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Minimizing topological entropy for continuous maps on graphs

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Abstract. We study the existence of models with minimal topological entropy among the class of all continuous maps from a given graph to itself with a fixed behavior on a given finite invariant set.

1. *Introduction*

A classical problem in dynamics is the following one. Assume that X is a topological space and that f is a map from X to itself. Assume also that we know f restricted to an invariant set $A \subset X$. What can be said about the dynamics of f? Moreover, does there exist a map exhibiting the same behavior as $f|_A$ and having minimal dynamics in some sense (i.e. minimal topological entropy, minimal set of periods or periodic orbits, minimal set of 'types' of orbits, ... ?).

When the space X is a closed interval of the real line and A is a finite set the answer to the above problem is well known (see [**10**]). Indeed, the 'connect-the-dots' map is the one exhibiting minimal dynamics in all senses mentioned above. A generalization of this problem when A is not invariant can be found in [**3**].

In [**4**] this problem is considered when the space is a tree and A is a finite invariant set. In that paper it was shown that there is an important difference between the interval and the tree case (see for instance [**4**, Example 1.1]). Namely, to obtain models with minimal dynamics it is necessary to modify the tree (here minimal dynamics means having a minimal set of 'types' of orbits and, consequently, minimal topological entropy).

From above, a natural question is the following. If we do not allow the space to be modified is there another criterion of minimal dynamics for which there exist minimal models?

The aim of this paper is to study this problem when the criterion for minimal dynamics is minimal topological entropy and the space is a graph. In particular, we prove (see the Main

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Theorem) that given a graph G, a continuous map f from G to itself and a finite invariant set A then there exists a 'simple' map having minimal entropy among all maps from G to itself which coincide with f on A and are homotopic to f relative to A. Moreover, when G is a tree our result is constructive.

In the rest of this section we introduce the basic definitions and notions and state in detail the main result of the paper.

A *connected graph* (or simply a *graph*) is a pair (X, V), where X is a connected compact Hausdorff space and $V \subset X$ is finite and such that $X \setminus V$ is the disjoint union of a finite number of open subsets of X, called *edges*, with the property that each of them is homeomorphic to an open interval of the real line. A uniquely arcwise connected graph will be called a *tree*. Given a point $x \in V$, the number of edges whose boundaries contain x (with the edges whose closures are homeomorphic to a circle counted twice) will be called the *valence of* x and denoted by $\mathrm{val}(x)$. If $x \in X \setminus V$ (i.e. it belongs to the interior of some edge) then we set $\mathrm{val}(x) = 2$. A point of valence 1 is called an *endpoint* and a point of valence larger than two is called a *branching point*. We will denote the sets of endpoints and branching points of X by $\mathrm{End}(X)$ and $B(X)$, respectively. It is known that any graph can be embedded in $\mathbb{R}^3$. So, in what follows we will consider any graph endowed with the topology induced by the topology of $\mathbb{R}^3$.

Let G be a graph. Every subset of G homeomorphic to a non-empty open interval of the real line will be called an *interval* (however, notice that an interval $I \subset G$ is not necessarily open in G). As usual, a *subinterval* of an interval I will be an interval contained in I. Each interval I will be naturally equipped with the linear ordering of the real line transported by a homeomorphism φ from I to $(0, 1)$. Note that φ^{-1} extends naturally to a continuous map $\widetilde{\varphi} : [0, 1] \to \mathrm{Cl}(I)$. Then, the *left endpoint* (respectively *right endpoint*) of I is defined to be $\widetilde{\varphi}(0)$ (respectively $\widetilde{\varphi}(1)$). We note that both endpoints of I may coincide. In such a case $\mathrm{Cl}(I)$ is homeomorphic to a circle. If I is an interval, to simplify the notation, we will denote $\mathrm{Cl}(I)$ by $\overline{I}$. Also, if $a, b \in \overline{I}$ then the closure of the (unique) subinterval of I having a and b as endpoints will be denoted by $\langle a, b\rangle$ (or by $\langle a, b\rangle_I$ if confusion seems possible). Note that, when a and b are the endpoints of I, then $\langle a, b\rangle = \overline{I}$. In this spirit we also will write $I = (a, b)$ if we want to specify the endpoints of I.

A *pointed graph* is a couple (G, A) where G is a graph and A is a finite subset of G. Any continuous map f from G into itself such that $f(A) \subset A$ will be called a *pointed graph map*. Given such a map, we are interested in studying the dynamics of f 'forced' by $f|_A$. So, each edge containing a point from $\mathrm{End}(G) \setminus A$ can be retracted without consequences for our purposes. By recursively doing this simplification we can assume that each pointed graph (G, A) satisfies $\mathrm{End}(G) \subset A$, and we will do so in what follows.

Let (G, A, f) and (G', A', f') be such that (G, A) and (G', A') are pointed graphs and $f : G \to G$ and $f' : G' \to G'$ are pointed graph maps. We say that (G, A, f) and (G', A', f') are *equivalent* if there exists a homeomorphism $\varphi : G \to G'$ with $\varphi(A) = A'$ such that f and $\varphi^{-1} \circ f' \circ \varphi$ are homotopic relative to A. Notice that then, $\varphi^{-1} \circ f' \circ \varphi|_A = f|_A$. Each equivalence class of the above relation will be called an *action*. As usual, the class of (G, A, f) will be denoted by $[G, A, f]$.

In [**4**] the notion of a *pattern* for tree maps was introduced and the representatives of a pattern having minimal dynamics (topological entropy) were found. The notion of a

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FIGURE 1. An example of a locally monotone map at x is shown in (a), while (b) shows an example of a map which is not locally monotone at y.

pattern can be extended straightforwardly to graph maps as follows. We say that (G, A, f) and (G', A', f') have the same *pattern* if there exists a continuous map $\psi : G \to G'$ such that:

(i) $\psi|_A$ is one-to-one and $\psi(A) = A'$;
(ii) the image under ψ of each connected component of $G \setminus A$ is a connected component of $G' \setminus A'$;
(iii) ψ restricted to the closure of each connected component of $G \setminus A$ is a homotopy equivalence relative to A;
(iv) $f' \circ \psi = \psi \circ f$.

Note that two representatives of an action have the same pattern but not conversely. Indeed, two representatives of the same pattern can occur at non-homeomorphic graphs while actions fix the space by homeomorphisms. Since the notion of a pattern does not fix the space, a natural question is whether there exist representatives with minimal topological entropy and how they look, when the space is fixed.

If (G, A, f) and (G', A', f') are two representatives of an action then $(G, A, \varphi^{-1} \circ f' \circ \varphi)$ (where $\varphi : G \to G'$ is the homeomorphism from the equivalence between (G, A, f) and (G', A', f')) is another representative of the same action. So, in what follows, to simplify the notation, we will assume that the graph G and the set A are fixed.

Let (G, A) be a pointed graph. Every connected component of $G \setminus A \cup B(G)$ will be called a *basic interval*.

Let G and G' be graphs and let $f : G \to G'$ be continuous. For $x \in G$ we denote by $K_f(x)$ (or $K(x)$ when no confusion is possible) the connected component of $f^{-1}(f(x))$ which contains x. Notice that if f does not have flat spots then all sets $K_f(x)$ are degenerate.

Let I be an interval of a graph and let G be a graph. Let $f : \overline{I} \to G$ be continuous and let $x \in \overline{I}$. We will say that f is *locally monotone at* x if there exists a neighborhood V of $K(x)$ such that $f(V)$ is homeomorphic to an interval of the real line (perhaps degenerate to a point) and $f|_V$ is (not necessarily strictly) monotone as an interval map (see Figure 1 for some examples). We note that $f(V)$ is degenerate to a point only when $K(x) = \overline{I}$ (and then $f|_{\overline{I}}$ is constant). We will say that f is *monotone* if it is locally monotone at each point of $\overline{I}$. Let J be a subinterval of I. We will say that $\overline{J}$ is a *trivial interval* if $f(\mathrm{Bd}(J))$ reduces to a point and $f|_{\overline{J}}$ is homotopic (relative to $\mathrm{Bd}(J)$) to a constant map. It is easy to see that f is monotone if and only if f restricted to any trivial interval is constant.

