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Critical Exponents, Colines, and Projective Geometries

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1. A conjecture about projective geometries

In [9, p. 469], Oxley made the following conjecture, which is a geometric analogue of a conjecture of Lovász (see [1, p. 290]) about complete graphs.

Conjecture 1.1. *Let G be a rank-n $\mathrm{GF}(q)$-representable simple matroid with critical exponent $n-\gamma$. If, for every coline X in G, $c(G|X;q) = c(G;q) - 2 = n-\gamma-2$, then G is the projective geometry* $\mathrm{PG}(n-1,q)$.

We shall call the rank n, the critical 'co-exponent' γ, and the order q of the field the *parameters* of Oxley's conjecture. We exhibit several counterexamples to this conjecture. These examples show that, for a given prime power q and a given positive integer γ, Oxley's conjecture holds for only finitely many ranks n. We shall assume familiarity with matroid theory and, in particular, the theory of critical problems. See [6] and [9].

A subset C of points of $\mathrm{PG}(n-1,q)$ is a (γ,k)-*cordon* if, for every k-codimensional subspace X in $\mathrm{PG}(n-1,q)$, the intersection $C \cap X$ contains a γ-dimensional subspace of $\mathrm{PG}(n-1,q)$. In this paper, our primary interest will be in constructing $(\gamma,2)$-cordons. With straightforward modifications, our methods will also yield (γ,k)-cordons.

Complements of counterexamples to Oxley's conjecture are $(\gamma,2)$-cordons.

Theorem 1.2. *The complement* $\mathrm{PG}(n-1,q)\backslash C$ *is a counterexample to Oxley's conjecture with parameters n, γ, and q if and only if the set C of points is a $(\gamma,2)$-cordon containing no $(\gamma+1)$-dimensional subspaces.*

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Proof. The critical exponent of the complement $\mathrm{PG}(n-1,q)\backslash C$ is the minimum co-dimension of a subspace contained in C. Because C contains γ-dimensional subspaces but no $(\gamma+1)$-dimensional subspaces, $\mathrm{PG}(n-1,q)\backslash C$ has critical exponent $n-\gamma$. Furthermore, as C is a $(\gamma,2)$-cordon, every coline of $\mathrm{PG}(n-1,q)\backslash C$ has critical exponent $n-\gamma-2$. Since C is non-empty, $\mathrm{PG}(n-1,q)\backslash C$ is a counterexample to Oxley's conjecture. □

The rest of this paper is devoted to finding or constructing cordons.

2. Cordons

A $(1,2)$-cordon is simply a 2-block, that is, a subset of $\mathrm{PG}(n-1,q)$ with critical exponent greater than 2. When $q \geqslant 3$ and $m \geqslant q^2+1$, the cycle matroid $M(K_m)$ has critical exponent at least 3. Moreover, all its lines have at most three points. Hence, when $m \geqslant q^2+1$, $M(K_m)$ is a $(1,2)$-cordon containing no $(q+1)$-point lines. Similarly, when $q \geqslant 3$ and $m \geqslant q+2$, the jointless Dowling geometries $Q'_m(\mathrm{GF}(q)^\times)$ are $(1,2)$-cordons containing no $(q+1)$-point lines ([10]; see also [6, Section 8.5]). When $q=2$, the cycle matroid $M(\Gamma)$, where Γ is a graph with no graphical triangles that is not 4-colourable, is a 1-cordon containing no lines. More specifically, one can take Γ to be the graph Γ_{23} on 23 vertices obtained from Mycielski's construction [8]. The idea of constructing examples by taking complements of blocks is also used in [7].

Although $(\gamma,2)$-cordons are $(\gamma+1)$-blocks, the converse does not hold in general. One exception is the tangential block $\mathrm{PG}(n-1,q)$. Because every coline intersects a $(\gamma+2)$-dimensional subspace at a subspace of dimension at least γ, a point set isomorphic to $\mathrm{PG}(m-1,q)$ is a $(\gamma,2)$-cordon when $m \geqslant \gamma+2$. In the rest of this section, we shall show how simple modifications of the idea behind the known infinite families of tangential blocks yield cordons.

