ON MANIFOLDS CONTAINING A SUBMANIFOLD WHOSE COMPLEMENT IS CONTRACTIBLE

By G. M. KELLY

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1. *Introduction.* The problem discussed here arose in the course of some reflections on the critical point theory of Lusternik and Schnirelmann (4). In (4) it is shown how it is possible to associate, with a suitably differentiable real-valued function f defined on a compact manifold M, a set of real numbers $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_c$, which are critical levels of f and which in certain respects are analogous to, and indeed generalizations of, the eigenvalues of a quadratic form. The number c depends on M and is called the category of M. If R^n is Euclidean n-space, S^n the unit sphere of R^{n+1}, and P^n the real projective n-space obtained from S^n by identifying opposite points, then a quadratic form ϕ in the $(n+1)$ coordinates of R^{n+1} defines a real function on S^n and, by passage to the quotient, on P^n. P^n has category $n+1$, and the numbers λ in this case are just the eigenvalues of the quadratic form.

Now it is well known that, if a new quadratic form ψ is obtained from ϕ by setting the last variable equal to 0, the eigenvalues of ψ separate those of ϕ. We are led to consider the relationship of the generalized eigenvalues of a function f on a manifold M with those of the restriction, g, of f to a submanifold N. When M is P^n and N is P^{n-1}, the expected generalization of the above result is obtained: the generalized eigenvalues of g separate those of f. It is by no means the same if M is the product of n circles and N the product of $n-1$ of them, although in both cases M is of dimension n and category $n+1$, N is of dimension $n-1$ and category n, and the inclusion map from N to M induces monomorphisms in homology mod 2. On reflexion, the matter is seen to turn on the nature of $M-N$; it is because $P^n - P^{n-1}$ is contractible that the above result is true (and easy to prove) in this case.

This leads to the question, what manifolds M have a submanifold N (of any dimension) such that $M-N$ is contractible? It is with this question that the present paper deals.

2. *Statement of results.* Let M be a compact manifold of dimension n, and N a compact manifold of dimension $n-k$ imbedded in M, where $1 \leqslant k \leqslant n$. (In this paper, 'manifold' means 'connected manifold without boundary'.) We suppose M and N to be separately triangulable at all times; when we further want the pair (M, N) to be triangulable we shall explicitly say so.

We can dispose at once of the case $k = n$. In this case N is a point, and if $M-N$ is contractible, M has the homotopy type of a closed spherical neighbourhood of N with its boundary identified to a point, that is M has the homotopy type of S^n.

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Conversely, if M has the homotopy type of S^n and N is a point, $M-N$ is contractible, being clearly simply connected and being homologically trivial by Poincaré duality. In future we shall suppose $1 \leqslant k \leqslant n-1$.

If $K_m = K^{2m}$ is complex projective space of m complex and $2m$ topological dimensions, and if $Q_m = Q^{4m}$ is quaternionic projective space of m quaternionic and $4m$ topological dimensions, we have the following simple cases in which $M-N$ is contractible: $M = P^n$, K^n, or Q^n and $N = P^{n-1}$, K^{n-2}, or Q^{n-4}, respectively, n being even in the second case and divisible by 4 in the third. In each of these cases $M-N$ is R^n.

Our results are summed up in the following two theorems.

THEOREM 1. *If $M-N$ is contractible, then n is a multiple of k and k is a power of 2. If it be further given that the pair (M, N) is triangulable, then there is a map $S^{2k-1} \to S^k$ of Hopf invariant* 1.†

THEOREM 2. *If (M, N) is triangulable and $M-N$ is contractible, then*

(i) *if $k = 1$, M has the homotopy type of P^n;*

(ii) *if $k = 2$, M has the homotopy type of K^n.*

Note. I have not been able to show, under these hypotheses alone, that, if $k = 4$, M has the homotopy type of Q^n.

3. *The cohomology ring* mod 2 *of* M. We use the Poincaré duality theorem and its extensions as described in (5), p. 140. In this section the coefficient group is Z_2, the cyclic group of order 2.

