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On the geodesic flow on CAT(0) spaces

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Abstract. Under certain assumptions on CAT(0) spaces, we show that the geodesic flow is topologically mixing. In particular, the Bowen–Margulis' measure finiteness assumption used by Ricks [Flat strips, Bowen–Margulis measures, and mixing of the geodesic flow for rank one CAT(0) spaces. *Ergod. Th. & Dynam. Sys.* **37** (2017), 939–970] is removed. We also construct examples of CAT(0) spaces that do not admit finite Bowen–Margulis measure.

Key words: geodesic flow, mixing, CAT(0) space, topological dynamics
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1. *Introduction*

Topological transitivity and topological mixing of the geodesic flow are two dynamical properties that have been studied extensively for Riemannian manifolds. Anosov [**1**] first proved topological transitivity of the geodesic flow for compact manifolds of negative curvature. Eberlein [**10**] proved topological mixing for a large class of manifolds. In particular, he established topological mixing for complete finite volume manifolds of negative curvature as well as for compact manifolds of non-positive curvature not admitting isometric, totally geodesic embedding of $\mathbb{R}^2$. The latter is the class of so-called visibility manifolds (see [**11**] and [**10**]) and, in modern terminology, it can equivalently be described as the class of compact CAT(0) manifolds that are hyperbolic in the sense of Gromov (see [**3**, Ch. II, Theorem 9.33]). For certain classes of quotients of CAT(-1) spaces by discrete groups of isometries, topological mixing was shown in [**5**]. All the above results are along the lines of Eberlein's approach where the following two properties of the universal covering were essential:

(u) uniqueness of geodesic lines joining two boundary points at infinity; and
(c) the distance of asymptotic geodesics tends, up to re-parametrization, to zero.

Recently Ricks (see [**14**]) made a significant generalization by proving mixing of the Bowen–Margulis measure under the geodesic flow on all rank-one CAT(0) spaces under

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the natural assumption that the Bowen–Margulis measure (also constructed in [**14**] for CAT(0) spaces) is finite. In this work, we extend the classical approach of Eberlein to show topological mixing of the geodesic flow for a class of spaces X that are quotients of a CAT(0) space $\widetilde{X}$ by a non-elementary discrete group of isometries Γ such that $\partial\widetilde{X}$ is connected and equal to the limit set $\Lambda(\Gamma)$. We impose certain conditions on the CAT(0) space $\widetilde{X}$ (see Standing Assumptions, after Definition 4), but we allow the Bowen–Margulis measure to be infinite. Observe that the action of Γ is not assumed to be co-compact.

In [**9**, Theorem 1.2], finite volume n-dimensional manifolds ($n \geq 2$) of pinched negative curvature were constructed whose fundamental group is convergent. The latter implies, in particular, that the Bowen–Margulis measure is infinite. From these examples, one can easily construct CAT(0) spaces that are hyperbolic in the sense of Gromov, not admitting finite Bowen–Margulis measure and satisfy our Standing Assumptions.

2. *Definitions and preliminaries*

Let Y be a proper metric space.

Definition 1. A geodesic segment in Y is an isometric map $h : [a, b] \to Y$. If $x = h(a)$ and $y = h(b)$, then a geodesic segment joining x and y will be denoted by $[x, y]$ and its interior by (x, y). Let $I = [0, +\infty)$ or $I = (-\infty, +\infty)$. A geodesic line (respectively, geodesic ray) in Y is a local isometric map $h : I \to Y$ where $I = (-\infty, +\infty)$ (respectively, $I = [0, +\infty)$). A closed geodesic is a local isometric map $h : I \to Y$ that is a periodic map. A metric space is called geodesic if every two points can be joined by a geodesic segment. A geodesic metric space is called geodesically complete if every geodesic segment extends to a geodesic line.

Definition 2. We say that the metric space Y is a Hadamard space if Y is simply connected, complete, geodesic and has curvature at most zero.

We refer the reader to [**2**] and [**3**] for a systematic treatment of Hadamard spaces.

Throughout this paper, X denotes the quotient space $\widetilde{X}/\Gamma$ where $\widetilde{X}$ is a Hadamard space and Γ a non-elementary discrete group acting freely by isometries on $\widetilde{X}$. In §4.2 X will, in addition, be a two-dimensional surface. Denote by $p : \widetilde{X} \to X$ the covering projection. Here Γ is isomorphic to $\pi_1(X)$ and we make no distinction between Γ and $\pi_1(X)$.

The visual boundary $\partial\widetilde{X}$ of $\widetilde{X}$ is defined by means of geodesic rays (see [**8**, Ch. 2, §3 p. 21]). Recall that two geodesic rays g_1, g_2 (or geodesics) in $\widetilde{X}$ are called *asymptotic* if $d(g_1(t), g_2(t))$ is bounded for all $t \in \mathbb{R}^+$. Equivalently, if $g(+\infty)$ denotes the boundary point determined by the geodesic ray $g|_{[0,+\infty)}$, two geodesic rays g_1, g_2 (or geodesics) in $\widetilde{X}$ are asymptotic if $g_1(+\infty) = g_2(+\infty)$. Since $\widetilde{X}$ is a CAT(0) space, geodesic lines and geodesic rays in $\widetilde{X}$ are global isometric maps. Note also that geodesic segments with given endpoints are unique. This is just the Hadamard Cartan theorem for CAT(0) spaces (see [**2**, Theorem 4.5, Ch. I]). Moreover, we have uniqueness of geodesic rays in the following sense: for any $x \in \widetilde{X}$, $\xi \in \partial\widetilde{X}$ there is a unique geodesic ray $r : [0, \infty) \to \widetilde{X} \cup \partial\widetilde{X}$ such that $r(0) = x$, $r(\infty) = \xi$ (see [**3**, Ch. II, Proposition 8.2]). The corresponding result for geodesic lines is not true. However, the following theorem holds (see [**2**, Corollary 5.8(ii), Ch. I]).

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THEOREM 3. (Flat strip theorem) *If $f, g : \mathbb{R} \to \widetilde{X}$ are two geodesics with $f(\infty) = g(\infty)$ and $f(-\infty) = g(-\infty)$, then f and g bound a flat strip, that is, a convex region isometric to the convex hull of two parallel lines in the flat plane.*

Definition 4. We say that $f : \mathbb{R} \to \widetilde{X}$ is a unique geodesic if for any geodesic $g : \mathbb{R} \to \widetilde{X}$ with $f(-\infty) = g(-\infty)$ and $f(\infty) = g(\infty)$, g is a re-parametrization of f. We say that $f : \mathbb{R} \to \widetilde{X}$ is a closed (respectively, non-closed) geodesic if $p(f)$ is closed, that is, periodic (respectively, non-closed; that is, not periodic) in X.

The *limit set* $\Lambda(\Gamma)$ of Γ is defined to be $\Lambda(\Gamma) = \overline{\Gamma x} \cap \partial\widetilde{X}$, where x is an arbitrary point in $\widetilde{X}$. Since the action of Γ on $\widetilde{X}$ is not assumed to be co-compact, it does not follow, in general, that $\Lambda(\Gamma) = \partial\widetilde{X}$. However, we assume throughout that $\Lambda(\Gamma) = \partial\widetilde{X}$.

For each non-trivial element $\varphi \in \Gamma$ and each $x \in \widetilde{X}$ the sequence $\varphi^n(x)$ (respectively, $\varphi^{-n}(x)$) has a limit point $\varphi(+\infty)$ (respectively, $\varphi(-\infty)$) in $\partial\widetilde{X}$ when $n \to +\infty$. This is equivalent to saying that Γ has no elliptic elements that hold as the action of Γ is assumed to be free (see [**2**, Ch. II, Proposition 3.2]). However, as Γ can contain parabolic elements, $\varphi(+\infty)$ and $\varphi(-\infty)$ may coincide. In the case ϕ is a hyperbolic element of Γ, the point $\varphi(+\infty)$ is called *attractive* and the point $\varphi(-\infty)$ is called the *repulsive* point of φ.

As Γ is a discrete group of isometries of $\widetilde{X}$ we have the following result from [**7**].

PROPOSITION 5. [**7**, Proposition 1.7, Ch. II] *Let φ be a hyperbolic element of Γ and ψ any element of Γ. If $\mathrm{Fix}(\psi)$ is the set of points in $\partial\widetilde{X}$ fixed by the action of ψ, then either*

$$\{\varphi(-\infty), \varphi(+\infty)\} \cap \mathrm{Fix}(\psi) = \emptyset \quad \text{or} \quad \{\varphi(-\infty), \varphi(+\infty)\} \subset \mathrm{Fix}(\psi).$$

It follows that f, g are two closed non-homotopic geodesics, then f and g cannot be asymptotic. Thus, if $F_h \subset \partial\widetilde{X}$ denotes the set of limit points of all hyperbolic elements of Γ, then F_h splits as the disjoint union

$$F_h = F_h^u \sqcup F_h^{nu},$$

where

$$F_h^{nu} := \{\xi \in \partial\widetilde{X} \mid \xi = g(+\infty) \text{ for some } g \text{ closed and non-unique}\}$$

and

$$F_h^{u} := \{\xi \in \partial\widetilde{X} \mid \xi = g(+\infty) \text{ for some } g \text{ closed and unique}\}.$$

Observe that F_h^u, F_h^{nu} are invariant under the action of Γ.

Standing Assumptions. Let $X = \widetilde{X}/\Gamma$ where $\widetilde{X}$ is a proper and geodesically complete CAT(0) space with $\partial\widetilde{X}$ connected and Γ a non-elementary discrete group of isometries acting freely on $\widetilde{X}$ with $\Lambda(\Gamma) = \partial\widetilde{X}$ such that $\widetilde{X}$ satisfies the following conditions:

(Δ) the space $\widetilde{X}$ is hyperbolic in the sense of Gromov;

(U) if f is a non-closed geodesic in $\widetilde{X}$, then f is unique;

(C) if f, g are asymptotic geodesics with $f(+\infty) = g(+\infty) \in \partial\widetilde{X} \setminus F_h^{nu}$, then for appropriate parametrizations of f, g

$$\lim_{t\to\infty} d(f(t), g(t)) = 0;$$

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(D) the set

$$\{(g(+\infty), g(-\infty)) : g \text{ is closed and unique}\}$$

is dense in $\partial^2 \widetilde{X}$.

The geodesic flow for a complete geodesic metric space X is defined by the map

$$\mathbb{R} \times GX \to GX,$$

where GX is the space of all local isometric maps $g : \mathbb{R} \to X$ (see §2.1 for a precise definition and properties) and the action of $\mathbb{R}$ is given by right translation, that is, for all $t \in \mathbb{R}$ and $g \in GX$, $(t, g) \to t \cdot g$ where $t \cdot g : \mathbb{R} \to X$ is the geodesic defined by $(t \cdot g)(s) = g(s + t)$, $s \in \mathbb{R}$.

Definition 6. The geodesic flow $\mathbb{R} \times GX \to GX$ is topologically transitive if given any non-empty open sets $\mathcal{O}$ and $\mathcal{U}$ in GX, there exists a sequence $t_n \to \infty$ such that $t_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for all n.

Definition 7. The geodesic flow $\mathbb{R} \times GX \to GX$ is topologically mixing if given any non-empty open sets $\mathcal{O}$ and $\mathcal{U}$ in GX, there exists a real number $t_0 > 0$ such that for all $|t| \geq t_0$, $t \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$.

The main theorem of this paper is the following.

THEOREM 8. *Let X be the quotient of a CAT(0) space $\widetilde{X}$ by a non-elementary discrete group of isometries Γ acting freely on $\widetilde{X}$ such that $\partial \widetilde{X}$ is connected and equal to the limit set $\Lambda(\Gamma)$. If conditions (Δ), (U), (C) and (D) stated above are satisfied, then the geodesic flow $\mathbb{R} \times GX \to GX$ is topologically mixing.*

We use the following results from [**7**]. Let Z be a proper δ-hyperbolic geodesic metric space and let Γ be a group of isometries of Z acting properly discontinuously on Z such that the cardinality of the limit set $\Lambda(\Gamma)$ is infinite (in fact, the results in the following are used in cases where $\Lambda(\Gamma) = \partial Z$).

PROPOSITION 9. [**7**, Corollaries 4.2 and 6.3, Ch. II] *There exists an orbit of Γ dense in $\Lambda(\Gamma) \times \Lambda(\Gamma)$. In particular, for every $\xi \in \Lambda(\Gamma)$, the orbit $\Gamma \cdot \xi$ is dense in $\Lambda(\Gamma)$.*

PROPOSITION 10. [**7**, Corollary 5.1, Ch. II] *The set*

$$\{(\phi(+\infty), \phi(-\infty)) : \phi \in \Gamma \textit{ is a hyperbolic element}\}$$

is dense in $\Lambda(\Gamma) \times \Lambda(\Gamma)$.

Recall also that the boundary ∂Y of a complete geodesic metric space Y can be defined, as a topological space, using Busemann functions as explained in [**2**, Ch. II, §1], where it is shown that the function $\alpha : Y \times Y \times Y \to \mathbb{R}$ given by

$$\alpha(y, x, x') := d(x', y) - d(x, y)$$

extends to a continuous function

$$\alpha : (Y \cup \partial Y) \times Y \times Y \to \mathbb{R},$$

which is Lipschitz with respect to the second and third variable.

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By [**2**, Lemma 2.2, Ch. II] and the discussion following it, we have that the topology given to $\partial\widetilde{X}$ via Busemann functions coincides with the compact-open topology (given to $\partial\widetilde{X}$ using geodesic rays and the fact that $\widetilde{X}$ is a CAT(0) space). Thus, we obtain a continuous function

$$\alpha : (\partial\widetilde{X} \cup \widetilde{X}) \times \widetilde{X} \times \widetilde{X} \to \mathbb{R}$$

given by

$$\alpha(y, x, x') := d(x', y) - d(x, y) \tag{1}$$

for $(y, x, x') \in \widetilde{X} \times \widetilde{X} \times \widetilde{X}$ and

$$\alpha(\xi, x, x') := \lim_{n\to\infty} \alpha(y_n, x, x') \tag{2}$$

for $(\xi, x, x') \in \partial\widetilde{X} \times \widetilde{X} \times \widetilde{X}$ where $y_n \to \xi$ (see [**2**, Ch. II, Proposition 2.5]).