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Let (G, A, f) be a representative of an action. We say that (G, A, f) is *monotone* if f restricted to any basic interval is monotone. Lastly, (G, A, f) will be called *simplicial* if it is monotone and $f(B(G)) \subset A \cup B(G)$.

Let (G, A, f) be a simplicial representative of an action. A point $v \in B(G)$ will be called *free* if $K(v) \cap A = \emptyset$. Moreover, we will say that f is *direct at* v if for every neighborhood U of $K(v)$ it follows that $f(v) \in \mathrm{Int}(f(U))$ (with the topology of $f(U)$). When f is direct at each free branching point we will say that (G, A, f) is *direct*.

Given a continuous map f from a compact topological space into itself we will denote by $h(f)$ the *topological entropy of* f (see [**1, 7**] for a definition of this notion).

Let Λ be an action. We define the *entropy of* Λ, denoted by $h(\Lambda)$, as

$$h(\Lambda) = \inf\{h(f) : (G, A, f) \text{ is a representative of } \Lambda\}.$$

MAIN THEOREM. *Given an action* Λ *there exists a direct representative* (G, A, f) *of* Λ *such that* $h(f) = h(\Lambda)$.

Notice that the Main Theorem is an existence theorem but does not give an algorithm to construct the map which gives the entropy of the action. However, in §5 we will give an effective algorithm to compute this map for an action $[T, A, f]$ when T is a tree. Also, we will show how to compute this entropy in a simple case (see Example 5.2).

The proof of the Main Theorem follows the strategy of [**3**]. In particular the proof is based in a sequence of reductions to smaller classes of maps. In §2 we make the reduction to the class of monotone maps, in §3 to the class of 'Markov' maps and in §4 we make the last reduction to the class of simplicial maps. Finally, in §5 we prove the Main Theorem.

2. *Monotone representatives*

PROPOSITION 2.1. *Let* (G, A, f) *be a representative of an action. Then there exists* (G, A, g), *a monotone representative of* $[G, A, f]$, *such that* $h(g) \leq h(f)$.

To prove Proposition 2.1 we follow the strategy that one would use in the context of interval or tree maps. In the context of graph maps the new feature is that the image of a basic interval can make turns to a circuit. We will use the three following technical lemmas.

LEMMA 2.2. *Let* $f : G \to G$ *and* $g : G \to G$ *be graph maps. Let* $M = \{x \in G : f(x) \neq g(x)\}$ *and assume that* g *restricted to each connected component of* M *is constant. Then* $h(g) \leq h(f)$.

Proof. We start by showing that it is enough to prove the lemma when M has a unique connected component. Indeed, if M contains a finite number of connected components $\{M_1, M_2, \ldots, M_n\}$ then, for each $k \in \{1, 2, \ldots, n\}$, we define $g_k = g$ on $\widetilde{M}_k = M_1 \cup M_2 \cup \cdots \cup M_k$ and $g_k = f$ in the complement of $\widetilde{M}_k$. Then $h(f) \geq h(g_1) \geq h(g_2) \geq \cdots \geq h(g_n) = h(g)$ and the lemma follows. If M contains an infinite number of connected components $\{M_1, M_2, \ldots, M_n \ldots\}$ we consider the sequence $\{g_k\}_{k=1}^{\infty}$ defined analogously. Clearly, g_n converges to g in the C^0-topology. Therefore, since $h(f) \geq h(g_1) \geq h(g_2) \geq \cdots \geq h(g_n) \geq \cdots$ and $h(\cdot)$ is lower semicontinuous (see, e.g., [**9**]), it follows that $h(g) \leq h(f)$.

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So we assume that M has a unique connected component. By using Bowen's formula of entropy [**5**] we have

$$h(g) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} (1/n) \log s_g(n, \varepsilon)$$

where $s_g(n, \varepsilon)$ is the largest possible cardinality of an (n, ε)-separated set for g. Recall that $E \subset G$ is (n, ε)-*separated for* g if $d_n(x, y) > \varepsilon$ for each $x, y \in E$, where $d_n(x, y)$ denotes the maximum of the distances from $g^i(x)$ to $g^i(y)$ for $i \in \{0, 1, \ldots, n-1\}$. Then, to show that $h(f) \geq h(g)$ it suffices to prove that

$$\limsup_{n \to \infty} (1/n) \log s_g(n, \varepsilon) \leq \limsup_{n \to \infty} (1/n) \log s_f(n, \varepsilon)$$

for every $\varepsilon > 0$.

Let $n \geq 2$ and let B be an (n, ε)-separated set for g. For $k = 0, 1, \ldots, n-2$, let

$$B_k = \{x \in B : k \text{ is the least integer such that } f^k(x) \in M\}$$

and let $B_{n-1} = B \setminus \cup_{k=0}^{n-2} B_k$. Note that, for $k = 0, 1, \ldots, n-1$, each of the sets B_k is $(k+1, \varepsilon)$-separated for g and f. Thus, $s_g(n, \varepsilon) \leq s_f(1, \varepsilon) + s_f(2, \varepsilon) + \cdots + s_f(n, \varepsilon)$. On the other hand, since each (k, ε)-separated set is also $(k+1, \varepsilon)$-separated, it follows that the sequence $\{s(n, \varepsilon)\}_{n \in \mathbb{N}}$ is non-decreasing. Consequently,

$$\begin{aligned}
\limsup_{n \to \infty} (1/n) \log s_g(n, \varepsilon) &\leq \limsup_{n \to \infty} (1/n) \log \left(\sum_{i=1}^{n} s_f(i, \varepsilon) \right) \\
&\leq \limsup_{n \to \infty} (1/n) \log \left(n \max_{1 \leq i \leq n} s_f(i, \varepsilon) \right) \\
&= \limsup_{n \to \infty} (1/n) \log \left(\max_{1 \leq i \leq n} s_f(i, \varepsilon) \right) \\
&= \limsup_{n \to \infty} (1/n) \log s_f(n, \varepsilon).
\end{aligned}$$

This ends the proof of the lemma. □

The strategy of the proof of the next results uses the ideas of the 'water functions' defined for circle maps (see for instance [**6**]).

LEMMA 2.3. *Let G be a graph, let $I = (a, b)$ be an interval of a graph and let $f : \overline{I} \to G$ be a continuous map. Assume that the interior of $f(I)$ is an interval of G. Then there exists a continuous map $g : \overline{I} \to G$ satisfying the following conditions:*

(a) $g|_{\{a,b\}} = f|_{\{a,b\}}$;
(b) *g and f are homotopic (relative to $\{a, b\}$);*
(c) *g is monotone;*
(d) *g is constant on each connected component of the set $\{x \in I : f(x) \neq g(x)\}$.*

Proof. Notice that $f(\overline{I})$ is homeomorphic either to a closed interval of the real line or to the circle. Assume first that $f(\overline{I})$ is homeomorphic to an interval. If $f(a) = f(b)$ then we define $g(x) = f(a)$ for all $x \in \overline{I}$ and we are done. Assume now that $f(a) \neq f(b)$. Then

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set

$$A_1 = \{x \in I : f(x) = f(a)\},$$
$$a_1 = \begin{cases} \max A_1, & \text{if } A_1 \neq \emptyset \\ a, & \text{otherwise,} \end{cases}$$
$$B_1 = \{x \in I : x > a_1 \text{ and } f(x) = f(b)\},$$
$$b_1 = \begin{cases} \min B_1, & \text{if } B_1 \neq \emptyset \\ b, & \text{otherwise.} \end{cases}$$

In $f(\overline{I})$ consider the linear ordering such that $f(a) < f(b)$ and define

$$g(t) = \begin{cases} f(a), & \text{if } t \in \langle a, a_1\rangle_I \\ \max\{f(z) : a_1 \le z \le t\}, & \text{if } t \in \langle a_1, b_1\rangle_I \\ f(b), & \text{if } t \in \langle b_1, b\rangle_I. \end{cases}$$

It is not difficult to see that g satisfies the required properties.

