SOME QUESTIONS ON ENTANGLED LINEAR ORDERS

RAPHAËL CARROY, MAXWELL LEVINE, AND LORENZO NOTARO

Abstract. Entangled linear orders were introduced by Abraham and Shelah in [2]. Todorčević [29] showed that these linear orders exist under CH. We prove the following results:

(1) If CH holds, then, for every $n > 1$, there is an n-entangled linear order which is not $(n+1)$-entangled.
(2) If CH holds, then there are two homeomorphic sets of reals $A, B \subseteq \mathbb{R}$ such that A is entangled but B is not 2-entangled.
(3) If $\mathbb{R} \subseteq \mathsf{L}$, then there is an entangled $\mathbf{\Pi}^1_1$ set of reals.
(4) If $\diamondsuit$ holds, then there is a 2-entangled non-separable linear order.

§1. Introduction. Cantor proved that all countable dense unbounded linear orders are isomorphic. The countability hypothesis in Cantor's result cannot be avoided in general. However, in the early 1970s, Baumgartner proved that consistently all $\aleph_1$-dense separable linear orders are isomorphic [4]. This statement is known as *Baumgartner's axiom*. In their work [2], Abraham and Shelah introduced entangled linear orders as combinatorial objects witnessing the failure of Baumgartner's axiom.

An uncountable linear order L is said to be *n-entangled*, for some positive integer $n > 0$, if for every uncountable collection F of increasing, pairwise disjoint n-tuples of elements of L and for every $t \in {}^n2$, there are $x, y \in F$ such that, for every $i < n$, $x_i < y_i$ if and only if $t_i = 0$. A linear order is called *entangled* if it is n-entangled for every $n > 0$ (see Definition 2.1 for a more general definition).

Entangled linear orders satisfy a strong form of rigidity[1]: they have no nontrivial self-embeddings[2] (a linear order with this property was first constructed in [10]); actually, they do not even have nontrivial monotone self-maps; furthermore, every two disjoint uncountable subsets of an entangled linear order have different order types (a linear order with this property was first constructed, under CH, in [11]).

Todorčević [29] showed that these linear orders provide a powerful method to construct counterexamples relative to various problems, such as the productivity of chain conditions and the square bracket partition relation. He also observed that every 2-entangled linear order is ccc and that every 3-entangled linear order is separable (see Section 2). In particular, every entangled linear order is isomorphic

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[1] Indeed, entangled linear orders are called *hyper-rigid* in [5].

[2] A linear order with this property is sometimes called *exact* in the literature (see, e.g., [14, 15]).

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to a suborder of the real line $\mathbb{R}$. Therefore, when studying entangled linear orders, it is not restrictive to only consider sets of reals that are entangled with respect to their natural linear ordering. For recent applications of entangled linear orders, see, e.g., [3, 7, 17, 24].

The existence of an entangled linear order is independent of ZFC: on the one hand, these linear orders exist under CH (actually, under $\mathrm{cf}(2^{\aleph_0}) = \omega_1$) [5, 29]; on the other hand, they do not exist under Baumgartner's axiom [2].

In this article, we address the following natural questions:

(1) Does an n-entangled linear order need to be $(n+1)$-entangled?
(2) Is entangledness a topological property? That is to say, if we have two homeomorphic sets of reals $A, B \subseteq \mathbb{R}$ with A entangled, does it follow that B is also entangled?
(3) How definable can an entangled set of reals be?
(4) Under which conditions is there a 2-entangled linear order which is not separable?

Our main results are as follows. The first two are instances of more general statements proved later, formulated with reference to cardinal characteristics of the continuum (see Theorems 3.1 and 4.1).

THEOREM A. *Assume* CH. *Then, for every* $n > 1$, *there exists an* n*-entangled set of reals which is not* $(n+1)$*-entangled.*

THEOREM B. *Assume* CH. *Then, there are two homeomorphic sets of reals* $A, B \subseteq \mathbb{R}$ *such that* A *is entangled but* B *is not* 2*-entangled.*

Recall that Π^0_1 sets are complements of semirecursive sets, also called effectively open sets. Projections of Π^0_1 sets are denoted Σ^1_1, and their complements are the Π^1_1 sets. As we argue in Section 5, every analytic set of reals is not entangled. In this sense, our next result is optimal.

THEOREM C. *Suppose that* $\mathbb{R} \subseteq \mathsf{L}$. *Then there exists a* Π^1_1 *set of reals which is entangled.*

Since, as we already noted, every 2-entangled linear order is ccc, a 2-entangled non-separable linear order is, in particular, a Suslin line. In particular, we conclude that CH—from which the existence of an entangled linear order follows—does not suffice to prove the existence of a 2-entangled non-separable linear order. Indeed, it is well-known that Suslin's Hypothesis is consistent with CH [9]. However, we prove that the existence of a 2-entangled non-separable linear order follows from $\diamondsuit$. Note that the existence of such a linear order is already known to be consistent by a forcing argument due to Krueger [21].

THEOREM D. *Suppose that* $\diamondsuit$ *holds. Then, there exists a* 2*-entangled non-separable linear order.*

The underlying general technique used to prove all four results is the diagonalization method. More precisely, the proofs of the first two theorems use a diagonalization argument *à la* Sierpiński (e.g., [27, 28]). The proof of our third result uses a diagonalization argument similar to the construction (originally due to Gödel [19, p. 170]) of a Π^1_1 set without the perfect set property. Finally, the

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diagonalization construction of our fourth result is similar to the construction due to Jensen of a Suslin line under $\diamondsuit$ [8, 22].

In Section 2 we provide a more general definition of entangled linear order and we give a brief account of what is known on this topic. Sections 3–6 are then devoted to the proofs of Theorems A–D. We provide some open questions along the way.

Notation. For most of our article the notation is standard (see, e.g., [18, 20]). Given two tuples s and t, we denote the concatenation of s with t by $s^\frown t$. In Section 5, following set-theoretic practice, we refer to elements of the Baire space ${}^\omega\omega$ as "reals," and we employ some notation from [19].

§2. Preliminaries. There exists a more general notion of entangled linear order than the one we gave in the introduction, one that extends to cardinalities larger than the continuum. This more general definition (Definition 2.1) will be useful in our discussion.

We say that two tuples x, y are *disjoint* if they have no element in common—that is, if $\mathrm{ran}(x) \cap \mathrm{ran}(y) = \emptyset$.

Given a linear order $(L, \leq)$ and two n-tuples e and e' of elements of L, we let $t(e, e') \in {}^n2$ be the binary n-tuple defined as follows: for every $i < n$, the i-th element of $t(e, e')$ is 1 if and only if $e'_i < e_i$.

For each $t \in {}^{<\omega}2$, we denote by $\neg t$ the tuple $\langle 1 - t_i \mid i < \mathrm{length}(t)\rangle$. Clearly, if the n-tuples e and e' are disjoint, then $t(e, e') = \neg t(e', e)$. We say that e and e' *realize* s if either $t(e, e') = s$ or $t(e, e') = \neg s$.

Given a set S of mutually disjoint n-tuples of elements of some linear order, and given a binary n-tuple t, we say that S *avoids* t if no couple of distinct elements of S realizes t. For each $n > 0$, we call the n-tuple $\langle 0\rangle^n = \langle 0, 0, \dots, 0\rangle \in {}^n2$ *the increasing type*.

We point out a trivial yet useful observation. The usual order $\leq$ on real numbers is closed but not open as a subset of $\mathbb{R}^2$ and it becomes clopen[3] on $\mathbb{R}^2$ minus the diagonal. Therefore, for every $t \in {}^n2$, the following (symmetric) binary relation R_t on $\mathbb{R}^{2n}$ is open: $e\ R_t\ e'$ if e and e' are disjoint and realize t.

Definition 2.1 [2, 29]. Given a linear order $(L, \leq)$, an infinite cardinal κ, and a positive integer $n > 0$, we say that L is (κ, n)-*entangled* if $|L| \geq \kappa$ and for every collection F of size κ of 1–1 mutually disjoint n-tuples of L and for every binary n-tuple t, there are two distinct elements of F that realize t. A linear order which is (κ, n)-entangled for every $n > 0$ is said to be κ-*entangled*. If κ is not mentioned, it is assumed that $\kappa = \aleph_1$.

We now state some known theorems regarding entangled linear orders. The first one is about some topological consequences of entangledness. In particular, it implies that every 3-entangled linear order is isomorphic to a set of reals. This theorem is stated without proof in [29]. For a proof, we refer the reader to [21].

Theorem 2.2 [29]. *Every* 2*-entangled linear order is ccc, and every* 3*-entangled linear order is separable.*

Corollary 2.3. *Every* 3*-entangled linear order is isomorphic to a suborder of* $\mathbb{R}$.

[3]That is, both open and closed.

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PROOF. Every 2-entangled linear order has countably many gaps[4]: otherwise, letting $\{(a_\xi, b_\xi) \mid \xi < \omega_1\}$ be an uncountable family of gaps, the map $a_\xi \mapsto b_\xi$ would be uncountable, 1–1 and increasing, against 2-entangledness of the underlying linear order.

Therefore, it follows from the classical Dedekind completion argument that every 3-entangled linear order, being separable and having countably many gaps, is isomorphic to a suborder of $\mathbb{R}$. ⊣

The next theorem is an example of the numerous applications of entangled linear orders as studied by Shelah, Todorčević, and others (see, e.g., [5, 6, 25, 26, 29]).

THEOREM 2.4 [29]. *If there is an entangled linear order, then there are two ccc partial orders whose product is not ccc.*

The next results are mainly concerned with the existence or nonexistence of entangled linear orders:

- There exists a $\mathrm{cf}(2^{\aleph_0})$-entangled linear order [5, 29].
- Adding a single Cohen real or random real adds an entangled linear order [30, p. 55].
- It follows from the Open Graph Axiom that there are no 2-entangled sets of reals [30, Proposition 8.4].
- It follows from $\mathsf{MA}(\aleph_1)$ that there are no entangled linear orders [2].
- It is consistent, relative to ZF, that $\mathsf{MA}(\aleph_1)$ holds and that, for every $n > 0$, there is an n-entangled set of reals [1].