This function, called the *generalized Busemann function*, in fact, generalizes the classical Busemann function whose definition makes sense in our context: for arbitrary $\xi \in \partial\widetilde{X}$ and $x \in \widetilde{X}$, the restriction

$$\alpha(\xi, x, \cdot) \equiv \alpha|_{\{\xi\}\times\{x\}\times\widetilde{X}}$$

is simply the Busemann function associated to the unique geodesic ray from x to ξ.

We use the following facts about the generalized Busemann function.

LEMMA 11.

(a) *The generalized Busemann function α is Lipschitz with respect to the second and third variable with Lipschitz constant* 1.

(b) *If $f, g \in G\widetilde{X}$ with $f(-\infty) = g(+\infty)$, then*

$$\alpha(g(+\infty), g(0), f(t)) = t + \alpha(g(+\infty), g(0), f(0)).$$

(c) *If $f, g \in G\widetilde{X}$ are asymptotic geodesics, then there exists a unique re-parametrization $\overline{f}$ of f such that $\alpha(f(+\infty), \overline{f}(0), g(0)) = 0$.*

A proof of (a) can be found in [**2**, Ch. II, §1] and the proof given for Lemma 2.3 in [**5**] holds verbatim for (b) and (c).

Definition 12. We say that a geodesic $h \in G\widetilde{X}$ belongs to the stable set $W^s(g)$ of a geodesic g if g, h are asymptotic. Two points $x, x' \in \widetilde{X}$ are said to be equidistant from a point $\xi \in \partial\widetilde{X}$ if $\alpha(\xi, x, x') = 0$.

We say that a geodesic $h \in G\widetilde{X}$ belongs to the strong stable set $W^{ss}(g)$ of a geodesic g if $h \in W^s(g)$ and $g(0), h(0)$ are equidistant from $g(\infty) = h(\infty)$.

Similarly, if $h, g \in GX$, we say that $h \in W^{ss}(g)$ (respectively, $W^s(g)$) if there exist lifts $\widetilde{h}, \widetilde{g} \in G\widetilde{X}$ of h, g such that $\widetilde{h} \in W^{ss}(\widetilde{g})$ (respectively, $W^s(\widetilde{g})$).

We next restate condition (C) using the terminology of strong stable sets.

PROPOSITION 13. *Let $f, g \in G\widetilde{X}$ with $f \in W^{ss}(g)$. Assume $f(+\infty) = g(+\infty) \in \partial\widetilde{X} \setminus F_h^{nu}$, that is, if $h \in W^s(g)$, then h is not a non-unique closed geodesic. Then* $\lim_{t\to\infty} d(f(t), g(t)) = 0$.

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The proof of the above proposition is identical to that given in [**5**, Proposition 2.2]. We conclude this section with the following result.

LEMMA 14. *Let $x, y \in \widetilde{X}$ and $\xi \in \partial\widetilde{X}$ with $\alpha(\xi, x, y) = 0$. For any open set O in $\widetilde{X}$ containing y, there exist open sets C and D of $\widetilde{X}$ and $\partial\widetilde{X}$, respectively, such that $(x, \xi) \in C \times D$ and for every $(x', \xi') \in C \times D$ there exists $y' \in O$ with $\alpha(\xi', x', y') = 0$.*

Proof. Given an open set O containing y, choose $\varepsilon > 0$ so that the open ball $B(y, \varepsilon) \subset O$. As $\alpha(\xi, x, y) = 0$, by continuity of α we may find open sets $C \subset \widetilde{X}$ and $D \subset \partial\widetilde{X}$ such that $(x, \xi) \in C \times D$ and

$$(x', \xi') \in C \times D \Rightarrow |\alpha(\xi', x', y)| < \varepsilon.$$

These are the desired open sets.

Given $(x', \xi') \in C \times D$, let r' be the geodesic ray with $r'(0) = y$ and $r'(+\infty) = \xi'$. Denote by f' any geodesic line that extends r'. By Lemma 11(b),

$$|\alpha(\xi', x', f'(t)) - \alpha(\xi', x', f'(0))| = |t|.$$

Let $t_0 = \alpha(\xi', x', y)$. Then $|t_0| < \varepsilon$ and $\alpha(\xi', x', f'(t_0)) = 0$. Since $f'(t) \in O$ for $|t| < \varepsilon$ we have $y' := f'(t_0) \in O$ and

$$\alpha(\xi', x', y') = \alpha(\xi', x', f'(t_0)) = 0. \qquad \square$$

2.1. *Properties of geodesics and geodesic rays.* Let GX be the space of all local isometric maps $g : \mathbb{R} \to X$. As usual, the image of such a g will be referred to as a geodesic in X. Consider also the space $G\widetilde{X}$ of all isometric maps $g : \mathbb{R} \to \widetilde{X}$. Both spaces GX and $G\widetilde{X}$ are equipped with the compact-open topology. Moreover, the space $G\widetilde{X}$ with the compact-open topology is metrizable (see [**12**, 8.3.B]) and second countable.

We denote both projections $\widetilde{X} \to X$ and $G\widetilde{X} \to GX$ by p. Denote by $R\widetilde{X}$ the set of all geodesic rays in $\widetilde{X}$, that is, the set of all isometric maps $r : [0, \infty) \to \widetilde{X}$ equipped with the compact open topology.

PROPOSITION 15. *The function $\varrho : R\widetilde{X} \to \widetilde{X} \times \partial\widetilde{X}$ given by*

$$\varrho(r) = (r(0), r(\infty)),$$

where $r(\infty)$ denotes the unique boundary point determined by r, is a homeomorphism.

Proof. By uniqueness of geodesic rays the inverse function ϱ^{-1} is well defined for all $(x, \xi) \in \widetilde{X} \times \partial\widetilde{X}$.

We first show continuity of ϱ^{-1}. Let $(x_0, \xi_0) \in \widetilde{X} \times \partial\widetilde{X}$ and let $\{x_n\} \subset \widetilde{X}$, $\{\xi_n\} \subset \partial\widetilde{X}$ be sequences with $x_n \to x_0$ and $\xi_n \to \xi_0$. Denote by r_n, $n \geq 0$, the unique geodesic ray with $r_n(0) = x_n$ and $r_n(\infty) = \xi_n$. Similarly, denote by q_n, $n \geq 1$ the unique geodesic ray with $q_n(0) = x_0$ and $q_n(\infty) = \xi_n$. The assumption $\xi_n \to \xi_0$ means, by definition, that

$$q_n \to r_0 \tag{3}$$

and we need to show $r_n \to r_0$. For each $n \in \mathbb{N}$, the geodesic rays q_n and r_n are asymptotic, hence, the distance function $t \to d(q_n(t), r_n(t))$, $t \geq 0$, is convex

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(see [**2**, Ch. I, Proposition 5.4]) and bounded. Therefore, it is decreasing and $d(x_0, x_n)$ is an upper bound for all $t \geq 0$ because

$$d(x_0, x_n) = d(q_n(0), r_n(0)) \geq d(q_n(t), r_n(t)). \tag{4}$$

Let $\mathcal{O}$ be a neighborhood of r_0 of the form

$$\mathcal{O}(r_0, K, \varepsilon) = \{r' \in R\widetilde{X} \mid d(r'(t), r_0(t)) < \varepsilon \text{ for all } t \in [0, K]\}. \tag{5}$$

Find $n_1 \in \mathbb{N}$ such that $d(x_0, x_n) < \varepsilon/2$ for all $n > n_1$, which, by (4), yields

$$d(q_n(t), r_n(t)) < \frac{\varepsilon}{2} \quad \text{for all } n > n_1 \text{ and } t \in [0, K].$$

As $q_n \to r_0$ we may find $n_2 \in \mathbb{N}$ such that $q_n \in \mathcal{O}(r_0, K, \varepsilon/2)$ for all $n > n_2$, which means

$$d(q_n(t), r_0(t)) < \frac{\varepsilon}{2} \quad \text{for all } n > n_2 \text{ and } t \in [0, K].$$

Combining the last two inequalities we have

$$d(r_n(t), r_0(t)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{3} \quad \text{for all } n > \max\{n_1, n_2\} \text{ and } t \in [0, K],$$

which shows that $r_n \in \mathcal{O}$ for n large enough. Thus, $r_n \to r_0$ as desired.

For the continuity of ϱ, let $\{r_n\} \subset R\widetilde{X}$ be a sequence converging to a geodesic ray r_0. Clearly, $r_n(0) \to r_0(0)$ and we need to check that $r_n(+\infty) \to r_0(+\infty)$. This amounts to verifying that

$$q_n \to r_0,$$

where q_n, $n \geq 1$, is the unique geodesic ray with $q_n(0) = x_0$ and $q_n(\infty) = r_n(+\infty)$. Since for each n, the geodesic rays r_n and q_n are asymptotic, an argument similar to that given above for ϱ^{-1}, shows that for an arbitrary neighborhood $\mathcal{O}$ of r_0, $q_n \in \mathcal{O}$ for n large enough. □

PROPOSITION 16. *Let f be a unique geodesic in $G\widetilde{X}$ and $\{f_n\} \subset G\widetilde{X}$ a sequence of geodesics with $f_n(+\infty) \to f(+\infty)$ and $f_n(-\infty) \to f(-\infty)$. Then we may re-parametrize $\{f_n\}$ such that $f_n \to f$.*

Proof. Fix $x_0 := f(0)$ as base point and choose sequences $\{x_n\}$, $\{y_n\}$ with $x_n, y_n \in \operatorname{Im} f_n$ for each n, such that

$$x_n \to f(+\infty), \quad y_n \to f(-\infty) \quad \text{and} \quad d(x_0, x_n) = d(x_0, y_n)$$

for all n. Denote by m_n the midpoint of the segment $[x_n, y_n] \subset \operatorname{Im} f_n$ and, by passing if necessary to a subsequence, we may assume that $\{m_n\}$ converges to $m \in \widetilde{X} \cup \partial\widetilde{X}$. Recall that equivalence classes of unbounded sequences can be used, via the Gromov product

$$(x, y)_{x_0} := \tfrac{1}{2}(d(x_0, x) + d(x_0, y) - d(x, y))$$

to define the boundary of a hyperbolic space (see [**8**, Ch. 2, §2]). We examine the following three cases:

- $\{m_n\}$ is bounded, that is, $m_n \to m \in \widetilde{X}$;
- $\{m_n\}$ is unbounded and equivalent to $\{x_n\}$ (and, hence, to $\{y_n\}$), that is, $(m_n, x_n)_{x_0} \to \infty$ as $n \to \infty$;
- $\{m_n\}$ is unbounded and $\{m_n\}$ is not equivalent to $\{x_n\}$ (and, hence, neither to $\{y_n\}$).

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In the first case, since f is unique, m must belong to Im f, hence, using the real numbers t_m where $f(t_m) = m$ and $f_n(t_{m_n}) = m_n$ with the appropriate signs according to orientation, we obtain the desired parametrization of each f_n. In the second case, by the choice of $\{x_n\}$, $\{y_n\}$, we have $(m_n, x_n)_{x_0} = (m_n, y_n)_{x_0}$. Thus, all three sequences $\{x_n\}$, $\{y_n\}$, and $\{m_n\}$ are equivalent, that is, they define the same boundary point, a contradiction since $f(+\infty) \neq f(-\infty)$. In the third case $\{m_n\}$ converges to a point $m \in \partial\widetilde{X}$ with $m \neq f(\pm\infty)$. This case cannot occur either. Indeed, as $\partial\widetilde{X}$ is metrizable, we may find neighborhoods O and U of $f(+\infty)$ and $f(-\infty)$, respectively, such that $m \notin O \cup U$. Set

$$Q(O, U) := \{x \in \widetilde{X} \mid x \in \text{Im } g, \text{ for some } g \in G\widetilde{X} \text{ with } g(\pm\infty) \in O \cup U\}.$$

Then, by [**7**, Part B, Lemma 3], $O \cup U$ is the accumulation set of $Q(O, U)$ in $\partial\widetilde{X}$, a contradiction, since $m \notin O \cup U$. □

3. *Mixing of the geodesic flow*

3.1. *Topological transitivity.* It is apparent that topological mixing implies topological transitivity. However, in the proof of topological mixing given in the following we need a property equivalent to topological transitivity, namely, that $\overline{W^s(f)} = GX$ for any $f \in GX$. We are omitting the proof of the equivalence since it will not be used in the following. In this section, we establish this property (see Proposition 21). We need the following result.

Duality Condition. For each $f \in G\widetilde{X}$, there exists a sequence of isometries $\{\phi_n\}_{n\in\mathbb{N}} \subset \Gamma \equiv \pi_1(X)$ such that $\phi_n(x) \to f(+\infty)$ and $\phi_n^{-1}(x) \to f(-\infty)$ for some (hence, any) $x \in \widetilde{X}$.

Topological transitivity for our class of spaces will then follow from the following theorem found in [**2**, Theorem 2.3, Ch. III].

THEOREM 17. *Let Y be a geodesically complete separable Hadamard space and Γ a group of isometries of Y satisfying the duality condition. Then the following are equivalent:*

(a) *the geodesic flow is topologically transitive mod Γ;*
(b) *for some $\xi \in \partial Y$, the orbit $\Gamma \cdot \xi$ is dense in ∂Y.*

LEMMA 18. *The space $\widetilde{X}$ satisfies the duality condition.*

Proof. If $f \in G\widetilde{X}$ is closed we may consider $\{\phi_n\}$ to be powers of the hyperbolic isometry corresponding to f. Then clearly $\phi_n(f(0)) \to f(+\infty)$ and $\phi_n^{-1}(f(0)) \to f(-\infty)$.

