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Eigenvalues and strong orbit equivalence

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Abstract. We give conditions on the subgroups of the circle to be realized as the subgroups of eigenvalues of minimal Cantor systems belonging to a determined strong orbit equivalence class. Actually, the additive group of continuous eigenvalues $E(X, T)$ of the minimal Cantor system (X, T) is a subgroup of the intersection $I(X, T)$ of all the images of the dimension group by its traces. We show, whenever the infinitesimal subgroup of the dimension group associated with (X, T) is trivial, the quotient group $I(X, T)/E(X, T)$ is torsion free. We give examples with non-trivial infinitesimal subgroups where this property fails. We also provide some realization results.

1. *Introduction*

Two dynamical systems are orbit equivalent if there is a bijection between their phase spaces that preserves their structures (measure preserving, topological, etc.) and induces a one-to-one correspondence between their orbits. The notion of orbit equivalence arises first in the context of probability measure preserving group actions (measurable orbit equivalence), as a consequence of the study of von Neumann algebras [**20**, **29**]. One of the most remarkable results in this theory establishes that there is only one orbit equivalence class among the free ergodic probability measure preserving actions of amenable groups [**8**, **23**].

Motivated by the measurable orbit equivalence results, in particular, the characterization of the orbit equivalence classes in terms of von Neumann algebras [**17**, **18**], Giordano *et al* obtain in [**12**] one of the most important results in the context of the orbit equivalence from a topological point of view: the orbit equivalence classes of the minimal $\mathbb{Z}$-actions

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on the Cantor set are characterized in terms of the K_0 group of the associated C^*-algebra (see [**26**, **32**] for an interplay between C^*-algebras and dynamics). As a consequence, they obtain that there are as many orbit equivalent classes as reduced simple dimension groups with distinguished order unit. Thus, unlike in the measurable setting, in the topological context it is natural to ask for the dynamical properties which are preserved under orbit (or strong orbit) equivalence. For instance, in [**15**] it is shown that the set of invariant probability measures of a given minimal Cantor system is affinely isomorphic to the set of traces of the associated dimension group. Thus, the set of invariant probability measures is preserved, up to affine homeomorphism, under strong orbit equivalence. On the contrary, within a strong orbit equivalence class it is possible to find a minimal Cantor system having any possible entropy (see [**2**, **22**] and [**30**] for the general case).

In this paper, we study the relation between (strong) orbit equivalence and the spectral properties of a system. We know from [**22**] that strong orbit equivalent minimal Cantor systems share the same subgroup of rational continuous eigenvalues. Thus, if the subgroup of rational continuous eigenvalues of a minimal Cantor system is not cyclic, then within its strong orbit equivalence class there are no mixing minimal Cantor systems. This does not hold if the orbit equivalence is not strong as shown again in [**22**]. Indeed, Ormes proved [**22**, Theorem 8.2] that in a prescribed orbit equivalence class it is possible to realize any countable subgroup of the circle as a group of measurable eigenvalues.

In this work we investigate the case of non-rational eigenvalues and whether the dimension groups induce restrictions (other than those due to the rational eigenvalues) on the groups of eigenvalues that can be realized within this given strong orbit equivalence class.

It happens that a first restriction has been shown in [**16**]: the additive group of eigenvalues $E(X, T)$ of a minimal Cantor system (X, T) is a subgroup of the intersection of all the images of the dimension group by its traces. Dynamically speaking, it is a subgroup of $I(X, T) = \bigcap_{\mu\in\mathcal{M}(X,T)}\{\int f\, d\mu \mid f \in C(X, \mathbb{Z})\}$, where $\mathcal{M}(X, T)$ is the set of T-invariant probability measures of (X, T) and $C(X, \mathbb{Z})$ is the set of continuous functions from X to $\mathbb{Z}$. Another proof of this observation can be found in [**5**] but it was not pointed out.

In this paper we prove the following strong restriction.

THEOREM 1. *Suppose that (X, T) is a minimal Cantor system such that the infinitesimal subgroup of the dimension group $K^0(X, T)$ is trivial. Then the quotient group $I(X, T)/E(X, T)$ is torsion free.*

To illustrate this result, take $K^0(X, T) = \mathbb{Z} + \alpha\mathbb{Z} = I(X, T)$, with α irrational. This is the case for a Sturmian subshift. Then, within the strong orbit equivalence class of (X, T), the only groups of continuous eigenvalues that can be realized are $\mathbb{Z}$, which will provide topologically weakly mixing minimal Cantor systems, and $\mathbb{Z} + \alpha\mathbb{Z}$. Moreover, both can be realized: in the first case using results from [**22**] and in the second case by a Sturmian subshift.

Relations between additive eigenvalues and topological invariants can be found in [**11**, **24**, **27**, **28**], but they do not apply to Cantor systems.

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In §2 we recall the concept of Kakutani–Rohlin partitions that will be necessary throughout this paper. The next section concerns the notions and definitions we will need. In particular, we recall the algebraic notions and dynamical interpretations of dimension group, trace, infinitesimal and rational subgroup. Section 3 is devoted to the proof of our main result: Theorem 1. To this aim we use a precise description of entrance times with respect to some well-chosen Kakutani–Rohlin partitions. We follow the approach proposed in [**3**, **4**] to tackle eigenvalue problems. Apart from Theorem 1, there are three results that could be of independent interest: we obtain a new necessary condition for being an eigenvalue (Proposition 9); we give an elementary proof of the fact that $E(X, T)$ is included in $I(X, T)$ (Proposition 11); and for every α in $I(X, T)$, we show there exists a continuous function $f : X \to \mathbb{Z}$ such that $\alpha = \int f \, d\mu$ for all T-invariant measure μ (Lemma 12).

In the last section we provide realization examples around Theorem 1. In particular, we give an example where the automorphism group (i.e., the group of self homeomorphisms of the space commuting with the map T) is not invariant under orbit equivalence.

2. *Definitions and background*

2.1. *Dynamical systems.* We introduce here the notation and recall some classical facts. We refer to [**25**] for a more detailed exposition. By a *topological dynamical system* we mean a couple (X, T) where X is a compact metric space and $T : X \to X$ is a homeomorphism. We say that it is a *Cantor system* if X is a Cantor space; that is, X has a countable basis of its topology which consists of closed and open sets (clopen sets) and does not have isolated points. It is *minimal* if it does not contain any non-empty proper closed T-invariant subset. A dynamical system (Y, S) is called a *factor* of (X, T) if there is a continuous and onto map $\phi : X \to Y$, called a *factor map*, such that $\phi \circ T = S \circ \phi$. If ϕ is one-to-one we call ϕ a *conjugacy* and say that (X, T) and (Y, S) are *conjugate*. If (X, T) is minimal and $\phi : X \to Y$ is a factor map for which there exists $x \in X$ such that $\sharp\phi^{-1}(\phi(x)) = 1$, we call ϕ an *almost one-to-one factor map* and (X, T) an *almost one-to-one extension* of (Y, S).

We denote by $\mathcal{M}(X, T)$ the set of all T-invariant probability measures μ, defined on the Borel σ-algebra $\mathcal{B}_X$ of X. For such a measure μ, the quadruple $(X, \mathcal{B}, \mu_X, T)$ is called a *measurable dynamical system*. This system is said to be *ergodic* if any T-invariant measurable set has measure 0 or 1. Two measurable dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{B}', \nu, S)$ are *measure theoretically conjugate* if we can find invariant subsets $X_0 \subset X$, $Y_0 \subset Y$ with $\mu(X_0) = \nu(Y_0) = 1$ and a bimeasurable bijective map $\psi : X_0 \to Y_0$ such that $S \circ \psi = \psi \circ T$ and $\mu(\psi^{-1}B) = \nu(B)$ for any $B \in \mathcal{B}'$.

A complex number λ is a *continuous eigenvalue* (respectively a *measurable eigenvalue*) of (X, T) if there exists a continuous (respectively integrable with respect to an invariant measure) function $f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called a *continuous eigenfunction* associated with λ. Of course, any continuous eigenvalue is a measurable one for any fixed measure. Hence, every eigenvalue is of modulus 1, i.e., belongs to the circle $\mathbb{S}^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, and every eigenfunction has a constant modulus. Notice that any continuous eigenfunction provides a factor map from (X, T) to a rotation.

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In this work we are mainly concerned with continuous eigenvalues $\lambda = \exp(2i\pi\alpha)$ of minimal Cantor systems. Such an α is called an *additive continuous eigenvalue* of (X, T), and the set of additive continuous eigenvalues, denoted by $E(X, T)$, is an additive subgroup of $\mathbb{R}$ called the *group of additive continuous eigenvalues*. It is well known that $E(X, T)$ is countable and contains $\mathbb{Z}$.

