SMOOTH TABOO DENSITY FOR ONE-DIMENSIONAL DIFFUSIONS

S. F. V. LO AND G. O. ROBERTS

ABSTRACT

The paper is concerned with finding sufficient smoothness conditions on the diffusion and drift coefficient of a one-dimensional stochastic diffusion to imply the existence and smoothness of a taboo density.

1. *Introduction*

In this paper, we are concerned with the existence and smoothness of taboo densities for one-dimensional diffusion processes. Consider the one-dimensional stochastic differential equation

$$dX_s = \sigma(X_s)\,dB_s + b(X_s)\,ds, \quad X_0 = x, \tag{1}$$

where X is a diffusion process in $\mathbb{R}$, B is a standard Brownian motion process in $\mathbb{R}$, $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ and $b : \mathbb{R} \longrightarrow \mathbb{R}$ are deterministic functions. If σ and b are sufficiently regular (for example, Lipschitz), the stochastic differential equation (1) has a unique strong solution from each starting point, and this solution is a time-homogeneous strong Markov process.

Let $\tilde{P}$ be the law of the solution X of (1), and τ_y be the time that X first exits from some constant boundary y so that

$$\tau_y = \inf\{s > 0 : X_s \geqslant y\}.$$

Denote also the taboo density by $\pi(t, x, x', y)$, that is

$$\pi(t, x, x', y) \equiv \tilde{P}_x(X_t \in dx', \tau_y > t)/dx', \quad x, x' < y.$$

Now we are interested in the existence of π and the way that the smoothness of σ, b is reflected in the smoothness of π. We have

$$\tilde{P}_x(\tau_y \in ds, X_t \in dx')/ds = \tilde{P}_x(\tau_y \in ds \,|\, X_t = x')/ds \cdot \tilde{p}(t, x, x'),$$

where $\tilde{p}(t, x, x')$ is the transition density for X, and $\tilde{P}_x(\tau_y \in ds \,|\, X_t = x')$ is the regular conditional probability of $\{\tau_y \in ds\}$ given $\{X_t = x'\}$ under $\tilde{P}_x$ and, for all $B \in \mathscr{B}$ with $\mathscr{B}$ being the smallest σ-field containing all intervals, satisfies

$$\tilde{P}_x(\tau_y \in ds, X_t \in B) = \int_B \tilde{P}_x(\tau_y \in ds \,|\, X_t = x')\,\tilde{p}(t, x, x')\,dx'.$$

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The joint continuity and the differentiability of $\tilde{p}(t,x,x')$ in all three variables has already been studied by Rogers [7]. Also for $s>t$, $\tilde{P}_x(\tau_y \in ds \,|\, X_t = x')/ds = \tilde{P}_{t,x'}(\tau_y \in ds)/ds$, the existence and smoothness of which has also been examined [4]. It therefore suffices to study the smoothness of

$$\tilde{P}_x(\tau_y \in ds \,|\, X_t = x')/ds, \quad 0 < s < t, x, x' < y,$$

that is the first passage density of the diffusion process X in the time interval $(0,t)$ given that $X_t = x'$ under $\tilde{P}_x$.

In proving our result, we demonstrate the existence of diffusion bridges, the smoothness of transition densities and first hitting time densities involving these bridges. The result seems interesting in its own right. Throughout this paper, our probabilistic method owes much to the earlier papers by Rogers [7] and Pauwels [4].

2. *The main result*

We shall establish the following existence and smoothness result for the taboo density of X the solution of (1).

THEOREM 1. *Suppose that $k \in \mathbb{Z}_+$ and that*

(i) *σ possesses bounded continuous derivatives of all orders up to and including $k+2$;*

(ii) *b possesses bounded continuous derivatives of all orders up to and including $k+1$;*

(iii) *there exists $\varepsilon > 0$ such that $\sigma(x) \geqslant \varepsilon$ for all x.*

Then the solution to (1) *has taboo density $\pi : (0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to (0,\infty)$ which is jointly continuous and possesses continuous partial derivatives in all variables of all orders up to and including k.*