Now assume that $f(\overline{I})$ is homeomorphic to a circle. Since $f(I)$ is homeomorphic to an interval we have $f(a) = f(b) \notin f(I)$. Now consider the linear ordering induced in $f(I)$ by $f(x) < f(y)$ for each x and y close enough to a and b respectively. Then, we define $g(a) = g(b) = f(a)$ and, for $t \in I$,

$$g(t) = \max\{f(z) : z \in I,\ z \le t\}.$$

As above, g satisfies the required properties. □

LEMMA 2.4. *Let G be a graph, let I be an interval of G, let $f : \overline{I} \to G$ be a continuous map and let $c \in I$. Assume that f is monotone in the closure of the connected components of $I \setminus \{c\}$. Then f is monotone if and only if f is locally monotone at c.*

Proof. From the definition it follows that if f is monotone then it is locally monotone at c. Assume now that f is locally monotone at c and let $y \in \overline{I}$. Let $I = (a, b)$. Without loss of generality we may assume that $y \in \langle a, c\rangle_I$. If $y \in K(c)$ then $K(y) = K(c)$ and f is locally monotone at y. If $y \notin K(c)$ then, since $f|_{\langle a,c\rangle}$ is monotone, we can take a neighborhood V of $K(y)$ in $\langle a, c\rangle$ such that $V \cap K(c) = \emptyset$ and $f|_V$ is monotone. Then f is locally monotone at y. □

Proof of Proposition 2.1. We will construct the map g by modifying f in such a way that the following conditions hold for each basic interval I of (G, A, f):

(1) $g|_{\mathrm{Bd}(I)} = f|_{\mathrm{Bd}(I)}$;
(2) $g|_{\overline{I}}$ is homotopic (relative to $\mathrm{Bd}(I)$) to $f|_{\overline{I}}$;
(3) $g|_{\overline{I}}$ is monotone;
(4) $h(g) \le h(f)$.

By Lemma 2.2 the last condition can be replaced by the following one:

(5) g is constant on each connected component of the set $\{x \in I : f(x) \neq g(x)\}$.

To construct such a map g, we fix a basic interval I of (G, A, f) and we will construct a map $g : \overline{I} \to G$ satisfying (1)–(3) and (5). If $f(I) \cap B(G) = \emptyset$ then, by Lemma 2.3, we are done. So, we assume that $f(I) \cap B(G) \neq \emptyset$.

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Let us denote by W the closed set $(f^{-1}(B(G)) \cap \overline{I}) \cup \mathrm{Bd}(I)$. By Lemma 2.3 there exists a map f_1 verifying that $f_1|_W = f|_W$, f_1 is monotone restricted to the closure of each connected component of $\overline{I} \setminus W$ and (1), (2) and (5) hold with g replaced by f_1.

Let ε be smaller than the minimum of the diameters of all basic intervals of (G, A). Since $\overline{I}$ and G are compact metric spaces and f_1 is continuous it follows that f_1 is uniformly continuous. Hence, there exists δ such that $d(f_1(x), f_1(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Note that if a connected subset of G has diameter less than δ then its image cannot contain any basic interval (and, in particular, any circuit) of (G, A).

Let C be a connected component of $I \setminus W$ having diameter smaller than δ and such that $\mathrm{Bd}(C) \cap \mathrm{Bd}(I) = \emptyset$. Then, since $f(\mathrm{Bd}(C)) \subset B(G)$, it follows that $\overline{C}$ is a trivial interval for f_1. Consequently, $f_1|_{\overline{C}}$ is constant. Let $\mathcal{D}$ be the set formed by the components C of $I \setminus W$ such that $f_1|_{\overline{C}}$ is not constant. It follows that each $C \in \mathcal{D}$ either has diameter larger than or equal to δ or $\mathrm{Bd}(C) \cap \mathrm{Bd}(I) \neq \emptyset$. Thus, $\mathcal{D}$ is finite. Moreover, f_1 is constant on each component of $I \setminus (\cup_{C \in \mathcal{D}} C)$. Thus, there exist $k > 1$ and points $s_1 < s_2 < \cdots < s_{k-1}$ in I such that $f(s_i) \in B(G)$ for $i = 1, 2, \ldots, k-1$ and, for each connected component C of $I \setminus \{s_1, s_2, \ldots, s_{k-1}\}$, $f_1|_{\overline{C}}$ is monotone and $f_1(\overline{C})$ is contained in the closure of a basic interval. Now set $I = (s_0, s_k)$. For consistency we also will write $s_0 < s$ (respectively $s < s_k$) for each $s \in I$ (note that there is no ambiguity in doing this even when $s_0 = s_k$).

To end the proof of the proposition it is enough to show that for $i = 1, 2, \ldots, k$ there exists a continuous map $f_i : \overline{I} \to G$ such that:

(i) $f_i|_{\mathrm{Bd}(I)} = f|_{\mathrm{Bd}(I)}$, f_i is homotopic (relative to $\mathrm{Bd}(I)$) to f and f_i is constant on each connected component of the set $\{x \in I : f_i(x) \neq f(x)\}$;

(ii) the map $f_i|_{\langle s_0, s_i \rangle}$ is monotone and $f_i|_{\langle s_i, s_k \rangle} = f|_{\langle s_i, s_k \rangle}$ when $i < k$;

(iii) if $i < k$ then for each $j \in \{1, 2, \ldots, k-1\}$ either $f_i(s_j) \in B(G)$ or $f_i(s_l) = f_i(s_0)$ for $l = 1, 2, \ldots, j$ and $j < i$;

(iv) for each connected component C of $I \setminus \{s_1, s_2, \ldots, s_{k-1}\}$, $f_i|_{\overline{C}}$ is monotone and $f_i(\overline{C})$ is contained in the closure of a basic interval.

Note that then, in view of (i) and (ii), the map f_k satisfies (1)–(3) and (5) and so, the proposition holds by taking $g = f_k$.

Now we will construct the above sequence of maps by induction. Observe that the map f_1 constructed above verifies the conditions (i)–(iv). Now assume that there exists $1 \leq i < k$ and a map f_i verifying conditions (i)–(iv). We will show that there exists a map f_{i+1} verifying conditions (i)–(iv).

If $f_i|_{\langle s_0, s_{i+1} \rangle}$ is monotone then we set $f_{i+1} = f_i$ and we are done. Otherwise, by (ii), (iv) and Lemma 2.4, $f_i|_{\langle s_0, s_{i+1} \rangle}$ is not locally monotone at s_i. Let

$$R_i = \{x \in \langle s_0, s_i \rangle : f_i(x) = f_i(s_{i+1})\}.$$

Note that if $R_i \neq \emptyset$ then it is a closed connected subset of $\langle s_0, s_i \rangle$. In this case we set

$$f_{i+1}(x) = \begin{cases} f_i(s_{i+1}), & \text{if } x \in \langle \min R_i, s_{i+1} \rangle \\ f_i(x), & \text{otherwise.} \end{cases}$$

Clearly, f_{i+1} verifies (i) and (iv). Property (ii) follows from the fact that $\min R_i \leq s_i$ and $f_i|_{\langle s_0, s_i \rangle}$ is monotone. Finally, if for some j we have $f_{i+1}(s_j) \neq f_i(s_j)$ then

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$\min R_i \leq s_j \leq s_i$. Hence, $f_{i+1}(s_j) = f_i(s_{i+1}) \in B(G)$. Otherwise, we have $f_{i+1}(s_j) = f_i(s_j)$ and, hence, (iii) holds.