$M-N$ being contractible, it follows from the exact homology sequence of $(M, M-N)$ that the injection $H_r(M) \to H_r(M, M-N)$ is an isomorphism for $r \neq 0$; if $i\colon N \to M$ is the inclusion, it follows by duality that $i^*\colon H^r(M) \to H^r(N)$ is an isomorphism for $r \neq n$. Let p_r be the rth Betti number mod 2 of M, and so also of N if $r \neq n$. Then duality in M and in N respectively implies

$$p_r = p_{n-r} \quad \text{for} \quad \text{all } r,$$

and

$$p_r = p_{n-k-r} \quad \text{for} \quad r \neq -k \text{ or } n,$$

whence

$$p_r = p_{r-k} \quad \text{for} \quad r \neq n+k \text{ or } 0.$$

Since $p_r = 0$ for $r < 0$ or $r > n$, while $p_0 = p_n = 1$, we deduce that

$$p_0 = p_k = p_{2k} = \ldots = p_{rk} = \ldots = 1, \quad \text{for} \quad rk \leqslant n,$$

while all other p_r are zero. Thus $n = mk$ for some integer m, proving the first part of Theorem 1.

If z_r $(r = 0, \ldots, m)$ represents the non-zero element of $H^{kr}(M)$, then $w_r = i^*z_r$ is the non-zero element of $H^{kr}(N)$ for $0 \leqslant r \leqslant m-1$. The orthogonality of the cup-product pairing of $H^r(M)$ and $H^{n-r}(M)$ to $H^n(M)$ implies $z_r z_{m-r} = z_m$, juxtaposition denoting the cup-product; and similarly $w_r w_{m-1-r} = w_{m-1}$. We can write the latter equation as

† *Note added in proof.* At the time of submitting this paper the author was not aware of the results of Adams, 'On the non-existence of elements of Hopf invariant one', *Ann. Math. Princeton*, 72 (1960), 20–104, announced earlier in *Bull. Amer. Math. Soc.* 64 (1958), 279–82. These results enable Theorem 1 to be sharpened to: *If $M-N$ is contractible, then n is a multiple of k and $k = 1, 2, 4$ or 8.* This may be deduced at once at the end of § 3 of the present paper by replacing the reference to Adem (1) equation 1·2 by one to Adams, *Ann. Math. Princeton*, 72 (1960), 20–104, §1·2.

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$i^*(z_r z_{m-1-r}) = i^* z_{m-1}$, for $0 \leqslant r \leqslant m-1$; i^* being isomorphic in these dimensions, we have $z_r z_{m-1-r} = z_{m-1}$ for $0 \leqslant r \leqslant m-1$. Then $z_1 z_r z_{m-1-r} = z_1 z_{m-1} = z_m$. Thus $z_1 z_r = z_{r+1}$ for $0 \leqslant r \leqslant m-1$. It follows that $z_r = z_1^r$ for $0 \leqslant r \leqslant m$, and so the cohomology ring of M is the ring of polynomials in z_1, modulo z_1^{m+1}.

From the results announced in (1) and proved, for example, in (2), it now follows that k is a power of 2. (See, in particular, (1), equation (1·2).) This proves the second assertion of Theorem 1.

4. *The neighourhood of N in M.* We shall suppose henceforth that (M, N) is a triangulable pair. Let C^∞-differentiable structures be given to M and N, so that N is a C^∞-submanifold of M.

For some sufficiently small ϵ the geodesics in M of length ϵ drawn normal to N from each point of N form a closed neighbourhood K of N in M; their end-points form its boundary L, and we denote $\overline{M-K}$ by J. J is a deformation retract of $M-N$ and so is contractible; N is a deformation retract of K. K and L are locally trivial fibre spaces over N with fibres E^k and S^{k-1} respectively, E^k being the k-disk. K, L, and J are triangulable.

LEMMA 3. *L is connected.*

Since L is a fibre space over the connected complex N, with fibre S^{k-1} which is connected if $k > 1$, the lemma is true if $k > 1$; for $k = 1$, L is a double covering of N which might, *a priori*, consist of two disjoint copies of N.

Since J is contractible, M has the homotopy type of M/J, that is, of M with J identified to a point; which is the same thing as K/L.

If, in the case $k = 1$, L were disconnected (that is, if N were two-sided in M), K would be a cylinder on N, L consisting of the top and bottom of this cylinder. K/L would then be homeomorphic to the union of two cones on N, their bases stuck together and their vertices identified.

