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PRICING VARIANCE SWAPS UNDER DOUBLE HESTON STOCHASTIC VOLATILITY MODEL WITH STOCHASTIC INTEREST RATE

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In this paper, we discuss the problem of pricing discretely sampled variance swaps under a hybrid stochastic model. Our modeling framework is a combination with a double Heston stochastic volatility model and a Cox–Ingersoll–Ross stochastic interest rate process. Due to the application of the T-forward measure with the stochastic interest process, we can only obtain an efficient semi-closed form of pricing formula for variance swaps instead of a closed-form solution based on the derivation of characteristic functions. The practicality of this hybrid model is demonstrated by numerical simulations.

Keywords: double Heston hybrid model, generalized Fourier transform, realized variance, variance swap

1. INTRODUCTION

Variance swaps are volatility derivatives which have been playing an important role in the financial markets. They are defined as a forward contract on the annualized realized variance of the returns of the underlying asset. Since the global financial crisis in 2008, variance swaps have become the most widespread accepted and popular traded products. These financial products were launched in the late 1990s with the breakthrough of volatility derivatives into the financial market. The value of variance swap depended on the future levels of volatility where forward contracts were written on the realized variance. Demeterfi *et al.* [11] showed that volatility derivatives were traded in order to aid decision-making between long and short positions, handle the spread between realized and implied volatility, and hedge against volatility risks.

In this field, researchers have been focusing on developing suitable methods for evaluating variance swaps. Based on Brownian-driven diffusion processes, the general solutions of pricing variance swaps generate continuous sample paths of asset price dynamics. Assuming that the underlying process evolves without jumps, there are two kinds of solutions of analytical methods for pricing variance swaps. The first solution replicates a variance swap by a portfolio of standard options, making a common assumption that the realized variance can be approximated by a continuously sampled variance. Carr *et al.* [7] used dynamic trading options to price and hedge the forward contract without requiring specification of a function

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for the volatility process. The second solution focuses on pricing variance swaps with the discretely sampled realized variance. When approximation was done with the large errors occured in some sampling periods, discrete sampling was better than continuous sampling. Based on the mean-reverting squared-root volatility process, Grunbichler and Longstaff [14] developed a pricing model for options on variance. Heston and Nandi [17] presented an analytical solution for both variance and volatility swaps with the GARCH volatility process. Little and Pant [23] presented the finite-difference method to obtain an efficient and accurate solution of discrete variance swaps. Zhu and Lian [24,25] managed to obtain closed-form formulas by combining Heston two-factor stochastic volatility with a constant interest rate. Rujivan and Zhu [19,20] simplified the partial differential equations by using iterated expectations as an extension of Zhu and Lian.

However, these studies based on the Heston model have a shortcoming that they fail to fit the implied volatility smile very well at short maturity. Carr and Wu [6] showed that a two-factor Heston model where the factor number was referring to the risk factors or the number of stochastic processes in a model was not flexible enough to take into account the risk component introduced by the variability of the skew.

Multifactor models are as critical to the equity option valuation literature as they are to the term structure literature. Da Fonseca *et al.* [10] showed that the multifactor volatility model was flexible enough to take into consideration correlations with the underlying asset returns. Their numerical result showed that the flexibility induced by the multifactor volatility process allowed a better fit of the smile-skew effect at both long and short maturities. Christoffersen *et al.* [8] presented that the double Heston model was a three-factor model involving a two-factor structure volatility process as an extension of the Heston model. The two factors had distinct correlations with market returns in the double Heston. Rouah [18] showed the implied volatility from the double Heston model, finding it to be much closer to the market implied volatility than the Heston model. The double Heston model allowed more flexibility in modeling conditional skewness and kurtosis for given levels of conditional variance. This result was consistent with the finding that the slope of the smile is largely independently of the level of the volatility. Sun [21,22] confirmed that the double Heston model with jumps had an improved market fitting and hedging performance compared with the double Heston model and the Bates model.