We say two dynamical systems (X, T) and (Y, S) are *orbit equivalent* (OE) whenever there exists a homeomorphism $\phi : X \to Y$ sending orbits to orbits: for all $x \in X$,

$$\phi(\{T^n x \mid n \in \mathbb{Z}\}) = \{S^n \phi(x) \mid n \in \mathbb{Z}\}.$$

This induces the existence of maps $\alpha : X \to \mathbb{Z}$ and $\beta : X \to \mathbb{Z}$ satisfying, for all $x \in X$,

$$\phi \circ T(x) = S^{\alpha(x)} \circ \phi(x) \text{ and } \phi \circ T^{\beta(x)}(x) = S \circ \phi(x).$$

When α and β have both at most one point of discontinuity, we say (X, T) and (Y, S) are *strongly orbit equivalent* (SOE). We recall below the seminal result in [**12**] that characterized these equivalences in terms of dimension groups.

THEOREM 2. [**12**] *Let (X, T) and (Y, S) be two minimal Cantor dynamical systems. The following are equivalent:*

(1) *(X, T) and (Y, S) are strong orbit equivalent; and*
(2) *$K^0(X, T)$ and $K^0(Y, S)$ are isomorphic as dimension groups with order units.*

The following are also equivalent:

(1) *(X, T) and (Y, S) are orbit equivalent; and*
(2) *$K^0(X, T)/\mathrm{Inf}(K^0(X, T))$ and $K(Y, S)/\mathrm{Inf}(K^0(Y, S))$ are isomorphic as dimension groups with order units.*

2.2. *Partitions and towers.* Sequences of partitions associated with minimal Cantor systems were used in [**15**] to build representations of such systems as adic transformations on ordered Bratteli diagrams. Here we do not introduce the whole formalism of Bratteli diagrams since we will only use the language describing the tower structure, even if both languages are very close. We recall some definitions and fix some notation.

For a minimal Cantor system (X, T), a *clopen Kakutani–Rohlin partition* (CKR partition) is a partition $\mathcal{P}$ of X given by

$$\mathcal{P} = \{T^j B(k) \mid 1 \le k \le C, 0 \le j < h(k)\}, \tag{2.1}$$

where $C, h(1), \ldots, h(k)$ are positive integers and $B(1), \ldots, B(C)$ are clopen subsets of X such that

$$\bigcup_{k=1}^{C} T^{h(k)} B(k) = \bigcup_{k=1}^{C} B(k).$$

The set $B = \bigcup_{1 \le k \le C} B(k)$ is called the *base* of $\mathcal{P}$. Let

$$\{\mathcal{P}_n = \{T^j B_n(k) \mid 1 \le k \le C_n, 0 \le j < h_n(k)\}\}_{n \in \mathbb{N}} \tag{2.2}$$

be a sequence of CKR partitions. For every $n \in \mathbb{N}$, we denote B_n the base of $\mathcal{P}_n$. To be coherent with the notation of [**3**], we assume that $\mathcal{P}_0$ is the trivial partition, that is, $B_0 = X$, $C_0 = 1$ and $h_0(1) = 1$, and for the partition $\mathcal{P}_1$, $h_1(k) = 1$ for any integer $1 \le k \le C_1$.

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We say that the sequence $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ is *nested* if it satisfies the following: for any integer $n \in \mathbb{N}$,

(KR1) $B_{n+1} \subseteq B_n$;

(KR2) $\mathcal{P}_{n+1} \succeq \mathcal{P}_n$; i.e., for any $A \in \mathcal{P}_{n+1}$ there exists an atom $A' \in \mathcal{P}_n$ such that $A \subseteq A'$;

(KR3) $\bigcap_{n\in\mathbb{N}} B_n$ consists of a unique point; and

(KR4) the sequence of partitions spans the topology of X.

In [**15**] it is proven that, given a minimal Cantor system (X, T), there exists a nested sequence of CKR partitions fulfilling **(KR1)**–**(KR4)** with the following additional technical conditions: for any integer $n \geq 0$,

(KR5) for any $1 \leq k \leq C_n$, $1 \leq l \leq C_{n+1}$, there exists an integer $0 \leq j < h_{n+1}(l)$ such that $T^j B_{n+1}(l) \subseteq B_n(k)$; and

(KR6) $B_{n+1} \subseteq B_n(1)$.

With the sequence $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ we associate the sequence of matrices $\{M_n\}_{n\geq 1}$, where $M_n = (m_n(l, k))_{1\leq l\leq C_n, 1\leq k\leq C_{n-1}}$ is given by

$$m_n(l, k) = \#\{0 \leq j < h_n(l) \mid T^j B_n(l) \subseteq B_{n-1}(k)\}.$$

Notice that **(KR5)** is equivalent to the condition that for any $n \geq 1$, the matrix M_n has positive entries.

For $n \geq 0$, we set $H_n = (h_n(l) \mid 1 \leq l \leq C_n)^T$. Since the sequence of partitions is nested, we have $H_n = M_n H_{n-1}$ for any $n \geq 1$. Notice also that, by convention,

$$M_1 = H_1 = (1, \ldots, 1)^T. \tag{2.3}$$

For $n > m \geq 0$, we define

$$P_{n,m} = M_n M_{n-1} \ldots M_{m+1},\ P_1 = M_1 \quad \text{and} \quad P_{n+1} = P_{n+1,1}. \tag{2.4}$$

Clearly, we have the relations

$$P_{n,m}(l, k) = \#\{0 \leq j < h_n(l) \mid T^j B_n(l) \subseteq B_m(k)\}, \tag{2.5}$$

for $1 \leq l \leq C_n$, $1 \leq k \leq C_m$, and

$$P_{n+1,m} H_m = H_{n+1} = P_{n+1} H_1. \tag{2.6}$$

Throughout the paper, we will strongly use a technique that we call *telescoping*; that is, starting from a sequence of CKR partitions $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ fulfilling **(KR1)**–**(KR6)**, we will consider an infinite subsequence of partitions satisfying an additional property. Actually, it is easy to check that any such subsequence of CKR partitions satisfies also **(KR1)**–**(KR6)**. Moreover, the sequences of the associated matrices of the type $\{M_n\}_{n\in\mathbb{N}}$ and $\{P_{n,m}\}_{n>m\geq 0}$ are subsequences of the previous ones.

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2.3. *Dimension groups, traces, infinitesimals and rational subgroups.*

2.3.1. *Dimension groups.* We recall here some basic definitions of the algebraic notion of dimension groups arising from the C^*-algebras. Relations with dynamical systems will be explain in the next subsection. Most of the notions arises from [**10**].

By an *ordered group* we shall mean a countable abelian group G together with a subset G^+, called the *positive cone*, such that $G^+ - G^+ = G$, $G^+ \cap (-G^+) = \{0\}$ and $G^+ + G^+ \subset G^+$. We shall write $a \leq b$ if $b - a \in G^+$. We say that an ordered group is *unperforated* if $a \in G$ and $na \in G^+$ for some $a \in G$ and $n \in \mathbb{N}$ implies that $a \in G^+$. Observe that an unperforated group is torsion free. We say (G, G^+) is *acyclic* whenever G is not isomorphic to $\mathbb{Z}$. By an *order unit* for (G, G^+) we mean an element $u \in G^+$ such that for every $a \in G$, $a \leq nu$ for some $n \in \mathbb{N}$.

Definition 1. A *dimension group* (G, G^+, u) with distinguished order unit u is an unperforated ordered group (G, G^+) satisfying the *Riesz interpolation property*, i.e., given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ $(i, j = 1, 2)$, there exists $c \in G$ with $a_i \leq c \leq b_j$, $(i, j = 1, 2)$.

We say that two dimension groups (G_1, G_1^+, u_1) and (G_2, G_2^+, u_2) are isomorphic whenever there exists an order isomorphism $\phi : G_1 \to G_2$, i.e., ϕ is a group isomorphism such that $\phi(G_1^+) = G_2^+$, and $\phi(u_1) = u_2$. An *order ideal* is a subgroup J such that $J = J^+ - J^+$ (where $J^+ = J \cap G^+$) and $0 \leq a \leq b \in J$ implies $a \in J$. A dimension group (G, G^+, u) is *simple* if it contains no non-trivial order ideals. It is easily seen that (G, G^+) is a simple dimension group if and only if every $a \in G^+ \backslash \{0\}$ is an order unit. Moreover, an unperforated simple ordered group is acyclic if and only if it satisfies the Riesz interpolation property (see [**9**]). Thus, the dimension groups are all acyclic.

2.3.2. *Traces.* Let (G, G^+, u) be a simple dimension group with distinguished order unit u. We say that a homomorphism $p : G \to \mathbb{R}$ is a *trace* (also called a *state*) if p is non-negative (i.e., $p(G^+) \geq 0$) and $p(u) = 1$. We denote the collection of all traces on (G, G^+, u) by $S(G, G^+, u)$. Now, $S(G, G^+, u)$ is a convex compact subset of the locally convex space $\mathbb{R}^G$ endowed with the product topology. In fact, one can show that $S(G, G^+, u)$ is a Choquet simplex. It is a fact (see [**15**]) that $S(G, G^+, u)$ determines the order on G. Actually,

$$G^+ = \{a \in G \mid p(a) > 0, \forall p \in S(G, G^+, u)\} \cup \{0\}.$$

As we will see later, the following group is fundamental in the study of continuous eigenvalues of minimal Cantor systems.