Suppose $f \in G\widetilde{X}$ is non-closed. By Proposition 10, there exists a sequence of closed geodesics c_n such that $c_n(+\infty) \to f(+\infty)$ and $c_n(-\infty) \to f(-\infty)$. Using Proposition 16 and changing appropriately the parametrizations of each c_n, we obtain $c_n \to f$. We may alter the period t_n of each c_n so that $t_n \nearrow +\infty$ as $n \to \infty$. Set ϕ_n to be the isometry, which corresponds to translating c_n by t_n. Then, $\phi_n(c_n(0)) \to f(+\infty)$ and, since $f(0)$ is at bounded distance from $c_n(0)$ for all n, it follows that $\phi_n(f(0)) \to f(+\infty)$. Similarly, we show that $\phi_n^{-1}(f(0)) \to f(-\infty)$. □

THEOREM 19. *There exists a geodesic γ in GX whose orbit $\mathbb{R}\gamma$ under the geodesic flow is dense in GX. Equivalently, the geodesic flow is topologically transitive.*

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Proof. Equivalence of the two statements is a general fact that follows from separability of $\widetilde{X}$ and second countability of the topology of $G\widetilde{X}$ (see [**2**, Remark 2.2, Ch. III]). By Proposition 9, for any $\xi \in \partial\widetilde{X}$ the orbit $\Gamma \cdot \xi$ is dense in $\partial\widetilde{X}$. Moreover, by the above lemma, $\widetilde{X}$ satisfies the duality condition, thus, by the above-mentioned Theorem 17 from [**2**], the geodesic flow is topologically transitive. □

Observe that, in particular, the image of such a geodesic γ is a dense subset of X. Therefore, the geodesic γ whose orbit is dense in GX cannot be a closed geodesic. We need the following result.

COROLLARY 20. *There exists a geodesic γ in GX whose orbit $\mathbb{R}\gamma$ under the geodesic flow is dense in GX and, in addition, $\widetilde{\gamma}(+\infty) \notin F_h^{nu}$ for some, hence any, lift $\widetilde{\gamma}$ of γ.*

Proof. We first show that the cardinality of the set $D = \{\gamma \mid \overline{\mathbb{R}\gamma} = GX\}$ is uncountably infinite. To check this, observe that D is just the intersection

$$D = \bigcap_{\mathcal{B} \in \underline{\mathfrak{B}}} \mathbb{R}\mathcal{B},$$

where $\underline{\mathfrak{B}}$ is a countable basis for GX with $\emptyset$ excluded. Since the geodesic flow is topologically transitive, each $\mathbb{R}\mathcal{B}$ is dense and, clearly, open and non-empty. By Baire's theorem, D is non-empty and if $D = \{\gamma_1, \gamma_2, \ldots\}$ were countable, then the countable intersection

$$\bigcap_{\mathcal{B} \in \underline{\mathfrak{B}}'} \mathbb{R}\mathcal{B}$$

where $\underline{\mathfrak{B}}' = \underline{\mathfrak{B}} \cup (\bigcup_{i=1}^{\infty} GX \setminus \{\gamma_i\})$ would be empty, contradicting Baire's theorem. The corollary now follows from the fact that $F_h^{nu} \times F_h^{nu}$ is countable, thus, there exist $\gamma \in D$ such that $(\widetilde{\gamma}(-\infty), \widetilde{\gamma}(+\infty)) \notin F_h^{nu} \times F_h^{nu}$. In other words, there exists a geodesic γ with dense orbit in GX whose lift to $G\widetilde{X}$ has at least one of its limit points in $\partial\widetilde{X} \setminus F_h^{nu}$. By replacing γ with $-\gamma$ we may assume that $\widetilde{\gamma}(+\infty) \notin F_h^{nu}$. □

PROPOSITION 21. *For any $f \in GX$, $\overline{W^s(f)} = GX$.*

Proof. Let $g \in GX$ be arbitrary and pick lifts $\widetilde{g} \in G\widetilde{X}$ of g and $\widetilde{f} \in G\widetilde{X}$ of f. Theorem 19 provides a geodesic $\widetilde{\gamma} \in G\widetilde{X}$ and sequences $\{t_n\}$ in $\mathbb{R}$ and $\{\phi_n\}$ in Γ such that $\phi_n(t_n \cdot \widetilde{\gamma}) \to \widetilde{g}$. Set $\widetilde{\gamma}_n := \phi_n(t_n \cdot \widetilde{\gamma})$.

Since the orbit $\Gamma \cdot \widetilde{f}(+\infty)$ is dense in $\partial\widetilde{X}$ we may pick, for each fixed n, a sequence $\{\phi_{n,k}\}_{k=1}^{\infty} \subset \Gamma$ such that $\phi_{n,k}(\widetilde{f}(+\infty)) \to \widetilde{\gamma}_n(+\infty)$. For all k, consider geodesics $\widetilde{g}_{n,k}$ with $\widetilde{g}_{n,k}(+\infty) = \phi_{n,k}(\widetilde{f}(+\infty))$ and $\widetilde{g}_{n,k}(-\infty) = \widetilde{\gamma}_n(-\infty)$. Clearly $g_{n,k} = p(\widetilde{g}_{n,k}) \in W^s(f)$ for all $k, n \in \mathbb{N}$ and by a diagonal argument we obtain a sequence $g_{n,k(n)} = p(\widetilde{g}_{n,k(n)})$ that, up to appropriate parametrization, converges to $g = p(\widetilde{g})$. □

3.2. *Proof of topological mixing.* For the proof of topological mixing for the geodesic flow on X, we closely follow the notation and the analogous proof for CAT(-1) spaces in [**5**], which, in fact, follows the steps of Eberlein's work (cf. [**10**]). For the proofs of the following lemmata, we refer to the corresponding proofs in [**5**] and deal only with the issues arising from the non-unique closed geodesics.

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LEMMA 22.

(a) *For any* $g \in GX$ *and* $t \in \mathbb{R}$, $\overline{W^{ss}(t \cdot g)} = t \cdot (\overline{W^{ss}(g)})$.

(b) *Let* $g, h \in GX$ *with* $h \in W^{ss}(g)$ *and let* $\mathcal{O} \subset GX$ *be an open neighborhood of* h. *Then there exists an open neighborhood* $\mathcal{A}$ *containing* g *such that for any* $g_1 \in \mathcal{A}$, $W^{ss}(g_1) \cap \mathcal{O} \neq \emptyset$.

(c) *If* $h \in \overline{W^{ss}(g)}$, *then* $\overline{W^{ss}(h)} \subset \overline{W^{ss}(g)}$.

Proof. (a) If $h \in \overline{W^{ss}(t \cdot g)}$, there exist a sequence $\{h_n\}_{n\in\mathbb{N}} \subset W^{ss}(t \cdot g)$ with $h_n \to h$. It is clear from the definitions that $(-t) \cdot h_n \to (-t) \cdot h$ and $\{(-t) \cdot h_n\}_{n\in\mathbb{N}} \subset W^{ss}(g)$. This shows that $(-t) \cdot h \in \overline{W^{ss}(g)}$ and, hence, $h = t \cdot ((-t) \cdot h) \in t \cdot (\overline{W^{ss}(g)})$. For the converse inclusion, let $h \in t \cdot (\overline{W^{ss}(g)})$. This means that there exist a sequence $\{h_n\}_{n\in\mathbb{N}} \subset W^{ss}(g)$ with $t \cdot h_n \to h$. Clearly, $t \cdot h_n \in W^{ss}(t \cdot g)$, hence $h \in \overline{W^{ss}(t \cdot g)}$.

(b) Lift g and h to geodesics $\widetilde{g}$ and $\widetilde{h}$ in $G\widetilde{X}$ such that $\widetilde{h} \in W^{ss}(\widetilde{g})$ and consider an open neighborhood $\widetilde{\mathcal{O}} \subset G\widetilde{X}$ of $\widetilde{h}$ such that $p(\widetilde{\mathcal{O}}) \subset \mathcal{O}$. We show that there exists an open neighborhood $\widetilde{\mathcal{A}}$ containing $\widetilde{g}$ such that for any $\widetilde{g}_1 \in \widetilde{\mathcal{A}}$, $W^{ss}(\widetilde{g}_1) \cap \widetilde{\mathcal{O}} \neq \emptyset$. Then $\mathcal{A} = p(\widetilde{\mathcal{A}})$ would be the desired neighborhood of $g = p(\widetilde{g})$.

We may assume that $\widetilde{\mathcal{O}}$ is of the form

$$\widetilde{\mathcal{O}}(\widetilde{h}, K, \varepsilon) = \{\widetilde{f} \in G\widetilde{X} \mid d(\widetilde{f}(t), \widetilde{h}(t)) < \varepsilon \text{ for all } t \in [-K, K]\}.$$

Consider the open neighborhood

$$\widetilde{\mathcal{O}}_R = \{r \in R\widetilde{X} \mid r = \widetilde{f}|_{[-K,\infty)} \text{ for some } \widetilde{f} \in \widetilde{\mathcal{O}}\}$$

of $R\widetilde{X}$. Clearly, $\alpha(\xi, \widetilde{h}(0), \widetilde{g}(0)) = 0$ where $\xi = \widetilde{g}(+\infty) = \widetilde{h}(+\infty)$. By Proposition 15 we may choose open sets A and B of $\widetilde{X}$ and $\partial\widetilde{X}$, respectively, such that $(\widetilde{h}(-K), \xi) \in A \times B$ and $\varrho^{-1}(A \times B) \subset \widetilde{\mathcal{O}}_R$ where ϱ is the function produced in Proposition 15. By Lemma 14, we may choose open sets C and D of $\widetilde{X}$ and $\partial\widetilde{X}$, respectively, such that $(\widetilde{g}(-K), \xi) \in C \times D$ and for every geodesic ray $r_g : [-K, \infty) \to \widetilde{X}$ with $r_g(-K) \in C$ and $r_g(+\infty) \in D$, there exists a geodesic ray $r_h : [-K, \infty) \to \widetilde{X}$ with $r_h(-K) \in A$, $r_g(+\infty) = r_h(+\infty)$ and $\alpha(r_g(+\infty), r_h(-K), r_g(-K)) = 0$, in other words, $r_h \in W^{ss}(r_g)$.

The inverse image $\varrho^{-1}(C \times (B \cap D))$ is an open neighborhood in $R\widetilde{X}$ containing $\widetilde{g}|_{[-K,\infty)}$. Extend all geodesic rays in $\varrho^{-1}(C \times (B \cap D))$ to geodesic lines in order to obtain $\widetilde{\mathcal{A}} \subset G\widetilde{X}$ containing $\widetilde{g}$.

Every geodesic $\widetilde{g}_1 \in \widetilde{\mathcal{A}}$ determines a geodesic ray $r_{g_1} \in \varrho^{-1}(C \times (B \cap D))$ for which we have shown that there exists a geodesic ray $r_{h_1} \in \varrho^{-1}(A \times B)$. Extending r_{h_1} to a geodesic line, we obtain a geodesic $\widetilde{h}_1 \in \widetilde{\mathcal{O}}$ with $\widetilde{h}_1 \in W^{ss}(\widetilde{g}_1)$, thus $W^{ss}(\widetilde{g}_1) \cap \widetilde{\mathcal{O}} \neq \emptyset$.

(c) Let g^* be an arbitrary element in $\overline{W^{ss}(h)}$ and $\mathcal{O}$ an arbitrary open neighborhood of g^*. We show that $g^* \in \overline{W^{ss}(g)}$. Since $W^{ss}(h) \cap \mathcal{O} \neq \emptyset$ we may choose, by part (b), an open neighborhood $\mathcal{A}$ of h such that, for every $f \in \mathcal{A}$, $W^{ss}(f) \cap \mathcal{O} \neq \emptyset$. Since $h \in \overline{W^{ss}(g)}$, there exists $g_1 \in W^{ss}(g) \cap \mathcal{A}$ and, thus, $W^{ss}(g_1) \cap \mathcal{O} = W^{ss}(g) \cap \mathcal{O} \neq \emptyset$. Since $\mathcal{O}$ was arbitrary, $g^* \in \overline{W^{ss}(g)}$ as required. □

LEMMA 23. *Let* $f \in G\widetilde{X}$ *be unique, that is, either* $p(f)$ *is non-closed or,* $p(f) \in GX$ *is closed and unique in its homotopy class, and* $\mathcal{O}$ *is a neighborhood of* f.

(a) *There exists a neighborhood* O *in* $\partial\widetilde{X}$ *of* $f(+\infty)$ *such that for any geodesic* g *with* $g(+\infty) \in O$ *and* $g(-\infty) = f(-\infty)$ *there exists a re-parametrization* $\overline{g}$ *of* g *with* $\overline{g} \in \mathcal{O}$.

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(b) *If* $\{\xi_n\}$ *is a sequence in* $\partial\widetilde{X}$ *with* $\xi_n \to f(+\infty)$ *and* $\{f_n\}$ *a sequence of geodesics with* $f_n(+\infty) = \xi_n$ *and* $f_n(-\infty) = f(-\infty)$, *we may re-parametrize* $\{f_n\}$ *such that* $f_n \to f$.

Proof. (a) The proof of part (a) follows from (b). To see this, assume the result does not hold. Then for a decreasing sequence of open neighborhoods $O_n \searrow f(+\infty)$ there must exist $\xi_n \in O_n$ such that any geodesic f_n with $f_n(+\infty) = \xi_n$ and $f_n(-\infty) = f(-\infty)$ has the property $f_n \notin \mathcal{O}$. In particular $\{f_n\}$ does not converge to f contradicting (b).