Meanwhile, researchers have also been developing methods for stochastic volatility–stochastic interest hybrid models. Although Bernard and Cui [2] obtained analytical and asymptotic results with three different stochastic volatility models, they assumed that the interest rate was a constant, which is unrealistic. Elliott and Siu [13] showed that the stochastic interest rate simulated the real fluctuation property better. Grzelak and Oosterlee [15] investigated the correlation effects of European product pricing with the Heston–Hull–White and Heston–CIR hybrid models. In recent studies, Cao *et al.* [5] derived a semi-analytical pricing formula for partially correlated Heston–CIR models of discretely sampled variance swaps. Their model framework was just a single stochastic volatility stochastic process which combined with the CIR stochastic interest rate process.

In this article, a hybridization of the double Heston stochastic volatility model and the CIR stochastic interest rate model is applied to investigate the problem of pricing variance swaps with discrete sampling. Our attention focuses on the pricing discrete sampling variance swaps among underlying asset, interest rate, and volatility. In Section 2, we not only introduce the concepts of realized variance, variance swaps, and the double Heston–CIR hybrid model, but further analysis is also done, such as Cholesky decomposition with the correlation matrix. Section 3 shows the derivation of a semi-closed analytical formula via the dimension-reduction technique and the characteristic functions. We can only obtain an efficient semi-closed form of pricing formula for variance swaps due to the application of the

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T-forward measure. In Section 4, numerical examples are presented, and the conclusion is given in Section 5.

2. MODEL FRAMEWORK

In this section, we introduce the concepts of realized variance, variance swaps, and the double Heston–CIR hybrid model. A hybrid model that combines the double Heston stochastic volatility model with the CIR stochastic interest rate model is developed. Cao *et al.* [5] showed that the combination of stochastic volatility and stochastic interest rate is better than the combination with the constant interest rate.

2.1. The Double Heston Hybrid Model

It is well known that the classical Black and Scholes [3] model may fail to reflect certain features of the reality of financial markets due to unrealistic assumptions such as constant volatility and constant interest rate. The Heston model was created to overcome the Black–-Scholes model's shortcoming of assuming constant volatility. It is given by

$$\begin{cases} dS(t) = \mu S(t)\,dt + \sqrt{v(t)}S(t)\,dB_1(t), \\ dv(t) = k(\theta - v(t))\,dt + \sigma\sqrt{v(t)}\,dB_2(t), \end{cases}$$

where μ is the expected return of the underlying asset, θ is the long-term mean of variance, k is a mean-reverting speed parameter of the variance, and σ is the so-called volatility of volatility. However, the Heston model sometimes fails to fit the implied volatility smile very well. Such failure often happens when short maturity is involved. To overcome this shortcoming, we adopt the double Heston model, which is a simple extension of the original Heston model. The double Heston model is a straightforward way to incorporate a stochastic correlation by using multiple stochastic volatility factors. It has much more flexibility in controlling the level and slope of the smile and modeling the volatility term structure.

Given $T > 0$, the underlying asset price $\{S(t) : t \in [0, T]\}$ is a stochastic process. The double Heston–-CIR hybrid model under the risk-neutral measure Q is given by

$$\begin{cases} dS(t) = r(t)S(t)\,dt + \sqrt{v_1(t)}S(t)\,dW_1(t) + \sqrt{v_2(t)}S(t)\,dW_2(t), \\ dv_1(t) = k_1^*(\theta_1^* - v_1(t))\,dt + \sigma_1^*\sqrt{v_1(t)}\,dW_3(t), \\ dv_2(t) = k_2^*(\theta_2^* - v_2(t))\,dt + \sigma_2^*\sqrt{v_2(t)}\,dW_4(t), \\ dr(t) = \alpha(\beta - r(t))\,dt + \eta\sqrt{r(t)}\,dW_5(t), \end{cases} \tag{2.1}$$

where $r(t)$ is the CIR stochastic interest rate process. To ensure that the square root process is always positive, we require that $2k_1^*\theta_1^* \geq (\sigma_1^*)^2$, $2k_2^*\theta_2^* \geq (\sigma_2^*)^2$, and $2\alpha\beta \geq \eta^2$ [9,16] . The Brownian motion under Q is denoted by $\{W_i(t) : t \in [0, T], i = 1, 2, 3, 4, 5\}$. We assume that the correlations between these Brownian motions are equal to zero except $[dW_1(t), dW_3(t)] = \rho_1\,dt, [dW_2(t), dW_4(t)] = \rho_2\,dt$. A general symmetric correlation matrix $\mathbf{C}$ is assumed, which is given by