Such a space, the union of two contractible subcomplexes, must have a trivial cohomology ring; for two cohomology classes of dimensions > 0 can be realized by two cocycles, the one vanishing on the simplexes of one subcomplex and the other on the simplexes of the other subcomplex, so that their product is zero.

But the cohomology ring of M is not trivial, and so the lemma is true.

LEMMA 4. *M is simply connected if $k > 1$, and $\pi_1(M) = Z_2$ if $k = 1$.*

Since $M = K \cup J$, while $K \cap J$ is the connected subcomplex L, $\pi_1(M)$ is the quotient group of the free product of $\pi_1(K)$ and $\pi_1(J)$ obtained by identifying the images in $\pi_1(K)$ and in $\pi_1(J)$ of each element of $\pi_1(L)$. J being contractible, this means

$$\pi_1(M) = \pi_1(K)/[\operatorname{im} \pi_1(L)],$$

the map $\pi_1(L) \to \pi_1(K)$ being induced by inclusion, and $[\operatorname{im} \pi_1(L)]$ denoting the normal subgroup generated by $\operatorname{im} \pi_1(L)$. N being a deformation retract of K, this becomes $\pi_1(M) = \pi_1(N)/[\operatorname{im} \pi_1(L)]$, $\pi_1(L) \to \pi_1(N)$ being induced by the projection. From the exact sequence of the fibre space L, which contains the section

$$\pi_1(L) \to \pi_1(N) \to \pi_0(S^{k-1}) \to 0,$$

we have $\pi_1(M) = \pi_0(S^{k-1})$, proving the lemma.

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Note. I do not know whether Lemma 4 remains true if M is replaced by N. It is easy to see that L, as the boundary of the contractible manifold-with-boundary J, is an homology $(n-1)$-sphere. It then follows that Lemma 4 is true of N if and only if $\pi_1(L) = 0$. It is possible that the boundary of a contractible manifold-with-boundary is necessarily simply connected, but I know of no proof of this.

5. *The integral cohomology ring of M, for $k > 1$.* For $k > 1$, M is orientable since simply connected; N is also orientable since $H^{n-k-1}(N; Z_2)$ is zero. Thus we can repeat the argument of § 3 with the infinite cyclic group Z or the cyclic group of Z_p of order p as coefficients. We find that i^*: $H^r(M; Z) \to H^r(N; Z)$ is an isomorphism for $r \neq n$, and that $H^{kr}(M; Z) = Z$ for $0 \leqslant r \leqslant m$, all the other integral cohomology groups of M being zero.

If z_r is a generator of $H^{kr}(M; Z)$ for $0 \leqslant r \leqslant m$ and $w_r = i^* z_r$ for $0 \leqslant r \leqslant m-1$, the duality theorems imply $z_r z_{m-r} = \pm z_m$ and $w_r w_{m-1-r} = \pm w_{m-1}$. We find as before that, after proper choices of the signs of the generators z_r, $z_1^r = z_r$ for $0 \leqslant r \leqslant m$. Thus the integral cohomology ring of M is the ring of polynomials in z_1 modulo z_1^{m+1}.

6. *The existence of a map $S^{2k-1} \to S^k$ of Hopf invariant* 1. We may suppose $k > 1$. The final assertion of Theorem 1 follows from the following two lemmas.

LEMMA 5. *If a simply connected complex M has the cohomology ring described in* § 5, *there exists a complex $Y = S^k \cup e^{2k}$, obtained by attaching a $2k$-cell to S^k, with $z^2 = u$, where z and u are generators of $H^4(Y; Z)$ and $H^8(Y; Z)$, respectively.*

LEMMA 6. *If z and u generate $H^4(Y; Z)$ and $H^8(Y; Z)$, where $Y = S^k \cup e^{2k}$ and the attaching map $S^{2k-1} \to S^k$ for e^{2k} has Hopf invariant γ, then $z^2 = \gamma u$.*

The mod 2 version of Lemma 6 is stated in (1), p. 722, but the proof is not given.†

Proof of Lemma 5. By the Hurewicz theorem, the first non-zero homotopy group of M is $\pi_k(M) = Z$. Let f: $S^k \to M$ be a cellular map representing a generator of $\pi_k(M)$, and let C be the mapping cylinder of f. The injection $H_k(S^k) \to H_k(C)$ is an isomorphism, and so the first non-vanishing homology group of (C, S^k) is $H_{2k}(C, S^k) = H_{2k}(C) = Z$. (The coefficients here, and throughout this section, are integral.) By the Hurewicz theorem, the first non-vanishing homotopy group of (C, S^k) is $\pi_{2k}(C, S^k) = Z$. Let g: E^{2k}, $S^{2k-1} \to C, S^k$ represent a generator of $\pi_{2k}(C, S^k)$, and define Y as $S^k \cup e^{2k}$, e^{2k} being attached by $g|S^{2k-1}$.