From now we shall assume that $\sigma(x) = 1$ for all x since the general problem can be reduced to this case by standard scale transformation as suggested by Rogers [7]. Define the strictly increasing function

$$F(x) \equiv \int_0^x \sigma(u)^{-1}\, du$$

and consider the process $Y_t \equiv F(X_t)$. Itô's formula gives

$$dY_t = dB_t + \left(\frac{b}{\sigma}(X_t) - \tfrac{1}{2}\sigma'(X_t)\right) dt = dB_t + \beta(Y_t)\, dt,$$

where

$$\beta(y) = \left(\frac{b}{\sigma} - \tfrac{1}{2}\sigma'\right) \circ F^{-1}(y)$$

which, under the hypotheses on b and σ in Theorem 1, has bounded continuous derivatives of all orders up to and including $k+1$. Also, we assume that Theorem 1(i)–(iii) hold throughout the rest of this paper.

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3. *Some technical lemmas*

We will use the following results. Some will be stated without proof, but Lemma 2 (the existence of a diffusion bridge process) is of independent interest. In this section, we assume that b possesses bounded continuous derivatives up to the first order.

PROPOSITION 1. *Let M be a stochastic process on $[0,t]$. If M is a left continuous non-negative martingale on $[0,t)$ with $M_0 = 1$, then the following are equivalent:*

(i) *M is a martingale on $[0,t]$ and is uniformly integrable on $[0,t)$;*

(ii) $E(M_t) = 1$.

LEMMA 1. *Let P_x be the law of Brownian motion started at x and $\tilde{P}_x$ be the law of the solution X of (1) with $\sigma = 1$. Then, since b possesses continuous first derivative,*

$$\frac{\tilde{P}_{s,X_s}(X_t \in dx')}{P_{s,X_s}(X_t \in dx')} \xrightarrow{\text{a.s. } (\tilde{P}_x)} 1 \quad \text{as} \quad s \uparrow t,$$

where a.s. = *almost surely, $\bar{P}_x$ is the law of the Brownian bridge process between $(0,x)$ and (t,x'), that is the solution of*

$$dZ_s = dB_s + \frac{(x' - Z_s)}{(t-s)}\,ds, \quad Z_0 = x, 0 \leqslant s < t.$$

Proof. We shall first obtain an expression for the density of X under the measure $\tilde{P}_{s,z}(s \in \mathbb{R}_+, z \in \mathbb{R})$. We have, for any bounded continuous function $l: \mathbb{R} \longrightarrow \mathbb{R}$,

$$\begin{aligned}
\tilde{E}_{s,z}(l(X_t)) &= E_{s,z}\left(\exp\left(\int_s^t b(X_u)\,dX_u - \frac{1}{2}\int_s^t b^2(X_u)\,du\right) l(X_t)\right) \\
&= E_{s,z}\left(E_{s,z}\left(\exp\left(\int_s^t b(X_u)\,dX_u - \frac{1}{2}\int_s^t b^2(X_u)\,du\right) l(X_t) \mid X_t\right)\right) \\
&= \int_{-\infty}^{\infty} l(x')\,P_{s,z}(X_t \in dx') \\
&\quad \times E_{s,z}\left(\exp\left(\int_s^t b(X_u)\,dX_u - \frac{1}{2}\int_s^t b^2(X_u)\,du\right) \mid X_t = x'\right).
\end{aligned}$$

The density of X therefore has an expression

$$\begin{aligned}
\tilde{P}_{s,z}(X_t \in dx')/dx' &= P_{s,z}(X_t \in dx')/dx' \\
&\quad \times E_{s,z}\left(\exp\left(\int_s^t b(X_u)\,dX_u - \frac{1}{2}\int_s^t b^2(X_u)\,du\right) \mid X_t = x'\right).
\end{aligned}$$

Equivalently,

$$\frac{\tilde{P}_{s,z}(X_t \in dx')}{P_{s,z}(X_t \in dx')} = \bar{E}_{s,z}\left(\exp\left(\int_s^t b(X_u)\,dX_u - \frac{1}{2}\int_s^t b^2(X_u)\,du\right)\right). \tag{2}$$