(b) This is a special case of Proposition 16. □

PROPOSITION 24. *There exists a geodesic* $g \in GX$ *such that* $\overline{W^{ss}(g)} = GX$.

Proof. We follow the line of proof of Proposition 4.1 in [**5**]. In that setup, geodesic lines are uniquely determined by their boundary points, so conditions (u) and (c) stated in the beginning of the introduction hold. The modification will consist of the following: every geodesic that comes into play will be replaced by a (unique) geodesic whose limit point belongs to $\partial\widetilde{X} \setminus F_h^{nu}$ so that conditions (U) and (C) can be applied.

To prove the proposition, it suffices to show that

$$\begin{aligned} &\text{for any open } \mathcal{O} \text{ and } \mathcal{U} \subseteq GX, \\ &\qquad \text{there exists } g \in \mathcal{O} \text{ such that } W^{ss}(g) \cap \mathcal{U} \neq \emptyset. \end{aligned} \tag{6}$$

Then, using a countable basis $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ for the topology of GX the proof is completed by a standard topological argument (cf. [**10**, Theorem 5.2]).

Let $\mathcal{O}, \mathcal{U} \subseteq GX$ be arbitrary open sets. Pick $f \in p^{-1}(\mathcal{O})$ such that f is non-closed. Similarly, choose $h \in p^{-1}(\mathcal{U})$ such that h is not closed.

By condition (U), f (respectively, h) is unique, thus, by Lemma 23(a), there exists connected open neighborhood $O_f \subset \partial\widetilde{X}$ of $f(+\infty)$ (respectively, U_h of $h(+\infty)$) such that for every $\xi \in O_f$ (respectively, $\xi \in U_h$) there exists a geodesic with boundary points ξ, $f(-\infty)$ (respectively, ξ, $h(-\infty)$) that belongs to $p^{-1}(\mathcal{O})$ (respectively, $p^{-1}(\mathcal{U})$).

By condition (D), there exists a closed and unique geodesic β such that

$$\{\beta(+\infty), \beta(-\infty)\} \subset F_h^u \tag{7}$$

and

$$(\beta(+\infty), \beta(-\infty)) \in O_f \times U_h.$$

By the choice of O_f (Lemma 23(a)) there exists a geodesic joining $\beta(+\infty)$ and $f(-\infty)$ that belongs to $p^{-1}(\mathcal{O})$, which by property (7) is unique. Replace f by this geodesic and, thus, we may assume that $f(+\infty) = \beta(+\infty)$. Similarly, we arrange so that $h(+\infty) = \beta(-\infty)$. Denote by ϕ the hyperbolic isometry corresponding to β.

For each n, by extending the geodesic segment joining $f(0)$ with $\phi^n(h(0))$ to a geodesic line, it follows that the function $\alpha(\cdot, f(0), \phi^n(h(0)))$ attains positive and negative values on $\partial\widetilde{X}$. As $\partial\widetilde{X}$ is assumed to be connected, there exists ξ_n in $\partial\widetilde{X}$ such that $\alpha(\xi_n, f(0), \phi^n(h(0))) = 0$. We claim that

$$\xi_n \to f(+\infty) \quad \text{as } n \to \infty. \tag{8}$$

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To see this assume, on the contrary, that $\{\xi_n\}$ (or, a subsequence of it) converges to $\xi \in \partial X$ with $\xi \neq f(+\infty)$. Let M be a positive real number. For each fixed n, using equation (2) and the fact that ξ_n is chosen so that $\phi^n(h(0))$ and $f(0)$ are equidistant from ξ_n, we may pick a sequence $\{x_m^n\}_{m\in\mathbb{N}}$ with the property $x_m^n \to \xi_n$ as $m \to \infty$ and

$$|\alpha(x_m^n,\, f(0),\, \phi^n(h(0)))| \leq M \quad \text{for all } m \text{ large enough},$$

with $M > 0$ being independent of n. It is well known (see, for example, [**7**, Ch. I, §4]) that $\widetilde{X} \cup \partial\widetilde{X}$ is metrizable, hence, by a diagonal argument we obtain a sequence $\{x_{m(n)}^n\}_{n\in\mathbb{N}}$ such that $x_{m(n)}^n \to \xi$ as $n \to \infty$ and

$$|\alpha(x_{m(n)}^n,\, f(0),\, \phi^n(h(0)))| \leq M \quad \text{for all } n. \tag{9}$$

For the hyperbolic product of the sequences $\{x_{m(n)}^n\}_{n\in\mathbb{N}}$ and $\{\phi^n(h(0))\}_{n\in\mathbb{N}}$ with base point $f(0)$ we have

$$\begin{aligned} 2(x_{m(n)}^n,\, \phi^n(h(0)))_{f(0)} &= d(f(0),\, x_{m(n)}^n) + d(f(0),\, \phi^n(h(0))) \\ &\quad - d(\phi^n(h(0)),\, x_{m(n)}^n) \\ &= -\alpha(x_{m(n)}^n,\, f(0),\, \phi^n(h(0))) + d(f(0),\, \phi^n(h(0))). \end{aligned}$$

It follows by (9) that $(x_{m(n)}^n,\, \phi^n(h(0)))_{f(0)} \to \infty$ as $n \to \infty$, hence, the sequences $\{x_{m(n)}^n\}_{n\in\mathbb{N}}$ and $\{\phi^n(h(0))\}_{n\in\mathbb{N}}$ define the same point at the boundary. This is a contradiction, since $\phi^n(h(0)) \to \beta(+\infty) = f(+\infty)$ and $\{x_{m(n)}^n\}_{n\in\mathbb{N}} \to \xi$.

Thus, equation (8) is proved. In a similar manner we show that

$$\phi^{-n}(\xi_n) \to h(+\infty) \quad \text{as } n \to \infty. \tag{10}$$

Choose now geodesics $f_n \in G\widetilde{X}$, $n \in \mathbb{N}$, such that $f_n(+\infty) = \xi_n$ and $f_n(-\infty) = f(-\infty)$ and by Lemma 23(b), we may parametrize f_n so that $f_n \to f$ or, equivalently, $f_n(0) \to f(0)$. Similarly, choose $h_n \in GX$ such that $h_n(+\infty) = \xi_n$ and $h_n(-\infty) = \phi^n(h(-\infty))$ and parametrize them so that

$$\alpha(\xi_n,\, f_n(0),\, h_n(0)) = 0. \tag{11}$$

It is apparent that for n large enough, $f_n \in p^{-1}(\mathcal{O})$ and $h_n \in W^{ss}(f_n)$. If we show that $\phi^{-n}(h_n) \in p^{-1}(\mathcal{U})$ for n large enough, then we would have

$$\begin{aligned} &p(f_n) \in \mathcal{O}, \\ &p(h_n) = p(\phi^{-n}(h_n)) \in \mathcal{U}, \quad \text{and} \\ &p(h_n) \in W^{ss}(p(f_n)). \end{aligned}$$

The above three properties imply that for n large enough, $W^{ss}(p(f_n)) \cap \mathcal{U} \neq \emptyset$, as required in equation (6). We conclude the proof of the proposition by showing that $\phi^{-n}(h_n) \in p^{-1}(\mathcal{U})$ for n large enough. In fact, we show that $\phi^{-n}h_n \to h$. Clearly,

$$(\phi^{-n}(h_n))(+\infty) = \phi^{-n}(h_n(\infty)) = \phi^{-n}(\xi_n) \to h(+\infty) \tag{12}$$

and

$$(\phi^{-n}(h_n))(-\infty) = \phi^{-n}(h_n(-\infty)) = h(-\infty). \tag{13}$$

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Use equations (12) and (13) to apply Lemma 23(b) for the unique geodesic h to obtain a re-parametrization, say $\overline{h}_n$, of each $\phi^{-n}(h_n)$ such that $\overline{h}_n \to h$. In particular, we have

$$d(h(0), \operatorname{Im} \overline{h}_n) \to 0,$$

which implies

$$d(h(0), \operatorname{Im} \phi^{-n}(h_n)) \to 0$$

as $n \to +\infty$. Therefore,

$$d(\phi^n(h(0)), \operatorname{Im} h_n) \to 0 \quad \text{as } n \to \infty. \tag{14}$$

Let $h_n(t_n)$, $t_n \in \mathbb{R}$ be the point on $\operatorname{Im} h_n$ that realizes the distance in equation (14). As the function α is Lipschitz with respect to the third variable (with Lipschitz constant 1) we have

$$|\alpha(\xi_n,\ f(0),\ \phi^n(h(0))) - \alpha(\xi_n,\ f(0),\ h_n(t_n))| \le d(\phi^n(h(0)),\ h_n(t_n)).$$

Using the defining property of ξ_n, that is, $\alpha(\xi_n,\ f(0),\ \phi^n(h(0))) = 0$, it follows that

$$\alpha(\xi_n,\ f(0),\ h_n(t_n)) \to 0 \quad \text{as } n \to \infty.$$

Similarly, using the fact that $f_n(0) \to f(0)$ as $n \to \infty$ and the Lipschitz property of α with respect to the second variable we have

$$\alpha(\xi_n,\ f_n(0),\ h_n(t_n)) \to 0 \quad \text{as } n \to \infty.$$

Since, by Lemma 11(c), there is a unique point on each $\operatorname{Im} h_n$ that is equidistant from $f_n(0)$ with respect to ξ_n, namely, $h_n(0)$ (cf. equation (11)), it follows that $t_n \to 0$, which, combined with equation (14), implies that

$$d(\phi^n(h(0)),\ h_n(0)) \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\phi^{-n}(h_n(0)) \to h(0)$ as $n \to \infty$. By Proposition 16 it follows that $\phi^{-n} h_n \to h$, which implies that $\phi^{-n}(h_n) \in p^{-1}(\mathcal{U})$ concluding the proof. □

PROPOSITION 25. *For every closed geodesic $c \in GX$, $\overline{W^{ss}(c)} = GX$.*

Proof. Let g be the geodesic produced in Proposition 24, that is, $\overline{W^{ss}(g)} = GX$ and let c be a closed geodesic. By Proposition 21, $\overline{W^s(c)} = GX$ so that $g \in \overline{W^s(c)} = GX$. Thus, there exists a sequence $\{g_n\} \subset W^s(c)$ such that $g_n \to g$. For each $n \in \mathbb{N}$, consider lifts $\widetilde{g}_n$ and $\widetilde{c}$ of g_n and c, respectively, satisfying $\widetilde{g}_n \in W^s(\widetilde{c})$ and use Lemma 11(c) to obtain a real number $\widetilde{t}_n$ such that $\widetilde{t}_n \cdot g_n \in W^{ss}(c)$. Each $\widetilde{t}_n$ may be written as

$$\widetilde{t}_n = k\omega + t_n,$$

where $k \in \mathbb{Z}$ and $t_n \in [0, \omega)$. By choosing, if necessary, a subsequence, $t_n \to t$ for some $t \in [0, \omega]$. Then $t_n \cdot g_n \to t \cdot g$ and $t_n \cdot g_n \in W^{ss}(c)$, which simply means that $t \cdot g \in \overline{W^{ss}(c)}$ and by Lemma 22(c) we have $\overline{W^{ss}(c)} \supset \overline{W^{ss}(t \cdot g)} = \overline{t \cdot W^{ss}(g)} = GX$. □

We need a point-wise version of topological mixing and a criterion for such a property.

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Definition 26. Let h and f be in GX and let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence converging to $+\infty$ or $-\infty$. We say that h is s_n-mixing with f (notation, $h \sim_{s_n} f$) if for every neighborhood $\mathcal{O}$ and $\mathcal{U}$ in GX of h and f, respectively, $s_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for all n sufficiently large.

For a geodesic h, denote by $-h$ the geodesic with the reverse orientation, that is, $(-h)(t) := h(-t)$, $t \in \mathbb{R}$. For a neighborhood $\mathcal{O}$ of h denote by $-\mathcal{O}$ the neighborhood of $-h$ defined by $-\mathcal{O} := \{-f \mid f \in \mathcal{O}\}$.

LEMMA 27. *Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence converging to $+\infty$ or $-\infty$. Then*

$$h \sim_{s_n} f \Leftrightarrow f \sim_{-s_n} h \Leftrightarrow -f \sim_{s_n} -h.$$

Proof. Let $\mathcal{O}$ and $\mathcal{U}$ in GX be arbitrary neighborhoods of h and f, respectively. The assumption $h \sim_{s_n} f$ means that for each $n \in \mathbb{N}$, there exists a geodesic $h'_n \in \mathcal{O}$ such that $s_n \cdot h'_n \in \mathcal{U}$ or, equivalently, $(-s_n) \cdot (s_n \cdot h'_n) \in (-s_n) \cdot \mathcal{U}$. In other words, $h'_n \in (-s_n) \cdot \mathcal{U}$, which implies that $(-s_n) \cdot \mathcal{U} \cap \mathcal{O} \neq \emptyset$. This shows that $f \sim_{-s_n} h$. The converse of the first equivalence is trivial as $\{-(-s_n)\} = \{s_n\}$.

Assuming $f \sim_{-s_n} h$ we have, by definition, that $(-s_n) \cdot \mathcal{U} \cap \mathcal{O} \neq \emptyset$ for all large n. Thus, for each large enough $n \in \mathbb{N}$, there exists a geodesic $f'_n \in \mathcal{U}$ such that $(-s_n) \cdot f'_n \in \mathcal{O}$ or, equivalently, $-[(-s_n) \cdot f'_n] \in -\mathcal{O}$. Since $-[(-s_n) \cdot f'_n] = s_n \cdot (-f'_n)$ we have $s_n \cdot (-f'_n) \in -\mathcal{O}$. Clearly, $-f'_n \in -\mathcal{U}$ and, thus, $s_n \cdot (-\mathcal{U}) \cap (-\mathcal{O}) \neq \emptyset$. This shows that $-f \sim_{s_n} -h$. The proof of the converse of the second equivalence is again trivial as $-(-f) = f$. □

Remark 28. The first equivalence in the previous lemma shows that the s_n-mixing relation is not a symmetric relation. In particular, it is not an equivalence relation.