If p: E^{2k}, $S^{2k-1} \to Y$, S^k is the identification map, so that $p|S^{2k-1} = g|S^{2k-1}$, g defines a map h: $Y, S^k \to C, S^k$ such that $hp = g$. It is clear that $h|S^k = 1$.

Since p_*: $H_{2k}(E^{2k}, S^{2k-1}) \to H_{2k}(Y, S^k)$ is an isomorphism

and g_*: $H_{2k}(E^{2k}, S^{2k-1}) \to H_{2k}(C, S^k)$ is an isomorphism,

h_*: $H_r(Y, S^k) \to H_r(C, S^k)$ is an isomorphism for $r = 2k$ and so for $r < 3k$. Further $h|S^k$: $S^k \to S^k$ induces homology isomorphisms in all dimensions, and so, by the five-

† *Note added in proof.* This lemma is, of course, due to Steenrod, 'Cohomology Invariants of Mappings', *Ann. Math. Princeton*, 50 (1949), 954–88. At the time of submitting this paper the author could not recall where, or whether, he had seen a proof, and devised the one given here; which, however, is perhaps not without interest, proving as it does directly and simply the equivalence of the two most used definitions of the Hopf invariant without referring each back to Hopf's original geometric definition.

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lemma, $h_*: H_r(Y) \to H_r(C)$ is an isomorphism for $r < 3k$. The same is therefore true of $h^*: H^r(C) \to H^r(Y)$, so that Y has the desired cohomology ring, and Lemma 5 is proved.

Proof of Lemma 6. We write S for S^k. Let $\Delta: (Y, S) \to (Y \times Y, S \times S)$ be the diagonal map. Choose a base point b in S, and let Y_1, Y_2, S_1, S_2 denote $Y \times b$, $b \times Y$, $S \times b$, $b \times S$, respectively. Let T denote $S_1 \vee S_2$ and let X denote the $2k$-skeleton of $Y \times Y$, which is $(S \times S) \cup Y_1 \cup Y_2$.

The group $H_{2k}(Y \times Y, Y_1 \vee Y_2)$ is naturally isomorphic to $H_k(Y) \otimes H_k(Y) = Z$. If the cohomology ring of Y is to be given by $z^2 = \gamma u$, the image of a generator of $H_{2k}(Y)$ under the composition of $\Delta_*: H_{2k}(Y) \to H_{2k}(Y \times Y)$ and the injection

$$H_{2k}(Y \times Y) \to H_{2k}(Y \times Y, Y_1 \vee Y_2)$$

must be γ times a generator of this last group.

By the cellular approximation theorem, there is a cellular map f with

$$f \simeq \Delta: (Y, S) \to (Y \times Y, S \times S);$$

because f is cellular, $f(Y) \subset X$ and $f(S) \subset T$. Thus if $h: (Y, S) \to (X, T)$ is the map defined by f and if $i: (X, T) \to (Y \times Y, S \times S)$ is the inclusion, $f = ih$. The map $h|S: S \to T$ is homotopic to, and so by the homotopy extension theorem may be taken to be, the map obtained by pinching the equator of S to a point and mapping its northern and southern hemispheres onto S_1 and S_2 respectively with degrees 1; this is because Δ and the pinching map both represent

$$(1, 1) \in \pi_k(S_1) + \pi_k(S_2) = \pi_k(T) = \pi_k(S \times S).$$

In the diagram

$$H_{2k}(Y) \to H_{2k}(Y, S) \xrightarrow[h_*]{} H_{2k}(X, T) \xrightarrow[j_*]{} H_{2k}(X, Y_1 \vee Y_2) \to H_{2k}(Y \times Y, Y_1 \vee Y_2),$$

the maps other than h_* being induced by inclusions, the first and the last map are isomorphisms, and so the image under $j_* h_*$ of a generator of $H_{2k}(Y, S)$ is γ times a generator of $H_{2k}(X, Y_1 \vee Y_2)$. Using the Hurewicz isomorphism, we pass to homotopy groups:

$$\pi_{2k}(Y, S) \xrightarrow[h_*]{} \pi_{2k}(X, T) \xrightarrow[j_*]{} \pi_{2k}(X, Y_1 \vee Y_2);$$

and the same is true.