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Define $\varphi(u) = \int_0^u b(v)\,dv$ and notice that $\varphi \in \mathscr{C}^2$ by hypothesis, thus by Itô's formula,

$$\varphi(X_t) - \varphi(X_s) - \frac{1}{2}\int_s^t \varphi''(X_u)\,du = \int_s^t b(X_u)\,dX_u. \tag{3}$$

Now the right-hand side of (2) is equal to

$$E_{s,z}\left(\exp\left(\varphi(X_t) - \varphi(X_s) - \frac{1}{2}\int_s^t \lambda(X_u)\,du\right) \mid X_t = x'\right)$$

$$= \exp(\varphi(x') - \varphi(z))\, E_{s,z}\left(\exp\left(-\frac{1}{2}\int_s^t \lambda(\gamma(u))\,du\right) \mid X_t = x'\right),$$

where

$$\lambda(X_u) = b'(X_u) + b^2(X_u), \tag{4}$$

and γ is a Brownian bridge on $[s,t)$ connecting time space points (s,z) and (t,x').

Since $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and $K \equiv \{\gamma(u) : s \leqslant u \leqslant t\}$ is compact, λ attains a minimum and a maximum on K. Also, λ is bounded on $\mathbb{R}$ due to the boundedness and continuity of b and b'. We have

$$\bar{E}(e^{-(1/2)(t-s)\bar{M}}) \leqslant \bar{E}_{s,z}\left(\exp\left(-\frac{1}{2}\int_s^t \lambda(\gamma(u))\,du\right)\right) \leqslant \bar{E}_{s,z}(e^{-(1/2)(t-s)\bar{m}}),$$

for some $\bar{m}$, $\bar{M}$ such that $|\bar{m}| < \infty$ and $|\bar{M}| < \infty$. By letting $s \uparrow t$ and using the bounded convergence theorem, we have

$$\lim_{s\uparrow t} E_{s,z}\left(\exp\left(-\frac{1}{2}\int_s^t \lambda(\gamma(u))\,du\right) \mid X_t = x'\right) = 1.$$

Moreover,

$$\lim_{s\uparrow t} X_s = X_t \overset{\text{a.s.}(\bar{P})}{=} x',$$

so

$$\lim_{s\uparrow t} \exp(\varphi(X_s)) \overset{\text{a.s.}(\bar{P})}{=} \exp(\varphi(X_t)).$$

Now by boundedness of λ, the fact that

$$X_s \xrightarrow{\text{a.s.}(\bar{P})} x'$$

and the continuity of ψ, the result follows. □

LEMMA 2. *Let $\tilde{P}_x$, $\bar{P}_x$ and P_x be the laws defined as in Lemma* 1 *and in the proof of Lemma* 1. *If we define, for $0 \leqslant s < t$,*

$$D_s \equiv \frac{P_x(X_t \in dx')}{\tilde{P}_x(X_t \in dx')} \frac{\tilde{P}_{s,X_s}(X_t \in dx')}{P_{s,X_s}(X_t \in dx')} \times \exp\left(\int_0^s b(X_u)\,dX_u - \frac{1}{2}\int_0^s b^2(X_u)\,du\right),$$

then D is a martingale on $[0,t]$ and is uniformly integrable on $[0,t)$ under $\bar{P}_x$ and there exists a unique probability measure $\tilde{\bar{P}}_{t,x}$ on $((C[0,t],\mathbb{R}),\mathscr{F}_t^0)$, where $\mathscr{F}_s^0 \equiv \sigma(X_r, r \leqslant s)$, such that $d\tilde{\bar{P}}_{t,x}/d\bar{P}|_{\mathscr{F}_s^0} = D_s$. Furthermore, $\tilde{\bar{P}}_{t,x}$ is a regular conditional probability of a diffusion X conditioned on $X_t = x'$, which we shall call a diffusion bridge.

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Proof. To prove Lemma 2, we shall make use of Proposition 1 by showing that $\bar{E}_x(D_t) = 1$ and that D is a non-negative martingale on $[0, t)$ with $D_0 = 1$.