Next, we note that it directly follows from the Erdős–Dushnik–Miller theorem [11] (see also [18, Theorem 9.7] for a proof) that the definition of (κ, n)-entangled linear order is equivalent to the seemingly stronger version in which we do not only require the existence of just two distinct elements of F to realize the given type t, but we ask for the existence of an *infinite* subset of F whose elements, mutually, realize t.

THEOREM 2.5 (Erdős–Dushnik–Miller). *Any infinite graph $G = (V, E)$ not containing an independent set of size $|V|$ contains an infinite complete subgraph.*

PROPOSITION 2.6. *Let L be a (κ, n)-entangled linear order. Then, for every collection F of size κ of 1–1 mutually disjoint n-tuples of L and for every binary n-tuple t, there exists an infinite $H \subseteq F$ such that for all distinct $e, e' \in H$, e and e' realize t.*

PROOF. Let us fix a (κ, n)-entangled linear order L, a collection F of size κ of 1–1 mutually disjoint n-tuples of L, and a binary n-tuple t.

Consider the binary relation R_t on F defined by: $e \; R_t \; e'$ if e and e' realize t. By entangledness, the graph (F, R_t) does not contain an independent set of size $|F| = \kappa$. Therefore, by the Erdős–Dushnik–Miller Theorem, the graph (F, R_t) contains an infinite complete subgraph. In other words, there exists an infinite $H \subseteq F$ such that for all distinct $e, e' \in H$, e and e' realize t. ⊣

[4]Given a linear order L, a *gap* of L is a pair $(a, b) \in L$ with $a < b$ such that there is no $c \in L$ with $a < c < b$.

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Finally, let us provide a proof of the following proposition from [29, Section 1] that provides a criterion to show that a set $E \subseteq \mathbb{R}$ is $(2^{\aleph_0}, n)$-entangled.

PROPOSITION 2.7. *Let $E \subseteq \mathbb{R}$ be a set of cardinality $2^{\aleph_0}$. Assume that there exists an enumeration $\langle r_\alpha \mid \alpha < 2^{\aleph_0}\rangle$ of E such that, for every continuous function f from a G_δ subset of $\mathbb{R}^{n-1}$ to $\mathbb{R}$, there exists an $\alpha < 2^{\aleph_0}$ with*

$$f\big[\{r_\gamma \mid \gamma < \beta\}^{n-1}\big] \cap E \subseteq \{r_\gamma \mid \gamma < \beta\}, \tag{1}$$

for all $\beta \geq \alpha$. Then E is $(2^{\aleph_0}, n)$-entangled.

LEMMA 2.8. *Let E be a $(2^{\aleph_0}, n)$-entangled set of reals, for some $n > 0$, and let $g : F \to E$ be a function with F being a subset of E^n of cardinality $2^{\aleph_0}$ whose elements are 1–1 and pairwise disjoint. If g has $2^{\aleph_0}$-many discontinuity points, then any $t \in {}^{n+1}2$ is realized by two distinct $x, y \in \operatorname{graph}(g)$.*

PROOF. Let us assume without loss of generality that $E \subseteq [0,1]$, and let $G \subseteq [0,1]^{n+1}$ be the closure of the graph of the map g.

It follows from the compactness of the unit interval that for every $u \in F$, u is a continuity point of g if and only if $|G_u| = 1$ or, equivalently, $G_u = \{g(u)\}$, where $G_u := \{y \in [0,1] \mid u^\frown y \in G\}$. We conclude from our hypothesis that there are $2^{\aleph_0}$-many $u \in F$ with $|G_u| > 1$. Therefore, there must be a rational $q \in \mathbb{Q}$ and a subset $D \subseteq F$ of cardinality $2^{\aleph_0}$ such that, for every $u \in D$, there are $a, b \in G_u$ with $a < q < b$.

Fix some $t \in {}^{n+1}2$ and assume without loss of generality that $t(n) = 0$. By entangledness, there are $u, v \in D$ such that $t(u,v) = t \restriction n$. Then, by definition of D, we can pick $x, y \in F$ sufficiently close to u and v, respectively, such that $g(x) < q < g(y)$ and $t(x,y) = t(u,v) = t \restriction n$. Therefore, $x^\frown g(x)$ and $y^\frown g(y)$ realize t. ⊣

PROOF OF PROPOSITION 2.7. We prove the proposition by induction on $n > 0$. The base case $n = 1$ follows from the fact that E has cardinality $2^{\aleph_0}$. Suppose that $n > 1$ and fix an enumeration $\langle r_\alpha \mid \alpha < 2^{\aleph_0}\rangle$ of E that satisfies the hypothesis of the proposition. By the induction hypothesis, E is $(2^{\aleph_0}, n-1)$-entangled.

Fix a collection F of $2^{\aleph_0}$-many 1–1 pairwise disjoint n-tuples of elements of E. By passing to a subset of F, we can assume without loss of generality that all the tuples of F are increasing with respect to the fixed enumeration of E.

Suppose, towards a contradiction, that F, seen as a function from a subset of E^{n-1} to E, has less than $2^{\aleph_0}$-many discontinuity points. Then, passing to a subset if necessary, we can suppose that F is continuous. By Kuratowski's extension theorem [20, Theorem 3.8], there exists a continuous map $\tilde{f}$ from a G_δ subset of $\mathbb{R}^{n-1}$ to $\mathbb{R}$ that extends F. Since F has cardinality $2^{\aleph_0}$, and its elements are increasing with respect to the fixed enumeration of E, we conclude that for cofinally many $\beta < 2^{\aleph_0}$, the set $\tilde{f}[\{r_\gamma \mid \gamma < \beta\}^{n-1}] \cap E$ is not included in $\{r_\gamma \mid \gamma < \beta\}$, against (1). Hence the contradiction.

Therefore, F has $2^{\aleph_0}$-discontinuity points. But then, by the $(2^{\aleph_0}, n-1)$-entangledness of E and Lemma 2.8, we conclude that every $t \in {}^n2$ is realized by a pair of elements of F. Thus, E is $(2^{\aleph_0}, n)$-entangled. ⊣

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§3. Separating *n*-entangledness. Theorem A is a direct consequence of the following theorem. Recall that $\mathrm{cov}(\mathcal{B})$ is the least cardinal κ such that a perfect Polish space (it does not matter which) can be expressed as the union of κ-many meager sets.

THEOREM 3.1. *Assume* $\mathrm{cov}(\mathcal{B}) = 2^{\aleph_0}$. *Then, for every* $n > 1$, *there exists a* $(2^{\aleph_0}, n)$*-entangled set of reals which is not* $(2^{\aleph_0}, n+1)$*-entangled.*

LEMMA 3.2. *For every* $n > 1$, *there exists a countable set* $D \subseteq \mathbb{R}^{n+1}$ *of* 1–1 *mutually disjoint* $(n+1)$*-tuples of reals such that*:

(1) *D avoids the increasing type.*
(2) *For every nonempty open* $U \subseteq \mathbb{R}^n$, *for every* $i < n$, *there are two disjoint* $e, e' \in D$ *such that*:
  (2a) $t(e, e') = \langle 0\rangle^i{}^\frown\langle 1\rangle^\frown\langle 0\rangle^{n-i}$.
  (2b) $e \restriction n$ *and* $e' \restriction n$ *belong to* U.
  (2c) *There are two disjoint* $w, w' \in U$ *that realize the increasing type and such that* $w_j = e_j$ *and* $w'_j = e'_j$ *for all* $j < n$ *with* $j \neq i$.

PROOF. We define by induction a sequence $\langle (r^0_k, r^1_k) \mid k \in \omega\rangle$ of couples of $(n+1)$-tuples of reals such that the set $D = \{r^i_k \mid k \in \omega \text{ and } i \in 2\}$ satisfies the desired properties.

Fix an enumeration $\langle U_k \mid k \in \omega\rangle$ of a countable basis of $\mathbb{R}^n$. Fix also a surjective map $k \mapsto \langle (k)_0, (k)_1\rangle$ from ω to $\omega \times n$.

Now we deal with the inductive construction of the sequence. Suppose we have defined (r^0_m, r^1_m) for every $m < k$, towards defining (r^0_k, r^1_k), with the induction hypothesis being that $D_k := \{r^i_m \mid m < k \text{ and } i \in 2\}$ is a set of 1–1 mutually disjoint $(n+1)$-tuples that avoids the increasing type.

For each n-tuple s of reals, let

$$u_k(s) := \min\{x \in \mathbb{R} \mid \exists w \in D_k\ (t(s, w \restriction n) = \langle 1\rangle^n \text{ and } w_n = x)\}$$
$$l_k(s) := \max\{x \in \mathbb{R} \mid \exists w \in D_k\ (t(s, w \restriction n) = \langle 0\rangle^n \text{ and } w_n = x)\},$$

where we use the convention that $\min \emptyset = +\infty$ and $\max \emptyset = -\infty$.

CLAIM 3.2.1. $l_k(s) < u_k(s)$ *for every* s.

PROOF. If there is no $w \in D_k$ such that $t(s, w \restriction n) = \langle 1\rangle^n$ the claim trivially follows, and similarly if there is no $w' \in D_k$ such that $t(s, w' \restriction n) = \langle 0\rangle^n$. So we can assume that such w and w' exist and fix them. Clearly, $w \neq w'$ and $t(w \restriction n, w' \restriction n) = \langle 0\rangle^n$. By the induction hypothesis, w and w' do not realize the increasing type. Thus, we must have $w'_n < w_n$. It directly follows from the definition of u and l that $l_k(s) < u_k(s)$. ⊣

Since D_k is finite, we can pick two 1–1 disjoint tuples $p, q \in U_{(k)_0}$ such that:

(a) $(\mathrm{ran}(p) \cup \mathrm{ran}(q)) \cap \bigcup_{w \in D_k} \mathrm{ran}(w) = \emptyset$.
(b) $t(p, q) = \langle 0\rangle^{(k)_1}{}^\frown\langle 1\rangle^\frown\langle 0\rangle^{n-1-(k)_1}$.
(c) $l_k(p) = l_k(q)$ and $u_k(p) = u_k(q)$.
(d) Letting $\tilde{p}, \tilde{q} \in \mathbb{R}^n$ be such that $\tilde{p}_j = p_j$ and $\tilde{q}_j = q_j$ for all $j \neq (k)_1$ and $\tilde{p}_{(k)_1} = q_{(k)_1}$ and $\tilde{q}_{(k)_1} = p_{(k)_1}$, we have $\tilde{p}, \tilde{q} \in U_{(k)_0}$.