If $g: (E^{2k}, S^{2k-1}) \to (Y, S)$ is the attaching map for e^{2k}, then the composition of the maps

$$\pi_{2k}(E^{2k}, S^{2k-1}) \xrightarrow[g_*]{} \pi_{2k}(Y, S) \xrightarrow[h_*]{} \pi_{2k}(X, T) \xrightarrow[j_*]{} \pi_{2k}(X, Y_1 \vee Y_2)$$

maps a generator of the first group onto γ times a generator of the last.

Now, if $r: (Y_1, S_1) \to (X, T)$, $s: (Y_2, S_2) \to (X, T)$, and $t: (S \times S, T) \to (X, T)$, are inclusions, we have

$$H_{2k}(X, T) = r_* H_{2k}(Y_1, S_1) + s_* H_{2k}(Y_2, S_2) + t_* H_{2k}(S \times S, T),$$

r_*, s_* and t_* being monomorphisms and the sum a direct one. By the Hurewicz isomorphism, we may write

$$\pi_{2k}(X, T) = r_* \pi_{2k}(Y_1, S_1) + s_* \pi_{2k}(Y_2, S_2) + t_* \pi_{2k}(S \times S, T).$$

We have $j_* r_* = 0$, $j_* s_* = 0$, and $j_* t_*: \pi_{2k}(S \times S, T) \to \pi_{2k}(X, Y_1 \vee Y_2)$ is an isomorphism—the last statement being true by excision for the homology groups, and so

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for π_{2k}. Thus if ρ, σ, τ generate respectively $\pi_{2k}(Y_1, S_1)$, $\pi_{2k}(Y_2, S_2)$, and $\pi_{2k}(S \times S, T)$, the map hg: $E^{2k}, S^{2k-1} \to X, T$ represents $\alpha r_* \rho + \beta s_* \sigma + \gamma t_* \tau \in \pi_{2k}(X, T)$ for some integers α, β. (In fact $\alpha = \beta = 1$ but we do not need this.)

Now let g', h', r', s' represent $g|S^{2k-1}, h|S, r|S_1$ and $s|S_2$, respectively, and let ∂ be the homotopy boundary operator. Then $h'g'$: $S^{2k-1} \to T$ represents

$$\alpha r'_* \partial\rho + \beta s'_* \partial\sigma + \gamma \partial\tau \in \pi_{2k-1}(T).$$

Referring to (3), Ch. VI, § 4, where the generalized Hopf invariant of g' is defined by just such an expression for $h'g'$, we see that the Hopf invariant of g' is indeed γ.

7. *The case* $k = 1$. Let P be the universal covering manifold of M, which by Lemma 4 is in fact a double covering of M; and let Q be the part of M over N. Two points of Q covering the same point of N can be joined by a path in P whose projection is a loop in M; since $\pi_1(N) \to \pi_1(M)$ is onto by § 4, this loop can be deformed, modulo its end-points, into N; covering this deformation we get a path in Q joining the two points, and so Q is connected. $P - Q$ is a double covering of $M - N$ which, since the latter is contractible, consists of two disjoint copies of $M - N$. P is orientable because simply connected, and Q is orientable because, $P - Q$ having two components, Q is two-sided in P.

With coefficients Z or Z_p for any p, the injection $H_r(P) \to H_r(P, P - Q)$ is an isomorphism for $r \neq 0$ or 1, since $H_r(P - Q) = 0$ for $r \neq 0$. If j: $Q \to P$ is the inclusion, we have by duality j^*: $H^r(P) \to H^r(Q)$ is an isomorphism for $r \neq n$ or $n - 1$. Then for the Betti numbers p_r of P we have

$$p_r = p_{n-r} \quad \text{for all } r$$

and $$p_r = p_{n-1-r} \quad \text{for } r \neq n, n-1, 0 \text{ or } -1.$$

Thus $$p_r = p_{r-1} \quad \text{for } r \neq 0, 1, n, \text{ or } n+1.$$

But $p_1 = 0$ since P is simply connected, whence $p_r = 0$ for $1 \leqslant r \leqslant n - 1$; while $p_0 = p_n = 1$.