Write $\mathscr{E}_{r,s} = \exp(\int_r^s b(X_u)\, dX_u - \frac{1}{2}\int_r^s b^2(X_u)\, du)$. Then for $0 \leqslant s < t$,

$$\bar{E}_x(D_s) = \frac{P_x(X_t \in dx')}{\tilde{P}_x(X_t \in dx')} \bar{E}_x\left(\mathscr{E}_{0,s} \frac{\tilde{P}_{s,X_s}(X_t \in dx')}{P_{s,X_s}(X_t \in dx')}\right)$$

in which

$$\begin{aligned}
&\bar{E}_x\left(\mathscr{E}_{0,s} \frac{\tilde{P}_{s,X_s}(X_t \in dx')}{P_{s,X_s}(X_t \in dx')}\right) \\
&\qquad = \bar{E}_x(\mathscr{E}_{0,s}\, \bar{E}_{s,X_s}(\mathscr{E}_{s,t})) \quad \text{(by (2))} \\
&\qquad = E_x(\mathscr{E}_{0,s}\, E_{s,X_s}(\mathscr{E}_{s,t} \mid X_t = x') \mid X_t = x') \\
&\qquad = E_x(E_x(\mathscr{E}_{0,s} \mid X_s, X_t = x')\, E_{s,X_s}(\mathscr{E}_{s,t} \mid X_t = x') \mid X_t = x') \\
&\qquad = E_x(E_x(\mathscr{E}_{0,s} \mid X_s, X_t = x')\, E_x(\mathscr{E}_{s,t} \mid X_s, X_t = x') \mid X_t = x') \\
&\qquad = E_x(E_x(\mathscr{E}_{0,s}\mathscr{E}_{s,t} \mid X_s, X_t = x') \mid X_t = x') \quad (\text{as } \mathscr{E}_{0,s} \perp \mathscr{E}_{s,t} \mid X_s) \\
&\qquad = E_x(\mathscr{E}_{0,t} \mid X_t = x') = \frac{\tilde{P}_x(X_t \in dx')}{P_x(X_t \in dx')} \qquad (5)
\end{aligned}$$

where the last equality is obtained using the fact (see, for example, [**2**, Theorem IV.4.2]) that $\tilde{P}_x \ll P_x$. By the Cameron–Martin–Girsanov theorem, we have for $s > 0$,

$$\frac{d\tilde{P}_x}{dP_x}|_{\mathscr{F}_s^0} = \exp\left(\int_0^s b(X_u)\, dX_u - \frac{1}{2}\int_0^s b(X_u)^2\, du\right)$$

and hence $\bar{E}_x(D_s) = 1$, for all $0 \leqslant s < t$. Now by Lemma 1, we have $\lim_{s \uparrow t} D_s = D_t$ exists, where

$$D_t \overset{\text{a.s.}(\bar{P}_x)}{=} \frac{P_x(X_t \in dx')}{\tilde{P}_x(X_t \in dx')} \exp\left(\int_0^t b(X_u)\, dX_u - \frac{1}{2}\int_0^t b^2(X_u)\, du\right)$$

and so $\bar{E}_x(D_t) = 1$. Furthermore, D is a non-negative martingale on $[0, t)$ since we have, for $r \leqslant s$ and $r, s \in [0, t)$,

$$\bar{E}_x(D_s \mid \mathscr{F}_r) = \frac{P_x(X_t \in dx')}{\tilde{P}_x(X_t \in dx')} \mathscr{E}_{0,r} \times \bar{E}_{r,X_r}\left(\mathscr{E}_{r,s} \frac{\tilde{P}_{s,X_s}(X_t \in dx')}{P_{s,X_s}(X_t \in dx')}\right),$$

where, similar to (5),

$$\bar{E}_{r,X_r}\left(\mathscr{E}_{r,s} \frac{\tilde{P}_{s,X_s}(X_t \in dx')}{P_{s,X_s}(X_t \in dx')}\right) = \frac{\tilde{P}_{r,X_r}(X_t \in dx')}{P_{r,X_r}(X_t \in dx')}.$$

Now from [**5**, Proposition VIII.1.13], there exists a unique probability measure $\tilde{\bar{P}}_{t,x}$ on $((C[0, t], \mathbb{R}), \mathscr{F}_t^0)$, where $\mathscr{F}_s^0 \equiv \sigma(X_r, r \leqslant s)$, such that $d\tilde{\bar{P}}_{t,x}/d\bar{P}|_{\mathscr{F}_s^0} = D_s$. D is a non-negative martingale on $[0, t)$ and it is left continuous (by the left continuity of the filtration $\mathscr{F}$) with $D_0 = 1$, by Proposition 1, D is a martingale on $[0, t]$ and is uniformly integrable on $[0, t)$ under $\bar{P}_x$.