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Indeed, by the finitude of D_k, we can always find a nonempty open set $V \subseteq U_{(k)_0}$ such that $\mathrm{ran}(s) \cap \bigcup_{w \in D_k} \mathrm{ran}(w) = \emptyset$ for all $s \in V$; furthermore, to satisfy (c) and (d) we just need to choose, in $U_{(k)_0}$, p and q so that they are close enough (e.g., with respect to the Euclidean metric on $\mathbb{R}^n$); thus, any two 1–1, disjoint, close enough p and q realizing $\langle 0\rangle^{(k)_1} {}^\frown \langle 1\rangle {}^\frown \langle 0\rangle^{n-1-(k)_1}$ taken inside V will satisfy (a)–(d).

Again by the finitude of D_k, we can pick $x, y \in (l_k(p), u_k(p))$ with $x < y$ such that $x, y \notin \bigcup_{w \in D_k} \mathrm{ran}(w) \cup \mathrm{ran}(p) \cup \mathrm{ran}(q)$. Set $r_k^0 = p {}^\frown \langle x\rangle$ and $r_k^1 = q {}^\frown \langle y\rangle$. Clearly, r_k^0 and r_k^1 realize $\langle 0\rangle^{(k)_1} {}^\frown \langle 1\rangle {}^\frown \langle 0\rangle^{n-(k)_1}$.

It follows from the choices of p, q, x, y that $D_{k+1} = D_k \cup \{r_k^0, r_k^1\}$ is still a set of 1–1 mutually disjoint $(n+1)$-tuples that avoids the increasing type, and (d) guarantees that $\tilde{p}$ and $\tilde{q}$ are in $U_{(k)_0}$ and realize the increasing type, thus ensuring $(2c)$. This finishes the inductive construction. It is straightforward to check that the set D as defined satisfies the desired properties.

Proof of Theorem 3.1. Fix a set D of $(n+1)$-tuples that satisfies the statement of Lemma 3.2. Fix also an enumeration $\langle f_\alpha \mid \alpha < 2^{\aleph_0}\rangle$ of the continuous functions from G_δ subsets of $\mathbb{R}^{n-1}$ to $\mathbb{R}$. We can assume that $f_0(s) = s_0$ for every $s \in \mathbb{R}^{n-1}$.

We define inductively a sequence $\langle r_\alpha \mid \alpha < 2^{\aleph_0}\rangle$ of 1–1 mutually disjoint $(n+1)$-tuples of reals that avoids the increasing type, but such that the set $E := \bigcup_{\alpha < 2^{\aleph_0}} \mathrm{ran}(r_\alpha)$ is $(2^{\aleph_0}, n)$-entangled. In particular, E is a $(2^{\aleph_0}, n)$-entangled set of reals which is not $(2^{\aleph_0}, n+1)$-entangled.

Suppose we have defined r_β for every $\beta < \alpha$ towards defining r_α, with the induction hypothesis being that the set

$$P_\alpha := D \cup \{r_\beta \mid \beta < \alpha\}$$

is a set of 1–1 mutually disjoint $(n+1)$-tuples that avoids the increasing type (for the base case $\alpha = 0$, this is guaranteed by Lemma 3.2). Let

$$Q_\alpha := \bigcup_{w \in P_\alpha} \mathrm{ran}(w),$$

and

$$A_\alpha := \big\{ s \in \mathbb{R}^n \mid \exists i < n \; \exists \beta \leq \alpha \; \exists u \in (Q_\alpha \cup \mathrm{ran}(s \restriction i))^{n-1} \text{ such that } s_i = f_\beta(u) \big\}.$$

Furthermore, for each $s \in \mathbb{R}^n$ let

$$u_\alpha(s) := \inf\{x \in \mathbb{R} \mid \exists w \in P_\alpha \; (t(s, w \restriction n) = \langle 1\rangle^n \text{ and } w_n = x)\},$$
$$l_\alpha(s) := \sup\{x \in \mathbb{R} \mid \exists w \in P_\alpha \; (t(s, w \restriction n) = \langle 0\rangle^n \text{ and } w_n = x)\}.$$

By an argument analogous to the one used in Claim 3.2.1, we can show that $l_\alpha(s) \leq u_\alpha(s)$ for every $s \in \mathbb{R}^n$. Next, we prove the following two claims, the second of which is key.

Claim 3.2.2. *A_α is the union of $<2^{\aleph_0}$-many nowhere dense sets.*

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PROOF. Consider the following map:

$$g : \mathbb{R}^n \times (Q_\alpha \cup n)^{n-1} \longrightarrow \mathbb{R}^{n-1}$$
$$(s, c) \longmapsto w, \text{ where } w_i := \begin{cases} c_i & \text{if } c_i \in Q_\alpha, \\ s_j & \text{if } c_i = j < n. \end{cases}$$

Clearly, for any fixed $c \in (Q_\alpha \cup n)^{n-1}$, the map $g(\cdot, c) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is continuous. It suffices to show that given any fixed $i < n, \beta < \alpha$, and $c \in (Q_\alpha \cup i)^{n-1}$, the set of all those $s \in \mathbb{R}^n$ such that $s_i = f_\beta \circ g(s, c)$ is nowhere dense. But this is a direct consequence of the continuity of f_β and of $g(\cdot, c)$. ⊣

CLAIM 3.2.3. *There exists an $s \in \mathbb{R}^n \setminus A_\alpha$ and an $x \in \mathbb{R}$ such that*

(1) $l_\alpha(s) \leq x \leq u_\alpha(s)$.
(2) *For all $u \in (Q_\alpha \cup \mathrm{ran}(s))^{n-1}$, for all $\beta \leq \alpha$, $f_\beta(u) \neq x$.*

PROOF. If there were an $s \in \mathbb{R}^n \setminus A_\alpha$ such that $l_\alpha(s) < u_\alpha(s)$ we would be done. Indeed, since there are only $<2^{\aleph_0}$-many possible $u \in (Q_\alpha \cup \mathrm{ran}(s))^{n-1}$ and $\beta \leq \alpha$, there would certainly be an $x \in (l_\alpha(s), u_\alpha(s))$ satisfying (2). Thus, we need to deal with the case in which $l_\alpha(s) = u_\alpha(s)$ for every $s \in \mathbb{R}^n \setminus A_\alpha$.

Towards a contradiction, suppose that for every $s \in \mathbb{R}^n \setminus A_\alpha$ there exists a $u \in (Q_\alpha \cup \mathrm{ran}(s))^{n-1}$ and a $\beta \leq \alpha$ such that $l_\alpha(s) = u_\alpha(s) = f_\beta(u)$. Consider the map g defined in Claim 3.2.2. Then, another way of stating our (absurd) assumption is that for every $s \in \mathbb{R}^n \setminus A_\alpha$ there exists a $c_s \in (Q_\alpha \cup n)^{n-1}$ and a $\beta_s \leq \alpha$ such that $l_\alpha(s) = u_\alpha(s) = f_{\beta_s} \circ g(s, c_s)$.

Since we are assuming $\mathrm{cov}(\mathcal{B}) = 2^{\aleph_0}$, and since, by Claim 3.2.2, A_α is the union of $<2^{\aleph_0}$-many nowhere dense sets, it follows that there must exist a nonempty open set U, a $\beta \leq \alpha$, and a $c \in (Q_\alpha \cup n)^{n-1}$ such that U is included in the closure of the set

$$R := \{s \in \mathbb{R}^n \setminus A_\alpha \mid (\beta_s, c_s) = (\beta, c)\}.$$

Now, since c has length $n-1$, there is an $i < n$ such that $i \notin \mathrm{ran}(c)$. Fix one such i and fix e, e' in D that satisfy (2a)–(2c) of Lemma 3.2 for U and i. Let w, w' witness (2c) for e, e'.

For every $s, s' \in R$ such that $t(s, e \restriction n) = \langle 1 \rangle^n$ and $t(s', e' \restriction n) = \langle 0 \rangle^n$, we have $u_\alpha(s) \leq e_n$ and $l_\alpha(s') \geq e'_n$. Thus,

$$f_\beta \circ g(s, c) = u_\alpha(s) \leq e_n < e'_n \leq l_\alpha(s') = f_\beta \circ g(s', c).$$

Note that the value of $f_\beta \circ g(s, c)$ does not depend on the i-th element of s, since $i \notin \mathrm{ran}(c)$. Therefore, by continuity of f_β and by the density of R in U, we can find $z, z' \in R \cap U$ close enough to w and w', respectively, such that

(a) $t(z, z') = \langle 0 \rangle^n$;
(b) $f_\beta \circ g(z, c) \leq e_n$;
(c) $f_\beta \circ g(z', c) \geq e'_n$.

By density of $\{a \restriction n \mid a \in D\}$ in $\mathbb{R}^n$, there must be $a, a' \in D$ (with $a \restriction n$ and $a' \restriction n$ close enough to z and z', respectively) with $a \restriction n$ and $a' \restriction n$ in U such that

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(i) $t(a \restriction n, a' \restriction n) = \langle 0 \rangle^n$;
(ii) $t(z, a \restriction n) = \langle 0 \rangle^n$;
(iii) $t(z', a' \restriction n) = \langle 1 \rangle^n$.

In particular, $a_n \leq l_\alpha(z) = f_\beta \circ g(z, c)$ and $a'_n \geq u_\alpha(z') = f_\beta \circ g(z', c)$. Since $f_\beta \circ g(z, c) < f_\beta \circ g(z', c)$, we conclude $a_n < a'_n$. But then a and a' realize the increasing type. In particular, we have found two distinct elements of D that realize the increasing type, contradiction. ⊣

Fix an $s \in \mathbb{R}^n \setminus A_\alpha$ and an x satisfying the statement of our claim, and set $r_\alpha = s^\frown\langle x \rangle$.