Thus P has no torsion, and no non-zero homology groups (with integer coefficients) but $H_0(P) = Z$ and $H_n(P) = Z$; and so P has the homotopy type of S^n.

It remains to show that a homotopy equivalence of S^n and P can be chosen which will, by passage to the quotient, induce one of P^n and M. Regard R^n as imbedded in R^{n+1} by setting the last coordinate equal to zero; this imbeds S^n in S^{n+1} and so P^n in P^{n+1}. Let different points of P covering the same point of M be called antipodal; points of S^n covering the same point of P^n are antipodal in the ordinary sense.

LEMMA 7. *If* f: $S^n \to P$ *sends antipodal points to antipodal points, then* f *has odd degree.*

A proof of Tucker's for maps f: $S^n \to S^n$ applies equally well here; we give it for completeness. We use homology mod 2, and we use a bar over an element to represent the antipodal element. We suppose S_n and P to have symmetric triangulations (obtained for example by lifting triangulations of P^n and M), and we suppose the S^r ($r \leqslant n$) to be subcomplexes of S^n. Let a symmetric simplicial approximation to f be taken, inducing a symmetric chain map f.

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Let a_r, $\overline{a_r}$ be the mod 2 chains of S^n represented by the two hemispheres of $S^r \subset S^n$, and let $f.a_r = b_r \in C_r(P)$. It follows that $f.\overline{a_r} = \overline{b_r}$; and since we clearly have

$$\partial a_r = \partial \overline{a_r} = a_{r-1} + \overline{a_{r-1}},$$

we have also $\partial b_r = \partial \overline{b_r} = b_{r-1} + \overline{b_{r-1}}$; both for $r \geqslant 1$.

We have to prove that the image $b_n + \overline{b_n}$ of the fundamental n-cycle $a_n + \overline{a_n}$ of S^n is the fundamental n-cycle of P. If not, then $b_n + \overline{b_n} = 0$, that is, $\overline{b_n} = b_n$. So b_n is symmetric, say $b_n = c_n + \overline{c_n}$. Then $\partial c_n + \partial \overline{c_n} = b_{n-1} + \overline{b_{n-1}}$; thus $\partial c_n + b_{n-1}$ is symmetric, say $\partial c_n + b_{n-1} = c_{n-1} + \overline{c_{n-1}}$. The induction proceeds, yielding eventually $\partial c_1 + b_0 = c_0 + \overline{c_0}$; but this is impossible, since the mod 2 Kronecker index of the left side is 1 (b_0 being a single point) while that of the right side is 0.

LEMMA 8. *There exist maps $f\colon S^n \to P$, sending antipodal points to antipodal points, of any odd degree.*

We begin by mapping the two points of S^0 to two antipodal points of P. If f is defined on S^r, $r < n$, there is no obstruction to extending f over one of the hemispheres of S^{r+1} bounded by S^r, since $\pi_r(P) = 0$. We then define f on the other hemisphere of S^{r+1} by mapping antipodal points to antipodal points.

If f, g are maps of the kind considered which agree on S^{n-1}, and if E^n_+ and E^n_- are the two hemispheres of S^n bounded by S^{n-1}, then $f|E^n_+$ and $g|E^n_+$ define a difference map $h = (f,g)^+\colon S^n \to P$. It is well known that, given g, f can be so chosen that h represents any given element of $\pi_n(P)$. Similarly $f|E^n_-$ and $g|E^n_-$ determine a difference map $k = (f,g)^-\colon S^n \to P$. Let f, g, h, k represent respectively $\alpha, \beta, \gamma, \delta \in \pi_n(P)$. Then by the sum-theorem for such maps, $\alpha - \beta = \gamma + \delta$. We shall determine δ in terms of γ.

