robustly transitive, there exists an open neighborhood $\mathcal{U} \subseteq \mathrm{PHI}^k(M)$ of F such that every $G \in \mathcal{U}$ is topologically transitive.

We first prove that there exist $p_0,\ q_0 \in \mathrm{Per}(F_0)$ such that $\lambda^c(p_0) < 0 < \lambda^c(q_0)$. A similar conclusion holds for F_1. For contradiction, without loss of generality, we assume that

$$\lambda^c(p) \le 0 \quad \text{for all } p \in \mathrm{Per}(F_0).$$

Then, by Lemma 4.2, there exists a C^k-perturbation $G \in \mathcal{U}$ such that M_0 is an attractor of G, which contradicts the topological transitivity of G.

Now, take $p_0, q_0 \in M_0$ and $p_1, q_1 \in M_1$ such that

$$\lambda^c(p_0) < 0 < \lambda^c(q_0) \quad \text{and} \quad \lambda^c(q_1) < 0 < \lambda^c(p_1).$$

It remains to show that $W^s(p_0) \cap W^u(p_1) \neq \varnothing$ and, similarly, $W^u(q_0) \cap W^s(q_1) \neq \varnothing$.

First, notice that

$$W^s(p_0) = \bigcup_{n\in\mathbb{N}} F^{-n}(\mathcal{F}^{cs}_{\mathrm{loc}}(p_0)) \quad \text{and} \quad W^u(p_1) = \bigcup_{n\in\mathbb{N}} F^{n}(\mathcal{F}^{cu}_{\mathrm{loc}}(p_1)).$$

Since F is topologically transitive, there exists $x \in M$ such that

$$\overline{\{F^i(x) : i \ge i_0\}} = M \quad \text{for all } i_0 \in \mathbb{Z}.$$

In particular, there exists $i_0 \in \mathbb{Z}$ such that $F^{i_0}(x) \in \mathcal{F}^s_{\varepsilon_0}(\mathcal{F}^{cu}_{\mathrm{loc}}(p_1))$ for some $\varepsilon_0 > 0$. Assume that $F^{i_0}(x) \in \mathcal{F}^s_{\varepsilon_0}(z_1)$, where $z_1 \in \mathcal{F}^{cu}_{\mathrm{loc}}(p_1)$.

By the density of $\{F^i(x) : i \ge i_0\}$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $F^{i_0+N}(x) \in \mathcal{F}^u_\varepsilon(\mathcal{F}^{cs}_\varepsilon(p_0))$. Moreover, for fixed $\varepsilon > 0$, N can be arbitrarily large. By taking $\varepsilon > 0$ small, we may assume that

$$\mathcal{F}^s_\varepsilon(F^{i_0+N}(x)) \subseteq \mathcal{F}^u_{\varepsilon_0}(\mathcal{F}^{cs}_{\mathrm{loc}}(p_0)).$$

Now, take N large so that $F^N(z_1) \in \mathcal{F}^s_\varepsilon(F^{i_0+N}(x))$. It follows that $F^N(z_1) \in \mathcal{F}^u_{\varepsilon_0}(z_0)$ for some $z_0 \in \mathcal{F}^{cs}_{\mathrm{loc}}(p_0)$. Since $W^u(p_1)$ is F-invariant and u-saturated, we have $z_0 \in W^s(p_0) \cap W^u(p_1)$.

This completes the proof of Theorem 4.1. □

5. *Intermingled basins*

In this section, we prove Theorem B. Recall that for $F \in \mathrm{PHI}^k_-(M)$ $(k \ge 2)$, there are boundary SRB measures μ_0 on M_0 and μ_1 on M_1. For $i = 0, 1$, since μ_i is the unique SRB measure of $F_i : M_i \to M_i$, we have the following results:

- $\mathrm{supp}(\mu_i) = M_i$;
- $B(\mu_i) \cap M_i$ has full Lebesgue measure in M_i;
- for any u-segment $\gamma^u \subseteq M_i$, $\gamma^u \cap B(\mu_i)$ has full Lebesgue measure in γ^u;