It remains to prove that the inductive hypothesis is still satisfied, i.e., that $D \cup \{r_\beta \mid \beta \leq \alpha\}$ is a set of 1–1 mutually disjoint $(n+1)$-tuples that avoids the increasing type.

Claim 3.2.4. *r_α is 1–1.*

Proof. By construction, $r_\alpha \restriction n \notin A_\alpha$. In particular, we have that $r_{\alpha,i} \neq f_0(\langle r_{\alpha,j} \rangle^{n-1}) = r_{\alpha,j}$ for all $j < i < n$. Moreover, by (2) of Claim 3.2.3, we know that $r_{\alpha,n} \neq f_0(\langle r_{\alpha,i} \rangle^{n-1}) = r_{\alpha,i}$ for all $i < n$. Thus, r_α is 1–1. ⊣

Claim 3.2.5. *For all $r \in P_\alpha$, r and r_α are disjoint.*

Proof. Fix some $r \in P_\alpha$. The argument is analogous to the one of the previous claim. Since $r_\alpha \restriction n \notin A_\alpha$, we know that $f_0(\langle r_i \rangle^{n-1}) = r_i \notin \mathrm{ran}(r_\alpha \restriction n)$ for every $i \leq n$. In other words, we know that $\mathrm{ran}(r_\alpha \restriction n) \cap \mathrm{ran}(r) = \emptyset$. Moreover, by (2) of Claim 3.2.3, we can also conclude that $r_{\alpha,n} \notin \mathrm{ran}(r)$. Thus, r and r_α are disjoint. ⊣

Claim 3.2.6. *For all $r \in P_\alpha$, r and r_α do not realize the increasing type.*

Proof. Fix $r \in P_\alpha$. If $r \restriction n$ and $r_\alpha \restriction n$ do not realize the increasing type, then a fortiori r and r_α do not. So we can assume that $r \restriction n$ and $r_\alpha \restriction n$ realize the increasing type. For the sake of simplicity, let us assume $t(r \restriction n, r_\alpha \restriction n) = \langle 0 \rangle^n$ (the other case is analogous).

It follows that $u_\alpha(r_\alpha \restriction n) \leq r_n$. By (1) of Claim 3.2.3, we know that $r_{\alpha,n} \leq u_\alpha(r_\alpha \restriction n)$. Thus, given that $r_{\alpha,n} \neq r_n$ by Claim 3.2.5, we conclude that $r_{\alpha,n} < r_n$. In particular, $t(r, r_\alpha) = \langle 0 \rangle^n{}^\frown\langle 1 \rangle$, which is clearly different from the increasing type. ⊣

This ends the inductive definition. The set E is not $(2^{\aleph_0}, n+1)$-entangled, since $\{r_\alpha \mid \alpha < 2^{\aleph_0}\}$ is a collection of size $2^{\aleph_0}$ of 1–1 mutually disjoint $(n+1)$-tuples of E that avoids the increasing type. Now note that if we order $2^{\aleph_0} \times (n+1)$ with the lexicographic ordering, then the enumeration $\langle r_{\alpha,i} \mid \alpha < 2^{\aleph_0}, i \leq n \rangle$ of E satisfies the hypothesis of Proposition 2.7 by construction, and hence E is $(2^{\aleph_0}, n)$-entangled.

Remark 3.3. Actually, a slight modification of the argument of Theorem 3.1 yields the following: if $\mathrm{cov}(\mathcal{B}) = 2^{\aleph_0}$, then, for every $n > 1$ and for every cardinal $\kappa \leq 2^{\aleph_0}$ with $\mathrm{cf}(\kappa) = \mathrm{cf}(2^{\aleph_0})$, there exists a (κ, n)-entangled set of reals which is not $(\kappa, n+1)$-entangled.

Question 3.4. *Is the hypothesis $\mathrm{cov}(\mathcal{B}) = 2^{\aleph_0}$ necessary in Theorem 3.1?*

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QUESTION 3.5. *Let $n \geq 2$. Can there be an n-entangled set but no $(n+1)$-entangled sets?*[5]

§4. Entangledness is not a topological property. This section is devoted to the proof of Theorem 4.1 and Lemma 4.3, from which Theorem B directly follows.

Given two distinct $x, y \in {}^{\omega}2$, we let $\Delta(x, y)$ be the least $n \in \omega$ such that $x(n) \neq y(n)$. A subset $S \subseteq {}^{\omega}2$ is *monochromatic* if the parity of $\Delta(x, y)$ is the same for all distinct $x, y \in S$.

The cardinal invariant $\mathfrak{hm}$, introduced in [13], is defined as the least cardinality of a family of monochromatic sets that cover ${}^{\omega}2$. A peculiar property of $\mathfrak{hm}$ is that it cannot fall far behind the continuum, as $\mathfrak{hm}^{+} \geq 2^{\aleph_0}$—we refer the reader to [12, 16] for more on this cardinal invariant.

Regarding the statement of the next theorem, note that proving the existence of an entangled set of reals with a nontrivial autohomeomorphism is somewhat trivial: you can simply take an entangled set of reals and add two isolated points to it; it is still entangled and has a nontrivial autohomeomorphism, the one that simply swaps the two isolated points. It becomes less trivial if we ask our entangled set to be sufficiently crowded. Recall that a set $A \subseteq \mathbb{R}$ is said to be *κ-crowded*, for some infinite cardinal κ, if $|U \cap A| \geq \kappa$ for every open subset $U \subseteq \mathbb{R}$ with $U \cap A \neq \emptyset$.

THEOREM 4.1. *Assume $\mathfrak{hm} = 2^{\aleph_0}$. Then, there exists a $2^{\aleph_0}$-crowded, $2^{\aleph_0}$-entangled set of reals with a nontrivial autohomeomorphism.*

Given a set $A \subseteq X \times Y$, we denote by $\mathrm{Proj}_X[A]$ the canonical projection of A onto X. Moreover, given some $x \in X$, we denote by A_x the set $\{y \in Y \mid (x, y) \in A\}$. If $A \subseteq X^n$, then we simply write $\mathrm{Proj}_k(A)$, with $k \leq n$, instead of $\mathrm{Proj}_{X^k}(A)$.

The following lemma plays a role similar to that of Kuratowski's extension theorem in the proof of Proposition 2.7.

LEMMA 4.2. *Given a Polish space X and a set $A \subseteq X \times {}^{\omega}2$ such that A is closed in $\mathrm{Proj}_X[A] \times {}^{\omega}2$ and, for all $x \in X$, A_x is monochromatic, then there exists a G_δ set $\tilde{A}$ such that $A \subseteq \tilde{A} \subseteq \mathrm{cl}(A)$ and, for all $x \in X$, $\tilde{A}_x$ is monochromatic.*

PROOF. Consider the set $D = \{x \in X \mid \mathrm{cl}(A)_x \text{ is monochromatic}\}$. We claim that D is G_δ. Indeed, if we let S_0 and S_1 be the sets of all $s \in {}^{<\omega}2$ of even length and odd length, respectively, then, for all $x \in X$, the following holds:

$$\begin{aligned} x \in D \iff & \forall s \in S_0\ (\mathrm{cl}(A)_x \cap N_{s^\frown 0} = \emptyset \text{ or } \mathrm{cl}(A)_x \cap N_{s^\frown 1} = \emptyset) \text{ or} \\ & \forall s \in S_1\ (\mathrm{cl}(A)_x \cap N_{s^\frown 0} = \emptyset \text{ or } \mathrm{cl}(A)_x \cap N_{s^\frown 1} = \emptyset). \end{aligned}$$

Here, N_s is the standard open set of the Cantor space induced by s—i.e., the set of all the infinite binary sequences extending s. Using the sequential compactness of the Cantor space, an easy argument shows that the set of x's such that $\mathrm{cl}(A)_x \cap N_s = \emptyset$ is open for every s. Therefore we conclude that D is G_δ.

Now simply let $\tilde{A} = \mathrm{cl}(A) \cap (D \times {}^{\omega}2)$. It is clearly G_δ. Since we assumed A to be closed in $\mathrm{Proj}_X[A] \times {}^{\omega}2$, we clearly have $A_x = \mathrm{cl}(A)_x$ for all $x \in \mathrm{Proj}_X[A]$. Furthermore, since A_x is assumed to be monochromatic for all x, we conclude that $\mathrm{Proj}_X[A] \subseteq D$. In particular, $A \subseteq \tilde{A}$. ⊣

[5] A few weeks after our preprint was made public, a positive answer was given by Cruz [6].

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PROOF OF THEOREM 4.1. We work with the Cantor space ${}^{\omega}2$, which we consider endowed with the lexicographic ordering. Note that the order-topology of the lexicographic ordering coincides with the standard topology of the Cantor space.

Consider the map $s_2 : {}^{\omega}2 \to {}^{\omega}2$ defined by: for every $x \in {}^{\omega}2$ and $n \in \omega$,

$$s_2(x)(n) = \begin{cases} x(n) + 1 \bmod 2 & \text{if } n \bmod 2 = 0, \\ x(n) & \text{otherwise.} \end{cases}$$

Note that a set $A \subseteq {}^{\omega}2$ is monochromatic if and only if $s_2 \restriction A$ is monotone.

We want to define a $2^{\aleph_0}$-crowded and $2^{\aleph_0}$-entangled set $E \subseteq {}^{\omega}2$ which is closed under the map s_2. If we manage to do such a construction we are done, as the map $s_2 \restriction E$ would be a nontrivial autohomeomorphism of E.

To ensure crowdedness, fix a surjective map $\psi : 2^{\aleph_0} \to \omega$ with $|\psi^{-1}(n)| = 2^{\aleph_0}$ for every $n \in \omega$ and fix also an enumeration $\langle I_n \mid n \in \omega \rangle$ of the basic open sets of ${}^{\omega}2$. Finally, fix an enumeration $\langle H^n_\alpha \mid n > 0 \text{ and } \alpha < 2^{\aleph_0} \rangle$ of the G_δ sets $H \subseteq ({}^{\omega}2)^{n+1}$ such that H_x is monochromatic for all $x \in ({}^{\omega}2)^n$. We can suppose for simplicity that $H^1_0 = \{(x, x) \mid x \in {}^{\omega}2\} \cup \{(x, s_2(x)) \mid x \in {}^{\omega}2\}$.