Assume now that $R_i = \emptyset$. Then, from (iv) it follows that if for some j we have $f_i(s_j)$, $f_i(s_{j+1}) \in B(G)$ and $f_i|_{\langle s_j, s_j+1\rangle}$ is not constant, then $f_i(\langle s_j, s_j+1\rangle)$ is the closure of a basic interval. Thus, since $f_i|_{\langle s_0, s_{i+1}\rangle}$ is not locally monotone at s_i, from (iii) and (iv) it follows that there exists $0 \leq l_i < i$ such that

$$f_i(\langle s_0, s_{l_i}\rangle) = \{f_i(s_0)\}, \quad f_i(\langle s_{l_i}, s_{l_i+1}\rangle) \subset f_i(\langle s_i, s_{i+1}\rangle), \quad f_i(\langle s_{l_i+1}, s_i\rangle) = \{f_i(s_i)\}.$$

Furthermore, $f_i((s_i, s_{i+1}))$ is contained in the closure of a basic interval. Hence, the interior of $f_i((s_0, s_{i+1}))$ is an interval. Now we set

$$\widetilde{R}_i = \{x \in \langle s_i, s_{i+1}\rangle : f_i(x) = f_i(s_0)\}.$$

Note that $\widetilde{R}_i$ is a non-empty closed connected set. Then, we set

$$f_{i+1}(x) = \begin{cases} f_i(s_0), & \text{if } x \in \langle s_0, \max \widetilde{R}_i\rangle \\ f_i(x), & \text{otherwise.} \end{cases}$$

Clearly, by Lemma 2.4 f_{i+1} verifies (i), (ii) and (iv). Property (iii) follows in a similar way as above. □

3. *Markov representatives*

Let (G, A, f) be a monotone representative of an action. We say that (G, A, f) is *Markov* if every element of $B(G)$ has a finite orbit. Note that if (G, A, f) is simplicial then it is Markov but not conversely.

PROPOSITION 3.1. *Let Λ be an action. Then*

$$h(\Lambda) = \inf\{h(g) : (G, A, g) \text{ is a Markov representative of } \Lambda\}.$$

To prove Proposition 3.1 we need some definitions and preliminary results. Assume that (G, f, A) is a monotone representative of an action. Set

$$C(A, f) = \cup_{x \in A \cup B(G)} K_f(x)$$

and denote the cover of G whose elements are the connected components of $C(A, f)$ and of $G \setminus C(A, f)$ by $\mathcal{A} = \mathcal{A}_f$. Also denote by $\mathcal{A}^n = \mathcal{A}_f^n$ the cover of G whose elements are $A_0 \cap f^{-1}(A_1) \cap \cdots \cap f^{-n+1}(A_{n-1})$ where $A_i \in \mathcal{A}$ for $i = 0, 1, \ldots, n-1$. Such an element will be denoted by $(A_0, A_1, \ldots, A_{n-1})$ (or $(A_0, A_1, \ldots, A_{n-1})_f$ if we need to specify the map).

We note that $\mathcal{A}$ is a partition of G. Moreover, it is not difficult to prove by induction that $\mathcal{A}^n$ is also a partition of G for each $n \geq 1$. Then, we have the following.

LEMMA 3.2. *Let (G, A, f) be a monotone representative of an action. Then*

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log \operatorname{Card} \mathcal{A}^n = \inf\left\{\frac{1}{n} \log \operatorname{Card} \mathcal{A}^n : n \in \mathbb{N}\right\}.$$

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Proof. Since $\mathcal{A}^n$ is a partition of G for each $n \geq 1$ we see that $\mathcal{N}(\mathcal{A}^n) = \operatorname{Card} \mathcal{A}^n$ for each $n \geq 1$ (recall that $\mathcal{N}(\mathcal{A}^n)$ is defined to be the minimal possible cardinality of a subcover chosen from $\mathcal{A}^n$). Then, from [**2**, equation (4.3) and Lemma 4.1.1] we see that

$$h(f, \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}(\mathcal{A}^n) = \lim_{n\to\infty} \frac{1}{n} \log \operatorname{Card} \mathcal{A}^n = \inf \left\{ \frac{1}{n} \log \operatorname{Card} \mathcal{A}^n : n \in \mathbb{N} \right\}.$$

Therefore, to end the proof of the lemma it is enough to show that $h(f) = h(f, \mathcal{A})$. This follows similarly to the proof of [**2**, Proposition 4.2.3]. The main changes come from the fact that in the interval, when we remove a point we obtain less than three connected components (this is the number 3 in [**2**, Lemmas 4.2.1 and 4.2.2]) while in a graph we obtain less than $\max\{\operatorname{val}(x) : x \in G\} + 1$ components. □

Let (G, A, f) be a monotone representative of an action. Denote

$$d_f = \operatorname{Card}\{x \in B(G) : \operatorname{Orb}_f(x) \text{ is infinite}\}.$$

Clearly $d_f = 0$ if and only if (G, A, f) is Markov.

The next lemma will be crucial in the proof of Proposition 3.1.

LEMMA 3.3. *Let (G, A, f) be a monotone representative of an action and assume that $d_f \neq 0$. Then for all $\varepsilon > 0$ there exists (G, A, g), a monotone representative of $[G, A, f]$, such that $d_g < d_f$ and $h(g) < h(f) + \varepsilon$.*

Proof. Fix $\varepsilon > 0$. By Lemma 3.2 we can choose an $m \in \mathbb{N}$ such that $\frac{1}{m} \log \operatorname{Card} \mathcal{A}_f^m < h(f) + \varepsilon$ and $m > \operatorname{Card} \operatorname{Orb}_f(x)$ for all $x \in A \cup B(G)$ with finite orbit. For all $j \in \mathbb{N}$ set

$$C_j(A, f) = \cup_{i=0}^{j} f^i(C(A, f)).$$

We claim that there exists $v \in B(G)$ such that $\operatorname{Orb}_f(v)$ is infinite and $f^k(v) \notin C_m(A, f)$ for all $k > m$. To prove the claim note that, since $d_f \neq 0$, there exists $x \in B(G)$ such that $\operatorname{Orb}_f(x)$ is infinite. Therefore, there exists $j \geq m$ such that $f^j(x) \in C_m(A, f)$ and $f^k(x) \notin C_m(A, f)$ for all $k > j$ because $f(C_m(A, f))$ is finite. Moreover, since A is finite and f-invariant we have that $f^j(x) \in f^m(B(G))$. That is, there exists $v \in B(G)$ such that $f^j(x) = f^m(v)$. Then $\operatorname{Orb}_f(f^m(v)) = \operatorname{Orb}_f(f^j(x))$, and hence $f^k(v) \notin C_m(A, f)$ for all $k > m$. This ends the proof of the claim.

Now let M be a connected component of $G \setminus C_m(A, f)$ which contains infinitely many points of $\operatorname{Orb}_f(v)$. Since $B(G) \subset C_m(A, f)$, M is an interval. So, set $M = (a, b)$ and let $c \in \overline{M}$ be an accumulation point of $\operatorname{Orb}_f(v) \cap \overline{M}$. Clearly either $\langle a, c\rangle_M$ or $\langle c, b\rangle_M$ contains infinitely many points of $\operatorname{Orb}_f(v)$. We suppose that $c \neq b$ and that $\langle c, b\rangle$ contains infinitely many points of $\operatorname{Orb}_f(z)$. The proof in the other case is analogous. Set

$$j_1 = \min\{j \in \mathbb{N} : f^j(v) \in \operatorname{Int}(\langle c, b\rangle_M)\},$$
$$j_2 = \min\{j \in \mathbb{N} : j > j_1 \text{ and } c < f^j(v) < f^{j_1}(v)\}$$

Also, fix a point $c < c^* < f^{j_2}(v)$. Now we define a continuous map $g : G \to G$ in the following way:

(g.1) $g|_{G\setminus \operatorname{Int}(\langle c^*, f^{j_1}(v)\rangle)} = f|_{G\setminus \operatorname{Int}(\langle c^*, f^{j_1}(v)\rangle)}$. In particular $g(c^*) = f(c^*)$ and $g(f^{j_1}(v)) = f(f^{j_1}(v))$;

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(g.2) $g(y) = g(f^{j_1}(v))$ for each $y \in \langle f^{j_2}(v), f^{j_1}(v)\rangle$;
(g.3) $g|_{\langle c^*, f^{j_2}(v)\rangle}$ is monotone.
Note that such a map is well defined, $f|_A = g|_A$ and g is homotopic to f relative to A. Hence, $[G, A, f] = [G, A, g]$. Moreover, the map g satisfies the following properties:
(p.1) $C_i(A, f) = C_i(A, g)$ for all $i = 0, 1, \ldots, m$;
(p.2) $f|_{C_m(A,f)} = g|_{C_m(A,f)}$;
(p.3) (G, A, g) is monotone.
Moreover, let $z \in B(G)$ be with finite orbit. Since $\operatorname{Card}\operatorname{Orb}_f(z) < m$ we have $\operatorname{Orb}_f(z) \subset C_m(A, f)$ and, hence, $\operatorname{Orb}_f(z) = \operatorname{Orb}_g(z)$ by (p.2). Thus, $d_g \leq d_f$. On the other hand observe that $g^i(v) = f^i(v)$ for all $i = 0, 1, \ldots, j_2$ and $g^{j_2+1}(v) = g^{j_1+1}(v)$. So $\operatorname{Orb}_g(v)$ is finite and, hence, $d_g \leq d_f - 1$.