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Furthermore, if $B \in \mathscr{B}$, where $\mathscr{B}$ is the Borel σ-field, then

$$\begin{aligned}\int_{X_s \in B} \tilde{P}_x(X_t \in dx')\, d\tilde{\bar{P}}_{t,x} &= \int_{X_s \in B} D_s\, \tilde{P}_x(X_t \in dx')\, d\bar{P}_x \\ &= \int_{X_s \in B} \tilde{P}_x(X_s)\, \tilde{P}_{s,X_s}(X_t \in dx')\, \tilde{P}_x(X_t \in dx') \\ &= \tilde{P}_x(X_s \in B, X_t \in dx').\end{aligned}$$

This shows that $\tilde{\bar{P}}_{t,x}$ is the law of the diffusion bridge process $[X \mid X_t = x']$ on $((C[0,t], \mathbb{R}), \mathscr{F}_t^0)$. □

Of course in specific situations, for example Bessel bridges [**5**] or linear bridges [**6**], explicit expressions of the bridge distributions are available.

Finally, we note the following proposition.

PROPOSITION 2. *Suppose Ψ is a function such that $\Psi(s,t)$ and $\partial\Psi/\partial t$ are continuous in both s and t for $(s,t) \in [0,T]^2$ $(T \in \mathbb{R}_+)$, then*

$$\frac{d}{dt}\int_0^t \Psi(s,t)\, ds = \Psi(t,t) + \int_0^t \frac{\partial}{\partial t}\Psi(s,t)\, ds.$$

4. *Proof of the main result*

It is enough to establish the existence and differentiability of π on $(0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ for arbitrary $T > 0$, so we may assume that we are working on the canonical sample space $\Omega = \mathscr{C}([0,T], \mathbb{R})$, with the canonical process $X_s(\omega) = \omega(s)$ $(\omega \in \Omega, s \in [0,T])$ and its canonical filtration $\mathscr{F}_s^0 \equiv \sigma(X_r; r \leqslant s)$. Let P_x denote the law on $(\Omega, \mathscr{F}_T^0)$ of Brownian motion started at x. With this setting, we shall first prove the following lemma from which the proof of Theorem 1 will easily follow.

LEMMA 3. *Suppose that b possesses the properties as in Theorem* 1. *Let $\lambda(X_u) = b'(X_u) + b^2(X_u)$, and*

$$Y_{y-x}(u) = y - \left(\frac{s-u}{s}\right) R_{y-x}\left(\frac{su}{s-u}\right)$$

be a modified Bessel (3)*-process on the time interval $[0,s]$ with R_z being a Bessel* (3)*-process started from z, and β is an independent standard Brownian bridge on $[s,t]$ connecting time space points (s,y) and (t,x'). Then*

(i) *the diffusion bridge* (*see Lemma* 2) *has first hitting density of a constant boundary*

$$\begin{aligned}\tilde{\bar{P}}_{t,x}(\tau_y \in ds)/ds = (\bar{P}_x(\tau_y \in ds)/ds)\, E\left(\exp\left(-\frac{1}{2}\int_0^s \lambda(Y_{y-x}(u))\, du\right)\right) \\ \times E\left(\exp\left(-\frac{1}{2}\int_s^t \lambda(\beta(u))\, du\right)\right) \exp(\varphi(x') - \varphi(x))\, \frac{P_x(X_t \in dx')}{\tilde{P}_x(X_t \in dx')}, \quad (6)\end{aligned}$$

where φ is defined as (3)*;*

(ii) *such density is jointly continuous and possesses continuous partial derivatives in all variables of all orders up to and including k.*