Definition 2. Let (G, G^+, u) be an ordered group with unit. We define the *image subgroup* of (G, G^+, u) to be the subgroup of $\mathbb{R}$ given by

$$I(G, G^+, u) = \bigcap_{\tau \in S(G, G^+, u)} \tau(G).$$

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2.3.3. *Infinitesimals.* Let (G, G^+) be a simple dimension group and let $u \in G^+\backslash\{0\}$. We say that an element $a \in G$ is *infinitesimal* if $-\epsilon u \leq a \leq \epsilon u$ for all $0 < \epsilon \in \mathbb{Q}^+$ (for $\epsilon = p/q$, $p, q \in \mathbb{N}$, then $a \leq \epsilon u$ means that $qa \leq pu$).

It is easy to see that the definition does not depend upon the particular order unit u. An equivalent definition is $a \in G$ is infinitesimal if $p(a) = 0$ for all $p \in S(G, G^+, u)$. The collection of infinitesimal elements of G forms a subgroup, the *infinitesimal subgroup of* G, which we denote by $\mathrm{Inf}(G)$.

Observe that the quotient group $G/\mathrm{Inf}(G)$ is also a simple dimension group for the induced order, and the infinitesimal subgroup of $G/\mathrm{Inf}(G)$ is trivial (see [**15**]). Furthermore, an order unit for G maps to an order unit for $G/\mathrm{Inf}(G)$. Moreover, the trace spaces of G and $G/\mathrm{Inf}(G)$ are isomorphic.

When $S(G, G^+, u)$ consists of a unique trace, notice that $G/\mathrm{Inf}(G)$ is isomorphic to $(I(G, G^+, u), I(G, G^+, u) \cap \mathbb{R}^+, 1)$, as ordered groups with unit.

2.3.4. *Rational subgroups.* By a *rational group* H we mean a subgroup of $\mathbb{Q}$ that contains $\mathbb{Z}$. We say that H is a *cyclic* rational group if H is isomorphic to $\mathbb{Z}$. Clearly, $(H, H \cap \mathbb{Q}^+, 1)$ is a simple dimension group with distinguished order unit 1. For a simple dimension group with order unit (G, G^+, u), we define the *rational subgroup* of G, denoted $\mathbb{Q}(G, G^+, u)$ (or $\mathbb{Q}(G, u)$ for short), as

$$\mathbb{Q}(G, u) = \{m/n \mid n \in \mathbb{N}^*, m \in \mathbb{Z}, \exists g \in G, ng = mu\}.$$

The notion of rational subgroup of a dimension group with distinguished order unit depends heavily upon the choice of the order unit.

Notice that, for $n, m \in \mathbb{Z}$ and $g \in G$ such that $ng = mu$, one gets, for any trace τ, $\tau(g) = m/n$. Consequently, $\mathbb{Q}(G, u)$ is a subgroup of $I(G, G^+, u)$.

2.4. *Dynamical interpretation of dimension groups, traces, infinitesimals and rational subgroups.* We consider here (X, T), a minimal Cantor dynamical system.

2.4.1. *'Dynamical' dimension groups.* We denote by $C(X, \mathbb{Z})$ the set of continuous maps from X to $\mathbb{Z}$. Consider the map $\beta : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ defined by $\beta f = f \circ T - f$ for all $f \in C(X, \mathbb{Z})$. The images of β are called *coboundaries*. Let $H(X, T)$ be the quotient group $C(X, \mathbb{Z})/\beta C(X, \mathbb{Z})$. The class of a function $f \in C(X, \mathbb{Z})$ in this quotient is denoted by $[f]$. We call the class $[1]$ of the constant function equal to 1 the *order unit*.

The positive cone, $H^+(X, T, \mathbb{Z})$, is the set of classes of non-negative functions $C(X, \mathbb{N})$. Finally, the triple

$$K^0(X, T) = (H(X, T, \mathbb{Z}), H^+(X, T, \mathbb{Z}), [1]).$$

is an ordered group with order unit. It is, moreover, a dimension group, but notice that a converse also holds.

THEOREM 3. [**15**] *If (X, T) is a minimal Cantor system, then $K^0(X, T)$ is a simple dimension group. Furthermore, if (G, G^+, u) is a simple dimension group then there is a minimal Cantor system (X, T) such that $K^0(X, T)$ and (G, G^+, u) are isomorphic.*

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2.4.2. *Traces are invariant measures.* Given any invariant probability measure μ of the system (X, T), we associate a trace τ_μ on $K^0(X, T)$ defined by $\tau_\mu([f]) := \int f\, d\mu$ for any $f \in C(X, \mathbb{Z})$. It is shown in [**15**] that the map $\mu \mapsto \tau_\mu$ is an affine isomorphism from the space of T-invariant probability measures $\mathcal{M}(X, T)$ to the traces space $S(K^0(X, T))$.

We denote by $I(X, T)$ the image subgroup $I(K^0(X, T))$. Rephrasing the definition of the image subgroup in dynamical terms, it is clear that

$$I(X, T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \int f\, d\mu \;\middle|\; f \in C(X, \mathbb{Z}) \right\}.$$

2.4.3. *Infinitesimals are functions with zero integral for all invariant measures.* We have seen that $\mathrm{Inf}(K^0(X, T)) = \{g \in K^0(X, T) \mid \tau(g) = 0\ \forall \text{ traces } \tau\}$. Thus, due to the identification described before, we also have

$$\mathrm{Inf}(K^0(X, T)) = \left\{ [f] \in K^0(X, T) \;\middle|\; \int f\, d\mu = 0\ \forall \mu \in \mathcal{M}(X, T) \right\}.$$

Observe that if (X, T) is uniquely ergodic, then $K^0(X, T)/\mathrm{Inf}(K^0(X, T))$ is isomorphic to $(I(X, T), I(X, T) \cap \mathbb{R}^+, 1)$, as ordered groups with unit.

2.4.4. *The rational subgroup is the group of rational continuous eigenvalues.* For any $m/n \in \mathbb{Q}(K^0(X, T))$, there exists a class $[f]$ such that $n[f] = m[1_X]$ and thus $\int f\, d\mu = m/n$ for any T-invariant measure μ. Thus, we have the inclusion

$$\mathbb{Q}(K^0(X, T)) \subset I(X, T) \cap \mathbb{Q}.$$

The following theorem gives a clear dynamical interpretation of such elements.

THEOREM 4. [**12**, **22**] *Let (X, T) be a minimal Cantor system and let μ be any T-invariant measure. Then a rational p/q is an additive continuous eigenvalue of (X, T), i.e., belongs to $E(X, T)$, if and only if $p/q = \int f\, d\mu$ for some $f \in C(X, \mathbb{Z})$. Or, equivalently,*

$$E(X, T) \cap \mathbb{Q} = \mathbb{Q}(K^0(X, T)).$$

As observed in [**12**] (see also [**22**]), this implies the following result.

COROLLARY 5. *Let (X, T) and (Y, S) be two strong orbit equivalent minimal Cantor systems (i.e., $K^0(X, T)$ is isomorphic to $K^0(Y, S)$). Then (X, T) and (Y, S) share the same rational continuous eigenvalues; that is,*

$$E(X, T) \cap \mathbb{Q} = E(Y, S) \cap \mathbb{Q}.$$

3. *Group of eigenvalues and images of traces*

In this section (X, T) stands for a given minimal Cantor system. We fix a sequence $\{\mathcal{P}_n\}_n$ of CKR partitions of (X, T) satisfying (**KR1**)–(**KR6**) and (2.3). We recall such a partition always exists. Once it is fixed, we freely use the notation of §2.2.

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3.1. *Some necessary conditions to be an eigenvalue.* The following results are fundamental in our study of eigenvalues of minimal Cantor systems. We also set some notation.