To end the proof of the proposition it suffices to show that $h(g) < h(f) + \varepsilon$. To see this we start by showing that

$$\operatorname{Card}\mathcal{A}_f^m = \operatorname{Card}\mathcal{A}_g^m.$$

By the construction of g we have $\mathcal{A}_f = \mathcal{A}_g$. So, to prove that $\operatorname{Card}\mathcal{A}_f^m = \operatorname{Card}\mathcal{A}_g^m$ it is enough to see that $(A_0, A_1, \ldots, A_{m-1})_f \neq \emptyset$ if and only if $(A_0, A_1, \ldots, A_{m-1})_g \neq \emptyset$ for any $A_0, A_1, \ldots, A_{m-1} \in \mathcal{A}_f$. To prove this we fix $A_0, A_1, \ldots, A_{m-1} \in \mathcal{A}_f$. Assume that A_i is a connected component of $G \setminus C(A, f)$ for each $i = 0, 1 \ldots, m-1$ (then, each A_i is an interval contained in a basic interval of (G, A)). For each $i = 0, 1 \ldots, m-1$ we claim:
(1) both $(A_0, A_1, \ldots, A_i)_f$ and $(A_0, A_1, \ldots, A_i)_g$ are either empty or intervals;
(2) if $(A_0, A_1, \ldots, A_i)_f \neq \emptyset$ then $f^{i+1}((A_0, A_1, \ldots, A_i)_f) = g^{i+1}((A_0, A_1, \ldots, A_i)_g)$ is an interval and its endpoints belong to $C_{i+1}(A, f)$. Moreover the maps $f^{i+1}|_{(A_0,A_1,\ldots,A_i)_f}$ and $g^{i+1}|_{(A_0,A_1,\ldots,A_i)_g}$ are monotone.

Now we prove the claim by induction. Statements (1) and (2) for $i = 0$ follow from the definition of g. Then we assume that (1) and (2) hold for $0 \leq i - 1 < m - 1$ and we prove them for i. If $(A_0, A_1, \ldots, A_{i-1})_f = \emptyset$ there is nothing to prove. So assume that $(A_0, A_1, \ldots, A_{i-1})_f \neq \emptyset$. Then $(A_0, A_1, \ldots, A_i)_f \neq \emptyset$ is equivalent to $A_i \cap f^i((A_0, A_1, \ldots, A_{i-1})_f) \neq \emptyset$ which, in turn, is equivalent to $A_i \cap g^i((A_0, A_1, \ldots, A_{i-1})_g) \neq \emptyset$ which, finally, is equivalent to $(A_0, A_1, \ldots, A_i)_g \neq \emptyset$. Since $f^i((A_0, A_1, \ldots, A_{i-1})_f)$ is an interval and A_i is connected it follows that $A_i \cap f^i((A_0, A_1, \ldots, A_{i-1})_f)$ is an interval. Thus, since

$$(A_0, A_1, \ldots, A_i)_f = (f^i|_{(A_0,A_1,\ldots,A_{i-1})_f})^{-1}(A_i \cap f^i((A_0, A_1, \ldots, A_{i-1})_f)),$$

it follows that $(A_0, A_1, \ldots, A_i)_f$ is an interval because $f^i|_{(A_0,A_1,\ldots,A_{i-1})_f}$ is monotone. In a similar way we can show that the set $(A_0, A_1, \ldots, A_i)_g$ is an interval. Hence, (1) holds for i.

To prove (2) note that

$$f^{i+1}((A_0, A_1, \ldots, A_i)_f) = f(A_i \cap f^i((A_0, A_1, \ldots, A_{i-1})_f)) \subset f(A_i) \tag{3.1}$$

which is an interval because (G, A, f) is monotone. Moreover, $f^i|_{(A_0,A_1,\ldots,A_{i-1})_f}$ is monotone and $f|_{A_i}$ is monotone. Thus, $f^{i+1}((A_0, A_1, \ldots, A_i)_f)$ is an interval because $f^{i+1}|_{(A_0,A_1,\ldots,A_i)_f}$ is monotone. The same argument holds for g^{i+1} and $(A_0, A_1, \ldots, A_i)_g$. Lastly, since the map $f|_{A_i \cap f^i((A_0,A_1,\ldots,A_{i-1})_f)}$ is monotone, from equation (3.1) we obtain

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that the endpoints of the set $f^{i+1}((A_0, A_1, \ldots, A_i)_f)$ are the f-images of the endpoints of $A_i \cap f^i((A_0, A_1, \ldots, A_{i-1})_f)$. By the induction hypotheses, these endpoints coincide with the endpoints of the set $A_i \cap g^i((A_0, A_1, \ldots, A_{i-1})_g)$. Let z be one of these endpoints. If z is an endpoint of A_i then $z \in C(A, f)$ and, hence, $f(z) = g(z) \in C_1(A, f)$. If z is an endpoint of $f^i((A_0, A_1, \ldots, A_{i-1})_f)$ then, by the induction hypotheses, $z \in C_i(A, f)$. Consequently, $f(z) = g(z) \in C_{i+1}(A, f)$. This ends the proof of the claim.

Now we are ready to prove that the set $(A_0, A_1, \ldots, A_{m-1})_f$ is non-empty if and only if the set $(A_0, A_1, \ldots, A_{m-1})_g$ is non-empty. If A_i is a connected component of $G \setminus C(A, f)$ for each $i = 0, 1, \ldots, m-1$ then the statement follows from (1) of the above claim. So assume that there exists $i \in \{0, 1, \ldots, m-1\}$ such that A_i is a connected component of $C(A, f) = C(A, g)$ and A_j is a connected component of $G \setminus C(A, f)$ for each $j = 0, 1, \ldots, i-1$. By (1), $(A_0, A_1, \ldots, A_{i-1})_f$ and $(A_0, A_1, \ldots, A_{i-1})_g$ are either empty or intervals. If they are empty, then both $(A_0, A_1, \ldots, A_{m-1})_f$ and $(A_0, A_1, \ldots, A_{m-1})_g$ are empty and we are done. Otherwise, by (2), $f^i((A_0, A_1, \ldots, A_{i-1})_f) = g^i((A_0, A_1, \ldots, A_{i-1})_g)$. If $f^i((A_0, A_1, \ldots, A_{i-1})_f) \cap A_i = \emptyset$ then, again, $(A_0, A_1, \ldots, A_{m-1})_f = (A_0, A_1, \ldots, A_{m-1})_g = \emptyset$. So we may assume that $f^i((A_0, A_1, \ldots, A_{i-1})_f) \cap A_i \neq \emptyset$. Then, since $f^l(A_i) = g^l(A_i)$ for each $l = 0, 1, \ldots, m-1$, $f^l((A_0, A_1, \ldots, A_i)_f) = g^l((A_0, A_1, \ldots, A_i)_g)$ for each $l = i, i+1, \ldots, m-1$. Consequently, $(A_0, A_1, \ldots, A_{m-1})_f \neq \emptyset$ if and only if $(A_0, A_1, \ldots, A_{m-1})_g \neq \emptyset$ and, hence, $\operatorname{Card} \mathcal{A}_f^m = \operatorname{Card} \mathcal{A}_g^m$.