We begin with cordons constructed by modifying $M(K_m)$. Let $e_1, e_2, \ldots, e_n$ be a basis in $\mathrm{PG}(n-1,q)$. For $m \leqslant n$, let $B(s,m;q)$ be the set of points in $\mathrm{PG}(n-1,q)$ of the form $c_1e_1 + c_2e_2 + \cdots + c_me_m$, where at most s of the coordinates $c_i, 1 \leqslant i \leqslant m$, are nonzero. In other words, $B(s,m;q)$ is the (Hamming) weight-s Dowling geometry on the first m coordinates embedded in rank-n projective space (see [2], [3], and [6, Section 6]). Let H be the hyperplane defined by the *overall parity check equation*:

$$x_1 + x_2 + \cdots + x_m + x_{m+1} + \cdots + x_n = 0,$$

where $x_1, x_2, \ldots, x_n$ are the coordinates of the point x. Finally, let $C(t,m;q)$ be the set of points in $\mathrm{PG}(n-1,q)$ defined by

$$C(t,m;q) = B(t+1,m;q) \cap H.$$

When $t=1$, $C(t,m;q)$ is the set of points in the 'standard' representation of $M(K_m)$. For higher values of t, $C(t,m;q)$ is a 'thickened' $M(K_m)$. To be precise, let $\mathrm{lin}(i_1, i_2, \ldots, i_{t+1})$ be the t-dimensional subspace spanned by the points

$$e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, \ldots, e_{i_{t-1}} - e_{i_t}, e_{i_t} - e_{i_{t+1}}.$$

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Then, $C(t,m;q)$ is the union of subspaces

$$\bigcup_{1\leqslant i_1<i_2<\cdots<i_t<i_{t+1}\leqslant m} \operatorname{lin}(i_1,i_2,\ldots,i_t,i_{t+1}),$$

the union being over all increasing sequences $i_1,i_2,\ldots,i_t,i_{t+1}$, with $i_{t+1}\leqslant m$. To see this, observe that the points e_i-e_j span H and a typical point in $B(t+1,m;q)\cap H$, say, $c_1e_1+c_2e_2+\cdots+c_te_t+c_{t+1}e_{t+1}$, can be rewritten in the form

$$\begin{aligned}&c_1(e_1-e_2)+(c_1+c_2)(e_2-e_3)+\cdots\\&\quad+(c_1+c_2+\cdots+c_{t-1})(e_{t-1}-e_t)+(c_1+c_2+\cdots+c_{t-1}+c_t)(e_t-e_{t+1})\end{aligned}$$

using the equation $c_1+c_2+\cdots+c_t=-c_{t+1}$.

When $q\geqslant 3$, the sets $C(t,m;q)$ are different for different parameters t, m, and q. In the binary case, being in H is equivalent to having even weight. Thus, $C(2s-1,m;2)=C(2s,m;2)$.

Theorem 2.1. *If $n\geqslant m\geqslant \gamma q^2+1$, then $C(\gamma,m;q)$ is a $(\gamma,2)$-cordon in* $\mathrm{PG}(n-1,q)$.

Proof. A coline X can be defined as the set of points satisfying two linearly independent linear equations

$$\begin{aligned}a_1x_1+a_2x_2+\cdots+a_nx_n&=0\\ b_1x_1+b_2x_2+\cdots+b_nx_n&=0.\end{aligned}\tag{2.1}$$

Because $m\geqslant \gamma q^2+1$, there exist, by the pigeonhole principle, at least $\gamma+1$ coordinates $i_1,i_2,\ldots,i_\gamma,i_{\gamma+1}$, $1\leqslant i_j\leqslant m$, such that $a_{i_1}=a_{i_2}=\cdots=a_{i_\gamma}=a_{i_{\gamma+1}}$ and $b_{i_1}=b_{i_2}=\cdots=b_{i_\gamma}=b_{i_{\gamma+1}}$. Thus, all the points in the γ-dimensional subspace $\operatorname{lin}(i_1,i_2,\ldots,i_\gamma,i_{\gamma+1})$ are in X. □