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- there exists $L_i \subseteq M_i$ with positive Lebesgue measure such that $\lambda^c(x) < 0$ for all $x \in L_i$, and consequently, $|W^s(x) \cap \mathcal{F}^c(x)| > 0$ for all $x \in L_i$;
- there exists $K_i \subseteq L_i$ with positive Lebesgue measure such that

$$\inf_{x \in K_i} |W^s(x) \cap \mathcal{F}^c(x)| > 0.$$

THEOREM 5.1. *If $F \in \mathrm{PHI}^2_-(M)$ is topologically transitive, then F admits intermingled basins.*

Proof of Theorem 5.1. We first prove the following claim.

CLAIM 5.2. *For any cu-box $\mathcal{B}$, $\mathcal{B} \cap B(\mu_0)$ has positive Lebesgue measure in $\mathcal{B}$, and further, for any cu-box $\mathcal{B}'$ that is C^1-close to $\mathcal{B}$,*

$$\mathrm{Leb}(\mathcal{B}' \cap B(\mu_0)) \geq \tfrac{1}{2}\mathrm{Leb}(\mathcal{B} \cap B(\mu_0)).$$

Proof of Claim 5.2. Consider the open set $U = \mathcal{F}^s_\varepsilon(\mathcal{B})$. Since F is topologically transitive, given any non-empty open set V, $F^N(U) \cap V \neq \varnothing$ for infinitely many $N \in \mathbb{N}$. It follows that $\bigcup_{n\in\mathbb{N}} F^n(\mathcal{B})$ is dense in M. In particular, there exists $N \in \mathbb{N}$ and a cu-box $\mathcal{P} \subseteq F^N(\mathcal{B})$ sufficiently close to M_0 such that $\pi_0(\mathcal{P}) \cap K_0$ has positive Lebesgue measure in $\pi_0(\mathcal{P})$ and $x \in W^s(\pi_0(x))$ for any $x \in \mathcal{P}$ with $\pi_0(x) \in K_0$. However, $\pi_0(\mathcal{P}) \cap B(\mu_0)$ has full Lebesgue measure in $\pi_0(\mathcal{P})$. It follows that $\mathcal{P} \cap B(\mu_0)$ has positive Lebesgue measure in $\mathcal{P}$ and, hence, $\mathcal{B} \cap B(\mu_0)$ also has positive Lebesgue measure in $\mathcal{B}$. Moreover, $B(\mu_0)$ is s-saturated, hence, for any cu-box $\mathcal{B}'$ that is C^1 close to $\mathcal{B}$, $\mathrm{Leb}(\mathcal{B}' \cap B(\mu_0))$ has a uniform lower bound away from zero, given as above. □

A similar result holds for $B(\mu_1)$. Therefore, we conclude that for any non-empty open subset $U \subseteq M$, $U \cap B(\mu_0)$ and $U \cap B(\mu_1)$ both have positive Lebesgue measure.

To show that $B(\mu_0) \cup B(\mu_1)$ has full Lebesgue measure, assume for contradiction that the complement $\mathcal{C}$ has positive Lebesgue measure. Take a density point $x^* \in M$ of $\mathcal{C}$. Since the basins are s-saturated, we have

$$\frac{\mathrm{Leb}(\mathcal{C} \cap \mathcal{F}^{cu}_\varepsilon(x^*))}{\mathrm{Leb}(\mathcal{F}^{cu}_\varepsilon(x^*))} \to 1 \quad \text{as } \varepsilon \to 0.$$

By the above argument, there exists $N_k \to +\infty$ and a cu-box $\mathcal{P}_k \subseteq F^{N_k}(\mathcal{F}^{cu}_{\varepsilon_0}(x^*))$ near M_0, with the same size, such that $\mathcal{P}_k \cap B(\mu_0)$ consists of central leaves. Moreover, we have that $F^{-N_k}(\mathcal{P}_k) \subseteq \mathcal{F}^c_{\varepsilon_0}(\mathcal{F}^u_{\varepsilon_k}(x^*))$, where $\varepsilon_k \to 0^+$. Define

$$\mathcal{P}^*_k = F^{N_k}(F^{-N_k}(\mathcal{P}_k) \cap \mathcal{F}^{cu}_{\varepsilon_k}(x^*)).$$