By Lemma 3.2,

$$h(g) \leq \frac{1}{m} \log \operatorname{Card} \mathcal{A}_g^m = \frac{1}{m} \log \operatorname{Card} \mathcal{A}_f^m < h(f) + \varepsilon.$$

This ends the proof of the lemma. □

Proof of Proposition 3.1. Using iteratively Lemma 3.3 with ε replaced by ε/d_f it follows that for each $\varepsilon > 0$ there exists (G, A, g), a Markov representative of $[G, A, f]$, such that $h(g) < h(f) + \varepsilon$. Clearly this implies that

$$\begin{aligned}&\inf\{h(g) : (G, A, g) \text{ is a Markov representative of } [G, A, f]\}\\ &\quad \leq \inf\{h(g) : (G, A, g) \text{ is a monotone representative of } [G, A, f]\}.\end{aligned}$$

Note that Proposition 2.1 implies that

$$h(\Lambda) = \inf\{h(g) : (G, A, g) \text{ is a monotone representative of } [G, A, f]\}.$$

Therefore, the result follows. □

4. *Simplicial representatives*

This section is devoted to prove the following.

PROPOSITION 4.1. *Let (G, A, f) be a Markov representative of an action. Then there exists a simplicial representative (G, A, g) of $[G, A, f]$ such that $h(g) \leq h(f)$.*

To prove Proposition 4.1 we need to introduce some notation and a technical lemma.

To compute the topological entropy of Markov maps, the standard technique is to use the so-called *Markov partition*. In our setting we will define it as follows. Let (G, A, f)

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be a representative of an action and let $P \supset A \cup B(G)$ be a finite set. The *Markov partition* $\mathcal{P}$ of G by P is defined to be the set of all connected components of $G \setminus P$. Note that then each element from $\mathcal{P}$ is an interval. Assume now that $f(P) \subset P$. Then, to $\mathcal{P}$ we shall associate a *transition matrix* $M = (m_{ij})$ as follows. Let $I_1, I_2, \ldots, I_n$ be the elements of $\mathcal{P}$. Then we set m_{ij} equal to the maximum number of pairwise disjoint subintervals K of I_i such that $f(\overline{K}) = \overline{I}_j$. Note that M is a non-negative integral matrix.

Let X be a positive (respectively non-negative) matrix. From the Perron–Frobenius theorem (see for instance [**8**]) it follows that $\rho(X)$, the spectral radius of X, is an eigenvalue of X which has positive (respectively non-negative) left and right eigenvectors.

If (G, A, f) is a Markov representative of an action and X is the transition matrix of the Markov partition of G by P, then $h(f) = \max(0, \log \rho(X))$ (this follows from [**2**, Theorem 4.4.5] with the obvious changes).

Next we state and prove a technical lemma.

LEMMA 4.2. *Let X and Y be non-negative matrices. Let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers such that $\lim_{n\to\infty} m_n = \infty$ and let X_n be the matrix $X + (1/m_n)Z$, where Z is any positive matrix. Denote by $v^{(n)}$ the positive left eigenvector corresponding to $\rho(X_n)$ and assume that $v^{(n)}Y \leq v^{(n)}X_n$ for all $n \geq 1$. Then $\rho(Y) \leq \rho(X)$.*

Proof. Since X_n converges to X, it follows that $\rho(X_n)$ converges to $\rho(X)$. Thus it suffices to show that $\rho(Y) \leq \rho(X_n)$ for all $n \geq 1$.

Let w be a non-negative right eigenvector of Y corresponding to $\rho(Y)$. Then

$$\rho(Y)v^{(n)}w = v^{(n)}\rho(Y)w = v^{(n)}Yw \leq v^{(n)}X_n w = v^{(n)}\rho(X_n)w = \rho(X_n)v^{(n)}w.$$

Since $v^{(n)}w > 0$, $\rho(Y) \leq \rho(X_n)$ and the lemma follows. □

Let (G, A, f) be a representative of an action. Then we define

$$P_f = A \cup \left(\bigcup_{n\in\mathbb{Z}^+} f^n(B(G)) \right) \quad \text{and} \quad E_f = P_f \setminus (A \cup B(G)).$$

Note that $f(P_f) \subset P_f$ and if (G, A, f) is Markov then P_f (and, hence, E_f) is finite. Moreover, $E_f = \emptyset$ if and only if (G, A, f) is simplicial.

PROPOSITION 4.3. *Let (G, A, f) be a Markov representative of an action such that $E_f \neq \emptyset$. Given $p \in E_f$ there exists a Markov representative (G, A, g_p) of $[G, A, f]$ such that $E_{g_p} \subset E_f \setminus \{p\}$ and $h(g_p) \leq h(f)$.*

Proof. Let $\mathcal{P} = \{I_1, I_2, \ldots, I_s\}$ be the Markov partition of G by P_f and let X be the transition matrix of $\mathcal{P}$.

Clearly, there exists $b \in P_f \setminus A$ such that $b \neq p$ and $f(b) = p$. Since $f(b) = p \notin B(G)$ there exist exactly two elements from $\mathcal{P}$ which are adjacent to $f(b)$. Without loss of generality we may assume that these elements are precisely the intervals I_1, I_2. Denote by e_1 (respectively e_2) the endpoint of I_1 (respectively I_2) which is different from $f(b)$. Also, denote by S_b the connected component of $f^{-1}(I_1 \cup I_2 \cup \{f(b)\})$ which contains b. Note that S_b is open in G and $f(P_f \cap S_b) = \{f(b)\}$. Hence, $S_b \cap A = \emptyset$. Denote by $\mathcal{A}$ the set of all connected components of $S_b \setminus P_f$. Notice that if $K \in \mathcal{A}$ then K is an interval and,

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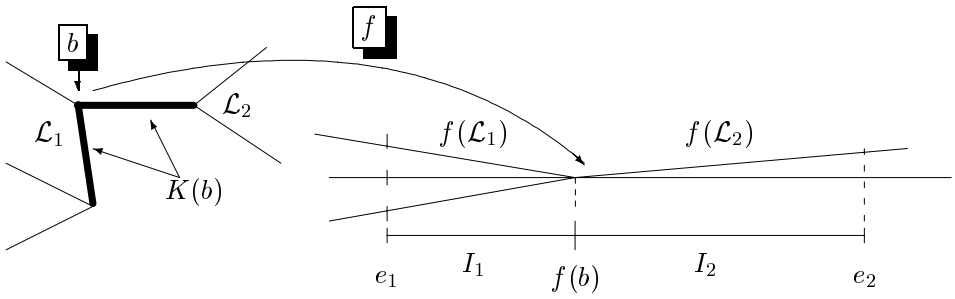


FIGURE 2. A schematic representation of the definition of I_1, I_2, e_1, e_2, $\mathcal{L}_1$ and $\mathcal{L}_2$.

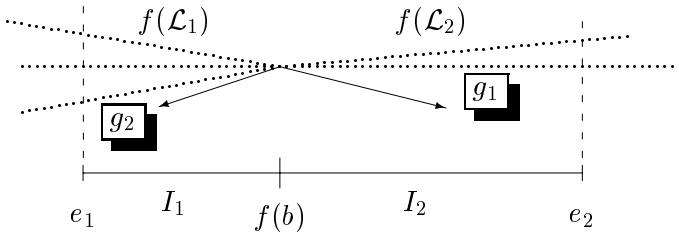


FIGURE 3. The point $f(b)$ is pushed towards e_2 (respectively e_1) by g_1 (respectively g_2).

for some $k \in \{1, 2\}$, either $f(K) = I_k$ or $f(K) = I_k \cup \{f(b)\}$ or $f(K) = \{f(b)\}$. Then, for $k = 1, 2$, we set

$$\mathcal{L}_k = \{K \in \mathcal{A} : I_k \subset f(K)\}$$

(see Figure 2). In particular, we have $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$.