The precise definition of h is

$$h(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_{n+1}) \quad \text{if} \quad x_{n+1} \geqslant 0,$$
$$h(x_1, \ldots, x_{n+1}) = g(x_1, \ldots, -x_{n+1}) \quad \text{if} \quad x_{n+1} \leqslant 0.$$

To define k, we need a rotation of S^n which will interchange E^n_+ and E^n_-; since P is simply connected and so n-simple, we need not have a base-point. Such a rotation is $r\colon S^n \to S^n$ given by $r(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, -x_n, -x_{n+1})$. Then the definition of k is

$$k(x_1, \ldots, x_{n+1}) = fr(x_1, \ldots, x_{n+1}) \quad \text{if} \quad x_{n+1} \geqslant 0,$$
$$k(x_1, \ldots, x_{n+1}) = gr(x_1, \ldots, -x_{n+1}) \quad \text{if} \quad x_{n+1} \leqslant 0.$$

Let $s\colon P \to P$ be the map which interchanges antipodal points. Then

$$k(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, -x_n, -x_{n+1}) \quad \text{if} \quad x_{n+1} \geqslant 0,$$
$$= sf(-x_1, \ldots, -x_{n-1}, x_n, x_{n+1}) \quad \text{if} \quad x_{n+1} \geqslant 0;$$

and

$$k(x_1, \ldots, x_{n+1}) = g(x_1, \ldots, -x_n, x_{n+1}) \quad \text{if} \quad x_{n+1} \leqslant 0,$$
$$= sg(-x_1, \ldots, -x_{n-1}, x_n, -x_{n+1}) \quad \text{if} \quad x_{n+1} \leqslant 0.$$

So

$$k(x_1, \ldots, x_{n+1}) = sh(-x_1, \ldots, -x_{n-1}, x_n, x_{n+1}),$$
$$= sht(x_1, \ldots, x_{n+1}),$$

where $t\colon S^n \to S^n$ is defined by

$$t(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n-1}, x_n, x_{n+1}).$$

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Since the degree of t is $(-1)^{n-1}$, we have $\delta = (-1)^{n-1}\sigma\gamma$, where σ is the degree of s. By the Lefschetz fixed-point theorem, since s has no fixed points, $1+(-1)^n\sigma = 0$, whence $\sigma = (-1)^{n-1}$. Thus $\delta = \gamma$.

So $\alpha - \beta = 2\gamma$; and thus, since γ is at our disposal, f may be chosen to have any odd degree, proving the lemma.

Now to prove the first assertion of Theorem 2, we need only take a map $f\colon S^n \to P$ which maps antipodal points to antipodal points and has degree 1. By a passage to the quotient f defines $g\colon P^n \to M$. g induces an isomorphism of $\pi_1(P^n)$ onto $\pi_1(M)$, since a loop in P^n whose covering arc interchanges the levels in S^n is mapped into a loop in M whose covering arc interchanges the levels in P. The homotopy groups of P^n and M vanish for dimensions 2 to $n-1$, and g induces a homotopy isomorphism in dimension n because f does so. Thus by Whitehead's theorem g is a homotopy equivalence.

8. *The case $k = 2$.* If $k = 2$, and $n = 2m$ say, then $\pi_1(M) = 0$, $H^r(M; Z) = 0$ for r odd, and $H^{2r}(M; Z) = Z$, generated by z_r say; and $z_1^r = z_r$. From these facts alone we can show that M has the homotopy type of K_m.

Now $\pi_2(K_m) = Z$ and $\pi_r(K_m) = 0$ for $r = 1$ and for $3 \leqslant r \leqslant 2m = n$. (See, for example, (3), Ch. V, Theorem 2·6). Further, since $K_m - K_{m-1}$ is contractible, K_m has the same cohomology ring as M.

Let M^r denote the r-skeleton of M. We construct a homotopy equivalence $f\colon M \to K_m$. Choose a base point $*$ in K_m and let $f(M^1) = *$. Let c be an integral two-cocycle of M whose cohomology class is z_1. If σ is an oriented 2-simplex of M, define f on σ so that $f\colon (\sigma, \dot{\sigma}) \to (K_m, *)$ represents $c(\sigma)\alpha \in \pi_2(K_m, *)$, α being a chosen generator of the latter group. The obstruction to extending f over M^3 is zero since $\delta c = 0$, and then there is no obstruction to extending f over M since $\pi_r(K_m) = 0$ for $3 \leqslant r \leqslant n$. We now prove that $f\colon M \to K_m$ is a homotopy equivalence.