LEMMA 6. [**4**, Theorems 3, 5] *Let (X, T) be a minimal Cantor system and let $\alpha \in E(X, T)$. Then there exist an integer $m > 1$, a real vector v_m and an integer vector w_m such that:*

(1) $\alpha P_m H_1 = v_m + w_m$*; and*

(2) $\sum_{n>m} \|P_{n,m} v_m\|_\infty < \infty$*, where $\|\cdot\|_\infty$ denotes the supremum norm.*

For any given T-invariant probability measure μ of (X, T), we set

$$\mu_n = (\mu(B_n(k))_{1\le k\le C_n}^T$$

and we call it the *measure vector* of (X, T). It is easy to check it fulfills, for all $1 \le m < n$, the following identities:

$$\mu_1^T H_1 = 1 \quad \text{and} \quad \mu_m^T = \mu_n^T P_{n,m}. \tag{3.7}$$

LEMMA 7. [**4**] *With the conditions and the notation of Lemma 6, for any T-invariant probability measure μ of (X, T), for any integer $n \ge m$,*

$$\alpha = \mu_m^T w_m \quad \textit{and} \quad 0 = \mu_n^T P_{n,m} v_m.$$

Proof. For any integer $n > m > 1$, we set $v_n = P_{n,m} v_m$ and $w_n = P_{n,m} w_m$. Observe that Relation (3.7) and Lemma 6 imply, for any integer $n > m$,

$$\alpha = \alpha \mu_1^T H_1 = \mu_m^T P_m H_1 \alpha = \mu_m^T w_m + \mu_m^T v_m = \mu_m^T w_m + \mu_n^T P_{n,m} v_m.$$

Since $\alpha - \mu_m^T w_m$ does not depend on n and $\lim_{n\to\infty} \|\mu_n^T P_{n,m} v_m\|_\infty = 0$, we deduce that $\mu_m^T v_m = \mu_n^T P_{n,m} v_m = 0$, for every $n > m$. □

The following proposition and lemma will provide key arguments in the proof of our main result, Theorem 1. For its proof we need to introduce some crucial quantities, as it can be seen in the series of papers [**3**–**5**] and [**7**].

For any integer $n \in \mathbb{N}$, we define the *entrance time* $r_n(x)$ of a point $x \in X$ to the base B_n by $r_n(x) = \min\{j \ge 0 \mid T^j x \in B_n\}$. The *suffix map of order n* is the map $s_n : X \to \mathbb{N}^{C_n}$ given by

$$(s_n(x))_k = \sharp\{j \in \mathbb{N} \mid 0 \le j < r_{n+1}(x),\ T^j x \in B_n(k)\}$$

for every $k \in \{1, \ldots, C_n\}$. A classical computation gives† (see, for example, [**3**])

$$r_n(x) = \sum_{k=1}^{n-1} \langle s_k(x),\ P_k H_1 \rangle, \tag{3.8}$$

where $\langle v, v' \rangle = v^T v'$ stands for the usual scalar product.

PROPOSITION 8. [**3**] *Let (X, T) be a minimal Cantor system and let $\alpha \in \mathbb{R}$. The following conditions are equivalent:*

† Observe that by the conventions on $\mathcal{P}_0$ and $\mathcal{P}_1$, we have $s_0(x) = 0$ for any $x \in X$.

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(1) *α belongs to $E(X, T)$; and*
(2) *the sequence of functions $(\exp(2i\pi\alpha r_n(\cdot)))_n$ converges uniformly.*

PROPOSITION 9. *With the conditions and the notation of Lemma 6, we have*

$$\max_{x\in X} |\langle s_n(x),\ P_{n,m}v_m\rangle| \to_{n\to+\infty} 0.$$

Proof. Let us denote by $|||\cdot|||$ the distance to the closest integer. The sequence $(|||\alpha r_n|||)_n$ is a uniform Cauchy sequence (Proposition 8) and $\alpha(r_{n+1} - r_n) = \langle s_n, \alpha P_n H_1\rangle$. Therefore $|||\langle s_n, \alpha P_n H_1\rangle|||$ converges to zero. We have $\alpha P_n H_1 = P_{n,m}v_m + P_{n,m}w_m$. Since $P_{n,m}w_m$ is an integer vector, we obtain $|||\langle s_n, \alpha P_n H_1\rangle||| = |||\langle s_n,\ P_{n,m}v_m\rangle|||$. Consequently, Lemma 6(2) ensures that for any $\epsilon \in (0, 1/8)$ there exists n_0 such that for all $n \geq n_0$ and all x,

$$\|P_{n,m}v_m\| < \epsilon < 1/8 \quad \text{and} \quad |||\langle s_n(x),\ P_{n,m}v_m\rangle||| < \epsilon.$$

We may write $\langle s_n(x),\ P_{n,m}v_m\rangle = \epsilon_n(x) + E_n(x)$ with $|\epsilon_n(x)| < \epsilon$ and $E_n(x)$ an integer. Notice that $(\epsilon_n)_n$ uniformly converges to 0.

Consider the set $A = \{x \in X \mid E_n(x) = 0\}$. Observe that $\bigcap_n B_n$ is contained in A so it is non-empty. It is not difficult to check that A is closed. Let us check it is T-invariant. We fix some $x \in A$. It is straightforward to verify that there are only three distinct possible cases for $s_n(x)$: $s_n(Tx) = s_n(x)$; $s_n(x) = 0$ and $s_n(Tx) \neq 0$; or $s_n(x) \neq 0$ with $s_n(Tx) = s_n(x) - e$ for some vector e from the canonical base. The first case is easy to handle. For the second one, notice it implies $x \in B_{n+1}$. So by the very definitions, there exist two vectors e_1, e_2 from the canonical base such that $s_n(Tx) = M_{n+1}^t e_1 - e_2$. It follows that

$$\begin{aligned}|\langle s_n(Tx),\ P_{n,m}v_m\rangle| &= |\langle M_{n+1}^t e_1,\ P_{n,m}v_m\rangle - \langle e_2,\ P_{n,m}v_m\rangle| \\ &= |\langle e_1,\ P_{n+1,m}v_m\rangle - \langle e_2,\ P_{n,m}v_m\rangle| \\ &\leq \|P_{n+1,m}v_m\| + \|P_{n,m}v_m\| < \tfrac{1}{4},\end{aligned}$$

so that $E_n(Tx) = 0$ and $Tx \in A$.

For the last case, consider

$$\begin{aligned}|E_n(Tx) - E_n(x)| &= |\epsilon_n(x) - \epsilon_n(Tx) + \langle s_n(Tx),\ P_{n,m}v_m\rangle - \langle s_n(x),\ P_{n,m}v_m\rangle| \\ &\leq \tfrac{1}{4} + |\langle e,\ P_{n,m}v_m\rangle| \leq \tfrac{1}{4} + \|P_{n,m}v_m\| \leq \tfrac{1}{2}.\end{aligned}$$

Therefore, we have $E_n(Tx) = E_n(x) = 0$ and $Tx \in A$. Finally, by minimality, we obtain $A = X$ which implies that

$$|\langle s_n(x),\ P_{n,m}v_m\rangle| = |\epsilon_n(x)| = |||\langle s_n(x),\ P_{n,m}v_m\rangle||| < \epsilon.$$

This completes the proof. □

LEMMA 10. *Let (X, T) be a minimal Cantor system. Then, for any $k \in \mathbb{Z}^*$ and $\alpha \in \mathbb{R}$ such that $k\alpha$ belongs to $E(X, T)$ and α does not, there exist an integer $m > 1$, a real vector v_m and an integer vector w_m such that:*
(1) *$k\alpha P_m H_1 = w_m + v_m$;*
(2) *$\sum_{n>m} \|P_{n,m}v_m\|_\infty < \infty$;*
(3) *for every measure $\mu \in \mathcal{M}(X, T)$ and integer $n \geq m$, $\langle \mu_n,\ P_{n,m}v_m\rangle = 0$; and*
(4) *the vector $(1/k)P_{n,m}w_m$ is not an integer vector for infinitely many integers $n > m$.*

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Proof. From Lemmas 6 and 7, there exist a positive integer m, a real vector v_m and an integer vector w_m satisfying the items (1)–(3). From Proposition 8, the sequence $(k\alpha r_n)_n$ converges uniformly (mod $\mathbb{Z}$). Moreover, the relation (3.8) gives us, for any integer $n > m+1$,

$$\begin{aligned}\alpha r_n(x) &= \sum_{i=1}^{n-1} \alpha\langle s_i(x),\, P_i H_1\rangle \\ &= \sum_{i=1}^{m} \langle s_i(x),\, \alpha P_i H_1\rangle + \sum_{i=m+1}^{n-1} \left\langle s_i(x),\, P_{i,m}\left(\frac{1}{k}v_m + \frac{1}{k}w_m\right)\right\rangle.\end{aligned}$$

Suppose that $(1/k)P_{i,m}w_m$ is an integer vector for any large enough integer i. To obtain a contradiction, by Proposition 8, it suffices to show that $(\alpha r_n)_n$ is a Cauchy sequence. As $(k\alpha r_n)_n$ converges uniformly (mod $\mathbb{Z}$), we deduce that $(\sum_{i=m+1}^{n}\langle s_i(x),\, P_{i,m}v_m\rangle)_n$ converges uniformly (mod $\mathbb{Z}$). Hence, given $\epsilon \in (0, \frac{1}{2})$, there exists an integer n_0 such that for any integer $n \geq n_0$, any integer $p \geq 0$ and $x \in X$, there exists an integer $E_p(x)$ such that

$$\left|\sum_{i=n}^{n+p}\langle s_i(x),\, P_{i,m}v_m\rangle - E_p(x)\right| < \frac{\epsilon}{4}.$$

By Proposition 9, we can assume that the integer n_0 is sufficiently large to have

$$\max_{x\in X} |\langle s_n(x),\, P_{n,m}v_m\rangle| < \frac{\epsilon}{4} \quad \text{for all } n \geq n_0. \tag{3.9}$$

Now fix $n \geq n_0$. Notice that

$$\begin{aligned}E_{p+1}(x) - E_p(x) = E_{p+1}(x) &- \sum_{i=n}^{n+p+1} \langle s_i(x),\, P_{i,m}v_m\rangle \\ &- \left(E_p(x) - \sum_{i=n}^{n+p}\langle s_i(x),\, P_{i,m}v_m\rangle\right) \\ &+ \langle s_{n+p+1}(x),\, P_{n+p+1,m}v_m\rangle.\end{aligned}$$

We deduce that $|E_{p+1}(x) - E_p(x)| < \epsilon < \frac{1}{2}$ and so $E_{p+1}(x) = E_p(x)$ for any $x \in X$, and $p \geq 0$.