In the rest of the proof we will use the following notation. Given $k \in \{1, 2\}$, we will denote by $\widehat{k}$ the unique element of $\{1, 2\} \setminus \{k\}$. Now, for $k = 1, 2$, we define g_k such that (G, A, g_k) is a Markov representative of $[G, A, f]$, $g_k|_{G \setminus S_b} = f|_{G \setminus S_b}$ and, for each $K \in \mathcal{A}$, $g_k|_{\overline{K}}$ is monotone and

$$g_k(K) = \begin{cases} I_1 \cup \{f(b)\} \cup I_2, & \text{if } K \in \mathcal{L}_k \\ \{e_{\widehat{k}}\}, & \text{otherwise,} \end{cases}$$

(see Figure 3). In other words, $g_k|_{S_b} = h_k \circ f|_{S_b}$ where h_k is a continuous map from $I_1 \cup \{f(b)\} \cup I_2$ to itself such that it expands I_k onto the entire $I_1 \cup \{f(b)\} \cup I_2$ and shrinks $I_{\widehat{k}}$ into $e_{\widehat{k}}$.

Note that g_k and f coincide on $P_f \setminus S_b$ and

$$g_k(P_f \cap S_b) \subset \{e_1, e_2\} \subset P_f \setminus \{p\}$$

for each $k = 1, 2$. Therefore, $g_k(P_f) \subset P_f$. So, since $A \cup B(G) \subset P_f$ we obtain that $P_{g_k} \subset P_f$ and, consequently, $E_{g_k} \subset E_f$. Also, $P_{g_k} \cap g_k^{-1}(p) \subset (P_f \cap f^{-1}(p)) \setminus \{b\}$. To see that (G, A, g_k) is a Markov representative of $[G, A, f]$ note that (G, A, g_k) is clearly

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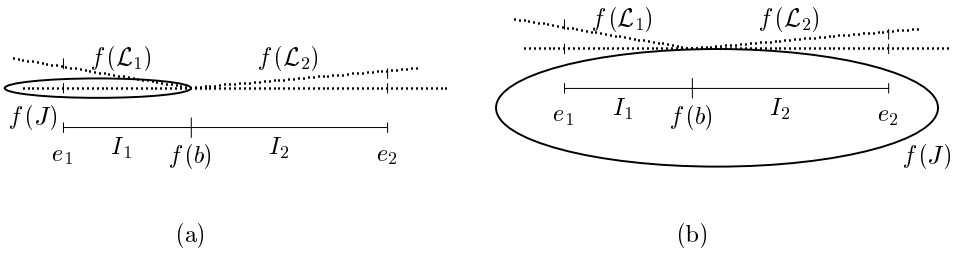


FIGURE 4. Cases (iv) and (v) for $J \cap S_b$ with $J \in \mathcal{P}$.

monotone, g_k and f coincide on A (recall that $A \cap S_b = \emptyset$) and P_{g_k} is finite because $P_{g_k} \subset P_f$.

To each of these maps g_k there corresponds a transition matrix $Y_k = (y_{ij}^k)$ of $\mathcal{P}$. Notice that, given $J \in \mathcal{P}$, the set $J \cap S_b$ is either:

(i) the empty set;
(ii) a subset of $K_f(b)$ (this happens when $f(J \cap S_b) = \{f(b)\}$);
(iii) an element of $\mathcal{L}_k$ for some k;
(iv) the union of two elements of $\mathcal{L}_k$ for some k (this happens when $\overline{J}$ is a circuit having b as a unique vertex and f is not locally one-to-one in b—see Figure 4(a));
(v) the union of one element of $\mathcal{L}_1$ with one element of $\mathcal{L}_2$ (this happens when $\overline{J}$ is a circuit having b as a unique vertex and f is locally one-to-one in b—see Figure 4(b)).

In cases (i), (ii) and (v) the rows of X and Y_k corresponding to J coincide. Moreover, for each $k \in \{1, 2\}$ and $i, j \in \{1, 2, \ldots, s\}$ we have $y_{ij}^k = x_{ij}$ when $j > 2$ and, for $j \in \{1, 2\}$, $y_{ij}^k = x_{ij} + \lambda_{ij}^k$, where

$$\lambda_{ij}^k = \begin{cases} 1, & \text{if } I_i \in \mathcal{L}_k^1 \\ 2, & \text{if } I_i \in \mathcal{L}_k^2 \\ -1, & \text{if } I_i \in \mathcal{L}_{\widehat{k}}^1 \\ -2, & \text{if } I_i \in \mathcal{L}_{\widehat{k}}^2 \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{L}_k^r$ denotes the set of elements of $\mathcal{P}$ whose intersection with S_b consists of r elements of $\mathcal{L}_k$.

Now, by using Lemma 4.2, we will prove that either $\rho(Y_1) \leq \rho(X)$ or $\rho(Y_2) \leq \rho(X)$. Clearly, for each $n \in \mathbb{N}$ there exists a $k_n \in \{1, 2\}$ such that

$$\sum_{i \in \mathrm{Ind}(\mathcal{L}_{k_n}^1)} w_i^{(n)} + 2 \sum_{i \in \mathrm{Ind}(\mathcal{L}_{k_n}^2)} w_i^{(n)} \leq \sum_{i \in \mathrm{Ind}(\mathcal{L}_{\widehat{k_n}}^1)} w_i^{(n)} + 2 \sum_{i \in \mathrm{Ind}(\mathcal{L}_{\widehat{k_n}}^2)} w_i^{(n)}, \tag{4.1}$$

where $w^{(n)} = (w_i^{(n)})$ denotes a positive left eigenvector corresponding to the spectral radius of the matrix $X + (1/n)Z$, where Z is a positive matrix, and

$$\mathrm{Ind}(\mathcal{L}_k^r) = \{i \in \{1, 2, \ldots, s\} : I_i \in \mathcal{L}_k^r\}.$$

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Clearly, for each $j \in \{1, 2, \ldots, s\}$, equation (4.1) gives

$$\sum_{i=1}^{s} w_i^{(n)} \lambda_{ij}^{k_n} \leq 0.$$

Thus, there exists a $k_0 \in \{1, 2\}$ and a sequence $\{m_n\}_{n\in\mathbb{N}}$ of positive integers such that $\lim_{n\to\infty} m_n = \infty$ and

$$\sum_{i=1}^{s} w_i^{(m_n)} y_{ij}^{k_0} \leq \sum_{i=1}^{s} w_i^{(m_n)} x_{ij}$$

for each $j \in \{1, 2, \ldots, s\}$. Therefore $v^{(n)} Y_{k_0} \leq v^{(n)} X_n$ where we have set $v^{(n)} = w^{(m_n)}$ and $X_n = X + (1/m_n)Z$. Then, by Lemma 4.2,

$$h(g_{k_0}) = \max(0, \log \rho(Y_{k_0})) \leq \max(0, \log \rho(X)) = h(f).$$

In summary, we have obtained a map $g_{p,1} = g_{k_0}$ such that $(G, A, g_{p,1})$ is a Markov representative of $[G, A, f]$, $E_{g_{p,1}} \subset E_f$, $P_{g_{p,1}} \cap g_{p,1}^{-1}(p) \subsetneq P_f \cap f^{-1}(p)$ and $h(g_{p,1}) \leq h(f)$.

Now, if p is yet in $E_{g_{p,1}}$ then there exists $b' \in P_{g_{p,1}} \setminus A$ such that $b' \neq p$ and $g_{p,1}(b') = p \in E_{g_{p,1}}$. So, we can repeat the above procedure with b' instead of b to obtain a Markov representative $(G, A, g_{p,2})$ of $[G, A, f]$ such that $E_{g_{p,2}} \subset E_{g_{p,1}}$, $P_{g_{p,2}} \cap g_{p,2}^{-1}(p) \subsetneq P_{g_{p,1}} \cap g_{p,1}^{-1}(p)$ and $h(g_{p,2}) \leq h(g_{p,1})$. Thus, after a finite number of steps we obtain a Markov representative (G, A, g_p) of $[G, A, f]$ such that $E(g_p) \subset E_f \setminus \{p\}$ and $h(g_p) \leq h(f)$. This ends the proof of the proposition. □

Proof of Proposition 4.1. If $P_f = A \cup B(G)$ (that is, $E_f = \emptyset$) then the proposition holds by taking $g = f$. Thus we assume that $E_f \neq \emptyset$. Since E_f is finite, by using Proposition 4.3 a finite number of times the proposition follows. □

5. *Proof of the Main Theorem*

Let (G, A, f) be a simplicial representative of an action Λ. We will denote by $\mathcal{M}(G, A)$ the set of all basic intervals of (G, A). Since (G, A, f) is simplicial, $A \cup B(G)$ is f-invariant and, hence, $\mathcal{M}(G, A)$ coincides with the Markov partition of G by $A \cup B(G)$. We note that the cardinality of $\mathcal{M}(G, A)$ does not depend on the choice of the representative of Λ and will be denoted by $p(\Lambda)$. We will denote by $\mathcal{S}(\Lambda)$ the set of all simplicial representatives of Λ and by $\mathcal{T}_n$ the set of all $(n \times n)$ non-negative integral matrices.