In addition to $\mathrm{PG}(\gamma+1,q)$ and $C(s,m;q)$, we construct two other families of $(\gamma,2)$-cordons. One family consists of the weight-$(\gamma+1)$ Dowling geometry $B(\gamma+1,m;q)$. The other is the family $B'(\gamma+1,m;q)$ of *jointless weight-$(\gamma+1)$ Dowling geometries*, where $B'(\gamma+1,m;q)$ is the set obtained by deleting the basis points $e_1,e_2,\ldots,e_m$ from $B(\gamma+1,m;q)$. As will be shown in Section 3, only the jointless Dowling geometries yield counterexamples to Oxley's conjecture.

Theorem 2.2. *If $n\geqslant m\geqslant \gamma(q+1)+1$, then $B(\gamma+1,m;q)$ is a $(\gamma,2)$-cordon in* $\mathrm{PG}(n-1,q)$.

Proof. Writing $a/0=\infty$, we partition the set $\{1,2,\ldots,m\}$ of coordinate positions in the system (2.1) of equations into $q+2$ mutually disjoint subsets, P and P_λ, $\lambda\in \mathrm{GF}(q)\cup\{\infty\}$. (These subsets may be empty.) The set P is $\{i:a_i=b_i=0\}$, the set of positions where the ratio is indeterminate. The other $q+1$ subsets are defined by

$$\begin{aligned}P_0&=\{i:a_i=0,b_i\neq 0\},\\ P_\infty&=\{i:a_i\neq 0,b_i=0\},\\ P_\lambda&=\{i:a_i/b_i=\lambda\},\end{aligned}$$

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where λ is a nonzero element in GF(q). Because $m \geqslant \gamma(q+1)+1$, at least one of the $q+1$ subsets $P \cup P_\lambda$, $\lambda \in \mathrm{GF}(q) \cup \{\infty\}$, contains at least $\gamma+1$ positions. By the pigeonhole principle, there are positions $i_1, i_2, \ldots, i_\gamma, i_{\gamma+1}$ such that the two equations

$$\begin{aligned} a_{i_1}x_{i_1} + a_{i_2}x_{i_2} + \cdots + a_{i_\gamma}x_{i_\gamma} + a_{i_{\gamma+1}}x_{i_{\gamma+1}} &= 0 \\ b_{i_1}x_{i_1} + b_{i_2}x_{i_2} + \cdots + b_{i_\gamma}x_{i_\gamma} + b_{i_{\gamma+1}}x_{i_{\gamma+1}} &= 0 \end{aligned} \tag{2.2}$$

are linearly dependent. Hence, the points satisfying the linear equations (2.1) and $x_j = 0$ if j is not one of the positions $i_1, i_2, \ldots, i_\gamma, i_{\gamma+1}$, form a γ-dimensional subspace in $B(\gamma+1, m; q)$ contained in X. □

Theorem 2.3. *If $n \geqslant m \geqslant \gamma(q+2)+1$, then $B'(\gamma+1, m; q)$ is a $(\gamma, 2)$-cordon in* PG($n-1, q$).

Proof. Observe that, because $m \geqslant \gamma(q+2)+1$, one of the $q+2$ subsets defined earlier has size at least $\gamma+1$. Hence, there are positions $i_1, i_2, \ldots, i_\gamma, i_{\gamma+1}$ such that the two equations (2.2) are linearly dependent. If that set is one of the sets P_λ, then (2.2) is equivalent to one equation with $\gamma+1$ unknowns in which all the $\gamma+1$ coefficients are nonzero. Hence, the points satisfying the equations (2.1) and $x_j = 0, j \neq i_1, i_2, \ldots, i_\gamma, i_{\gamma+1}$ form a γ-dimensional subspace U in X. Furthermore, as all the coefficients in the equation equivalent to (2.2) are nonzero, every nonzero solution has at least two nonzero coordinates. In particular, no basis point e_i is in U, and hence U is in $B'(\gamma+1, m; q)$.