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \rho_1 & 0 & 0 \\ 0 & 1 & 0 & \rho_2 & 0 \\ \rho_1 & 0 & 1 & 0 & 0 \\ 0 & \rho_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.2}$$

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Applying the Cholesky decomposition to decompose the matrix, the formula (2.2) can be rewritten as $\mathbf{C} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$, where $\mathbf{L}$ is a lower triangular matrix with strictly positive entries:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \rho_1 & 0 & \sqrt{1-\rho_1^2} & 0 & 0 \\ 0 & \rho_2 & 0 & \sqrt{1-\rho_2^2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

Thus, the system of SDEs (2.1) can be rewritten in terms of the independent Brownian motions, $\{\tilde{W}_i(t) : t \in [0,T], i = 1,2,3,4,5\}$, with the help of the lower triangular matrix $\mathbf{L}$:

$$\begin{pmatrix} dS(t)/S(t) \\ dv_1(t) \\ dv_2(t) \\ dr(t) \end{pmatrix} = \mu^Q dt + \Sigma \times \mathbf{L} \times \begin{pmatrix} d\tilde{W}_1(t) \\ d\tilde{W}_2(t) \\ d\tilde{W}_3(t) \\ d\tilde{W}_4(t)) \\ d\tilde{W}_5(t) \end{pmatrix}, \quad t \in [0,T] \tag{2.4}$$

where

$$\mu^Q = \begin{pmatrix} r(t) \\ k_1^*(\theta_1^* - v_1(t)) \\ k_2^*(\theta_2^* - v_2(t)) \\ \alpha(\beta - r(t)) \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sqrt{v_1(t)} & \sqrt{v_2(t)} & 0 & 0 & 0 \\ 0 & 0 & \sigma_1^*\sqrt{v_1(t)} & 0 & 0 \\ 0 & 0 & 0 & \sigma_2^*\sqrt{v_2(t)} & 0 \\ 0 & 0 & 0 & 0 & \eta\sqrt{r(t)} \end{pmatrix},$$

and $\{\tilde{W}_i(t) : t \in [0,T], i = 1,2,3,4,5\}$ are mutually independent under the risk-neutral measure Q and satisfy

$$\begin{pmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \\ dW_4(t)) \\ dW_5(t) \end{pmatrix} = \mathbf{L} \times \begin{pmatrix} d\tilde{W}_1(t) \\ d\tilde{W}_2(t) \\ d\tilde{W}_3(t) \\ d\tilde{W}_4(t)) \\ d\tilde{W}_5(t) \end{pmatrix}, \quad t \in [0,T]$$

2.2. Variance Swaps

The realized variance (RV) provides a relative accurate measure of volatility. Suppose $S(t)$ is the price of the underlying asset over a finite time horizon $[0,T]$ and observed N times during $[0,T]$: $0 \le t_1 < t_2 < \cdots < t_N \le T$. Then RV is given by

$$\mathrm{RV} = \frac{\mathrm{AF}}{N} \sum_{i=1}^{N} \left[\frac{S(t_i) - S(t_{i-1})}{S(t_{i-1})} \right]^2 \times 100^2, \tag{2.5}$$

where $S(t_i)$ is the price of the underlying asset at t_i. AF is the annualized factor, and the size of AF depends on the sampling frequency. If the sampling frequency is every business day, then AF is equal to 252. If the sampling frequency is every week or every month, then AF is 52 or 12, respectively. Equally spaced discrete observations are assumed here so that the annualized factor has a simple expression $\mathrm{AF} = 1/\Delta t = N/T$.