The inequality (3.9) ensures that $E_0(x) = 0$, and thus $E_p(x) = 0$ for any $p \geq 0$. It follows that $(\sum_{i=m+1}^{n}\langle s_i(x),\, P_{i,m}v_m\rangle)_n$ is a uniform Cauchy type sequence in x, so the sequence $(\alpha r_n)_n$ converges uniformly (mod $\mathbb{Z}$). This gives a contradiction. □

3.2. *Group of eigenvalues versus image group of dimension group.* A fundamental fact for this work is the following proposition (Proposition 11). This has been previously shown in [**16**], but has also been obtained in [**5**, Proposition 11] without being claimed.

PROPOSITION 11. *Let (X, T) be a minimal Cantor system. Then the set of additive continuous eigenvalues $E(X, T)$ is a subgroup of the image subgroup $I(X, T)$.*

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Proof. It suffices to show that $E(X, T)$ is a subset of $I(X, T)$. From Lemma 7, there exist a positive integer m and a vector $w_m \in \mathbb{Z}^{C_m}$ such that for every invariant measure μ one gets

$$\alpha = \mu_m^T w_m = \int f \, d\mu,$$

where $f = \sum_{k=1}^{C_m} w_m(k) 1_{B_m(k)}$. This shows that α is in $I(X, T)$. □

For any $\alpha \in I(X, T)$, by definition, for every invariant measure μ there exists $f_\mu \in C(X, \mathbb{Z})$ (thus depending on μ) such that $\alpha = \int f_\mu \, d\mu$. In the next lemma, we show this function can be chosen independently of the invariant measures.

LEMMA 12. *Let (X, T) be a minimal Cantor system. If α belongs to the image subgroup $I(X, T)$, then there exists a function $g \in C(X, \mathbb{Z})$ such that $\int g \, d\mu = \alpha$, for any measure $\mu \in \mathcal{M}(X, T)$.*

Proof. If $\mathcal{M}(X, T)$ is a singleton then the result is obvious. From now on, we will assume that $\mathcal{M}(X, T)$ contains at least two elements. For any $g \in C(X, \mathbb{Z})$, we define

$$\mathcal{M}_g = \left\{ \mu \in \mathcal{M}(X, T) \;\middle|\; \int g \, d\mu = \alpha \right\}.$$

Observe that $\mathcal{M}_g$ is convex and closed with respect to the weak* topology in $\mathcal{M}(X, T)$. From the definition of $I(X, T)$, it is clear we have

$$\mathcal{M}(X, T) = \bigcup_{g \in C(X, \mathbb{Z})} \mathcal{M}_g.$$

Since $C(X, \mathbb{Z})$ is countable, Baire's theorem implies there exists a map $g_0 \in C(X, \mathbb{Z})$ such that $\mathcal{M}_{g_0}$ has a non-empty interior. It follows that $I : \mu \mapsto \int g_0 \, d\mu$ is an affine map which is constant on an open set of $\mathcal{M}(X, T)$.

We get the conclusion by showing this map is constant. To prove this, let μ_0 be in the interior of $\mathcal{M}_{g_0}$, and let μ_1 be another measure in $\mathcal{M}(X, T)$. The map $t \in [0, 1] \mapsto I(t\mu_0 + (1 - t)\mu_1) \in \mathbb{R}$ is an affine map taking at least two times the same value α. So, it is a constant map and $I(\mu_0) = I(\mu_1) = \alpha$. Since the measure μ_1 is arbitrary, this concludes the proof. □

Remark 13. Observe that from [**14**, Lemma 2.4], Lemma 12 implies that for any $\alpha \in I(X, T) \cap (0, 1)$ there exists a clopen set U such that $\alpha = \mu(U)$ for any T-invariant probability measure μ. In particular, this is true when α is in $E(X, T)$.

By Theorem 3, this lemma can of course be rephrased in terms of the dimension group (G, G^+, u). Let $\tilde{G}$ denote the group $\{g \in G \mid \tau(g) = \tau'(g)$ for every trace $\tau, \tau' \in S(G, G^+, u)\}$. Notice that the unit u and any infinitesimal in $\mathrm{Inf}(G)$ belong to $\tilde{G}$.

COROLLARY 14. *Let (G, G^+, u) be a simple dimension group. Then for any trace $\tau \in S(G, G^+, u)$, the morphism*

$$\tau : \tilde{G} \to I(G, G^+, u)$$

is a surjective order preserving morphism. In particular, the dimension groups $(\tilde{G}/\mathrm{Inf}(G), \tilde{G} \cap G^+/\mathrm{Inf}(G), [u])$, with $[u]$ the class of the unit, and $(I(G, G^+, u), I(G, G^+, u) \cap \mathbb{R}^+, 1)$ are isomorphic.

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We are now able to prove our main theorem (Theorem 1).

Proof of Theorem 1. Suppose $I(X, T)/E(X, T)$ is not torsion free. Then there exist $\alpha \in I(X, T) \setminus E(X, T)$ and an integer $k > 1$ such that $k\alpha \in E(X, T)$. From Lemma 10, this implies there exist a positive integer m and vectors $w_m \in \mathbb{Z}^{C_m}$ and $v_m \in \mathbb{R}^{C_m}$ such that $k\alpha P_m H_1 = v_m + w_m$. We set $v_n = P_{n,m} v_m$ and $w_n = P_{n,m} w_m$ for every $n \geq m$, and the vector v_n is orthogonal to the vector of measures μ_n.

Since α is not a continuous eigenvalue, Lemma 10 implies there must be infinitely many integers n such that the following set is not empty:

$$I_n = \{i \in \{1, \ldots, C_n\} : w_n(i) \text{ is not divisible by } k\}.$$

Telescoping the sequence of CKR partitions if needed, we can assume that $I_n \neq \emptyset$ for every sufficiently large n. For $1 \leq i \leq C_n$, we write

$$w_n(i) = ka_n(i) + b_n(i),$$

where $a_n(i) \in \mathbb{Z}$ and $b_n(i)$ is some integer in $\{0, \ldots, k-1\}$. Thus, the index i is in I_n if and only if $b_n(i) \neq 0$. Observe that for any $n \geq m$

$$\alpha = \alpha \mu_1^T H(1) = \mu_n^T P(n) H(1) \alpha = \frac{1}{k} \mu_n^T (v_n + w_n) = \frac{1}{k} \mu_n^T w_n = \frac{1}{k} \mu_n^T b_n + \mu_n^T a_n. \tag{3.10}$$

Since α is in $I(X, T)$, Lemma 12 implies there exists a function $f_1 \in C(X, \mathbb{Z})$ such that for every invariant probability measure μ,

$$\alpha = \int f_1 \, d\mu.$$

On the other hand,

$$\mu_n^T a_n = \int f_2 \, d\mu,$$

where $f_2 = \sum_{i=1}^{C_n} a_n(i) 1_{B_n(i)}$. Thus equation (3.10) implies that for every invariant probability measure μ,

$$\frac{1}{k} \mu_n^T b_n = \int (f_1 - f_2) \, d\mu = \int f \, d\mu,$$

where $f = f_1 - f_2$. Thus, we obtain

$$\int kf \, d\mu = \mu_n^T b_n = \int \sum_{i=1}^{C_n} b_n(i) 1_{B_n(i)} \, d\mu.$$

Hence, the map

$$h = kf - \sum_{i=1}^{C_n} b_n(i) 1_{B_n(i)}$$

belongs to $\mathrm{Inf}(X, T)$, which is assumed to be trivial. Consequently, there exists a map $g \in C(X, \mathbb{Z})$ such that $h = g - g \circ T$.