Let w_r be generators of $H^{2r}(K_m; Z)$ with $w_r = w_1^r$. We can take a cell decomposition of K_m of the form $K_m = S^2 \cup e^4 \cup \ldots \cup e^{2m}$, where S^2 is in fact K_1. f can be replaced by a homotopic cellular map without altering f on M^2 where it is already cellular. α can be represented by the inclusion $S^2 \to K_m$. Then, if u is the generator of the 2-cochains of K_m, f induces the cochain map $f^\cdot$ with $(f^\cdot u)(\sigma) = c(\sigma)$, so that $f^\cdot u = c$, and thus $f^* w_1 = z_1$.

It now follows that $f^* w_r = f^*(w_1^r) = (f^* w_1)^r = z_1^r = z_r$, so that f induces cohomology isomorphisms in all dimensions. M and K_m being simply connected, it follows from Whitehead's theorem that f is a homotopy equivalence.

9. *Remarks on the case $k = 4$.* I have not been able to show that, if $k = 4$, M has the homotopy type of Q^n. Certainly this does not follow from the known cohomology ring of M and the fact that M is simply connected, in contrast to the case $k = 2$.

To see this, suppose that M is of dimension 8, simply connected, with $H^0(M; Z)$, $H^4(M; Z)$, and $H^8(M; Z)$ cyclic infinite with generators respectively 1, z, and u, where $z^2 = u$, the other cohomology groups being zero.

As in the proof of Lemma 5, we construct from M a space $Y = S^4 \cup e^8$ which, in the present case, is in fact of the same homotopy type as M.

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If $g: S^7 \to S^4$ is the attaching map for e^8, the homotopy type of Y is determined by the homotopy class of g. Y will have the given cohomology ring if and only if g has Hopf invariant 1 (by Lemma 6). Now $\pi_7(S^4) = \pi_7(S^7) + \pi_6(S^3) = Z + Z_{12}$; and if α and β generate Z and Z_{12}, respectively, the Hopf invariant of $m\alpha + n\beta$ is m.

So Y can have the right cohomology ring if g belongs to any of the classes $\pm\alpha + n\beta$, $0 \leqslant n \leqslant 11$.

On the other hand, if Y_1 and Y_2 are two spaces of the form $S^4 \cup e^8$, with attaching maps respectively g_1 and g_2, with classes γ_1 and $\gamma_2 \in \pi_7(S^4)$, they are of the same homotopy type only if $\gamma_1 = \pm\gamma_2$ or $\gamma_1 = \pm(-\iota) \circ \gamma_2$, where ι is the class of the identity map $S^4 \to S^4$.

For if $h: Y_1 \to Y_2$ is a homotopy equivalence, we can suppose it cellular, and then $h: Y_1, S^4 \to Y_2, S^4$. If $h': Y_2 \to Y_1$ is a homotopy inverse of h, we can take it to be cellular and also take the homotopies $hh' \simeq 1$ and $h'h \simeq 1$ to be cellular. We then see that $h: (Y_1, S^4) \to (Y_2, S^4)$ is in fact a homotopy equivalence of pairs. In the diagram

$$\begin{array}{ccc} \pi_8(Y_1, S^4) & \xrightarrow{\partial} & \pi_7(S^4) \\ \downarrow h_* & & \downarrow (h|S^4)_* \\ \pi_8(Y_2, S^4) & \xrightarrow[\partial]{} & \pi_7(S^4) \end{array}$$

if ϵ_1 and ϵ_2 generate $\pi_8(Y_1, S^4)$ and $\pi_8(Y_2, S^4)$, we have $\partial\epsilon_1 = \gamma_1$, $\partial\epsilon_2 = \gamma_2$, and, because h_* is an isomorphism, $h_*\epsilon_1 = \pm\epsilon_2$. Thus $(h|S^4)_*\gamma_1 = \pm\gamma_2$. But since $h|S^4$ is a homotopy equivalence, its class is $\pm\iota$, so that either $\gamma_1 = \pm\gamma_2$ or $(-\iota) \circ \gamma_1 = \pm\gamma_2$.

So, among the 24 classes γ giving the right cohomology ring, there are at least 6 different homotopy types.

It may be possible to show that M has the homotopy type of Q^n by using the further facts that M has the homotopy type of K/L, while L is an homology sphere and a $(k-1)$-sphere bundle over N, but it seems difficult to extract much information without knowing that the base space N is simply connected.

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The University of Sydney
Australia

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