A variance swap is defined as a forward contract on the future RV of the returns of an underlying asset. The short position of a variance swap pays the floating amounts of

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annualized realized and receives the fixed delivery price at expiry, whereas the long position is the opposite. The payoff of the long position of a variance swap at the maturity T is considered:

$$V(T) = (\mathrm{RV} - K) \times L, \tag{2.6}$$

where L is the notional amount of the swap in dollars per annualized volatility point squared, and K is the annualized delivery price of the variance swap.

In a risk-neutral world, the value of a variance swap with the stochastic interest rate at time $t \in [0, T]$ is the present expected value of the future payoff, which can be written as

$$V(T) = E_t^Q[e^{-\int_t^T r(s)\,ds}(\mathrm{RV} - K) \times L],$$

where $E_t^Q(\cdot)$ denotes the expectation with respect to the risk-neutral measure Q at t.

The above expectation is complicated to evaluate since it involves the joint distribution of the stochastic interest rate and the future payoff. The evaluation can be simplified by using the bond price as the numeraire since the price of a T-maturity zero-coupon bond at $t = 0$ is given by $P(0, T) = E_t^Q[e^{-\int_t^T r(s)\,ds}]$. To determine the value of K, the risk-neutral measure Q is transformed into the forward measure Q^T in the above expectation. We get

$$E_0^Q[e^{-\int_t^T r(s)\,ds}(\mathrm{RV} - K) \times L] = E_0^Q[e^{-\int_t^T r(s)\,ds}]E_0^T[(\mathrm{RV} - K) \times L], \tag{2.7}$$

where $E_t^T(\cdot)$ denotes the expectation with respect to Q^T at t. To ensure that parties of the forward contract are treated fairly, this value should be zero at $t = 0$. Thus, the fair delivery price of the variance swap is given by

$$\begin{aligned} K &= E_0^T[\mathrm{RV}], \\ &= E_0^T\left[\frac{\mathrm{AF}}{N}\sum_{i=1}^{N}\left[\frac{S(t_i) - S(t_{i-1})}{S(t_{i-1})}\right]^2 \times 100^2\right], \\ &= \frac{100^2}{T}\sum_{i=1}^{N} E_0^T\left(\frac{S(t_i)}{S(t_{i-1})} - 1\right)^2. \end{aligned}$$

Thus, the valuation of the fair delivery price of the variance swap can be reduced to the calculation of N expectations expressed in the forms of

$$E_0^T\left(\frac{S(t_i)}{S(t_{i-1})} - 1\right)^2 \tag{2.8}$$

for the equal time period $\Delta t = T/N$ and $t_i = i\Delta t, (i = 1, 2, \ldots, N)$, and here $t_0 = 0$.

Remark 2.1: *There are two cases should be considered, $i = 1$ and $i > 1$. For the first case $i = 1$, we have $t_{i-1} = t_0 = 0$ and $S(t_{i-1}) = S(0)$ is a known constant. However, for the case $i > 1$, the value of $S(t_{i-1})$ is unknown. In the process of calculating this expectation, i is regarded as a constant unless otherwise stated. Hence, both t_{i-1} and t_i are known constants.*

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Based on the property of conditional expectations, the calculation of Formula (2.8) can be divided into two parts

$$E_0^T\left(\frac{S(t_i)}{S(t_{i-1})}-1\right)^2 = E_0^T\left[E_{t_{i-1}}^T\left[\frac{S(t_i)}{S(t_{i-1})}-1\right]^2\right]. \tag{2.9}$$