Now we define inductively a sequence $\langle x_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ of elements of ${}^{\omega}2$ as follows: suppose we have defined the x_β's for all $\beta < \alpha$, then we pick x_α inside $I_{\psi(\alpha)}$ and outside the set

$$\bigcup \left\{ (H^n_\gamma)_x \mid n > 0 \text{ and } \gamma < \alpha \text{ and } x \in \{x_\beta \mid \beta < \alpha\}^n \right\}. \tag{2}$$

Note that the set in (1), being the union of less than $2^{\aleph_0}$-many monochromatic sets, does not cover $I_{\psi(\alpha)}$, since we are assuming $\mathfrak{hm} = 2^{\aleph_0}$.

Let $E = \{x_\alpha \mid \alpha < 2^{\aleph_0}\} \cup \{s_2(x_\alpha) \mid \alpha < 2^{\aleph_0}\}$. Clearly, E is $2^{\aleph_0}$-crowded. We are left to prove that E is $2^{\aleph_0}$-entangled.

To each pair (α, i) in $2^{\aleph_0} \times 2$, we associate the real $r_{\alpha,i}$ which is x_α if $i = 0$ and $s_2(x_\alpha)$ if $i = 1$. Note that the map $(\alpha, i) \mapsto r_{\alpha,i}$ is a bijection between $2^{\aleph_0} \times 2$ and E.

CLAIM 4.2.1. *E is $2^{\aleph_0}$-entangled.*

PROOF. We prove that E is $(2^{\aleph_0}, n)$-entangled for every $n > 0$ by induction on n. The base case $n = 1$ is trivial, hence we assume that E is $(2^{\aleph_0}, n)$-entangled and fix a set F of 1–1, mutually disjoint $(n+1)$-tuples of elements of E with $|F| = 2^{\aleph_0}$ and a $t \in {}^{n+1}2$ towards showing that t is realized by two elements of F.

Seeing F as a set of $(n+1)$-tuples of elements of $2^{\aleph_0} \times 2$, we can also assume without loss of generality that the $(n+1)$-tuples of F are increasing with respect to the lexicographic ordering on $2^{\aleph_0} \times 2$. We want to find two elements of F realizing t.

Suppose first that there exists a subset $F' \subseteq F$ of size $2^{\aleph_0}$ such that for all $r \in F'$ there is a β and an $(n-1)$-tuple s such that $r = s^\frown(x_\beta, s_2(x_\beta))$. Fix one such F'. We divide the argument in two cases, depending on whether $n = 1$:

n = 1: We claim that $\mathrm{Proj}_1[F']$ is not monochromatic. Indeed, suppose towards a contradiction that it is. Then, there exists an α such that $H^1_\alpha = {}^{\omega}2 \times \mathrm{cl}(\mathrm{Proj}_1[F'])$. By construction, $x_\beta \notin \mathrm{Proj}_1[F']$ for all $\beta > \alpha$, which contradicts F' having cardinality $2^{\aleph_0}$. Hence, $\mathrm{Proj}_1[F']$ is not monochromatic, or, equivalently, $s_2 \restriction \mathrm{Proj}_1[F'] = F'$ is not monotone. Hence, every $t \in {}^2 2$ is realized by two elements of F', as we wanted to show.

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n > 1: Denote by J the closure of $\mathrm{Proj}_n[F']$. Let D be the set of all those $x \in \mathrm{Proj}_{n-1}[F']$ such that J_x is monochromatic. Suppose towards a contradiction that $|D| = 2^{\aleph_0}$. By Lemma 4.2, there exists an α such that $H^{n-1}_\alpha \supseteq J \cap (D \times {}^\omega 2)$. Now pick some n-tuple $s^\frown x_\beta \in \mathrm{Proj}_n[F']$ with $\beta > \alpha$ and $s \in D$—it exists because we are assuming $|D| = 2^{\aleph_0}$. In particular, $x_\beta \in (H^{n-1}_\alpha)_s$, since $H^{n-1}_\alpha \supseteq J \cap (D \times {}^\omega 2) \supseteq \mathrm{Proj}_n[F'] \cap (D \times {}^\omega 2)$. But as $\alpha < \beta$, it follows directly from our construction that $x_\beta \notin (H^{n-1}_\alpha)_s$, hence the contradiction.

Thus, $|D| < 2^{\aleph_0}$. In particular, there exists a set $B \subseteq \mathrm{Proj}_{n-1}[F']$ of size $2^{\aleph_0}$ such that J_x is not monochromatic for all $x \in B$. We can assume without loss of generality that t is either equal to $u^\frown\langle 0,0\rangle$ or to $u^\frown\langle 1,0\rangle$ for some binary $(n-1)$-tuple u. If $t = u^\frown\langle 0,0\rangle$ (resp. $u^\frown\langle 1,0\rangle$), then let $C \subseteq B$ be such that $|C| = 2^{\aleph_0}$ and, for some finite binary sequence v of odd (resp. even) length, J_x has nonempty intersection with both $N_{v^\frown 0}$ and $N_{v^\frown 1}$ for all $x \in C$. By the induction hypothesis, E is $(2^{\aleph_0}, n-1)$-entangled. Hence, there are two distinct elements $w, z \in C$ such that $t(w,z) = u$. Since J_w and J_z have nonempty intersection with both $N_{v^\frown 0}$ and $N_{v^\frown 1}$, we can find $x, y \in F'$, with $x \restriction (n-1)$ and $y \restriction (n-1)$ sufficiently close to w and z respectively, such that:

(1) $t(x \restriction (n-1), y \restriction (n-1)) = u$;
(2) if $t = u^\frown\langle 0,0\rangle$, then $x(n-1) \in N_{v^\frown 0}$ and $y(n-1) \in N_{v^\frown 1}$; otherwise (i.e., if $t = u^\frown\langle 1,0\rangle$), $x(n-1) \in N_{v^\frown 1}$ and $y(n-1) \in N_{v^\frown 0}$.

By the way we chose F', we conclude that $t(x,y) = t$.

Now suppose that there is no such F'. This case is dealt with as in the standard construction of an entangled set. It follows from our hypothesis that there are $2^{\aleph_0}$-many $r \in F$ for which there is a γ and some n-tuple $s \in \{x_\beta, s_2(x_\beta) \mid \beta < \gamma\}^n$ such that r is either $s^\frown x_\gamma$ or $s^\frown s_2(x_\gamma)$. We just consider the case in which there exists a subset $F'' \subseteq F$ of size $2^{\aleph_0}$ such that, for all $r \in F''$, $r = s^\frown s_2(x_\gamma)$ for some γ and $s \in \{x_\beta \mid \beta < \gamma\}^n$, as the other cases are analogous. Let us consider F'' as a map from ${}^\omega 2)^n$ to ${}^\omega 2$, and assume towards a contradiction that it has less than $2^{\aleph_0}$-many discontinuity points. Then, by restricting it if necessary, we can suppose that F'' is continuous. By Lemma 4.2 (or directly by Kuratowski's extension theorem) there is an α such that $s_2^{-1} \circ F'' \subseteq H^n_\alpha$. Since F'' has cardinality $2^{\aleph_0}$, there is a γ with $\alpha < \gamma$ and an $s \in \{x_\beta \mid \beta < \gamma\}^n$ such that $s^\frown s_2(x_\gamma) \in F''$ and, a fortiori, $s^\frown x_\gamma \in H^n_\alpha$. But by construction $x_\gamma \notin (H^n_\alpha)_s$, hence the contradiction. Thus, F'' has $2^{\aleph_0}$-many discontinuity points. Since, by induction hypothesis, E is $(2^{\aleph_0}, n)$-entangled, we conclude by Lemma 2.8 that t is realized by two elements of $F'' \subseteq F$. ⊣

This finishes the proof. ⊣

Theorem B follows directly from Theorem 4.1 together with the following fact.

LEMMA 4.3. *Suppose that there exists a κ-crowded, κ-entangled set of reals with a nontrivial autohomeomorphism. Then, there are two homeomorphic sets of reals A, B such that A is κ-entangled but B is not $(\kappa, 2)$-entangled.*

PROOF. Fix a κ-crowded, κ-entangled $E \subseteq \mathbb{R}$ and a nontrivial autohomeomorphism $h : E \to E$. In particular there exist two nonempty disjoint open intervals I, J such that $|E \cap I| = |E \cap J| = \kappa$ and $h[E \cap I] = E \cap J$. Fix a nonempty open

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interval W disjoint from both I and J, and fix an increasing homeomorphism $f : J \to W$.

Let $A = E \cap (I \cup J)$ and $B = (E \cap J) \cup f[E \cap J]$. The set A, being a subset of E of cardinality κ, is κ-entangled. The set B, on the other hand, is not $(\kappa, 2)$-entangled, as witnessed by the increasing map f. Consider the following map:

$$\varphi : A \longrightarrow B$$
$$x \longmapsto \begin{cases} h(x) & \text{if } x \in I, \\ f(x) & \text{otherwise.} \end{cases}$$

The map φ is a homeomorphism by disjointness of I and J, and so we are done. ⊣

REMARK 4.4. As for Theorem 3.1, a slight modification of the argument of Theorem 4.1 yields the following: if $\mathfrak{hm} = 2^{\aleph_0}$, then, for every $n > 1$ and for every cardinal $\kappa \leq 2^{\aleph_0}$ with $\mathrm{cf}(\kappa) = \mathrm{cf}(2^{\aleph_0})$, there exists a κ-crowded, κ-entangled set of reals with a nontrivial autohomeomorphism.

QUESTION 4.5. *Is* $\mathfrak{hm} = 2^{\aleph_0}$ *necessary in Theorem* 4.1?

QUESTION 4.6. *Does it follow from* CH *that there exists an entangled set that is homogeneous as a topological space*?

§5. Definable entangled sets. In this section we prove Theorem C. Let us first observe that Theorem C is, arguably, optimal, since no analytic set can be 2-entangled. Indeed, 2-entangled sets do not satisfy the perfect set property.

PROPOSITION 5.1. *No* 2*-entangled set of reals contains an uncountable closed set*.