If that set is P, then (2.2) is a system of two identically zero equations. Hence, the $(\gamma+1)$-dimensional subspace V spanned by $e_{i_1}, e_{i_2}, \ldots, e_{i_\gamma}, e_{i_{\gamma+1}}$ is in X. Its intersection $V \cap H$ with the hyperplane H defined by the overall parity check equation (or any equation in which all the coefficients of $x_i, 1 \leqslant i \leqslant m$, are nonzero) contains no basis points e_i. Hence, $V \cap H$ is a γ-dimensional subspace in both X and $B'(\gamma+1, m; q)$. □

3. Subspaces in higher-weight Dowling geometries

In this section, we shall show that, except when $q = 2$ and γ is odd, the sets $C(\gamma, m; q)$ contain no $(\gamma+1)$-dimensional subspaces. To do this, we use the following result about subspaces in the weight-s Dowling geometry $B(s, m; q)$. This result is implicit in Bonin [2]. The *support* of a point in PG($n-1, q$) is the set $\{i : x_i \neq 0\}$ of (coordinate) positions. The *support* of a set A of points is the union of the supports of the points in A.

Theorem 3.1.

(a) *For all prime powers q, $B(s, m; q)$ contains no $(s+1)$-dimensional subspaces.*

(b) *For $q \geqslant 3$, every s-dimensional subspace in $B(s, m; q)$ has a basis consisting of s basis points. In other words, every s-dimensional subspace is the subspace of all the points whose support lies in a given s-element set of positions.*

(c) *Every s-dimensional subspace U in $B(s, m; 2)$ is associated with a collection S_0, S_1, $S_2, \ldots, S_k$, of mutually disjoint subsets of positions in $\{1, 2, \ldots, m\}$, satisfying the conditions*

(1) *for $1 \leqslant j \leqslant k$, $|S_j|$ is odd and at least 3,*

(2) $|S_0| + |S_1| + |S_2| + \cdots + |S_k| = s + k$, *and*

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(3) *U is the subspace of all the points in* $\mathrm{PG}(n-1,q)$ *satisfying the equations: for* $1 \leqslant j \leqslant k$,

$$\sum_{i:i\in S_j} x_i = 0,$$

and $x_j = 0$ *for every position j not in the union* $S_0 \cup S_1 \cup S_2 \cup \ldots \cup S_k$.

In the terminology of coding theory, s-dimensional subspaces in $B(s,m;2)$ are linear codes with s information bits and parity checks on disjoint even subsets of information bits.

For example, there are four types of subspace in $B(4,m;q)$. They can be described as follows:

Basis	Parity check equations
e_1, e_2, e_3, e_4	none
$e_1+e_5, e_2+e_5, e_3, e_4$	$x_1+x_2=x_5$
$e_1+e_5, e_2+e_5, e_3+e_5, e_4+e_5$	$x_1+x_2+x_3+x_4=x_5$
$e_1+e_5, e_2+e_5, e_3+e_6, e_4+e_6$	$x_1+x_2=x_5, x_3+x_4=x_6$.

Proof of Theorem 3.1. Let U be a subspace in $B(s,m;q)$ and let M be a rectangular matrix whose rows form a basis for U. By row operations (that is, taking linear combinations of basis points) and relabelling coordinate positions, we can suppose that $M=(I|N)$, where I is an identity matrix. If U has dimension $s+1$, then adding the basis points yields a point having weight at least $s+1$, a contradiction. This proves part (a).

If $q \geqslant 3$, M has s rows, and N has a nonzero entry, then it is easy to find a linear combination of the basis points with coefficients ± 1 having weight at least $s+1$. This proves part (b).