The following criterion for the s_n-mixing of h, f holds.

LEMMA 29. *If h and $f \in GX$, then $h \sim_{s_n} f$ if and only if for each subsequence $\{s'_n\}$ of $\{s_n\}$ there exists a subsequence $\{r_n\}$ of $\{s'_n\}$ and a sequence of non-closed geodesics $\{h_n\} \subset GX$ such that $h_n \to h$, $r_n \cdot h_n \to f$ and $\widetilde{h}_n(+\infty) \notin F_h^{nu}$ for some, hence any, lift $\widetilde{h}_n$ of h_n.*

Proof. If $h \sim_{s_n} f$ for some h and $f \in GX$, then using decreasing sequences of open neighborhoods of h and f it is easily shown that for each subsequence $\{s'_n\}$ of $\{s_n\}$ there exists a subsequence $\{r_n\}$ of $\{s'_n\}$ and a sequence $\{h_n\} \subset GX$ such that $h_n \to h$ and $r_n \cdot h_n \to f$. We proceed to show that we may replace $\{h_n\}$ by a sequence $\{g_n\}$ of non-closed geodesics so that $g_n \to h$ and $r_n \cdot g_n \to f$.

Let γ be the geodesic posited in Theorem 19, that is, its orbit $\mathbb{R} \cdot \gamma$ is dense in GX. As observed at the end of the proof of Theorem 19, γ is non-closed. Thus, there exists a sequence $\{t_i^1\}_{i\in\mathbb{N}}$ such that $t_i^1 \cdot \gamma \to h_1$. Set $g_i^1 = t_i^1 \cdot \gamma$ and, clearly, all g_i^1 are non-closed. Similarly, for each h_n, we may find a sequence of non-closed geodesics $g_i^n = t_i^n \cdot \gamma$ converging to h_n. By a diagonal argument we obtain a sequence of non-closed geodesics $\{g_n\}$ converging to h and, clearly, $\lim_{n\to\infty} r_n \cdot g_n = \lim_{n\to\infty} r_n \cdot h_n = f$.

By Lemma 20, the geodesic γ having dense orbit can be chosen so that $\widetilde{\gamma}(+\infty) \notin F_h^{nu}$. Since the non-closed geodesics $\{g_n\}$ constructed above are all translates of γ the last requirement of the lemma is fulfilled.

The proof of the converse statement is elementary. □

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Remark 30. For a geodesic $f \in GX$ and a sequence $s_n \to \infty$ the set

$$\{h \in GX : h \sim_{s_n} f\}$$

is a closed set.

Proof. Assume $\{h_k\}$ is a sequence with $h_k \to h$ and $h_k \sim_{s_n} f$ for all $k \in \mathbb{N}$. We show that $h \sim_{s_n} f$. Let $\mathcal{O}$ and $\mathcal{U}$ be arbitrary neighborhoods of h and f, respectively. Find k_0 such that $h_{k_0} \in \mathcal{O}$. Then, as $h_{k_0} \sim_{s_n} f$, we have $s_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for all n sufficiently large. The latter means, by definition, that $h \sim_{s_n} f$. □

We next show the following lemma that asserts that point-wise topological mixing is transferred via the strong stable relation of geodesics.

LEMMA 31. *Let f, g and $g' \in GX$ so that $f \in \overline{W^{ss}(g)}$, f is non-closed, g is closed and unique with $\widetilde{g}(+\infty) \in F_h^u$ for some, hence any, lift $\widetilde{g}$ of g. Then, if $g \sim_{s_n} g'$ for some sequence $s_n \to \infty$, then $f \sim_{s_n} g'$.*

Proof. Fix a sequence $s_n \to \infty$. By Remark 30, it suffices to prove the assertion of the lemma for f non-closed and $f \in W^{ss}(g)$. The rest of the proof follows the line of proof given in [**5**, Lemma 4.4], which we include here since several restrictions apply in our setup.

In order to use Lemma 29 above for showing that $f \sim_{s_n} g'$, let $\{t_n\}$ be arbitrary subsequence of $\{s_n\}$. As $g \sim_{s_n} g'$ there exists (again by Lemma 29) a subsequence $\{r_n\}$ of $\{t_n\}$ and a sequence $\{g_n\}$ such that

$$g_n \to g \quad \text{and} \quad r_n \cdot g_n \to g'.$$

Lift g and f to geodesics $\overline{g}$ and $\overline{f}$ in $G\widetilde{X}$ such that $\overline{f}(+\infty) = \overline{g}(+\infty)$ and

$$\alpha(\overline{f}(+\infty), \overline{f}(0), \overline{g}(0)) = 0.$$

Lift each g_n to a geodesic $\overline{g_n}$ such that $\overline{g_n} \to \overline{g}$. Since g is unique, the latter is equivalent to $\overline{g_n}(+\infty) \to \overline{g}(+\infty)$, $\overline{g_n}(-\infty) \to \overline{g}(-\infty)$, and $\overline{g_n}(0) \to \overline{g}(0)$. We may assume (cf. Lemma 29) that

$$\{\overline{g_n}(+\infty) \mid n \in \mathbb{N}\} \subset \partial\widetilde{X} \setminus F_h^{nu}.$$

Use Lemma 23(b) to define a sequence of geodesics $\{\overline{f_n}\}_{n\in\mathbb{N}}$ such that $\overline{f_n} \to \overline{f}$ with $\overline{f_n}(+\infty) = \overline{g_n}(+\infty)$ and $\overline{f_n}(-\infty) = \overline{f}(-\infty)$. By the continuity of the α function we have that

$$\lim_{n\to\infty} \alpha(\xi_n, \overline{f_n}(0), \overline{g_n}(0)) = \alpha(\xi, \overline{f}(0), \overline{g}(0)) = 0,$$

hence, by passing, if necessary, to a subsequence of $\{\overline{f_n}\}_{n\in\mathbb{N}}$, we may assume that

$$\alpha(\xi_n, \overline{f_n}(0), \overline{g_n}(0)) < 1/n \quad \textit{for all } n \in \mathbb{N}.$$

By Lemma 11(c) we may choose the parametrization of each $\overline{f_n}$ so that

$$\alpha(\xi_n, \overline{f_n}(0), \overline{g_n}(0)) = 0 \quad \textit{for all } n \in \mathbb{N}. \tag{15}$$

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As the change of parametrization tends to 0 as $n \to \infty$ we may assume that the sequence $\{\overline{f_n}\}_{n\in\mathbb{N}}$ satisfies equation (15) and $\overline{f_n} \to \overline{f}$. Moreover, if we set $f_n := p(\overline{f_n})$, then $f_n \to f$. We proceed now to show that $r_n \cdot f_n \to g'$. Let K be an arbitrary compact subset of $\mathbb{R}$ and ε arbitrary positive. By construction, $\overline{f_n} \in W^{ss}(\overline{g_n})$ for all $n \in \mathbb{N}$ and $\overline{f} \in W^{ss}(\overline{g})$. Since $\overline{g_n}(+\infty) \in \partial\widetilde{X} \setminus F_h^{nu}$, condition (C) applies for all pairs $\overline{g_n}$, $\overline{f_n}$ and $\overline{g}$, $\overline{f}$. Therefore, by Proposition 13,

$$\begin{gathered}\lim_{t\to\infty} d(\overline{f_n}(t), \overline{g_n}(t)) = 0, \quad \text{and} \\ \lim_{t\to\infty} d(\overline{f}(t), \overline{g}(t)) = 0.\end{gathered} \tag{16}$$

Choose a positive real T such that

$$d(\overline{f}(T), \overline{g}(T)) < \varepsilon/6.$$

The above equation holds for all $t > T$. This follows by convexity of the distance function (see [**2**, Ch. I, Proposition 5.4]) and equation (16). As $\overline{f_n} \to \overline{f}$ and $\overline{g_n} \to \overline{g}$ we may choose $N \in \mathbb{N}$ such that for all $n \geq N$

$$\begin{gathered}d(\overline{f_n}(T), \overline{f}(T)) < \varepsilon/6, \quad \text{and} \\ d(\overline{g_n}(T), \overline{g}(T)) < \varepsilon/6.\end{gathered}$$

Thus, $d(\overline{f_n}(T), \overline{g_n}(T)) < \varepsilon/2$ and, as before, it follows that

$$d(\overline{f_n}(t), \overline{g_n}(t)) < \varepsilon/2 \quad \text{for all } t > T.$$

As $r_n \to +\infty$, there exists n_0 such that $r_n \geq T - \min K$ for all $n \geq n_0$. Now for all n sufficiently large, namely, $n \geq \max\{N, n_0\}$, we have

$$d(\overline{f_n}(r_n + t), \overline{g_n}(r_n + t)) < \varepsilon/2 \quad \text{for all } t \in K,$$

which implies that

$$d(r_n \cdot f_n(t), r_n \cdot g_n(t)) < \varepsilon/2 \quad \text{for all } t \in K.$$

As $r_n \cdot g_n \to g'$, we have that for all n sufficiently large

$$d(r_n \cdot g_n(t), g'(t)) < \varepsilon/2 \quad \text{for all } t \in K.$$

Combining the last two inequalities we obtain that

$$d(r_n \cdot f_n(t), g'(t)) < \varepsilon \quad \text{for all } t \in K.$$

As K, ε were arbitrary, we have shown that for all n sufficiently large, $r_n \cdot f_n$ lies in any neighborhood of g'. Therefore, $r_n \cdot f_n \to g'$ as required. □

Proof of Theorem 8. It suffices to show that

$$\begin{gathered}\text{for all } h,\ f \in GX \text{ and for all } \{t_n\} \quad \text{with } t_n \to \infty, \\ \text{there exists sub-sequence } \{s_n\} \subset \{t_n\} \text{ such that } h \sim_{s_n} f.\end{gathered}$$

To see that this property is sufficient, assume it holds and, on the contrary, the geodesic flow is not mixing. Then, there would exist neighborhoods $\mathcal{O}$ and $\mathcal{U}$ in GX such that: for each $n \in \mathbb{N}$, there exists $T_n > n$ so that $T_n \cdot \mathcal{O} \cap \mathcal{U} = \emptyset$. Clearly, for any subsequence $\{s_n\}$

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of $\{T_n\}$ we have $s_n \cdot \mathcal{O} \cap \mathcal{U} = \emptyset$. Thus, for any $h \in \mathcal{O}$ and $f \in \mathcal{U}$, the above property does not hold.

Since the notion of s_n-mixing is defined via neighborhoods it suffices to show the above property only for geodesics h, f, which are not closed. This will allow the use of Lemma 31.

By condition (D), let c be a closed and unique geodesic with

$$\{\widetilde{c}(-\infty), \widetilde{c}(+\infty)\} \subset F_h^u \subset \partial\widetilde{X} \setminus F_h^{nu}$$

for some, hence any, lift $\widetilde{c}$ of c. By Proposition 25 we have $\overline{W^{ss}(c)} = GX$. Clearly, for all $t \in \mathbb{R}$, $\overline{W^{ss}(t \cdot c)} = GX$. Let $\{s_n\}$ be a subsequence of $\{t_n\}$ such that $s_n \cdot c \to t \cdot c$ for some $t \in [0, \omega]$, where ω is the period of c. Clearly, for any neighborhood $\mathcal{U}$ of $t \cdot c$, $s_n \cdot c \in \mathcal{U}$ for large enough $n \in \mathbb{N}$. In other words, $c \sim_{s_n} t \cdot c$. As $f \in GX = \overline{W^{ss}(c)}$ is non-closed we may apply Lemma 31 to the geodesics f, c and $t \cdot c$ to obtain $f \sim_{s_n} t \cdot c$. The latter is, by Lemma 27, equivalent to $-t \cdot c \sim_{s_n} -f$. As $-h \in \overline{W^{ss}(-t \cdot c)} = GX$ we apply Lemma 31 to the geodesics $-h$, $-t \cdot c$ and $-f$ to obtain $-h \sim_{s_n} -f$. By Lemma 27, the latter is equivalent to $f \sim_{s_n} h$, as required. □

4. *Examples and applications*

4.1. *Euclidean surfaces and their properties.* We start by recalling the notion of a Euclidean surface with conical singularities.

Let S be a surface equipped with a Euclidean metric with finitely many *conical singularities* (or *conical points*), say $s_1, \ldots, s_n$ in its interior. Every point that is not conical is called a *regular point* of S. Denote by $\theta(s_i)$ the angle at each s_i and we assume that $\theta(s_i) \in (2\pi, +\infty)$.

We write $C(v, \theta)$ for the standard cone with vertex v and cone angle θ, namely, $C(v, \theta)$ is the set $\{(r, t) : 0 \le r, t \in \mathbb{R}/\theta\mathbb{Z}\}$ equipped with the metric $ds^2 = dr^2 + r^2dt^2$.

Definition 32. A Euclidean surface with conical singularities $s_1, \ldots, s_n$ is a surface S equipped with a length metric $d(\cdot, \cdot)$ such that:

- every point $p \in S \setminus \{s_1, \ldots, s_n\}$ has a neighborhood isometric to a disk or half disk in the Euclidean plane;
- each $s_i \in \{s_1, \ldots, s_n\} \subset S \setminus \partial S$ has a neighborhood isometric to a neighborhood of the vertex v of the standard cone $C(v, \theta(s_i))$.

Clearly, the metric on S is a length metric and the surface S will be written *e.s.c.s.* for brevity. Note that for genus $g \ge 2$, such Euclidean structures exist, see [**15**]. Let $\widetilde{S}$ be the universal covering of S and let $p : \widetilde{S} \to S$ be the covering projection. Obviously, the universal covering $\widetilde{S}$ is homeomorphic to $\mathbb{R}^2$ and by requiring p to be a local isometric map we may lift d to a metric on $\widetilde{S}$, denoted again by d, so that $(\widetilde{S}, d)$ becomes a *e.s.c.s.*

We use the following result.