PROOF. It suffices to show that no closed set of reals is 2-entangled. Fix a closed uncountable $C \subseteq \mathbb{R}$, and fix a countable $D \subset C$ which is (topologically) dense in C and such that D is unbounded and dense as a linear order. By Cantor's isomorphism theorem, there exists a decreasing map $f : D \to D$, and, by the closure of C, f can be extended to a decreasing map from C to C. Therefore, C is not 2-entangled. ⊣

COROLLARY 5.2. *No analytic set of reals is* 2*-entangled.*

PROOF. Analytic sets satisfy the perfect set property—i.e., every uncountable analytic set contains an uncountable closed set. ⊣

In what follows we employ the notation of [19, Section 13]. Any real $x \in {}^{\omega}\omega$ encodes the binary relation $R_x = \{\langle m, k\rangle \mid x(\langle\!\langle m, k\rangle\!\rangle) = 0\}$ on ω, where $\langle\!\langle \cdot, \cdot \rangle\!\rangle : \omega \times \omega \to \omega$ is a recursive bijection, and consequently it also encodes the structure

$$M_x = \langle \omega, R_x \rangle$$

for the language of set theory. Also, each real $x \in {}^{\omega}\omega$ encodes the sequence of reals $\langle (x)_i \mid i \in \omega \rangle$ defined by $(x)_i(j) = x(\langle\!\langle i, j \rangle\!\rangle)$ for each $i, j \in \omega$.

THEOREM 5.3. *Suppose that* $\mathbb{R} \subseteq \mathsf{L}$. *Then there exists a* $\mathbf{\Pi}^1_1$ *set of reals which is entangled.*

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PROOF. Assume that $\mathbb{R} \subseteq \mathsf{L}$ and denote by $<_{\mathsf{L}}$ the canonical well-ordering of L. Consider the set $\mathcal{A}$ of all the countable admissible[6] ordinals α such that $\mathsf{L}_\alpha \vDash$ "Every set is countable." Note that $\mathcal{A}$ is unbounded in $\omega_1^{\mathsf{L}} = \omega_1$. The idea is to get a Π^1_1 set of unique codes for the structures $\langle \mathsf{L}_\alpha, \in \rangle$ with $\alpha \in \mathcal{A}$ (c.f. [19, Theorem 13.12]).

Consider the set $E \subseteq {}^{\omega}\omega$ of all reals x that satisfy the conjunction of the following formulae:

(1) $M_{(x)_0}$ is well-founded and extensional and $M_{(x)_0} \cong \langle \mathsf{L}_\alpha, \in \rangle$ for some $\alpha \in \mathcal{A}$.
(2) For every $i \in \omega$, $M_{(x)_{i+1}}$ is well-founded and extensional and $M_{(x)_{i+1}} \cong \langle \mathsf{L}_{\delta_{i+1}}, \in \rangle$, with δ_{i+1} being the least limit $\delta < \omega_1$ such that $(x)_i \in \mathsf{L}_\delta$.
(3) For every $i \in \omega$, for every $z \in {}^{\omega}\omega \cap \mathsf{L}_{\delta_{i+1}}$, if $\mathsf{L}_{\delta_{i+1}} \vDash$ "$z <_{\mathsf{L}} (x)_i$," then $M_z \not\cong M_{(x)_i}$.

By rewriting (1)–(3) in the language of second-order arithmetic, we find that the set E is Π^1_1—again, see [19, Theorem 13.12]. For each $x \in E$, let α_x be such that $M_{(x)_0} \cong \langle \mathsf{L}_{\alpha_x}, \in \rangle$.

CLAIM 5.3.1. *For each $x \in E$, $x \notin \mathsf{L}_{\alpha_x}$.*

PROOF. Suppose towards a contradiction that $x \in \mathsf{L}_{\alpha_x}$. Since $M_{(x)_0}$ is well-founded and extensional, a fortiori $\mathsf{L}_{\alpha_x} \vDash$ "$M_{(x)_0}$ is well-founded and extensional." But then, since α_x is admissible, the transitive collapse of $M_{(x)_0}$ also belongs to L_{α_x}. Since the transitive collapse of $M_{(x)_0}$ is L_{α_x}, we have reached a contradiction. ⊣

Note that for each $\alpha \in \mathcal{A}$, there exists exactly one $x \in E$ with $\alpha_x = \alpha$. Moreover, note that, for each $\beta > \alpha$ with $\beta \in \mathcal{A}$, the (unique) real $x \in E$ with $\alpha_x = \alpha$ can already be found in L_β—this is where we use that $\mathsf{L}_\beta \vDash$ "Every set is countable," to ensure that L_α is countable in L_β.

CLAIM 5.3.2. *E is entangled.*

PROOF. Since for every $\alpha \in \mathcal{A}$ there exists a unique $x \in E$ with $\alpha_x = \alpha$, the well-order on $\mathcal{A}$ naturally induces a well-order on E. More precisely, let $\langle \alpha_\nu \mid \nu < \omega_1 \rangle$ be the increasing enumeration of $\mathcal{A}$, and let $\langle x_\nu \mid \nu < \omega_1 \rangle$ be the enumeration of E such that $\alpha_{x_\nu} = \alpha_\nu$ for every $\nu < \omega_1$.

It suffices to show that for every continuous map f from a G_δ subset of $\mathbb{R}^k$ into $\mathbb{R}$ (for some $k \in \omega$), there exists a $\nu < \omega_1$ such that, for all $\mu \geq \nu$,

$$f\left[\{x_\eta \mid \eta < \mu\}^k\right] \cap E \subseteq \{x_\eta \mid \eta < \mu\}, \tag{3}$$

as then it follows from Proposition 2.7 that E is entangled.

Pick a continuous map f from a G_δ subset of $\mathbb{R}^k$ into $\mathbb{R}$ (for some $k \in \omega$). The map f is coded by some real $\bar{f} \in \mathbb{R}$. Let $\nu < \omega_1$ be large enough so that $\bar{f} \in \mathsf{L}_{\alpha_\nu}$—here we are using the hypothesis $\mathbb{R} \subseteq \mathsf{L}$. Now pick some $\rho > \mu \geq \nu$ and some $s \in \{x_\eta \mid \eta < \mu\}^k$. Since both $\bar{f}$ and s belong to L_{α_ρ}, it follows that $f(s)$ also belong to L_{α_ρ}. But then, since by Claim 5.3.1 $x_\rho \notin \mathsf{L}_{\alpha_\rho}$, it follows $f(s) \neq x_\rho$. Thus, (1) holds. ⊣

This finishes the proof. ⊣

[6] An ordinal α is *admissible* if L_α is a model of KP. We just need that L_α is a model of a sufficiently large fragment of ZFC.

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§6. A 2-entangled non-separable linear order. In this section we prove Theorem D. As we already noted in the introduction, every 2-entangled non-separable linear order is, in particular, a Suslin line. It is well-known that the existence of a Suslin line follows from $\diamondsuit$ (e.g., [8, 22]). We show that from the same hypothesis we can construct a 2-entangled Suslin line. Recall that the existence of a 2-entangled Suslin line is already known to be consistent by a forcing argument due to Krueger [21].

Recall that a partial order $(T, \leq)$ is a *tree* (in the order-theoretic sense) if for every $x \in T$, the set of the predecessors of x—i.e., the set $\{y \in T \mid y \leq x\}$—is well-ordered with respect to $\leq$. Given a node $x \in T$, the *height* of x is the order type of $\{y \in T \mid y < x\}$. For every tree $(T, \leq)$ and ordinal α, we let $\mathrm{Lev}_\alpha(T, \leq)$ be the set of all the nodes $x \in T$ of height α.

THEOREM 6.1. *Suppose that $\diamondsuit$ holds. Then, there exists a 2-entangled non-separable linear order.*

PROOF. The construction is similar to the classical construction of a Suslin line under $\diamondsuit$. We define a tree ordering $\trianglelefteq$ on $\omega_1 \times \omega$ with $\mathrm{Lev}_\alpha(\omega_1 \times \omega, \trianglelefteq) = \{\alpha\} \times \omega$ for every α. For every $x = (\alpha, k) \in \omega_1 \times \omega$, we denote α (resp. k) by $\mathrm{ht}(x)$ (resp. $n(x)$). Moreover, given $x, y \in \omega_1 \times \omega$, we write $x \top y$ (resp. $x \perp y$) when x and y are comparable (resp. incomparable) with respect to $\trianglelefteq$.

We want to define the tree $(\omega_1 \times \omega, \trianglelefteq)$ such that the following lexicographic-like linearization of $\trianglelefteq$ is 2-entangled and non-separable: for all $x, y \in \omega_1 \times \omega$,

$$x \preceq y \text{ if either } x \trianglelefteq y \text{ or } \big(x \perp y \text{ and } n(\min\{z \trianglelefteq x \mid z \not\trianglelefteq y\}) > n(\min\{z \trianglelefteq y \mid z \not\trianglelefteq x\})\big),$$

where the minimum is taken with respect to $\trianglelefteq$. From now on, the minima and maxima between nodes of our tree are taken with respect to $\preceq$.

We now introduce a useful binary relation induced by $\preceq$: given $x, y \in \omega_1 \times \omega$, we let $x \prec^+ y$ if $x \preceq y$ and $x \perp y$. The following straightforward observations are often implicitly used afterward: if $x \prec^+ y$, then $z \prec^+ y$ for all $z \trianglerighteq x$; moreover, if $y \preceq x$, then $y \preceq z$ for all $z \trianglerighteq x$.