In the first phase, the expectation $E_{t_{i-1}}^T[S(t_i)/S(t_{i-1})-1]^2$ is calculated and denotes the term $E_{t_{i-1}}^T[S(t_i)/S(t_{i-1})-1]^2$ by $H_i(v_1(t_{i-1}), v_2(t_{i-1}), r(t_{i-1}))$ for notational convenience. In the second phase, calculate the expectation

$$E_0^T[H_i(v_1(t_{i-1}), v_2(t_{i-1}), r(t_{i-1}))]. \tag{2.10}$$

Note that the numeraire under Q is $N_{1,t} = e^{\int_0^t r(s)\,ds}$, while the numeaire under Q^T is the price at time t of a zero-coupon bond with maturity T, which is given by

$$N_{2,t} = A(t,T)\,e^{-B(t,T)r(t)}. \tag{2.11}$$

LEMMA 2.2: *The functions $A(t,T)$ and $B(t,T)$ in expression (2.11) is given by*

$$\begin{aligned} B(t,T) &= \frac{2(e^{\phi(T-t)}-1)}{2\phi+(\alpha+\phi)(e^{\phi(T-t)}-1)}, \\ A(t,T) &= \left[\frac{2\phi\, e^{(\alpha+\phi)(T-t)/2}}{2\phi+(\alpha+\phi)(e^{(T-t)\phi}-1)}\right]^{2\alpha\beta/\eta^2}, \end{aligned} \tag{2.12}$$

where $\phi = \alpha^2 + 2\eta^2$. The parameters α, β, and η are defined in expression (2.1).

PROOF: The result can be found in Brigo [4]. ■

We can transform Formula (2.4) under Q to the T-forward measure Q^T:

$$\begin{pmatrix} dS(t)/S(t) \\ dv_1(t) \\ dv_2(t) \\ dr(t) \end{pmatrix} = \mu^T\,dt + \Sigma \times \mathbf{L} \times \begin{pmatrix} d\tilde{W}_1(t) \\ d\tilde{W}_2(t) \\ d\tilde{W}_3(t) \\ d\tilde{W}_4(t)) \\ d\tilde{W}_5(t) \end{pmatrix}, \quad t \in [0,T] \tag{2.13}$$

where

$$\mu^T = \begin{pmatrix} r(t) \\ k_1^*(\theta_1^* - v_1(t)) \\ k_2^*(\theta_2^* - v_2(t)) \\ \alpha(\beta - r(t)) - B(t,T)\eta^2 r(t) \end{pmatrix}.$$

3. PRICING VARIANCE SWAPS UNDER THE HYBRID MODEL

In this part, a semi-closed form solution for the delivery price under the double Heston–CIR hybrid model is derived. The solution for the $H_i(v_1(t_{i-1}), v_2(t_{i-1}), r(t_{i-1}))$ is considered firstly. In this part, the generalized Fourier transform and the Heston's [16] solution procedure are applied. Then the solution of the Formula (2.10) is considered and the final solution is derived.

A contingent claim $U_i(S(t), v_1(t), v_2(t), r(t))$ is considered in the first part.

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3.1. Computation of the First Phase

In this part, the term $H_i(v_1(t_{i-1}), v_2(t_{i-1}), r(t_{i-1}))$ is found.

The contingent claim $U_i(S(t), v_1(t), v_2(t), r(t))$ has a European-style payoff function at expiry t_i, which is denoted by

$$H_i(s) = \left(\frac{S}{S(t_{i-1})} - 1\right)^2. \tag{3.1}$$

Under the T-forward measure Q^T, a PDE system for H_i over $[t_{i-1}, t_i]$ is obtained as

$$\begin{cases} \dfrac{\partial U_i}{\partial t} + rs\dfrac{\partial U_i}{\partial S} + k_1^*(\theta_1^* - v_1)\dfrac{\partial U_i}{\partial v_1} + k_2^*(\theta_2^* - v_2)\dfrac{\partial U_i}{\partial v_2} \\ \quad + \left[\alpha(\beta - r) - B(t,T)\eta^2 r\right]\dfrac{\partial U_i}{\partial r} + \dfrac{1}{2}\left(v_1 S^2 + v_2 S^2\right)\dfrac{\partial^2 U_i}{\partial S^2} \\ \quad + \dfrac{1}{2}(\sigma_1^*)^2 v_1 \dfrac{\partial^2 U_i}{\partial v_1^2} + \dfrac{1}{2}(\sigma_2^*)^2 v_2 \dfrac{\partial^2 U_i}{\partial v_2^2} + \dfrac{1}{2}\eta^2 r\dfrac{\partial^2 U_i}{\partial r^2} + S\sigma_1^* v_1 \rho_1 \dfrac{\partial^2 U_i}{\partial S \partial v_1} \\ \quad + S\sigma_2^* v_2 \rho_2 \dfrac{\partial^2 U_i}{\partial S \partial v_2} = 0, \\ U_i(S, v_1, v_2, r, t_i) = H_i(S). \end{cases} \tag{3.2}$$