Choose $p \geq n$ such that f is constant on any atom of the partition $\mathcal{P}_p$ and such that the function g is constant on the base B_p. This is always possible as the sequence $\{\mathcal{P}_j\}_j$

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satisfies (**KR1**–**KR6**). Let x be an element in $B_p \subset X$ and let $1 \le i \le C_p$ be such that $x \in B_p(i)$. We then have

$$\begin{aligned}
0 &= g(x) - g(T^{h_p(i)}x) \\
&= \sum_{j=0}^{h_p(i)-1} h(T^j x) \\
&= \sum_{j=0}^{h_p(i)-1} kf(T^j x) - \sum_{j=0}^{h_p(i)-1} \sum_{l=1}^{C_n} b_n(l) 1_{B_n(l)}(T^j x) \\
&= \left(k \sum_{j=0}^{h_p(i)-1} f(T^j x) \right) - \sum_{l=1}^{C_n} b_n(l) \sum_{j=0}^{h_p(i)-1} 1_{B_n(l)}(T^j x) \\
&= \left(k \sum_{j=0}^{h_p(i)-1} f(T^j x) \right) - \sum_{l=1}^{C_n} b_n(l) P_{p,n}(i,l) \\
&= \left(k \sum_{j=0}^{h_p(i)-1} f(T^j x) \right) - (P_{p,n} b_n)(i).
\end{aligned}$$

It follows that all the coordinates of $P_{p,n}b_n$ are divisible by k. On the other hand, for every $i \in I_p$ we have

$$w_p(i) = (P_{p,n}w_n)(i) = P_{p,n}(ka_n + b_n)(i) = k(P_{p,n}a_n)(i) + (P_{p,n}b_n)(i),$$

which contradicts that I_p is non-empty.

4. *Examples*

In this section we will construct various examples of minimal Cantor systems starting with an 'abstract' simple dimension group (G, G^+, u) having some fixed properties, and then we will make use of Theorem 3 to show the existence of a minimal Cantor system (X, T) having this prescribed simple dimension group. Thus, we will mostly avoid mentioning this theorem and we will identify $K^0(X, T)$ with (G, G^+, u).

In some examples we will need some classical definitions which we will not recall here. They can be found in any book on ergodic theory; we refer the reader to [**25**] and [**13**].

4.1. *(Measurable) eigenvalues are not related to (strong) orbit equivalence.*

At the difference of continuous additive eigenvalues, irrational additive measurable eigenvalues cannot be interpreted in terms of dimension group. Counterexamples mainly come from the powerful result obtained by Ormes [**22**, Theorem 6.1], generalizing the Jewett–Krieger theorem to strong orbit equivalence classes.

THEOREM 15. [**22**] *Let (X, T) be a minimal Cantor system and μ an ergodic S-invariant Borel probability measure. Let (Y, S, ν) be an ergodic measurable dynamical system of a non-atomic Lebesgue probability space (Y, ν) such that $\exp(2i\pi/p)$ is an eigenvalue of (Y, S, ν) for any element $1/p \in \mathbb{Q}(K^0(X, T))$. Then there exists a minimal Cantor system*

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(X, T') strongly orbit equivalent to (X, T) such that (X, S', μ) is measurably conjugate to (Y, S, ν).

In the same paper Ormes obtained the following remarkable generalization of Dye's theorem [**8**].

THEOREM 16. [**22**] *Let (Y_1, S_1, ν_1) and (Y_2, S_2, ν_2) be ergodic dynamical systems of non-atomic Lebesgue probability spaces. There are minimal Cantor systems (X, T_1) and (X, T_2) and a Borel probability measure μ on X which is T_1 and T_2 invariant such that:*
(1) *(X, T_i, μ) is measurably conjugate to (Y_i, S_i, ν_i), for $i = 1, 2$; and*
(2) *(X, T_1) is strongly orbit equivalent to (X, T_2) by the identity map.*

For example, one can take (Y_1, S_1, ν_1) and (Y_2, S_2, ν_2) with any countable groups (eventually trivial) G_1 and G_2 of measurable eigenvalues. The theorem asserts that there are strongly orbit equivalent minimal Cantor systems (X, T_1) and (X, T_2) that are measurably conjugate to the two previous dynamical systems, respectively. Of course the groups of measurable eigenvalues of (X, T_1) and (X, T_2) are, respectively, G_1 and G_2.

4.2. *Rational eigenvalues are not preserved under orbit equivalence.* Let $G = \mathbb{Z} \times \mathbb{Q}$, $u = (1, 1)$ and $a \in (0, 1) \cap \mathbb{Q}$. We set $G^+ = \{v \in G : \tau_a(v) > 0\} \cup \{0\}$, where $\tau_a(v) = av(1) + (1 - a)v(2)$, for every $v = (v(1), v(2)) \in G$. It is straightforward to check that (G, G^+, u) is a simple dimension group verifying the following:

- $\mathbb{Q}(G, G^+, u) = \mathbb{Z}$;
- $I(G, G^+, u) = \mathbb{Q}$;
- $S(G, G^+, u) = \{\tau_a\}$; and
- $G/\mathrm{Inf}(G)$, $I(G, G^+, u)$ and $\mathbb{Q}$ are isomorphic.

From [**15**, Theorem 6.2] there exist minimal Cantor systems having (G, G^+, u) as a dimension group (up to isomorphism).

Proposition 11 implies that $E(X, T) \subset I(X, T) = I(G, G^+, u) = \mathbb{Q}$. Using Theorem 4 one obtains $E(X, T) = \mathbb{Q}(G, G^+, u) = \mathbb{Z}$. Thus, for every minimal Cantor system (X, T) such that $K^0(X, T)$ is isomorphic to (G, G^+, u), the group of eigenvalues $E(X, T)$ is equal to $\mathbb{Z}$.

Nevertheless, $I(X, T) = \mathbb{Q}$, so $I(X, T)/E(X, T) = \mathbb{Q}/\mathbb{Z}$ is a torsion group. It is not difficult to see that $\mathrm{Inf}(K^0(X, T))$ is not trivial.

Remark 17. Thus, Theorem 1 is not true when $\mathrm{Inf}(K^0(X, T))$ is not trivial.

As (X, T) is uniquely ergodic, then, as observed in §2.4.3, $K^0(X, T)/\mathrm{Inf}(K^0(X, T))$ is isomorphic to $(I(X, T), I(X, T) \cap \mathbb{R}^+, 1) = (\mathbb{Q}, \mathbb{Q}^+, 1)$.

On the other hand, since every minimal Cantor system (Y, S) whose simple dimension group is isomorphic to $(\mathbb{Q}, \mathbb{Q}^+, 1)$ verifies $E(Y, S) = \mathbb{Q}$ (due to Proposition 11 and Theorem 4) and is orbit equivalent to (X, T) (by [**12**, Theorem 2.2]), we deduce that continuous rational eigenvalues are not invariant under orbit equivalence, unlike strong orbit equivalence.

Remark 18. This example also shows that the groups of eigenvalues which are realizable among the class associated with G (i.e., a class of strong orbit equivalence) are

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not necessarily realizable by systems in the class determined by $G/\mathrm{Inf}(G)$ (i.e., the corresponding class of orbit equivalence) and *vice versa*.

4.3. *The group $I(X, T)/E(X, T)$ can be a torsion group, even if the rational subgroup of $I(X, T)$ is cyclic.* Let $\alpha \in (0, 1)$ be an irrational number such that $\alpha^2 \in \mathbb{Z} + \alpha\mathbb{Z}$, e.g. the inverse of the golden mean. Let (X, T) be the Sturmian subshift with angle 2α (we refer to [**19**] for the definition). Let us recall that its dimension group is isomorphic to $(\mathbb{Z} + 2\alpha\mathbb{Z}, \mathbb{Z} + 2\alpha\mathbb{Z} \cap \mathbb{R}^+, 1)$. Let (Y, S) be any system such that its simple dimension group is isomorphic to $(\mathbb{Z} + \alpha\mathbb{Z}, \mathbb{Z} + \alpha\mathbb{Z} \cap \mathbb{R}^+, 1)$. These groups having a unique trace, (Y, S) and (X, T) are uniquely ergodic.

The rational subgroup of $\mathbb{Z} + \alpha\mathbb{Z}$ being $\mathbb{Z}$, Theorem 16 ensures that we can choose (Y, S) having no non-trivial measurable eigenfunctions; that is, (Y, S) is *weakly mixing*. We consider the product system $(X \times Y, T \times S)$ with the product action, i.e., $T \times S(x, y) = (Tx, Sy)$ for any $x \in X$, $y \in Y$.

We will show that:

(1) for the image subgroup, we have $I(X \times Y, T \times S) = \mathbb{Z} + \alpha\mathbb{Z}$; and

(2) for the set of additive eigenvalues, $E(X \times Y, T \times S) = \mathbb{Z} + 2\alpha\mathbb{Z}$.