For the sake of clarity, we denote by L_α the set $\{\alpha\} \times \omega$ for each α. Fix a $\diamondsuit$-sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$. Fix also a bijection $\psi : \omega_1 \to (\omega_1 \times \omega)^2$. We define $\trianglelefteq$ inductively such that:

(1) $\trianglelefteq$ is a tree ordering on $\omega_1 \times \omega$ and $\mathrm{Lev}_\alpha(\omega_1 \times \omega, \trianglelefteq) = L_\alpha$ for every $\alpha < \omega_1$;
(2) for every n and $\alpha < \omega_1$, there are infinitely many m's such that $(\alpha, n) \trianglelefteq (\alpha + 1, m)$;
(3) if $\beta < \alpha < \omega_1$ and $x \in L_\beta$, then there is a $y \in L_\alpha$ with $x \triangleleft y$;
(4) Suppose that α is limit and $\psi : A_\alpha \subseteq (\alpha \times \omega)^2$ is a map, denoted for clarity by f_α, which is maximal among the 1–1 and monotone maps f from a subset of $\alpha \times \omega$ to $\alpha \times \omega$ such that $f(x) \neq x$ for all $x \in \mathrm{dom}(f)$. Then, for all $x, y \in (\alpha + 1) \times \omega$, if either
(4a) $\mathrm{ht}(x) = \mathrm{ht}(y) = \alpha$, or
(4b) $\mathrm{ht}(x) < \alpha$ and $\mathrm{ht}(y) = \alpha$ and $x \notin \mathrm{dom}(f_\alpha)$, or
(4c) $\mathrm{ht}(x) = \alpha$ and $\mathrm{ht}(y) < \alpha$ and $y \notin \mathrm{ran}(f_\alpha)$,
then the map $f_\alpha \cup \{(x, y)\}$ is not monotone.

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In (4) and onward, monotonicity is always with respect to $\preceq$. Suppose for the moment that we have already defined $\trianglelefteq$ satisfying (1)–(4). We claim that the linear order $(\omega_1 \times \omega, \preceq)$ is 2-entangled and non-separable.

CLAIM 6.1.1. *$(\omega_1 \times \omega, \preceq)$ is non-separable.*

PROOF. Fix a countable $D \subseteq \omega_1 \times \omega$. Then, there is an $\alpha < \omega_1$ such that $D \subseteq \alpha \times \omega$. By (2), we can pick $x \in L_\alpha$ and $y \in L_{\alpha+1}$ with $x \lhd y$. Then, the open interval $]x, y[$ in $(\omega_1 \times \omega, \preceq)$ is nonempty as, by (2) again, it contains infinitely many elements of $L_{\alpha+1}$. Moreover, it is straightforward that $]x, y[\subseteq (\omega_1 \setminus \alpha) \times \omega$. Thus, no element of D belongs to $]x, y[$. We conclude that $(\omega_1 \times \omega, \preceq)$ is non-separable. ⊣

CLAIM 6.1.2. *$(\omega_1 \times \omega, \preceq)$ is 2-entangled.*

PROOF. Fix a 1–1 monotone map $f : A \to \omega_1 \times \omega$, with $A \subseteq \omega_1 \times \omega$, such that $f(x) \neq x$ for all $x \in A$ towards showing that A is countable.

Let $\tilde{f}$ be a maximal 1–1 and monotone extension of f such that $\tilde{f}(x) \neq x$ for all $x \in \mathrm{dom}(\tilde{f})$.

There is a club $C \subseteq \omega_1$ such that, for all $\alpha \in C$, $\alpha \times \omega$ is closed under both $\tilde{f}$ and $\tilde{f}^{-1}$ and $\tilde{f} \restriction \alpha \times \omega$ is maximal among the maps from a subset of $\alpha \times \omega$ to $\alpha \times \omega$ that satisfy the same relevant properties of $\tilde{f}$—that is, being 1–1, monotone, and pointwise different from the identity. By the properties of the $\diamondsuit$-sequence there is an $\alpha \in C$ with $f_\alpha = \tilde{f} \restriction \alpha \times \omega$. We claim that $\mathrm{dom}(\tilde{f}) \subseteq \alpha \times \omega$, from which the countability of A directly follows.

Suppose towards a contradiction that there exists some $x \in \mathrm{dom}(\tilde{f}) \setminus (\alpha \times \omega)$. Fix one such x. Note that, since $\alpha \times \omega$ is closed under $\tilde{f}^{-1}$, we also have $\tilde{f}(x) \notin \alpha \times \omega$. Let $x \restriction \alpha$ be the $\trianglelefteq$-predecessor of x of height α, and define $\tilde{f}(x) \restriction \alpha$ analogously for $\tilde{f}(x)$. By (4), the map $(\tilde{f} \restriction \alpha \times \omega) \cup \{(x \restriction \alpha, \tilde{f}(x) \restriction \alpha)\}$ is not monotone.

It is straightforward that for every $z, w \in \omega_1 \times \omega$ with $\mathrm{ht}(z) > \mathrm{ht}(w)$, $z \prec w$ if and only if $z \prec^+ w$. Therefore, it follows directly from the useful observation made after the definition of $\prec^+$ and from $(\tilde{f} \restriction \alpha \times \omega) \cup \{(x \restriction \alpha, \tilde{f}(x) \restriction \alpha)\}$ not being monotone that also $(\tilde{f} \restriction \alpha \times \omega) \cup \{(x, \tilde{f}(x))\}$ is not monotone, which is a contradiction. ⊣

We define $\trianglelefteq$ by induction on its levels. Suppose that we have defined $\trianglelefteq$ on $\alpha \times \omega$ so that (1)–(4) holds below α, towards extending $\trianglelefteq$ to $(\alpha + 1) \times \omega$.

We deal with the only nontrivial case: α is limit, and the hypotheses of (4) are satisfied. Fix an increasing sequence $(\xi_m)_{m \in \omega}$ of countable ordinals such that $\sup_m \xi_m = \alpha$ and fix also an enumeration $(b_n)_{n \in \omega}$ of $\alpha \times \omega$.

We define a family $\langle z^n_m \mid n, m \in \omega \rangle$ of elements of $\alpha \times \omega$ such that $\mathrm{ht}(z^n_m) \geq \xi_m$ and $z^n_m \trianglelefteq z^n_{m+1}$ for every $n, m \in \omega$. Once we have defined this family, we extend $\trianglelefteq$ as follows: for every $x \in \alpha \times \omega$ and $n \in \omega$, we let $x \lhd (\alpha, n)$ if $x \trianglelefteq z^n_m$ for some m.

We describe how to define the z^n_ms and only then we show that this definition works—that is, the resulting extension of $\trianglelefteq$ on $(\alpha + 1) \times \omega$ still satisfies (1)–(4) below $\alpha + 1$.

Fix an $n \in \omega$ and suppose that we have defined z^k_m for all $k < n$ and all m, towards defining z^n_m for all m. Fix an enumeration $(w_m, i_m)_{m \in \omega}$ of the set $((\alpha \times \omega) \cup (\{\alpha\} \times n)) \times 2$.

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First let us define z^n_0. Suppose first that there is no $z \in \mathrm{dom}(f_\alpha) \cup \mathrm{ran}(f_\alpha)$ such that $b_n \lhd z$: in this case let z^n_0 be such that $b_n \lhd z^n_0$ and $\mathrm{ht}(z^n_0) = \max(\mathrm{ht}(b_n), \xi_0) + 2$—we can find such z^n_0 by (3).

Now suppose that there is a $z \in \mathrm{dom}(f_\alpha) \cup \mathrm{ran}(f_\alpha)$ such that $b_n \lhd z$. Fix one such z. Let $z^\star = f_\alpha(z)$ if $z \in \mathrm{dom}(f_\alpha)$ and $z^\star = f^{-1}_\alpha(z)$ otherwise. We need to consider the following cases:

(i) $z \top z^\star$ and f_α is increasing:
By the way we chose z and by case assumption, $z, z^\star$, and b_n are all $\trianglelefteq$-compatible and, moreover, $\max(\min(z, z^\star), b_n) \lhd \max(z, z^\star)$. Let $\delta = \mathrm{ht}(\max(\min(z, z^\star), b_n)) + 1$, and let $i \in \omega$ be such that $(\delta, i) \trianglelefteq \max(z, z^\star)$. By (2), we can pick $j > i$ such that $\max(\min(z, z^\star), b_n) \lhd (\delta, j)$. By (3) we can pick z^n_0 such that $(\delta, j) \trianglelefteq z^n_0$ and $\mathrm{ht}(z^n_0) \geq \xi_0$. In particular, we have $\min(z, z^\star) \prec z^n_0 \prec^+ \max(z, z^\star)$.

(ii) $z \top z^\star$ and f_α is decreasing:
By (3) we can pick z^n_0 such that $\max(z, z^\star) \trianglelefteq z^n_0$ and $\mathrm{ht}(z^n_0) \geq \xi_0$. In particular, we have $z, z^\star \preceq z^n_0$.

(iii) $z \prec^+ z^\star$ and $b_n \not\trianglelefteq z^\star$ and f_α is increasing:
By (3) we can pick z^n_0 such that $z \trianglelefteq z^n_0$ and $\mathrm{ht}(z^n_0) \geq \xi_0$. In particular, we have $z \preceq z^n_0 \prec^+ z^\star$.

(iv) $z^\star \prec^+ z$ and $b_n \not\trianglelefteq z^\star$ and f_α is increasing:
Let $i \in \omega$ be such that $(\mathrm{ht}(b_n) + 1, i) \trianglelefteq z$. By (2), we can fix some $j > i$ such that $b_n \lhd (\mathrm{ht}(b_n) + 1, j)$. By (3), we can pick z^n_0 such that $(\mathrm{ht}(b_n) + 1, j) \trianglelefteq z^n_0$ and $\mathrm{ht}(z^n_0) \geq \xi_0$. In particular, we have $z^\star \prec z^n_0 \prec^+ z$.

(v) $z \perp z^\star$ and $b_n \trianglelefteq z, z^\star$ and f_α is increasing:
By (3), we can pick z^n_0 such that $\min(z, z^\star) \trianglelefteq z^n_0$ and $\mathrm{ht}(z^n_0) \geq \xi_0$. In particular, we have $\min(z, z^\star) \preceq z^n_0 \prec^+ \max(z, z^\star)$.

(vi) $z \prec^+ z^\star$ and $b_n \not\trianglelefteq z^\star$ and f_α is decreasing:
Pick z^n_0 as in (iv). In particular, we have $z^n_0 \prec^+ z, z^\star$.

(vii) $z^\star \prec^+ z$ and $b_n \not\trianglelefteq z^\star$ and f_α is decreasing:
Pick z^n_0 as in (iii). In particular, we have $z, z^\star \preceq z^n_0$.