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Proof. (i) Fix t such that $0 \leqslant t \leqslant T$. To obtain the expression of the density of τ_y under $\tilde{\bar{P}}_{t,x}$, we consider $\tilde{\bar{E}}_{t,x}(f(\tau_y))$, where $f: \mathbb{R}_+ \longrightarrow \mathbb{R}$ is any bounded continuous function supported in $[0, t]$. We have

$$\begin{aligned}\tilde{\bar{E}}_{t,x}(f(\tau_y)) &= \bar{E}_x\left(f(\tau_y)\frac{d\tilde{\bar{P}}_{t,x}}{d\bar{P}_x}|_{\mathcal{F}_t^0}\right)\\ &= \bar{E}_x\left(f(\tau_y)\exp\left(\int_0^t b(X_u)\,dX_u - \frac{1}{2}\int_0^t b^2(X_u)\,du\right)\cdot\frac{P_x(X_t\in dx')}{\tilde{P}_x(X_t\in dx')}\right).\end{aligned}$$

Hence

$$\begin{aligned}\tilde{\bar{E}}_{t,x}(f(\tau_y)) &= \bar{E}_x\left(f(\tau_y)\exp\left(\varphi(X_t)-\varphi(X_0)-\frac{1}{2}\int_0^t \lambda(X_u)\right)\frac{P_x(X_t\in dx')}{\tilde{P}_x(X_t\in dx')}\right)\\ &= \bar{E}_x\left(f(\tau_y)\exp\left(\varphi(x')-\varphi(x)-\frac{1}{2}\int_0^t \lambda(X_u)\right)\frac{P_x(X_t\in dx')}{\tilde{P}_x(X_t\in dx')}\right).\end{aligned}$$

Conditioning on τ_y gives

$$\begin{aligned}\tilde{\bar{E}}_{t,x}(f(\tau_y)) = \bar{E}_x\Big(f(\tau_y)\,\bar{E}_x\Big(\exp\Big(-\frac{1}{2}\int_0^t \lambda(X_u)\Big)\,|\,\tau_y\Big)\Big)\\ \times\exp(\varphi(X_t)-\varphi(X_0))\frac{P_x(X_t\in dx')}{\tilde{P}_x(X_t\in dx')}.\end{aligned}$$

Now by [**4**, Corollary 1] a regular conditional distribution of $\{X_u, 0 \leqslant u \leqslant t \,|\, \tau_y = s$ and $X_t = x'$ and $X_0 = x\}$, where X is a Brownian motion process and $0 < s < t$, is provided by the law of the process $\{A(u), 0 \leqslant u \leqslant t\}$, where

$$A(u) \equiv \begin{cases} y-\left(\dfrac{s-u}{s}\right)R_{y-x}\left(\dfrac{su}{s-u}\right) & (0 \leqslant u < s),\\[2ex] y+(x'-y)\dfrac{u-s}{t-s}+\left(W(u-s)-\dfrac{u-s}{t-s}W(t-s)\right) & (s \leqslant u < t)\end{cases}$$

where R_{y-x} is a Bessel (3)-process started at $y-x$ independent of the Brownian motion W. Then we get

$$\begin{aligned}&\bar{E}_x\left(\exp\left(-\frac{1}{2}\int_0^t \lambda(X_u)\,du\right)|\,\tau_y = s\right)\\ &\quad= E_x\left(\exp\left(-\frac{1}{2}\int_0^t \lambda(X_u)\,du\right)|\,X_t = x', \tau_y = s\right)\\ &\quad= E_x\left(\exp\left(-\frac{1}{2}\left(\int_0^s+\int_s^t\right)\lambda(X_u)\,du\right)|\,X_t = x', \tau_y = s\right)\\ &\quad= E_x\left(\exp\left(-\frac{1}{2}\int_0^s \lambda(Y_{y-x}(u))\,du\right)\right)E_x\left(\exp\left(-\frac{1}{2}\int_s^t \lambda(\beta(u))\,du\right)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{\bar{E}}_{t,x}(f(\tau_y)) = \int_0^t f(s)\,\bar{P}_x(\tau_y\in ds)\,E\left(\exp\left(-\frac{1}{2}\int_0^s \lambda(Y_{y-x}(u))\,du\right)\right)\\ \times E\left(\exp\left(-\frac{1}{2}\int_s^t \lambda(\beta(u))\,du\right)\right)\exp(\varphi(x')-\varphi(x))\frac{P_x(X_t\in dx')}{\tilde{P}_x(X_t\in dx')}.\end{aligned}$$

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Thus the $\tilde{\tilde{P}}_{t,x}-$law of τ_y has a density (6).