THEOREM 33. [**6**, Theorem 12] *Let g be a non-closed geodesic or geodesic ray in a closed e.s.c.s. S with genus at least two. Then $d(\mathrm{Im}\, g, \{s_1, \ldots, s_n\}) = 0$.*

COROLLARY 34. *Let Q be a compact e.s.c.s. and g be a non-closed geodesic or geodesic ray in $Q \setminus \partial Q$. Then $d(\mathrm{Im}\, g, \{s_1, \ldots, s_n\}) = 0$.*

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Proof. Let Q^+ be a copy of Q and glue Q and Q^+ along their boundaries to obtain a closed surface S with $2n$ conical singularities $s_1, \ldots, s_n, s_1^+, \ldots, s_n^+$. By Theorem 33,

$$d(\operatorname{Im} g, \{s_1, \ldots, s_n, s_1^+, \ldots, s_n^+\}) = 0.$$

Since $\operatorname{Im} g \subset Q$, it is clear that $d(\operatorname{Im} g, \{s_1^+, \ldots, s_n^+\}) > 0$, hence,

$$d(\operatorname{Im} g, \{s_1, \ldots, s_n\}) = 0. \qquad \square$$

4.2. *Examples of CAT(0) surfaces.* We give an example of a two-dimensional CAT(0) surface X and we show that it satisfies all four assumptions of Theorem 8.

Let M be a finite area surface of genus $g \geq 2$ with pinched negative curvature. Let c_M be a simple closed separating geodesic in M such that the closure of at least one of the components of $M \setminus \operatorname{Im} c_M$ is compact. Denote by M_1 the compact subsurface of M and by M_2 the closure of the other component. Observe that M_2 may contain finitely many cusps, hence, M_2 may not be compact. Clearly, $M = M_1 \bigcup_{\operatorname{Im} c_M} M_2$.

Consider a compact *e.s.c.s.* S_1 of the same topological type as M_1 and with its boundary component ∂S_1 isometric to $\operatorname{Im} c_M$. Set

$$X = S_1 \bigcup_{\operatorname{Im} c_M} M_2 \tag{17}$$

to be the surface obtained by gluing S_1 with M_2 along their boundaries. Such a surface X is a CAT(0) space and [**14**, Theorem 11.6] applies to prove mixing provided that the Bowen–Margulis measure on $\partial \widetilde{X}$ is finite. However, the class of surfaces as defined by (17) includes examples where the Bowen–Margulis measure is not finite. For example, take the surface M to be the surface with one cuspidal end constructed in [**9**, Theorem 1.2] whose fundamental group is exotic and convergent, thus, the corresponding Bowen–Margulis measure is infinite.

The above construction can be performed by using, instead of a single simple closed separating geodesic, a collection $c_1, \ldots, c_k$ of pairwise disjoint simple closed geodesics such that the union $\operatorname{Im} c_1 \cup \cdots \cup \operatorname{Im} c_k$ splits M into two components with the closure of at least one of them being compact.

The rest of this sub-section is devoted to showing that a surface X as defined in (17) satisfies all four assumption of Theorem 8 and, thus, establish the following.

Application 35. If X is a surface of the form $X = S_1 \bigcup_{\operatorname{Im} c_M} M_2$ constructed in (17) above, then the geodesic flow $\mathbb{R} \times GX \to GX$ is topologically mixing.

We start with condition (Δ) by showing the following result.

PROPOSITION 36. *The space $\widetilde{X}$ is hyperbolic in the sense of Gromov.*

We need the following lemma.

LEMMA 37. *Let $[x, y]$ and $[x, z]$ be geodesic segments in a CAT(0) geodesic metric space Y with $d(y, z) \leq C_0$ for some $C_0 > 0$. Then, for every point $A \in [x, y]$, there exists $B \in [x, z]$ such that $d(A, B) \leq C_0$.*

Proof. The desired property holds for triangles in $\mathbb{R}^2$ hence, by CAT(0) inequality, the proof follows. $\square$

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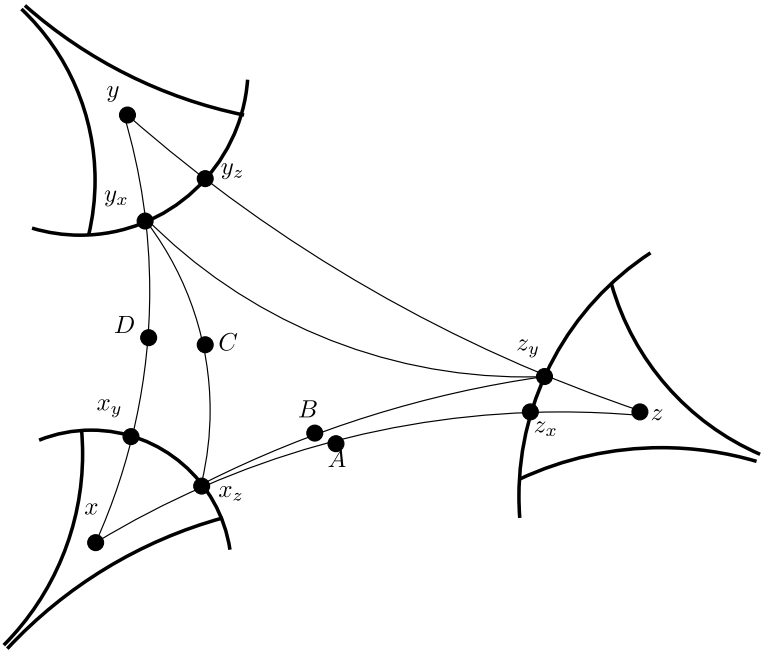


FIGURE 1. The triangle (x, y, x) in $\widetilde{X}$ and the thin triangle (x_z, z_y, y_x) in $\widetilde{X}_0$.

Proof of Proposition 36. If X does not have any cusps, X is closed and $\pi_1(X)$ hyperbolic, in which case we have nothing to show. Assume now that X contains at least one cusp. Let X_0 be the subsurface of X obtained as follows: consider a horoball based at each cusp of X and remove its interior. Then, X_0 is a compact surface with as many geodesic boundary components as the number of cusps in X. We may assume that the length of all boundary components is bounded by $C_0 > 0$.

Clearly, the universal cover $\widetilde{X}_0$ of X_0 is a subsurface of $\widetilde{X}$ whose fundamental domain is a polygon that can be obtained from the ideal fundamental domain of $\widetilde{X}$ by cutting off all its ideal vertices by horocycles. As X_0 is compact, hence hyperbolic, and the fundamental group $\pi_1(X_0) = \pi_1(X)$ is free, it follows that $\widetilde{X}_0$ is $\delta_{\widetilde{X}_0}$-hyperbolic for some $\delta_{\widetilde{X}_0} > 0$. We use the hyperbolicity of $\widetilde{X}_0$ to prove Proposition 36.

Let (x, y, z) be a geodesic triangle in $\widetilde{X}$. We show that every point $A \in [x, z]$ satisfies

$$d(A, [x, y] \cup [y, z]) \leq \delta_{\widetilde{X}_0} + 2C_0, \tag{18}$$

thus, showing that $\widetilde{X}$ is hyperbolic in the sense of Gromov.

Clearly, since $\widetilde{X}_0$ is $\delta_{\widetilde{X}_0}-$hyperbolic, if $x, y, z \in \widetilde{X}_0$ we have nothing to show. We treat the case $x, y, z \in \widetilde{X} \setminus \widetilde{X}_0$, and the other cases can be treated similarly.

Denote by $[x_y, y_x]$ the intersection$[x, y] \cap \partial\widetilde{X}_0$ and, similarly, $[x_z, z_x]$ and $[y_z, z_y]$ (see Figure 1). Note that both points x_y, x_z (respectively, y_x, y_z and z_x, z_y) belong to a single horocycle side of $\widetilde{X}_0$. Since the length of the boundary components of X_0 are assumed to be bounded by C_0 we have

$$d(x_y, x_z) \leq C_0, d(z_x, z_y) \leq C_0 \quad \text{and} \quad d(y_x, y_z) \leq C_0. \tag{19}$$

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If $A \in [x, x_z]$, then, since $d(x_z, x_y) \leq C_0$, by Lemma 37 we have

$$d(A, [x, x_y]) \leq C_0.$$

Similarly, if $A \in [z_x, z]$ we obtain $d(A, [z, z_y]) \leq C_0$. It follows that if $A \in [x, x_y] \cup [z_x, z]$ then

$$d(A, [x, y] \cup [y, z]) \leq C_0$$

hence, the desired inequality (18) holds.

If $A \in [x_z, z_x]$, apply Lemma 37 to the segments $[x_z, z_x]$ and $[x_z, z_y]$ using (19) to obtain a point

$$B \in [x_z, z_y] \quad \text{with } d(A, B) \leq C_0.$$

As $\widetilde{X}_0$ is $\delta_{\widetilde{X}_0}$-hyperbolic there exists a point

$$C \in [x_z, y_x] \cup [y_x, z_y] \quad \text{with } d(B, C) \leq \delta_{\widetilde{X}_0}.$$

Without loss of generality we may assume that $C \in [x_z, y_x]$. Again by Lemma 37 applied to the segments $[x_z, y_x]$ and $[x_y, y_x]$ we find a point

$$D \in [x_y, y_x] \quad \text{with } d(C, D) \leq C_0.$$

Combining the last three inequalities we obtain $d(A, D) \leq \delta_{\widetilde{X}_0} + 2C_0$, thus, we have shown (18) for all $A \in [x, y]$. □

PROPOSITION 38. *The space $\widetilde{X}$ satisfies condition (U).*

Proof. Assume not, that is, assume there exist non-closed geodesics f, f' in $G\widetilde{X}$ with $f(+\infty) = f'(+\infty)$ and $f(-\infty) = f'(-\infty)$ so that $\operatorname{Im} f \neq \operatorname{Im} f'$. Then by the flat strip theorem, $\operatorname{Im} f$ and $\operatorname{Im} f'$ bound a flat strip. Pick any geodesic g in the interior of the flat strip. Clearly, $\operatorname{Im} g$ has positive distance from the set of conical points in $\widetilde{X}$ and $p(g)$ does not intersect M_2. That is, $p(g)$ in contained in the compact *e.s.c.s.* S_1 and

$$d(\operatorname{Im} p(g), \{s_1, \ldots, s_n\}) > 0.$$

Since g is homotopic to both f, f' it projects to a non-closed geodesic, thus, the above inequality contradicts Corollary 34. □

We proceed now to show Condition (C).

PROPOSITION 39. *Let g_1, g_2 be asymptotic geodesics with*

$$\xi = g_1(+\infty) = g_2(+\infty) \in \partial\widetilde{X} \setminus F_h^{nu}.$$

Then for appropriate parametrizations of g_1, g_2 we have

$$\lim_{t\to\infty} d(g_1(t), g_2(t)) = 0.$$

We first show the following.

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LEMMA 40. *Let g_1, g_2 be two geodesics with*

$$\xi = g_1(+\infty) = g_2(+\infty) \in \partial \widetilde{X} \setminus F_h^{nu}$$

all as in the above proposition. Assume that

$$d(\mathrm{Im}g_1, \mathrm{Im}g_2) := \inf\{d(x, y) \mid x \in \mathrm{Im}g_1,\ y \in \mathrm{Im}g_2\} = 0.$$

Then there exists a unique re-parametrization $\overline{g}_1$ of g_1 such that

$$\lim_{t\to\infty} d(\overline{g}_1(t), g_2(t)) = 0.$$

Proof. As g_1, g_2 are asymptotic, the distance function

$$t \to d(g_1(t), g_2(t)),\ t \geq 0$$

is convex (see [**2**, Ch. I, Proposition 5.4]) and bounded. Therefore, it is decreasing with a global infimum, say, $C \geq 0$. Clearly, if $C = 0$ we have nothing to show. Assume $C > 0$. For each point $g_2(t)$ on $\mathrm{Im}g_2$, denote by $g_1(s(t))$ the unique point on $\mathrm{Im}g_1$ realizing the distance $d(g_2(t), \mathrm{Im}g_1)$. By [**2**, Ch. I, Corollary 5.6], the function $t \to d(g_1(s(t)), g_2(t))$, $t \geq 0$, is convex and by assumption it decreases to 0. As $d(g_1(t), g_2(t)) \searrow C$ it follows that

$$d(g_1(s(t)), g_1(t)) \to C \quad \text{as } t \to \infty. \tag{20}$$

Since g_1 is a geodesic, $|t - s(t)| \to C$ as $t \to \infty$. There exists a sequence $t_n \to \infty$ such that

$$t_n - s(t_n) \to \delta C \quad \text{with } |\delta| = 1. \tag{21}$$

Define the geodesic $\overline{g}_1$ by $\overline{g}_1(t) := g_1(t - \delta C)$ and we show that

$$d(\overline{g}_1(t_n), g_2(t_n)) \to 0 \quad \text{as } n \to \infty.$$

By the triangle inequality, we have

$$\begin{aligned} d(\overline{g}_1(t_n), g_2(t_n)) &\leq d(\overline{g}_1(t_n), g_1(s(t_n))) + d(g_1(s(t_n)), g_2(t_n)) \\ &= d(g_1(t_n - \delta C), g_1(s(t_n))) + d(g_1(s(t_n)), g_2(t_n)) \\ &= |t_n - \delta C - s(t_n)| + d(g_1(s(t_n)), g_2(t_n)) \\ &\to 0 + 0 \end{aligned}$$

where the first equality holds by the definition of $\overline{g}_1$, the second equality holds since g_1 is a geodesic and the limit is obtained by (21) and the fact that $d(g_1(s(t)), g_2(t))$, $t \geq 0$, decreases to 0. □