Omit the subscript i for notational convenience and the expiry t_i is replaced with T. Then, do a transform by letting:

$$\begin{cases} \tau = T - t, \\ x = \ln S. \end{cases}$$

After the transformation, the new PDE system is obtained as

$$\begin{cases} \dfrac{\partial U}{\partial \tau} = \left[r - \dfrac{1}{2}(v_1 + v_2)\right]\dfrac{\partial U}{\partial x} + k_1^*(\theta_1^* - v_1)\dfrac{\partial U}{\partial v_1} + k_2^*(\theta_2^* - v_2)\dfrac{\partial U}{\partial v_2} \\ \quad + \left[\alpha(\beta - r) - B(T - \tau, T)\eta^2 r\right]\dfrac{\partial U}{\partial r} + \dfrac{1}{2}(v_1 + v_2)\dfrac{\partial^2 U}{\partial x^2} \\ \quad + \dfrac{1}{2}(\sigma_1^*)^2 v_1 \dfrac{\partial^2 U}{\partial v_1^2} + \dfrac{1}{2}(\sigma_2^*)^2 v_2 \dfrac{\partial^2 U}{\partial v_2^2} + \dfrac{1}{2}\eta^2 r\dfrac{\partial^2 U}{\partial r^2} \\ \quad + \sigma_1^* v_1 \rho_1 \dfrac{\partial^2 U}{\partial x \partial v_1} + \sigma_2^* v_2 \rho_2 \dfrac{\partial^2 U}{\partial x \partial v_2}, \\ U(x, v_1, v_2, r, 0) = H(e^x). \end{cases} \tag{3.3}$$

LEMMA 3.1: *The solution of the PDE system (3.3) can be expressed in a semi-closed form as*

$$U(x, v_1, v_2, r, \tau) = \mathcal{F}^{-1}[e^{C(\omega,\tau) + D(\omega,\tau)v_1 + E(\omega,\tau)v_2 + F(\omega,\tau)r}\mathcal{F}[H(e^x)]]. \tag{3.4}$$

Applying the generalized Fourier transform, where $j = \sqrt{-1}$, ω is the Fourier transform variable, the solution system is obtained as:

$$\begin{cases} D(\omega, \tau) = \dfrac{a_1 + b_1}{(\sigma_1^*)^2}\left[\dfrac{1 - e^{b_1 \tau}}{1 - g_1 e^{b_1 \tau}}\right], \\ a_1 = k_1^* - \rho_1 \sigma_1^* \omega j, \quad b_1 = \sqrt{a_1^2 + (\sigma_1^*)^2 \omega(\omega + j)}, \quad g_1 = \dfrac{a_1 + b_1}{a_1 - b_1}. \end{cases} \tag{3.5}$$

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$$\begin{cases} E(\omega,\tau) = \dfrac{a_2+b_2}{(\sigma_2^*)^2}\left[\dfrac{1-e^{b_2\tau}}{1-g_2e^{b_2\tau}}\right], \\ a_2 = k_2^* - \rho_2\sigma_2^*\omega j, \quad b_2 = \sqrt{a_2^2+(\sigma_2^*)^2\omega(\omega+j)}, \quad g_2 = \dfrac{a_2+b_2}{a_2-b_2}. \end{cases} \tag{3.6}$$