Let U be an s-dimensional subspace in $B(s,m;2)$ and let M be an $s \times m$ matrix of the form $(I|N)$ constructed from a basis of U. If we sum all the rows of M, we obtain a row vector r in which the first s coordinates equal 1. Since U has weight at most s, all the remaining coordinates in r equal 0. Thus, the sum of all the entries in every column of the submatrix N is 0, or equivalently, every column of N contains an even number of 1 s.

Next, we will show that there is at most one 1 in any row of N. Suppose this is false and the first row, say, of N contains at least two 1 s. Let r_1 be the first row of M. When we take the sum $r+r_1$, we remove one 1 (the first coordinate of r) and insert at least two 1 s (those 1 s in the first row of N). Hence, $r+r_1$ has weight at least $s+1$, a contradiction.

Finally, let $c_1, c_2, \ldots, c_k$ be the indices of all the nonzero columns N and let S'_j be the set of row indices of the entries which are 1 s in the c_jth column. Let S_0 be the indices of all the rows in M with exactly one 1 and, for $1 \leqslant j \leqslant k$, let $S_j = S'_j \cup \{c_j\}$. Then the sets $S_0, S_1, \ldots, S_k$ satisfy the conditions given in Theorem 3.1 and uniquely determine the subspace U. This completes the proof of part (c). □

Corollary 3.2. *Suppose that $q \geqslant 3$, or $q=2$ and γ is even. Then,*

(a) *$C(\gamma,m;q)$ contains no $(\gamma+1)$-dimensional subspace in* $\mathrm{PG}(n-1,q)$, *and*

(b) *$B'(\gamma+1,m;q)$ contains no $(\gamma+1)$-dimensional subspace in* $\mathrm{PG}(n-1,q)$.

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Proof. By Theorem 3.1, a $(\gamma+1)$-subspace in $B(\gamma+1,m;q)$ contains a point not in the hyperplane H defined by the overall parity check equation. This proves part (a). By Theorem 3.1 again, $(\gamma+1)$-dimensional subspaces in $B(\gamma+1,m;q)$ contain at least one basis point and, hence, are not contained in $B'(\gamma+1,m,q)$. This proves part (b). □

Combining our results so far, we have found or constructed $(\gamma,2)$-cordons containing no $(\gamma+1)$-dimensional subspaces for the following parameters:

$$q \geqslant 3, \quad n \geqslant \gamma(q+2)+1;$$
$$q = 2, \quad \gamma = 1,\ n \geqslant 22;$$
$$q = 2, \quad \gamma \text{ even}, n \geqslant 4\gamma+1.$$

It remains to consider the cases when $q=2$ and γ is an odd integer greater than 1.

4. Cat's cradles

If C and D are two sets of points in $\mathrm{PG}(n-1,q)$ with disjoint supports, then the *cat's cradle* $C * D$ is the union of all the lines of the form $x \vee y$, where $x \in C$ and $y \in D$. Cat's cradles are generalizations of q-lifts [11]. We shall use cat's cradles to construct new cordons from old.

Lemma 4.1. *Let C be a (γ,k)-cordon and D be a (δ,k)-cordon in* $\mathrm{PG}(n-1,q)$ *with disjoint supports. Then $C * D$ is a $(\gamma+\delta,k)$-cordon.*

Proof. Use the fact that, if U_1 is a γ-dimensional subspace and U_2 is a δ-dimensional subspace in $\mathrm{PG}(n-1,q)$ with disjoint supports, then the cat's cradle $U_1 * U_2$ is the $(\gamma+\delta)$-dimensional subspace spanned by U_1 and U_2. □

Lemma 4.2. *Let C and D be two sets of points in* $\mathrm{PG}(n-1,q)$ *with disjoint supports. If C contains no $(\gamma+1)$-dimensional subspaces and D contains no $(\delta+1)$-dimensional subspaces, then $C * D$ contains no $(\gamma+\delta+1)$-dimensional subspaces.*