(ii) Under the conditions on b in Theorem 1, the smoothness of $\tilde{P}_x(X_t \in dx')$ (in x, x', t) up to order k and $E(\exp(-1/2\int_s^t \lambda(\beta(u))\,du))$ (in $y, x', t-s$ or in y, x', s, t) were proved by Rogers [**7**]. Also, the smoothness of $E(\exp(-1/2\int_0^s \lambda(Y_{y-x}(u)\,du)))$ up to order k (in x, y, s) was given by Pauwels [**4**]. Since the law of the process

$$x+(x'-x)\frac{u}{t}+(t-u)\,W^*_{(u/(t(t-u)))}$$

provides the regular conditional P_x-distribution for $\{X_u; 0 \leqslant u \leqslant t\}$ given $X_0 = x$ and $X_t = x'$, then

$$\bar{P}_x(\tau_y \in ds) = P_x(\tau_y \in ds \mid X_t = x') = P_0(\tau^*_\psi \in ds^*),$$

where

$$\tau^*_\psi = \inf\{u : W^*_u \geqslant \psi(u)\}$$

with

$$\psi(u) = (y-x)(1+ut)/t-(x'-x)\,u = y/t+(y-x')\,u,$$

and W^* is a standard Brownian motion process, $s^* = s/(t(t-s)) \in (0, \infty)$. By the Bachelier–Lévy formula,

$$P_0(\tau^*_\psi \in ds^*)/ds^* = \frac{y/t}{(s^*)^{3/2}}\phi\left(\frac{y/t+(y-x')\,s^*}{\sqrt{s^*}}\right).$$

Thus,

$$\bar{P}_x(\tau_y \in ds)/ds = \frac{y/t}{(s^*)^{3/2}}\phi\left(\frac{y/t+(y-x')\,s^*}{\sqrt{s^*}}\right)\left(\frac{1+s^*t}{t}\right)^2$$

which is bounded and infinitely continuously differentiable in all variables x, y, x' $(x' \neq y), s^*$ that is $s \in (0, t)$. The result therefore follows. □

Proof of Theorem 1. Now, $\tilde{P}_x(\tau > t, X_t \in dx') = \tilde{P}_x(X_t \in dx') - \tilde{P}_x(\tau \leqslant t, X_t \in dx')$ and $\tilde{P}_x(\tau \leqslant t, X_t \in dx') = \tilde{\tilde{P}}_{t,x}(\tau \leqslant t)\,\tilde{P}_x(X_t \in dx')$ in which $\tilde{\tilde{P}}_{t,x}(\tau \leqslant t)$, following from (6), is given by

$$\tilde{\tilde{P}}_{t,x}(\tau \leqslant t) = \int_0^t \bar{P}_x(\tau_y \in ds)\,E\left(\exp\left(-\frac{1}{2}\int_0^s \lambda(Y_{y-x}(u))\,du\right)\right)$$
$$\times E\left(\exp\left(-\frac{1}{2}\int_s^t \lambda(\beta(u))\,du\right)\right)\exp(\varphi(x')-\varphi(x))\frac{P_x(X_t \in dx')}{\tilde{P}_x(X_t \in dx')},$$

where φ is defined as (3) and λ is given by (4). Now we let

$$\Psi(x, y, x', s, t) \equiv \bar{P}_x(\tau_y \in ds)\,E\left(\exp\left(-\frac{1}{2}\int_0^s \lambda(Y_{y-x}(u))\,du\right)\right)$$
$$\times E\left(\exp\left(-\frac{1}{2}\int_s^t \lambda(\beta(u))\,du\right)\right)\exp(\varphi(x')-\varphi(x))\frac{P_x(X_t \in dx')}{\tilde{P}_x(X_t \in dx')}.$$