We need a Gauss–Bonnet formula stated for a simply connected non-positively curved surface P with one piece-wise geodesic boundary component ∂P and finitely many conical points in its interior and/or its boundary. For each conical point $s \in \partial P$, denote by $\theta_P(s)$ the cone angle at s inside P. Then the following holds (see [**4**, Proposition 8])

$$2\pi \leq \sum_{s\in P\setminus\partial P} (2\pi - \theta(s)) + \sum_{s\in\partial P} (\pi - \theta_P(s)). \tag{22}$$

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Proof of Proposition 39. If $\mathrm{Im}g_1 \cap \mathrm{Im}g_2 \neq \emptyset$, then there must exist a $K \in \mathbb{R}$ such that

$$\mathrm{Im}g_1|_{[K,+\infty)} \subset \mathrm{Im}g_2 \tag{23}$$

otherwise uniqueness of geodesic rays would be violated. Clearly, if property (23) holds, the result follows trivially, so we may assume that $\mathrm{Im}g_1 \cap \mathrm{Im}g_2 = \emptyset$. If $d(\mathrm{Im}g_1, \mathrm{Im}g_2) = 0$, the result follows from Lemma 40. We assume

$$d(\mathrm{Im}g_1, \mathrm{Im}g_2) = C > 0 \tag{24}$$

and we will reach a contradiction. Consider the convex region P bounded by

$$\mathrm{Im}g_1|_{[0,+\infty)} \cup \mathrm{Im}g_2|_{[0,+\infty)} \cup [g_1(0), g_2(0)] \equiv \partial P.$$

We claim that there exist finitely many conical points in the interior of P. Assume, in contrast, that there exist infinitely many conical points in the interior of P. Then, for any positive integer N we may find $T_N \in (0, +\infty)$ so that the bounded convex region P_N bounded by

$$\mathrm{Im}g_1|_{[0,T_N]} \cup \mathrm{Im}g_2|_{[0,T_N]} \cup [g_1(0), g_2(0)] \cup [g_1(T_N), g_2(T_N)] \equiv \partial P_{T_N}$$

contains at least N conical points in its interior. This can be done because g_1, g_2 are asymptotic and, thus, the distance between the segments $[g_1(0), g_2(0)]$, $[g_1(T_N), g_2(T_N)]$ tends to $+\infty$ as $T_N \to \infty$, which implies that $P = \bigcup_{N\in\mathbb{N}} P_{T_N}$.

Since there are finitely many conical points in S, there exists $\theta_0 > 0$ such that $2\pi - \theta(s) < -\theta_0 < 0$ for all conical points s in S and, hence,

$$2\pi - \theta(\widetilde{s}) < -\theta_0 < 0 \quad \text{for all conical points } \widetilde{s} \text{ in } \widetilde{S}.$$

In particular, all terms $(2\pi - \theta(\widetilde{s}))$ in the first summand on the right-hand side of formula (22) for P_{T_N} are negative and bounded by $-\theta_0$. The terms $(\pi - \theta_P(\widetilde{s}))$ are non-positive for all $\widetilde{s} \neq g_1(0), g_2(0), g_1(T_N), g_2(T_N)$. It follows that for N large enough, say $N > 3(2\pi/\theta_0)$, the right-hand side of formula (22) is negative, a contradiction. This shows that, in fact, there exist finitely many conical points in the interior of P. Let M_1 be a bound on the distance between conical points in P from $[g_1(0), g_2(0)]$ and

$$M_2 = \sup_t \{d(g_1(t), g_2(t)) \mid t \in [0, +\infty)\}.$$

Then the geodesic segment $[g_1(2M_1 + 2M_2), g_2(2M_1 + 2M_2)]$ splits P into two subsurfaces; a bounded one containing all conical points of P and an unbounded one not containing conical points. Hence, up to re-parametrization, we may assume that $P \setminus \partial P$ does not contain any conical points. Moreover, we may assume that

$$p(g_1(0)),\ p(g_2(0)) \in S_1, \tag{25}$$

Clearly, this assumption can be made if both $p(\mathrm{Im}g_1)$ and $p(\mathrm{Im}g_2)$ intersect S_1 (by appropriately restricting g_1 and g_2). We need to verify or exclude the following four additional cases:

(i) if both $p(\mathrm{Im}g_1)$ and $p(\mathrm{Im}g_2)$ are contained in M_2, then the desired result follows as M_2 has strictly negative curvature;

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(ii) if $p(\mathrm{Im}g_1) \subset M_2$ and $p(\mathrm{Im}g_2) \subset S_1$ (or *vice versa*), then g_1 and g_2 cannot be asymptotic;

(iii) if $p(\mathrm{Im}g_1) \subset M_2$ and $p(g_2)$ intersects $\mathrm{Im}c$ finitely many times, then by appropriate restriction of g_2 this case is reduced to either case (i) or (ii);

(iv) if $p(\mathrm{Im}g_1) \subset M_2$ and $p(g_2)$ intersects $\mathrm{Im}c$ infinitely many times, we may pick a lift $\widetilde{c}$ of c such that $d(g_2(0), \mathrm{Im}\widetilde{c}) > d(g_2(0), g_1(0))$, then $\mathrm{Im}\widetilde{c}$ splits $\widetilde{X}$ into two subsurfaces and $\partial\widetilde{X} \setminus \{\widetilde{c}(-\infty), \widetilde{c}(\infty)\}$ consists of two components one containing $g_1(\infty)$ and the other $g_2(\infty)$; this is a contradiction since g_1, g_2 are assumed asymptotic.

Enumerate the conical points on $\mathrm{Im}g_1|_{[0,+\infty)}$ by

$$\widetilde{s}_0^1 = g_1(0), \widetilde{s}_1^1, \widetilde{s}_2^1, \widetilde{s}_3^1, \ldots, \widetilde{s}_j^1, \ldots$$

according to their distance from $g_1(0)$, that is, $\widetilde{s}_j^1 \in [\widetilde{s}_0^1, \widetilde{s}_{j'}^1]$ for all $j < j'$. We exclude from the enumeration any conical point whose angle inside P is π. We also allow the case where $\mathrm{Im}g_1|_{[0,+\infty)}$ contains finitely many conical points, that is, the above sequence being finite. Similarly for $\mathrm{Im}g_2|_{[0,+\infty)}$. As the angle at each $\widetilde{s}_j^1$ is $> \pi$ we may extend each geodesic segment $[\widetilde{s}_{j+1}^1, \widetilde{s}_j^1]$, $j = 1, 2, \ldots$, to a geodesic segment $[\widetilde{s}_{j+1}^1, x_j^1] \ni \widetilde{s}_j^1$ so that $x_j^1 \in \partial P$ and the angle at $\widetilde{s}_j^1$ inside P is π. We claim that x_j^1 cannot belong to $\mathrm{Im}g_2|_{[0,+\infty)}$ and, thus, $x_j^1 \in [g_1(0), g_2(0)]$. Assume, in contrast, that x_j^1 for some j belongs to $\mathrm{Im}g_2|_{[0,+\infty)}$. Denote by $[x_j^1, \xi)_{g_2}$ the geodesic sub-ray of g_2 emanating from x_j^1. Similarly, denote by $[\widetilde{s}_j^1, \xi)_{g_1}$ the geodesic sub-ray of g_1 emanating from $\widetilde{s}_j^1$. Then, the union

$$[x_j^1, \widetilde{s}_j^1] \cup [\widetilde{s}_j^1, \xi)_{g_1}$$

is also a geodesic ray from x_j^1 to ξ because the angle at $\widetilde{s}_j^1$ is at least π on both sides. This contradicts uniqueness of geodesic rays in the CAT(0) space $\widetilde{X}$. For any $j < j'$, the geodesic segments $[\widetilde{s}_j^1, x_j^1]$ and $[\widetilde{s}_{j'}^1, x_{j'}^1]$ cannot intersect, otherwise there would exist two geodesic segments joining the intersection point with $\widetilde{s}_{j'}^1$ contradicting uniqueness of geodesic segments. Hence, x_j^1 and $x_{j'}^1$ are distinct. Moreover, as the geodesic triangle formed by $\widetilde{s}_{j'}^1$, $g_1(0)$ and $x_{j'}^1$ contains the segment $[\widetilde{s}_j^1, x_j^1]$, it follows that $d(g_1(0), x_j^1) < d(g_1(0), x_{j'}^1)$. We denote the latter property by the symbol $\prec$ and we have shown that

$$x_j^1 \prec x_{j'}^1 \quad \text{for all } j < j'. \tag{26}$$

Do the same with the segments $[\widetilde{s}_{j+1}^2, \widetilde{s}_j^2]$, $j = 1, 2, \ldots$, to obtain points

$$\{x_j^2 \mid j = 1, 2, \ldots\} \subset [g_1(0), g_2(0)]$$

satisfying

$$x_{j'}^2 \prec x_j^2 \quad \text{for all } j < j'.$$

As above, for any j, j' the geodesic segments $[\widetilde{s}_{j+1}^1, x_j^1]$ and $[\widetilde{s}_{j'+1}^2, x_{j'}^2]$ cannot intersect, otherwise there would exist two geodesic rays joining the intersection point with ξ contradicting uniqueness of geodesic rays. Therefore,

$$x_j^1 \prec x_{j'}^2 \quad \text{for all } j, j'. \tag{27}$$

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Let x^1 (respectively, x^2) be the unique accumulation point of the set $\{x_j^1 | j = 1, 2, \ldots\}$ (respectively, $\{x_j^2 | j = 1, 2, \ldots\}$). In the case $\mathrm{Im} g_1|_{[0,+\infty)}$ contains finitely many conical points, x^1 is simply $\max_j\{x_j^1\}$ and similarly for x^2. Moreover, by (27),

$$x_j^1 \prec x^1 \quad \text{and} \quad x^2 \prec x_j^2 \quad \text{for all } j. \tag{28}$$

Case A: $x^1 = x^2$.

To reach a contradiction, pick points x_j^1 and $x_{j'}^2$ such that

$$d(x_j^1, x_{j'}^2) < C.$$

Denote by r_j (respectively, $r_{j'}$) the geodesic ray emanating from x_j^1 (respectively, $x_{j'}^2$) with $r_j(+\infty) = \xi$ (respectively, $r_{j'}(+\infty) = \xi$). By construction, r_j and g_1 have a common sub-ray and so do $r_{j'}$ and g_2. It follows from (24) that for large enough T, $d(r_j(T), r_{j'}(T)) \geq C$ and, by choice, $d(r_j(0), r_{j'}(0)) < C$. Therefore, the distance function $t \to d(r_j(t), r_{j'}(t))$, $t \geq 0$, is not decreasing. This is a contradiction because the distance function is convex (see [**2**, Ch. I, Proposition 5.4]) and, as $r_j(+\infty) = r_{j'}(+\infty)$, it is also bounded.

Case B: $x^1 \neq x^2$. In this case it is easily seen that

$$d(x^1, x^2) \geq C. \tag{29}$$

For, if $d(x^1, x^2) < C$ we may find points x_j^1 and $x_{j'}^2$ such that

$$d(x_j^1, x_{j'}^2) < C$$

and proceed to reach a contradiction as above.

Moreover, it can be seen that

$$d(\mathrm{Im}\, r_1, \mathrm{Im}\, r_2) \geq C, \tag{30}$$

where $\mathrm{Im}\, r_i$, $i = 1, 2$, is the geodesic ray from x^i to ξ.

To check this, assume $d(\mathrm{Im}\, r_1, \mathrm{Im}\, r_2) \leq C - c_0$ for some $c_0 > 0$. We may find points x_j^1, $x_{j'}^2$ such that

$$d(x_j^1, x^1) = d(x^2, x_{j'}^2) = c_0/3$$

and, thus, by convexity

$$d(\mathrm{Im}\, r_j, \mathrm{Im}\, r_1) \leq c_0/3 \quad \text{and} \quad d(\mathrm{Im}\, r_2, \mathrm{Im}\, r_{j'}) \leq c_0/3.$$

Since r_j (respectively, $r_{j'}$) and g_1 (respectively, g_2) have a common sub-ray

$$d(\mathrm{Im}\, g_1, \mathrm{Im}\, g_2) = d(\mathrm{Im}\, r_j, \mathrm{Im}\, r_{j'}).$$

It follows that

$$\begin{aligned} C &= d(\mathrm{Im}\, g_1, \mathrm{Im}\, g_2) = d(\mathrm{Im}\, r_j, \mathrm{Im}\, r_{j'}) \\ &\leq d(\mathrm{Im}\, r_j, \mathrm{Im}\, r_1) + d(\mathrm{Im}\, r_1, \mathrm{Im}\, r_2) + d(\mathrm{Im}\, r_2, \mathrm{Im}\, r_{j'}) \\ &\leq c_0/3 + (C - c_0) + c_0/3 \\ &= C - c_0/3. \end{aligned}$$

We distinguish two sub-cases as follows.

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Subcase B1: $p^{-1}(\operatorname{Im} c_M) \cap P$ has finitely many components.

Then, by considering sub-rays of g_1 and g_2, we may assume that

$$\text{either } p(\operatorname{Im} g_1) \cup p(\operatorname{Im} g_2) \subset M_2 \quad \text{or,} \quad p(\operatorname{Im} g_1) \cup p(\operatorname{Im} g_2) \subset S_1.$$

In the former case, $\operatorname{Im} g_1$, $\operatorname{Im} g_2$ are contained in a subsurface of $\widetilde{X}$ that has strictly negative curvature. Thus, as they are asymptotic, their distance $d(\operatorname{Im} g_1, \operatorname{Im} g_2)$ must be zero contradicting (24). For the latter case, pick distinct points y_1, y_2 in the interior of the segment $[x^1, x^2]$ and denote by q_1 (respectively, q_2) the geodesic ray emanating from y_1 (respectively, y_2) with $q_1(+\infty) = \xi$ (respectively, $q_2(+\infty) = \xi$). Clearly, $\operatorname{Im} q_1$, $\operatorname{Im} q_2$ are disjoint, thus they are contained in a flat subsurface of $\widetilde{X} \cap (P \setminus \partial P)$ and, being at bounded distance, they are parallel. Pick a geodesic ray r in the interior of the flat half strip bounded by q_1, q_2. Clearly, $r(+\infty) = \xi$ and $p(r)$ is contained in the compact *e.s.c.s.* S_1 with

$$d(\operatorname{Im} p(r), \{s_1, \ldots, s_n\}) > 0.$$

By assumption, $\xi \in \partial \widetilde{X} \setminus F_h^{nu}$, which implies that r cannot be closed. Then the above inequality contradicts Corollary 34.