Given an action Λ we define a map $M : \mathcal{S}(\Lambda) \to \mathcal{T}_{p(\Lambda)}$ as follows. Let $(G, A, f) \in \mathcal{S}(\Lambda)$. Then we define $M(G, A, f)$ to be the transition matrix of $\mathcal{M}(G, A)$ by f. Recall that $h(f) = \max(0, \log \rho(M(G, A, f)))$.

Let $\mathcal{V}_n = (\mathbb{Z}^+)^n$. For each $A = (a_i)$, $B = (b_i) \in \mathcal{V}_n$ we will write $A \leq B$ if and only if $a_i \leq b_i$ for all $i = 1, 2, \ldots, n$. This relation is a partial ordering in $\mathcal{V}_n$. In what follows, for each action Λ, we will consider in $M(\mathcal{S}(\Lambda))$ the (partial) ordering induced by the ordering in $\mathcal{T}_{p(\Lambda)} = \mathcal{V}_{p(\Lambda)^2}$.

The following lemma shows that in $M(\mathcal{S}(\Lambda))$ there are only a finite number of minimal elements.

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LEMMA 5.1. *Any subset of $\mathcal{V}_n$ has a finite number of minimal elements.*

Proof. We use induction on n. The statement is clearly true for $n = 1$. So, assume that it holds for $n - 1 \geq 1$. Let $\mathcal{B}$ be the set of minimal elements of a given subset of $\mathcal{V}_n$ and let $A = (a_i) \in \mathcal{B}$. For $i \in \{1, 2, \ldots, n\}$ and $k < a_i$ we define

$$\mathcal{B}_A(i) = \{(t_r) \in \mathcal{B} : t_i < a_i\}$$

and

$$\mathcal{B}_A^k(i) = \{(t_r) \in \mathcal{B}_A(i) : t_i = k\}.$$

Clearly

$$\mathcal{B} \setminus \{A\} = \cup_{i=1}^n \mathcal{B}_A(i) = \cup_{i=1}^n \cup_{k=0}^{a_i - 1} \mathcal{B}_A^k(i).$$

By the induction hypothesis we have that $\mathcal{B}_A^k(i)$ is finite and, hence, so is $\mathcal{B}$. □

Proof of the Main Theorem. In view of Propositions 3.1 and 4.1 it follows that

$$\begin{aligned} h(\Lambda) &= \inf\{h(g) : (G, A, g) \in \mathcal{S}(\Lambda)\} \\ &= \max(0, \log \inf\{\rho(M(G, A, g)) : (G, A, g) \in \mathcal{S}(\Lambda)\}). \end{aligned}$$

Let $A, B \in \mathcal{T}_{p(\Lambda)}$. We recall that the spectral radius of A is equal to the limit when n tends to infinity of the n-root of the sum of the absolute values of the entries of A^n (see [**11**]). Hence, if $A \leq B$ then $\rho(A) \leq \rho(B)$. Consequently, by Lemma 5.1,

$$\inf\{\rho(M(G, A, g)) : (G, A, g) \in \mathcal{S}(\Lambda)\} = \min\{\rho(M(G, A, g)) : (G, A, g) \in \mathcal{S}^*(\Lambda)\},$$

where $\mathcal{S}^*(\Lambda)$ denotes the set of all $(G, A, g) \in \mathcal{S}(\Lambda)$ such that $M(G, A, g)$ is a minimal element of $M(\mathcal{S}(\Lambda))$. Let $(G, A, f) \in \mathcal{S}^*(\Lambda)$ be such that

$$\rho(M(G, A, f)) = \min\{\rho(M(G, A, g)) : (G, A, g) \in \mathcal{S}^*(\Lambda)\}.$$

Therefore, $h(f) = \max(0, \log \rho(M(G, A, f))) = h(\Lambda)$. Thus, to end the proof of the theorem, we only have to see that (G, A, f) is direct. Indeed, we will show that if $(G, A, g) \in \mathcal{S}(\Lambda)$ is not direct then $(G, A, g) \notin \mathcal{S}^*(\Lambda)$.

Assume that $(G, A, g) \in \mathcal{S}(\Lambda)$ is not direct. Then there is a point $v \in B(G)$ which is free, and a neighborhood U of $K(v)$ such that $g(v)$ belongs to the boundary of $g(U)$ (with the topology of $g(U)$). Therefore, there exists a unique basic interval J such that $g(v) \in \overline{J}$ and $J \cap g(U) \neq \emptyset$. Denote the endpoint of J which is different from $g(v)$ by e and let S_v be the connected component of $g^{-1}(\overline{J})$ containing v. Now we define a new map $\widetilde{g} : G \to G$ by

$$\widetilde{g}(x) = \begin{cases} g(x), & \text{if } x \in \mathrm{Cl}(G \setminus S_v) \\ \{e\}, & \text{otherwise.} \end{cases}$$

Observe that $(G, A, \widetilde{g}) \in \mathcal{S}(\Lambda)$ and that $M(G, A, \widetilde{g}) \lneqq M(G, A, g)$. Thus, $(G, A, g) \notin \mathcal{S}^*(\Lambda)$. □

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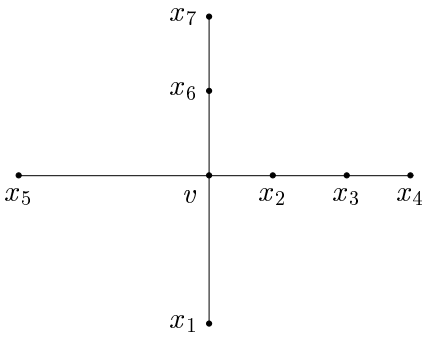


FIGURE 5. The tree T of Example 5.2.

In the rest of this section we will consider the problem of obtaining a finite algorithm to compute the entropy of an action. We start by considering the case of an action $[T, A, f]$ where T is a tree. We note that, since a tree is uniquely arcwise connected, then (T, A, g) is a representative of $[T, A, f]$ if and only if $g|_A = f|_A$. Moreover, the transition matrix of a simplicial representative (T, A, g) of $[T, A, f]$ is uniquely determined by the choice of g on $B(T) \setminus A$. Since there are finitely many possibilities for $g|_{B(T)\setminus A}$ in order that (T, A, g) is simplicial, the set of transition matrices of all direct representatives of $[T, A, f]$ is finite. Thus, the Main Theorem gives in practice a finite algorithm to compute $h([T, A, f])$.

Example 5.2. Consider the action $[T, A, f]$ where T is the tree shown in Figure 5, $A = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $f(x_i) = x_{i+1}$ for $i = 1, \ldots, 6$ and $f(x_7) = x_1$. Let (T, A, g) be a direct representative of $[T, A, f]$. Clearly, $g(v) \in \{x_3, x_2, v, x_6, x_7\}$. So, the set of transition matrices of all direct representatives of $[T, A, f]$ has five elements. The entropy of g turns out to be $\log 1.465571232\ldots$ when $g(v) \in \{x_3, x_7\}$ and $\log 1.754877667\ldots$ when $g(v) \in \{x_2, v, x_6\}$. So, $h([T, A, f]) = \log 1.465571232\ldots$ and there are (essentially) two direct models attaining this minimal entropy.

In the case of graphs the situation is different and we do not know any finite algorithm to compute the entropy of an action. However, proving the following conjecture could be a progress in this direction.

CONJECTURE. *For any action Λ, the set of transition matrices of all direct representatives of Λ is finite.*

Remark. Despite the above conjecture, it is easy to see that the set of transition matrices of all *simplicial* representatives of an action is not finite.

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