(viii) $z \perp z^\star$ and $b_n \trianglelefteq z, z^\star$ and f_α is decreasing:
By (3), we can pick z^n_0 such that $\max(z, z^\star) \trianglelefteq z^n_0$ and $\mathrm{ht}(z^n_0) \geq \xi_0$. In particular, we have $z, z^\star \preceq z^n_0$.

Suppose that we have defined z^n_m towards defining z^n_{m+1}. If either $i_m = 0$ and $w_m \in \mathrm{dom}(f_\alpha)$, or $i_m = 1$ and $w_m \in \mathrm{ran}(f_\alpha)$, then simply let z^n_{m+1} be such that $z^n_m \trianglelefteq z^n_{m+1}$ and $\mathrm{ht}(z^n_{m+1}) \geq \xi_{m+1}$—we can find such a z^n_{m+1} by (3).

Now assume that either $i_m = 0$ and $w_m \notin \mathrm{dom}(f_\alpha)$, or $i_m = 1$ and $w_m \notin \mathrm{ran}(f_\alpha)$. If there is no z with $z^n_m \lhd z$ such that $z \in \mathrm{ran}(f_\alpha)$ if $i_m = 0$ and $z \in \mathrm{dom}(f_\alpha)$ if $i_m = 1$, then let z^n_{m+1} be such that $z^n_m \lhd z^n_{m+1}$ and $\mathrm{ht}(z^n_{m+1}) \geq \xi_{m+1}$ and $z^n_{m+1} \neq w_m$. Otherwise, fix one such z and let $z^\star$ denote $f^{-1}_\alpha(z)$ if $i_m = 0$, and $f_\alpha(z)$ if $i_m = 1$. Now we need to consider the following cases:

(a) f_α is increasing and $z^\star \prec w_m$:
Let $i \in \omega$ be such that $(\mathrm{ht}(z^n_m) + 1, i) \trianglelefteq z$. By (2), there we can fix some $j > i$ such that $z^n_m \lhd (\mathrm{ht}(z^n_m) + 1, j)$. By (3), we can pick z^n_{m+1} such that $(\mathrm{ht}(z^n_m) + 1, j) \trianglelefteq z^n_{m+1}$ and $\mathrm{ht}(z^n_{m+1}) \geq \xi_{m+1}$. In particular, we have $z^n_{m+1} \prec^+ z$

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(b) f_α is decreasing and $w_m \prec z^\star$:
As in (a).
(c) f_α is increasing and $w_m \prec z^\star$:
By (3), we can pick z^n_{m+1} such that $z \trianglelefteq z^n_{m+1}$ and $\mathrm{ht}(z^n_{m+1}) \geq \xi_{m+1}$.
(d) f_α is decreasing and $z^\star \prec w_m$:
As in (c).

This ends the inductive definition of the z^n_ms. We are left to prove that (1)–(4) are still satisfied below $\alpha + 1$.

Since we chose the z^n_ms such that $\sup_m \mathrm{ht}(z^n_m) = \sup_m \xi_m = \alpha$ for every n, we conclude that $\mathrm{Lev}_\alpha((\alpha+1)\times\omega, \trianglelefteq) = L_\alpha$—that is, (1) is satisfied.

Condition (2) trivially holds since it holds below α by the induction hypothesis, and α is limit.

Moreover, since $b_n \trianglelefteq z^n_0 \triangleleft (\alpha, n)$ for every n by choice of z^n_0, (3) also holds—that is, every element of $\alpha \times \omega$ has an $\trianglelefteq$-successor in L_α.

Finally, we need to check that (4) is satisfied. We prove it by induction in the following sense: we suppose that for all $x, y \in (\alpha\times\omega) \cup (\{\alpha\}\times n)$ satisfying one among (4a)–(4c), the map $f_\alpha \cup \{(x,y)\}$ is not monotone, towards showing that the same can be said for all x, y taken in $(\alpha\times\omega)\cup(\{\alpha\}\times(n+1))$.

We will check only some representative cases, as the other are analogous. From now on we assume that f_α is increasing.

Let us show that $f_\alpha \cup \{((\alpha,n),(\alpha,n))\}$ is not increasing. Suppose that there is a $z \in \mathrm{dom}(f_\alpha) \cup \mathrm{ran}(f_\alpha)$ such that $b_n \triangleleft z$. Fix the z satisfying this property we chose when defining z^n_0—we assume for simplicity $z \in \mathrm{dom}(f_\alpha)$, as the case $z \in \mathrm{ran}(f_\alpha)$ is analogous. Then, looking at the possible cases (i.e., (i) and (iii)–(v)), either $z^n_0 \prec^+ z$ and $f_\alpha(z) \preceq z^n_0$, or $z^n_0 \prec^+ f_\alpha(z)$ and $z \preceq z^n_0$. In particular, since $z^n_0 \triangleleft (\alpha,n)$, either $(\alpha,n) \prec z$ and $f_\alpha(z) \prec (\alpha,n)$, or $z \prec (\alpha,n)$ and $(\alpha,n) \prec f_\alpha(z)$. In other words, the pairs $((\alpha,n),(\alpha,n))$ and $(z, f_\alpha(z))$ realize the type $\langle 1,0\rangle$.

Now suppose instead that there is no $z \in \mathrm{dom}(f_\alpha) \cup \mathrm{ran}(f_\alpha)$ such that $b_n \triangleleft z$. By definition of z^n_0 and case assumption, we have $\mathrm{ht}(z^n_0) \geq \mathrm{ht}(b_n) + 2$. Therefore, we can fix some t such that $b_n \triangleleft t \triangleleft z^n_0$. By case assumption, both t and z^n_0 do not belong to $\mathrm{dom}(f_\alpha) \cup \mathrm{ran}(f_\alpha)$. By maximality of f_α, we conclude that there is an $x \in \mathrm{dom}(f_\alpha)$ such that $(x, f_\alpha(x))$ and (t, z^n_0) realize $\langle 1,0\rangle$. From the latter observation and our case assumption, it follows that either $t \prec^+ x$ and $f_\alpha(x) \preceq z^n_0$, or $x \preceq t$ and $z^n_0 \prec^+ f_\alpha(x)$. Since $t \triangleleft z^n_0 \triangleleft (\alpha,n)$, we conclude that the pairs $((\alpha,n),(\alpha,n))$ and $(x, f_\alpha(x))$ realize the type $\langle 1,0\rangle$.

We have shown that $f_\alpha \cup \{((\alpha,n),(\alpha,n))\}$ is not increasing. Now fix some k with $k < n$ and let us show that $f_\alpha \cup \{(\alpha,k),(\alpha,n)\}$ is not increasing. Fix an m such that $w_m = (\alpha,k)$ and $i_m = 0$. Clearly $w_m = (\alpha,k) \notin \mathrm{dom}(f_\alpha)$.

Suppose first that there is a z with $z^n_m \triangleleft z$ such that $z \in \mathrm{ran}(f_\alpha)$. Fix the z satisfying this property we chose when defining z^n_{m+1}. Looking at the cases (a) and (c) in the definition of z^n_{m+1}, it is easy to see that either $f^{-1}_\alpha(z) \prec (\alpha,k)$ and $z^n_{m+1} \prec^+ z$, or $(\alpha,k) \prec f^{-1}_\alpha(z)$ and $z \preceq z^n_{m+1}$. Hence, the pairs $((\alpha,k),(\alpha,n))$ and $(f^{-1}_\alpha(z), z)$ realize $\langle 1,0\rangle$.

Now suppose instead that there is no z with $z^n_m \triangleleft z$ such that $z \in \mathrm{ran}(f_\alpha)$. Clearly, this means that $z^n_{m+1} \notin \mathrm{ran}(f_\alpha)$. By the induction hypothesis, there is some $x \in \mathrm{dom}(f_\alpha)$ such that the pairs $(x, f_\alpha(x))$ and $((\alpha,k), z^n_{m+1})$ realize $\langle 1,0\rangle$. But now

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note that, again by case assumption, we must have either $z^n_{m+1} \prec^+ f_\alpha(x)$ or $f_\alpha(x) \prec z^n_{m+1}$. Therefore, the pairs $(x, f_\alpha(x))$ and $((\alpha,k),(\alpha,n))$ also realize $\langle 1,0\rangle$.

We have shown that $f_\alpha \cup \{((\alpha,k),(\alpha,n))\}$ is not increasing. Now fix some $y \in \alpha \times \omega$ with $y \notin \operatorname{ran}(f_\alpha)$ towards showing that $f_\alpha \cup \{((\alpha,n),y)\}$ is not increasing. Fix an m such that $w_m = y$ and $i_m = 1$.

If there is a z with $z^n_m \lhd z$ such that $z \in \operatorname{dom}(f_\alpha)$, the argument used to show that there is some $x \in \operatorname{dom}(f_\alpha)$ such that $(x, f_\alpha(x))$ and $((\alpha,n),y)$ realize $\langle 1,0\rangle$ is the same as in the previous case. So, let us assume that there is no such z. In particular, we must have $z^n_{m+1} \notin \operatorname{dom}(f_\alpha)$. Moreover, by the way we chose z^n_{m+1}, we also have $z^n_{m+1} \neq w_m = y$. By maximality of f_α, there is some $x \in \operatorname{dom}(f_\alpha)$ such that the pairs $(x, f_\alpha(x))$ and (z^n_{m+1}, y) realize $\langle 1,0\rangle$. But now note that, again by case assumption, we must have either $z^n_{m+1} \prec^+ x$ or $x \prec z^n_{m+1}$. Therefore, the pairs $(x, f_\alpha(x))$ and $((\alpha,n),y)$ also realize $\langle 1,0\rangle$.

The other cases are analogous. ⊣

QUESTION 6.2. *Can there be a* 2*-entangled linear order but no separable* 2*-entangled linear order*?[7]

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[7] A positive answer was given by Martínez-Ranero and Polymeris [23].

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UNIVERSITÀ DEGLI STUDI DI TORINO
ITALY
E-mail: raphael.carroy@unito.it

UNIVERSITY OF FREIBURG
GERMANY
E-mail: maxwell.levine@mathematik.uni-freiburg.de

UNIVERSITY OF VIENNA
AUSTRIA
E-mail: lorenzo.notaro@univie.ac.at

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