Thus the quotient group $I(X \times Y, T \times S)/E(X \times Y, T \times S)$ will be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We call π_X and π_Y the projections of $X \times Y$ on X and Y respectively, and μ and ν the unique invariant probability measures of (X, T) and (Y, S) respectively. Let λ be the Lebesgue measure in $\mathbb{S}^1$. We denote by $(\mathbb{S}^1, R_{2\alpha})$ the rotation by angle $2\pi 2\alpha$ on the circle $\mathbb{S}^1$.

Since (Y, S) is weakly mixing, it is disjoint from $(\mathbb{S}^1, R_{2\alpha})$ (see [**13**, Theorem 6.27]) and thus the product measure $\lambda \times \nu$ is the unique invariant probability measure of $(\mathbb{S}^1 \times Y, R_{2\alpha} \times S)$. Since (X, T) and $(\mathbb{S}^1, R_{2\alpha})$ are measure theoretically conjugate, clearly $\mu \times \nu$ is the unique invariant probability measure of $(X \times Y, T \times S)$. Thus $(X \times Y, T \times S)$ is uniquely ergodic. Notice, moreover, that every open set has a positive measure. The ergodic theorem ensures then that the system $(X \times Y, T \times S)$ is minimal.

One can check, for example using [**14**, Lemma 2.6], that

$$\{\mu(A) \mid A \text{ clopen subset of } X\} = (\mathbb{Z} + 2\alpha\mathbb{Z}) \cap [0, 1] \quad \text{and}$$
$$\{\nu(B) \mid B \text{ clopen subset of } Y\} = (\mathbb{Z} + \alpha\mathbb{Z}) \cap [0, 1].$$

Also notice that any clopen set $C \subseteq X \times Y$ is a finite union of clopen sets of the kind $A \times B$, where A and B are clopen subsets of X and Y, respectively. Hence, by the very definition of α, we get

$$I(X \times Y, T \times S) = \langle \{\mu(A)\nu(B) : A \subseteq X, B \subseteq Y \text{ clopen subsets}\} \rangle = \mathbb{Z} + \alpha\mathbb{Z}.$$

Remark 19. By Theorem 2, the system $(X \times Y, T \times S)$ is orbit equivalent to the initial Sturmian subshift (X, T). Observe their automorphism groups are nevertheless distinct: at the difference of the Sturmian subshift [**21**], the product system $(X \times Y, T \times S)$ admits automorphisms, namely $\mathrm{Id} \times S$ and $T \times \mathrm{Id}$, that are not a power of the action $T \times S$.

Fundamental properties of Sturmian subshifts ensure there exists $\phi : X \to \mathbb{S}^1$, an almost one-to-one factor map from (X, T) to $(\mathbb{S}^1, R_{2\alpha})$. The function $\phi \circ \pi_X$ is then a factor map of the product system. This shows that $2\alpha \in E(X \times Y, T \times S)$.

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We will show that α is not an additive eigenvalue of the system $(X \times Y, T \times S)$.

Suppose there exists a continuous eigenfunction $f : X \times Y \to \mathbb{S}^1$, such that $f \circ T \times S = e^{2i\pi\alpha} f$. Since the map $\mathrm{Id} \times S$, product of the identity with the map S, commutes with the product action, the map $f \circ \mathrm{Id} \times S$ is also a continuous eigenfunction associated with the same eigenvalue. So there is a constant $\lambda \in \mathbb{S}^1$ such that $f \circ \mathrm{Id} \times S = \lambda f$. It follows that for any $x \in X$, the map $y \mapsto f(x, y)$ is a continuous eigenfunction of the system (Y, S) associated with the eigenvalue λ. The system (Y, S) being weakly mixing, we get that $\lambda = 1$ and $f(x, y)$ does not depend on y; we denote this last value $f(x)$. So the map $x \mapsto f(x)$ is a continuous eigenfunction of the system (X, T) associated with the eigenvalue $e^{2i\pi\alpha}$. This is impossible because this Sturmian subshift is an almost one-to-one extension of $(\mathbb{S}^1, R_{2\alpha})$.

We conclude that $E(X \times Y, T \times S) = \mathbb{Z} + 2\alpha\mathbb{Z}$.

According to Theorem 1, the infinitesimal subgroup of $K^0(X \times Y, T \times S)$ must be non-trivial. Let us give an example of a non-trivial infinitesimal element in the system $(X \times Y, T \times S)$.

From Lemma 12, there exists a function $g \in C(Y, \mathbb{Z})$ such that $\int g \, d\nu = 2\alpha$.

Claim. For any function $f \in C(X, \mathbb{Z})$ such that $\int f \, d\mu = \int g \, d\nu$, the function $F : (x, y) \in X \times Y \mapsto f(x) - g(y)$ is a non-trivial infinitesimal.

A standard computation shows us that $\int_{X\times Y} f(x) - g(y) \, d\mu \times \nu = 0$. Therefore, it remains to prove that it is not a coboundary of $C(X \times Y, \mathbb{Z})$.

Let us assume that the function $f - g$ is such a coboundary. Then there exists a function $H \in C(X \times Y)$ such that

$$f(x) - g(y) = H(x, y) - H(Tx, Sy) \quad \text{for all } x \in X, y \in Y.$$

By taking the integral of the former equality for the measure μ, we obtain

$$\begin{aligned} g(y) &= \int f \, d\mu - \int H(x, y) \, d\mu(x) + \int H(Tx, Sy) \, d\mu(x) \\ &= \int g \, d\nu - \int H(x, y) \, d\mu(x) + \int H(x, Sy) \, d\mu(x). \end{aligned}$$

By Lebesgue's dominated convergence theorem, the function $h : y \mapsto \int H(x, y) \, d\mu(x)$ is continuous. So $g - \int g \, d\nu = h \circ S - h$ is a real coboundary. Then, g taking integer values, the function $y \mapsto \exp(2i\pi h(y))$ defines a continuous eigenfunction associated with the additive eigenvalue $-\int g \, d\nu = -2\alpha$ for the system (Y, S). This is impossible because this system is weakly mixing. This proves our claim.

5. *Some results about realization*

Definition 3. Let (G, G^+, u) be a simple dimension group with distinguished order unit. We define $\mathcal{E}(G, G^+, u)$ as the collection of all the subgroups Γ of $\mathbb{R}$ for which there exists a minimal Cantor system (X, T) such that $K^0(X, T)$ and $E(X, T)$ are isomorphic to (G, G^+, u) and Γ, respectively.

In this section we are interested in a characterization of the family $\mathcal{E}(G, G^+, u)$ for a given simple dimension group. Most of our results are based on [**31**].

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Remark 20. Proposition 11 implies that the elements in $\mathcal{E}(G, G^+, u)$ are subgroups of $I(G, G^+, u)$. If, in addition, $\mathrm{Inf}(G) = \{0\}$, from Theorem 1 we get the following:

$$\mathcal{E}(G, G^+, u) \subseteq \{\Gamma : \text{ subgroup of } I(G, G^+, u) \text{ such that } I(G, G^+, u)/\Gamma \text{ is torsion free}\}.$$

5.1. *Basic example.* Let α be an irrational number. Consider $G = \mathbb{Z} + \alpha\mathbb{Z}$, $G^+ = G \cap \mathbb{R}^+$ and $u = 1$. Since the infinitesimal subgroup of (G, G^+, u) is trivial, the collection $\mathcal{E}(G, G^+, u)$ is a subfamily of $\{\mathbb{Z}, \mathbb{Z} + \alpha\mathbb{Z}\}$. It is known that the dimension group associated with the Sturmian subshift with angle α is isomorphic to (G, G^+, u) (see [**6**, **19**]). Moreover, it is an almost one-to-one extension of the rotation with angle α. Hence its subgroup of eigenvalues equals G. On the other hand, Theorem 15 implies there exists a minimal Cantor system having no non-trivial eigenvalues whose dimension group is isomorphic to (G, G^+, u). Thus, we get

$$\mathcal{E}(G, G^+, u) = \{\mathbb{Z}, \mathbb{Z} + \alpha\mathbb{Z}\}.$$

5.2. *Eigenvalues and dimension subgroups.* Let (G, G^+, u) and (H, H^+, w) be two simple dimension groups with distinguished order unit. An *order embedding* is a monomorphism $i : H \to G$ such that $i(h) \in G^+$ if and only if $h \in H^+$ and $i(w) = u$. An order embedding always induces an affine homomorphism $i^* : S(G) \to S(H)$ by

$$i^*(\tau)(h) = \tau(i(h)) \quad \text{for every } \tau \in S(G) \text{ and } h \in H.$$

We use the next two lemmas to show that a minimal Cantor system and any of its almost one-to-one extensions share their maximal equicontinuous factor.

The second lemma is a converse of the first one. A proof of the next result, in a more general context, can be found in [**1**, Chapter 9].