Proof. Suppose that U is a $(\gamma+\delta+1)$-dimensional subspace contained in $C * D$. Taking linear combinations, we can find a basis

$$u_1+v_1, u_2+v_2, \ldots, u_j+v_j, v_{j+1}, v_{j+2}, \ldots, v_{\gamma+\delta+1}$$

for U where $u_1,u_2,\ldots,u_j$ are independent points in C, $v_1,v_2,\ldots,v_{\gamma+\delta+1}$ are points in D (the points $v_1,v_2,\ldots,v_j$ may be zero), and $v_{j+1},v_{j+2},\ldots,v_{\gamma+\delta+1}$ are independent. As D contains no $(\delta+1)$-dimensional subspaces, the points $v_{j+1},v_{j+2},\ldots,v_{\gamma+\delta+1}$ span a subspace of dimension at most δ. Hence $j \geqslant \gamma+1$. Since C contains no $(\gamma+1)$-dimensional subspace, at least one linear combination $\sum a_i u_i$ of the points $u_1,u_2,\ldots,u_j$ is not in C. Hence, because every point in $C * D$ is uniquely a linear combination $u+v$, where $u \in C$, $v \in D$, the linear combination $\sum a_i(u_i+v_i)$, which equals $(\sum a_i u_i)+(\sum a_i v_i)$, is not in $C * D$, a contradiction. □

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Let γ be odd and greater than 1 and $n \geqslant m \geqslant 4(\gamma-1)+1$. Let M be a rank-r binary 2-block containing no 3-point lines, represented as a set of points supported on the set $\{m+1, m+2, \ldots, m+r\}$ of coordinate positions in $\mathrm{PG}(n-1,2)$. Then, by Lemmas 4.1 and 4.2, the cat's cradle $C(\gamma-1,m;2) * M$ is a $(\gamma,2)$-cordon containing no $(\gamma+1)$-dimensional subspaces. We can replace $C(\gamma-1,m;q)$ by $B'(\gamma,m;2)$ in this construction, but since $q^2 = q+2$ when $q=2$, the lower bound is unchanged.

Taking M to be the cycle matroid $M(\Gamma_{23})$, we obtain γ-cordons containing no $(\gamma+1)$-dimensional subspaces for the parameters

$$q = 2, \gamma \text{ odd and greater than } 1, n \geqslant 4(\gamma-1)+23.$$

Binary 2-blocks containing no 3-point lines seem to be rare. For example, by a theorem of Jaeger [5], no such block can be cographic.

Because $(\gamma,2)$-cordons are $(\gamma+1)$-blocks, there are no rank-n $(\gamma,2)$-cordons when $n \leqslant \gamma+1$. The only rank-$(\gamma+2)$ $(\gamma+1)$-block is $\mathrm{PG}(\gamma+1,n)$. Hence, there are no $(\gamma,2)$-cordons containing no $(\gamma+1)$-dimensional subspaces when $n \leqslant \gamma+2$.

5. Cordons and the polymatroid critical problem

We end with a brief discussion of the relationship between cordons and the polymatroid critical problem. The polymatroid critical problem is formulated in [4] and [13]. A unified account can be found in Section 4.3 of [6]. A γ-polymatroid is a polymatroid in which every element has the same rank γ; in particular, a $\mathrm{GF}(q)$-representable γ-polymatroid is essentially a collection of γ-dimensional subspaces in $\mathrm{PG}(n-1,q)$.

Proposition 5.1. *A set C of points in $\mathrm{PG}(n-1,q)$ is a (γ,k)-cordon if and only if the γ-polymatroid of all the γ-dimensional subspaces of $\mathrm{PG}(n-1,q)$ contained in C has critical exponent greater than k.* □

For example, the cordons $C(\gamma,m;q)$ are associated with the γ-polymatroids of the complete hypergraph on m vertices with hyperedges of size $\gamma+1$ ([12]; see also Section 4.3 of [6]).

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