Since for each $k \in \mathbb{N}$, $\det(DF^{N_k}|_{E^{cu}})$ is nearly constant on $\mathcal{F}^{cu}_{\varepsilon_0}(x^*)$ when ε_0 is small, we conclude that

$$\frac{\mathrm{Leb}(\mathcal{C} \cap \mathcal{P}^*_k)}{\mathrm{Leb}(\mathcal{P}^*_k)} \to 1 \quad \text{as } k \to +\infty,$$

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and consequently,

$$\frac{\mathrm{Leb}(B(\mu_0)\cap \mathcal{P}_k^*)}{\mathrm{Leb}(\mathcal{P}_k^*)}\to 0 \quad \text{as } k\to+\infty.$$

Noting that $\mathcal{P}_k\cap B(\mu_0)$ consists of central leaves, we also have

$$\frac{\mathrm{Leb}(B(\mu_0)\cap \mathcal{P}_k)}{\mathrm{Leb}(\mathcal{P}_k)}\to 0 \quad \text{as } k\to+\infty.$$

By taking a subsequence, we may assume that $\mathcal{P}_k$ converges to a cu-box $\mathcal{P}$. This contradicts the fact that a cu-box near $\mathcal{P}$ intersects $B(\mu_0)$ on a subset whose Lebesgue measure has a lower bound away from zero, see Claim 5.2. □

THEOREM 5.3. *If $F\in \mathrm{PHI}_-^k(M)$ $(k\geq 2)$ admits C^k-robustly intermingled basins, then F is boundary interconnected.*

Proof of Theorem 5.3. Since F admits C^k-robustly intermingled basins, there exists an open neighborhood $\mathcal{U}\subseteq \mathrm{PHI}_-^k(M)$ of F such that every $G\in\mathcal{U}$ admits intermingled basins.

We first show that there exist $q_0\in \mathrm{Per}(F_0)$ and $p_1\in \mathrm{Per}(F_1)$ admitting positive central Lyapunov exponents.

Assume for contradiction that $\lambda^c(p)\leq 0$ for all $p\in \mathrm{Per}(F_0)$. By Lemma 4.2, there exists $G\in\mathcal{U}$ such that M_0 is an attractor of G. Moreover, G acts in the same way as F on M_0 and M_1, hence, the boundary SRB measures of G are also μ_0 and μ_1. In this case, $B_G(\mu_1)$ cannot be dense near M_0, which contradicts the fact that G admits intermingled basins.

Now, we have $q_0\in \mathrm{Per}(F_0)$ and $p_1\in \mathrm{Per}(F_1)$ with positive central Lyapunov exponents. Fix $p_0\in \mathrm{Per}(F_0)$ and $q_1\in \mathrm{Per}(F_1)$ with negative central Lyapunov exponents. Since F admits intermingled basins, $B(\mu_0)\cap \mathcal{F}^{cu}_{\mathrm{loc}}(p_1)$ has positive Lebesgue measure in $\mathcal{F}^{cu}_{\mathrm{loc}}(p_1)$. By forward iteration, we know that there exists a cu-box $\mathcal{P}\subseteq W^u(p_1)$ near p_0, since we have $\mathrm{supp}(\mu_0)=M_0$. Therefore, $\mathcal{P}\pitchfork \mathcal{F}^{cs}_{\mathrm{loc}}(p_0)\neq\varnothing$, and hence, $W^u(p_1)\pitchfork W^s(p_0)\neq\varnothing$. A similar argument shows that $W^s(q_1)\pitchfork W^u(q_0)\neq\varnothing$. Hence, F is boundary interconnected. □

Proof of Theorem B. Theorem 5.1 shows that (2) $\Longrightarrow$ (1); Theorem 5.3 shows that (1) $\Longrightarrow$ (3). Finally, Corollary 3.7 shows that (3) $\Longrightarrow$ (2). □

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