LEMMA 21. *Let $\pi : (X, T) \to (Y, S)$ be a proximal extension of minimal Cantor systems. Then (X, T) and (Y, S) have the same maximal equicontinuous factor.*

LEMMA 22. *Let $\pi : (X, T) \to (Y, S)$ be an almost one-to-one extension of compact systems, such that (Y, S) is minimal. Then π is a proximal extension.*

Proof. Let $y \in Y$ be an element having only one pre-image by π. If π is injective, then the result is trivial. We can assume, then, that there exist $x' \neq x''$ in X such that $\pi(x') = \pi(x'') = y' \in Y$. Suppose that x' and x'' are not proximal. This means there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$d(T^n(x'), T^n(x'')) > \varepsilon. \tag{5.11}$$

Since (Y, S) is minimal, there exists a subsequence $(S^{n_i}(y'))_{i\geq 0}$ of the orbit of y' that converges to y. By compactness, taking subsequences if needed, we can suppose $(T^{n_i}(x'))_{i\geq 0}$ and $(T^{n_i}(x''))_{i\geq 0}$ converging to some z' and z'', respectively. Inequality (5.11) ensures that $z' \neq z''$, and since π is continuous, we have $\lim_{i\to\infty} \pi(T^{n_i}(x')) = \pi(z')$ and $\lim_{i\to\infty} \pi(T^{n_i}(x'')) = \pi(z'')$. On the other hand, the choice of $(n_i)_{i\geq 0}$ implies that

$$\pi(z') = \lim_{i\to\infty} \pi(T^{n_i}(x')) = \lim_{i\to\infty} S^{n_i}(y') = y,$$
$$\pi(z'') = \lim_{i\to\infty} \pi(T^{n_i}(x'')) = \lim_{i\to\infty} S^{n_i}(y'') = y,$$

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which contradicts the fact that y has only one pre-image. □

Let us recall a consequence of [**31**, Theorem 1.1] (see [**31**, Corollary 1.2]).

THEOREM 23. *Suppose that (Y, S) is a uniquely ergodic minimal Cantor system and (G, G^+, u) is a simple dimension group with distinguished order unit satisfying the following assumptions:*

(i) *there is an order embedding $i : K^0(Y, S) \to (G, G^+, u)$; and*

(ii) *$G/i(K^0(Y, S))$ is torsion free.*

Then there exists a minimal Cantor system (X, T) such that $K^0(X, T)$ is isomorphic to (G, G^+, u) and such that there is an almost one-to-one factor map $\pi : (X, T) \to (Y, S)$.

The next result is a direct consequence of Lemmas 21, 22 and Theorem 23, but stated in terms of dimension groups.

PROPOSITION 24. *Let (G, G^+, u) and (H, H^+, w) be two simple dimension groups with distinguished order unit, such that (H, H^+, w) has a unique trace. Suppose there exists an order embedding $i : H \to G$ with $G/i(H)$ torsion free. Then $\mathcal{E}(H, H^+, w)$ is a subfamily of $\mathcal{E}(G, G^+, u)$.*

Recall that for any countable dense subgroup Γ of $\mathbb{R}$ containing $\mathbb{Z}$, the dimension group $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ has only one trace and no non-trivial infinitesimal (see [**10**]).

PROPOSITION 25. *Let (G, G^+, u) be a simple dimension group with distinguished order unit and with a trivial infinitesimal subgroup. Then, for any subgroup Γ of $I(G, G^+, u)$ with $\mathbb{Z} \subseteq \Gamma$ and $I(G, G^+, u)/\Gamma$ torsion free, the family $\mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ is contained in $\mathcal{E}(G, G^+, u)$.*

Proof. Let $\tau : \tilde{G} \to I(G, G^+, u)$ be the morphism given by Corollary 14. Since the infinitesimal subgroup is trivial, it is an isomorphism. Let (H, H^+, u) be the dimension group image of $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ by the inverse τ^{-1}. It is easy to check the group $G/\tilde{G}$ is torsion free, so, by hypothesis, the quotient group G/H is also torsion free. Moreover, $S(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ has only one element because $S(H, H^+, u) \simeq S(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$. The desired result follows from Proposition 24. □

As a consequence of these results, we obtain the following characterization.

COROLLARY 26. *The following are equivalent.*

(1) *For any countable dense subgroup Γ of $\mathbb{R}$ containing $\mathbb{Z}$, there is a Cantor minimal system (X, T) with $E(X, T) = \Gamma$, and $K^0(X, T) \simeq (\Gamma, \Gamma \cap \mathbb{R}^+, 1)$, i.e.,*

$$\Gamma \in \mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1).$$

(2) *For any simple dimension group with distinguished order unit (G, G^+, u) and with no non-trivial infinitesimal,*

$$\mathcal{E}(G, G^+, u) = \{\Gamma \mid \Gamma \text{ is a subgroup of } I(G, G^+, u) \text{ such that } I(G, G^+, u)/\Gamma \text{ is torsion free}\}.$$

Proof. It is obvious that (1) is a consequence of (2). Proposition 25 and (1) imply (2). □

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PROPOSITION 27. *Let (G, G^+, u) be a simple dimension group with distinguished order unit and let Γ be a subgroup of $I(G, G^+, u)$ verifying the following:*

- *Γ is generated by a family of rationally independent numbers containing 1; and*
- *$I(G, G^+, u)/\Gamma$ is torsion free.*

Then $\mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ is contained in $\mathcal{E}(G, G^+, u)$.

Remark that Proposition 27 includes the case where Γ is finitely generated. Example 4.2 shows that Proposition 27 becomes false whenever the group Γ is not generated by a rationally independent family.

Proof. Let $\{\alpha_i\}_{i\geq 0}$ be a rationally independent family generating Γ. Without loss of generality, we can assume that $\alpha_0 = 1$. From Lemma 12, for every $i \in \mathbb{N}$ there exists a $g_i \in G$ such that $\tau(g_i) = \alpha_i$ for every $\tau \in S(G, G^+, u)$. We choose $g_0 = u$.

For any $\alpha \in \Gamma$, there exists a unique sequence of integers $(m_i)_{i\geq 0}$ with $m_i = 0$ except for a finite number of indices i, such that $\alpha = \sum_{i\geq 0} m_i \alpha_i$. This implies that the function $\phi : \Gamma \to G$ given by $\phi(\alpha) = \sum_{i\geq 0} m_i g_i$ is a well-defined one-to-one homomorphism. It is not difficult to see that this is an order embedding such that $G/\phi(\Gamma)$ is torsion free. Since $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ has only one trace, from Proposition 24 we obtain the desired property. □

5.3. *Necessary conditions for* $\Gamma \in \mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$. Let (G, G^+, u) be a simple dimension group with distinguished order unit having no non-trivial infinitesimal and with a unique trace. So it is isomorphic to the dimension group $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$, where $\Gamma = I(G, G^+, u)$. We suppose here that $\Gamma \in \mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$. That is, there exists a uniquely ergodic minimal Cantor system (X, T) whose dimension group is isomorphic to $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ and such that $E(X, T) = \Gamma$.

We use the notation of §§2.2 and 3.1. Recall that for every $n \geq 1$, $\mu_n = (\mu_n(k))_{k=1}^{C_n}$ denotes the vector of measures of the bases of the partition $\mathcal{P}_n$, corresponding to the unique invariant probability measure μ of (X, T) (see §2.2 for definitions). Since (X, T) is uniquely ergodic, the group Γ is generated by $\{\mu_n(k) : 1 \leq k \leq C_n, n \geq 1\}$, which implies that for every $m \geq 1$ and every $1 \leq k \leq C_m$ the number $\mu_{m,k}$ is in $E(X, T)$. From Lemma 6 we get that for every n sufficiently large, there exist an integer vector w_n and a real vector v_n such that

$$P_n H_1 \mu_m(k) = v_n + w_n \quad \text{and} \quad \sum_{l\geq n} \|P_{l,n} v_n\|_\infty < \infty. \tag{5.12}$$

Multiplying the first equation by μ_n^T we get the normalization conditions (see equation (3.7) and Lemma 7)

$$\mu_m(k) = \mu_n^T w_n.$$

On the other hand, we have

$$\mu_m(k) = \mu_n^T P_{n,m}(\cdot, k).$$

Since the infinitesimal subgroup is trivial, the previous two equations imply that for n sufficiently large,

$$w_n = P_{n,m}(\cdot, k). \tag{5.13}$$

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Equations (5.12) and (5.13) imply

$$\sum_{n\geq 1} \max_i \|h_{n,i}\mu_m - P^T_{n,m}(\cdot, i)\|_\infty < \infty. \tag{5.14}$$

On the other hand, the unique ergodicity of the system (X, T) implies that the rows of the matrix $P_{n,m}$ converge with n (after normalization) to μ_m. That is

$$\lim_{n\to\infty} \max_i \left\| \mu_m - \frac{1}{h_{n,i}} P^T_{n,m}(\cdot, i) \right\|_\infty = 0.$$

Thus, from (5.14), we deduce that if $E(X, T) = \Gamma$, then the rate of convergence of the rows of $P_{n,m}$ to the direction generated by μ_m has to be extremely fast.

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