Since $\Psi(x, y, x', s, t)$ and $\partial\Psi/\partial t$ are continuous in all variables under the hypotheses, so by Proposition 2,

$$\frac{\partial}{\partial t}\int_0^t \Psi(x, y, x', s, t)\,ds = \Psi(x, y, x', t, t)+\int_0^t \frac{\partial}{\partial t}\Psi(x, y, x', s, t)\,ds,$$

which is (jointly) bounded continuously differentiable up to and including order $k-1$ in all variables under the hypotheses (by Lemma 3), but also $\Psi(x, y, x', s, t)$ is bounded and continuous in all variables, and Theorem 1 is therefore proved. □

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5. *Some final remarks*

REMARK 1. Suppose that Z is a stochastic process in $\mathbb{R}$ satisfying the one-dimensional stochastic differential equation,

$$dZ_s = \sigma(Z_s)\,dB_s + b(Z_s, s)\,ds, \quad Z_0 = z, \tag{7}$$

where B is a standard Brownian motion process in $\mathbb{R}$, $\sigma: \mathbb{R} \longrightarrow \mathbb{R}$ and $b: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are deterministic functions. This is the case of a stochastic process with time-dependent drift.

The time-dependent case can be treated by similar techniques. The existence and smoothness result for the taboo density of Z can be established similarly to that of X, in which case we have the following theorem.

THEOREM 2. *Suppose that* $k \in \mathbb{Z}_+$ *and that*

(i) σ *possesses bounded continuous derivatives of all orders up to and including* $k+2$;

(ii) b *possesses bounded continuous derivatives with respect to both arguments of all orders up to and including* $k+1$;

(iii) *there exists* $\varepsilon > 0$ *such that* $\sigma(x) \geqslant \varepsilon$ *for all* x.

Then the solution to (7) *has taboo density which is jointly continuous and possesses continuous partial derivatives in all variables of all orders up to and including* k.

REMARK 2. We have considered the existence and smoothness of the taboo density of a one-dimensional time-homogeneous diffusion process to some constant boundary. Indeed, we may extend the results for constant boundaries to that for time-dependent boundaries since the first hitting time of a time-homogeneous diffusion process to a constant boundary can be considered as that of a stochastic process with time-dependent diffusion coefficient and time-dependent drift to some time-independent boundary.

Suppose that X is a diffusion process in $\mathbb{R}$ satisfying the one-dimensional stochastic differential equation:

$$dX_s = \sigma(X_s)\,dB_s + b(X_s)\,ds, \quad X_0 = x,$$

where B is a standard Brownian motion process in $\mathbb{R}$, $\sigma: \mathbb{R} \longrightarrow \mathbb{R}$ and $b: \mathbb{R} \longrightarrow \mathbb{R}$ are deterministic functions. By Itô's formula, we have the transformed process $Y = X - \psi(X_0 < \psi(0))$ satisfying the stochastic differential equation:

$$dY_s = \sigma(Y_s, s)\,dB_t + b(Y_s, s)\,ds, \quad Y_0 = x - \psi(0),$$

where $\sigma(Y_s, s) = \sigma(Y_s + \psi(s)), b(Y_s, s) = b(Y_s + \psi(s)) - \psi'(s)\,(s > 0)$.

Then the first hitting time of X to a time-dependent boundary $\psi(X_0 < \psi(0))$, τ_ψ, is given by

$$\begin{aligned} \tau_\psi^X &\equiv \inf\{s > 0 : X_s \geqslant \psi(s)\} \\ &= \inf\{s > 0 : X_s - \psi(s) \geqslant 0\} \equiv \tau_0^Y. \end{aligned}$$

Therefore, we may consider the first crossing problem of a time-homogeneous diffusion process X to a time-dependent boundary ψ as that of a stochastic process Y which has time-dependent diffusion coefficient and time-dependent drift to some constant boundary. The existence and smoothness result for the taboo density of Y can be established similarly but we shall not pursue it here.

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Tokyo-Mitsubishi International
6 Broadgate
London EC2M 2AA

violet.lo@t-mi.com

Department of Mathematics and Statistics
Lancaster University
Lancaster LA1 4YF

g.o.roberts@lancaster.ac.uk

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