Subcase B2: $p^{-1}(\operatorname{Im} c_M) \cap P$ has infinitely many components.

We may assume that all such components are segments with one endpoint on $\operatorname{Im} g_1$ and the other on $\operatorname{Im} g_2$. In this subcase, the convex region bounded by $\operatorname{Im} r_1$ and $\operatorname{Im} r_2$ consists of infinitely many Euclidean and hyperbolic quadrilaterals formed by sub-segments of the components of $p^{-1}(\operatorname{Im} c_M) \cap P$ and sub-segments of r_1, r_2. To fix notation, let

$$\{[A_k, B_k] \mid k = 1, 2, \ldots\}$$

be an enumeration of the components of $p^{-1}(\operatorname{Im} c_M) \cap P$ such that, for all k, $A_k \in \operatorname{Im} r_1$, $B_k \in \operatorname{Im} r_2$ and

$$d(A_k, r_1(0)) < d(A_{k+1}, r_1(0)) \quad \text{and} \quad d(B_k, r_2(0)) < d(B_{k+1}, r_2(0)).$$

Each segment $[A_k, A_{k+1}]$ (respectively, $[B_k, B_{k+1}]$) has length bounded below by some constant depending on the geometry of M and S_1. In other words,

$$\text{there exists } C' \quad \text{such that } d(A_k, A_{k+1}) > C' \quad \text{and} \quad d(B_k, B_{k+1}) > C', \tag{31}$$

Set $A_0 = r_1(0)$, $B_0 = r_2(0)$, and denote by Q_k, $k = 1, 2, \ldots$, the quadrilateral formed by the segments $[A_{k-1}, A_k]$, $[A_k, B_k]$, $[B_{k-1}, B_k]$, and $[A_{k-1}, B_{k-1}]$. We may assume (cf. (25)) that Q_0 is a Euclidean quadrilateral and so is Q_k for all k even. Consequently, Q_k is a hyperbolic quadrilateral for all k odd. Clearly,

$$P = \bigcup_{k=0}^{\infty} Q_k$$

and for all $m \neq n$,

$$Q_m \cap Q_n = \begin{cases} [A_{\min\{m,n\}}, B_{\min\{m,n\}}] & \text{if } |m-n| = 1 \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

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Denote by a_k, β_k, γ_k, and δ_k the angles of Q_k, that is,

$$\begin{aligned}
a_k &= \measuredangle_{A_k}([A_k, A_{k+1}], [A_k, B_k]) \\
\beta_k &= \measuredangle_{B_k}([A_k, B_k], [B_k, B_{k+1}]) \\
\gamma_k &= \measuredangle_{B_{k+1}}([B_k, B_{k+1}], [A_{k+}, B_{k+1}]) \\
\delta_k &= \measuredangle_{A_{k+1}}([A_k, A_{k+1}], [A_{k+1}, B_{k+1}]).
\end{aligned}$$

We have the following relations

$$\begin{aligned}
a_{k+1} + \delta_k = \pi = \beta_{k+1} + \gamma_k \\
a_k + \beta_k + \gamma_k + \delta_k = 2\pi \quad \text{if } k \text{ is even,} \\
a_k + \beta_k + \gamma_k + \delta_k < 2\pi \quad \text{if } k \text{ is odd.}
\end{aligned}$$

It follows that for all k,

$$a_{2k-1} + \beta_{2k-1} \lneqq a_{2k} + \beta_{2k} = a_{2k+1} + \beta_{2k+1} \lneqq a_{2k+2} + \beta_{2k+2}. \tag{32}$$

In particular, the sequence

$$\{a_k + \beta_k\}_{k\in\mathbb{N}} \text{ is increasing.} \tag{33}$$

Denote by $A(Q_{2k+1})$ the area of the quadrilateral Q_{2k+1}.

Claim: the sequence $\{A(Q_{2k+1})\}$ is bounded below by some $\Lambda > 0$.

Assume, in contrast, that $\lim_{k_n\to\infty} A(Q_{2k_n+1}) = 0$ for some subsequence $\{A(Q_{2k_n+1})\}$ of $\{A(Q_{2k+1})\}$.

Consider the geodesic segment $[A_{2k_n+1}, B_{2k_n+2}]$ that splits Q_{2k_n+1} into two triangles, say T_{2k_n+1} and T'_{2k_n+1}. By assumption, the area of T'_{2k_n+1} tends to 0 as $k_n \to \infty$. By (30) the side $[A_{2k_n+1}, B_{2k_n+1}]$ of T'_{2k_n+1} is bounded below by C and, by (31), the side $[B_{2k_n+1}, B_{2k_n+2}]$ is bounded below by C'. It follows that $\beta_{2k_n+1} \to 0$. In a similar way, we use the geodesic segment $[A_{2k+2}, B_{2k+1}]$ to show that $\alpha_{2k+1} \to 0$. Thus, $\{a_{2k_n+1} + \beta_{k_n+1}\} \to 0$, contradicting (33). This completes the proof of the claim.

Note that as M has negative curvature bounded away from 0, there exists a constant λ_M such that

$$A(Q_{2k+1}) \le \lambda_M(2\pi - (a_{2k+1} + \beta_{2k+1} + \gamma_{2k+1} + \delta_{2k+1})).$$

Combining this inequality with the claim, we have

$$\begin{aligned}
2\pi - (a_{2k+1} + \beta_{2k+1} + \gamma_{2k+1} + \delta_{2k+1}) &\ge \frac{\Lambda}{\lambda_M} \\
a_{2k+1} + \beta_{2k+1} + \gamma_{2k+1} + \delta_{2k+1} &\le 2\pi - \frac{\Lambda}{\lambda_M} \\
a_{2k+1} + \beta_{2k+1} + (\pi - \beta_{2k+2}) + (\pi - a_{2k+2}) &\le 2\pi - \frac{\Lambda}{\lambda_M} \\
a_{2k+1} + \beta_{2k+1} + \frac{\Lambda}{\lambda_M} &\le \alpha_{2k+2} + \beta_{2k+2}.
\end{aligned} \tag{34}$$

By (32) it follows that $a_k + \beta_k \to \infty$ as $k \to \infty$, a contradiction.

Therefore, the assumption $d(\mathrm{Im}g_1, \mathrm{Im}g_2) = C > 0$ (cf. (24)) leads to a contradiction and the result follows from Lemma 40. □

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We proceed now to show Condition (D). We need the following result.

LEMMA 41. *Let φ be a hyperbolic element of Γ and let $\eta = \varphi(-\infty)$ and $\xi = \varphi(\infty)$ be the repulsive and attractive points of φ in $\partial\widetilde{X}$. Then any geodesic line c joining η and ξ projects to a closed geodesic in X.*

Proof. By [**2**, Proposition 3.3, p. 31], there is an axis c_0 of φ in $\widetilde{X}$ that projects to a closed geodesic in X. Let c be a geodesic line of $\widetilde{X}$ joining the points η, ξ. Then, by the flat strip theorem, c and c_0 are parallel in $\widetilde{X}$, that is, they bound a flat strip in $\widetilde{X}$. Therefore, c is also an axis of φ and, thus, it projects to a closed geodesic in S. □

PROPOSITION 42. *The set*

$$\{(g(+\infty), g(-\infty)) \mid p(g) \text{ is closed and unique}\}$$

is dense in $\partial^2\widetilde{X}$.

Proof. Let $O \times U$ be open in $\partial^2\widetilde{X}$ where O, U are disjoint intervals in $\partial\widetilde{X}$. By Proposition 10, there exists a hyperbolic $\phi \in \Gamma$ such that

$$(\phi(+\infty), \phi(-\infty)) \in O \times U.$$

If the closed geodesic β corresponding to the axis $(\phi(+\infty), \phi(-\infty))$ is unique we have nothing to show. Suppose β is not unique. Then it is contained in a flat strip, hence, by parallel translation we may assume that β contains a conical point $\widetilde{s}$ with cone angle $\theta(\widetilde{s}) > 2\pi$. The conical point $\widetilde{s}$ splits β into two geodesic rays, denote them by r_2 and r_3, which form an angle π inside the flat strip bounded by β and the other angle being equal to $\theta(\widetilde{s}) - \pi$. Clearly, $r_2(+\infty) = \beta(+\infty) \in O$ and $r_3(+\infty) = \beta(-\infty) \in U$. Let r_1, r_4 be geodesic rays with $r_1(0) = r_4(0) = \widetilde{s}$ such that:

- $r_1(+\infty) \in O \setminus \{\beta(+\infty)\}$ and $r_4(+\infty) \in U \setminus \{\beta(-\infty)\}$;
- r_1 and r_4 do not intersect the interior of the flat strip bounded by β; and
- $\sphericalangle_{\widetilde{s}}(r_4, r_1) > \pi$.

Pick a number θ_0 satisfying $0 < \theta_0 < (\theta(\widetilde{s}) - 2\pi/2)$. If $\sphericalangle_{\widetilde{s}}(r_1, r_2) > \theta_0$ we may replace r_1 by a geodesic ray r_1' emanating form $\widetilde{s}$ satisfying the above three properties and such that, in addition, $\sphericalangle_{\widetilde{s}}(r_1', r_2) < \theta_0$. We similarly replace, if necessary, r_4. Hence, we may assume that the (clockwise) angles formed by r_i at $\widetilde{s}$ satisfy the following relations

$$\begin{aligned} 0 \le \sphericalangle_{\widetilde{s}}(r_1, r_2) &< \theta_0, \\ \sphericalangle_{\widetilde{s}}(r_2, r_3) &= \pi, \\ 0 \le \sphericalangle_{\widetilde{s}}(r_3, r_4) &< \theta_0, \\ \sphericalangle_{\widetilde{s}}(r_4, r_1) &> \pi. \end{aligned} \tag{35}$$

Observe that equality in any of the above inequalities holds if and only if the images of the corresponding geodesics rays have a geodesic segment in common. Let $(r_1(+\infty), r_2(+\infty))$ be the (open) interval on the boundary $\partial\widetilde{X}$ between these two points contained in O and $(r_3(+\infty), r_4(+\infty))$ the corresponding interval in U. Clearly, these intervals are disjoint. By Proposition 10 and Lemma 41, there exist boundary points

$$\eta \in (r_1(+\infty), r_2(+\infty)), \quad \zeta \in (r_3(+\infty), r_4(+\infty)), \tag{36}$$

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such that $\eta = g(+\infty)$, $\zeta = g(-\infty)$ for some closed geodesic g in $G\widetilde{X}$. Clearly, $(g(+\infty), g(-\infty)) \in O \times U$ and we show that g is a unique (closed) geodesic.

We first show that Img must contain $\widetilde{s}$. $\widetilde{X}$ is homeomorphic to an open disk and the images of the geodesic rays r_i, $i = 1, 2, 3, 4$, split $\widetilde{X}$ into four open convex regions denoted by P_{12}, P_{23}, P_{34}, P_{41} bounded by

$$\mathrm{Im}r_1 \cup \mathrm{Im}r_2 \setminus \mathrm{Im}r_1 \cap \mathrm{Im}r_2, \quad \mathrm{Im}r_2 \cup \mathrm{Im}r_3, \quad \mathrm{Im}r_3 \cup \mathrm{Im}r_4 \setminus \mathrm{Im}r_3 \cap \mathrm{Im}r_4, \quad \mathrm{Im}r_4 \cup \mathrm{Im}r_1,$$

respectively. By its definition (cf. equation (36)) Img intersects P_{12} and P_{34}. Assume, in contrast, that $\widetilde{s} \notin \mathrm{Im}g$. Then Im$g$ must intersect either P_{23} or P_{41}. If Img intersects P_{23}, then Img intersects the boundary lines r_2 and r_3, which are sub-rays of β. This contradicts the uniqueness of geodesic segments. Assume now that Img intersects P_{41}. Then, Img intersects the boundary lines r_1 and r_4, that is, there exists $x = r_1(t_x) \in \mathrm{Im}g \cap \mathrm{Im}r_1$ and $y = r_4(t_y) \in \mathrm{Im}g \cap \mathrm{Im}r_4$. Since $\sphericalangle_{\widetilde{s}}(r_4, r_1) > \pi$ (cf. (35)), $r_1|_{[0,t_x]} \cup r_4|_{[0,t_y]}$ is a geodesic segment containing $\widetilde{s}$ with endpoints x, y. As $x, y \in \mathrm{Im}g$ and $\widetilde{s} \notin \mathrm{Im}g$ the geodesic g provides a geodesic segment with endpoints x, y distinct from $r_2|_{[0,t_x]} \cup r_3|_{[0,t_y]}$. This also contradicts uniqueness of geodesic segments.

To see that g is unique, let g' be a (closed) geodesic with $p(g')$ freely homotopic to $p(g)$. Then $g'(-\infty) = \eta = g(-\infty)$ and $g'(+\infty) = \zeta = g(+\infty)$. By the above argument Img' also contains $\widetilde{s}$ and by the uniqueness of geodesic rays, g and g' coincide. □

It is plausible to believe that the techniques used in this section to prove that the CAT(0) surface X satisfies the assumptions (Δ), (U), (C), and (D) can be applied for the class of multipolyhedra of piecewise constant curvature $\chi \leq 0$ (see [**13**, §3.2]).

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