The functions $F(\omega,\tau)$ and $C(\omega,\tau)$ satisfy the given ODE system

$$\begin{cases} \dfrac{dF}{d\tau} = \dfrac{1}{2}\eta^2F^2 - (\alpha+B\eta^2)F + j\omega, \\ \dfrac{dC}{d\tau} = k_1^*\theta_1^*D + k_2^*\theta_2^*E + \alpha\beta r, \end{cases} \tag{3.7}$$

with the initial conditions:

$$C(\omega,0)=0, \quad D(\omega,0)=0, \quad E(\omega,0)=0, \quad F(\omega,0)=0.$$

Proof: The generalized Fourier transform with respect to x is used to obtain a transformed PDE system of $\widetilde{U}(x,v_1,v_2,r,\tau)=\mathcal{F}[U(x,v_1,v_2,r,\tau)]$:

$$\begin{cases} \dfrac{\partial\widetilde{U}}{\partial\tau} = \dfrac{1}{2}(\sigma_1^*)^2v_1\dfrac{\partial\widetilde{U}^2}{\partial v_1^2} + \dfrac{1}{2}(\sigma_2^*)^2v_2\dfrac{\partial\widetilde{U}^2}{\partial v_2^2} + \dfrac{1}{2}\eta^2r\dfrac{\partial\widetilde{U}^2}{\partial r^2} \\ \quad + [\alpha(\beta-r) - B\eta^2r]\dfrac{\partial\widetilde{U}}{\partial r} + [k_1^*(\theta_1^*-v_1) + j\omega\sigma_1^*\rho_1v_1]\dfrac{\partial\widetilde{U}}{\partial v_1} \\ \quad + [k_2^*(\theta_2^*-v_2) + j\omega\sigma_2^*\rho_2v_2]\dfrac{\partial\widetilde{U}}{\partial v_2} + \left[j\omega r - \dfrac{v_1+v_2}{2}(j\omega+\omega^2)\right]\widetilde{U}, \\ \widetilde{U}(\omega,v_1,v_2,r,0) = \mathcal{F}[H(e^x)]. \end{cases} \tag{3.8}$$

The affine property for the double Heston hybrid model can be extended. Rouah [18] showed that the original double Heston model can be written in affine form. Combine with Heston's [17] assumption, Duffie and Singleton *al.* [12] showed that the PDE system (3.8) has an affine form solution. We assume that the solution form is given by:

$$\widetilde{U}(x,v_1,v_2,r,\tau) = e^{C(\omega,\tau)+D(\omega,\tau)v_1+E(\omega,\tau)v_2+F(\omega,\tau)r}\widetilde{U}(x,v_1,v_2,r,0). \tag{3.9}$$

Substituting the above Formula (3.9) into the PDE system (3.8), an ODE system can be obtained as:

$$\begin{cases} \dfrac{dC}{d\tau} = k_1^*\theta_1^*D + k_2^*\theta_2^*E + \alpha\beta F, \\ \dfrac{dD}{d\tau} = \dfrac{1}{2}(\sigma_1^*)^2D^2 + (j\rho_1\sigma_1^*\omega - k_1^*)D - \dfrac{1}{2}(j\omega+\omega^2), \\ \dfrac{dE}{d\tau} = \dfrac{1}{2}(\sigma_2^*)^2E^2 + (j\rho_2\sigma_2^*\omega - k_2^*)D - \dfrac{1}{2}(j\omega+\omega^2), \\ \dfrac{dF}{d\tau} = \dfrac{1}{2}\eta^2F^2 - (\alpha+B\eta^2)F + j\omega. \end{cases} \tag{3.10}$$

with the initial conditions $C(\omega,0)=0$, $D(\omega,0)=0$, $E(\omega,0)=0$, $F(\omega,0)=0$.

The functions $D(\omega,\tau)$ and $E(\omega,\tau)$ can be solved for an analytical form as Formula (3.5) and (3.6), respectively. The approximate solutions of the functions $C(\omega,\tau)$ and $F(\omega,\tau)$ can

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be found by numerical integrations using a standard mathematical software package (e.g., Matlab).

Performing the inverse Fourier transform, we obtain:

$$\begin{aligned} U(x, v_1, v_2, r, \tau) &= \mathcal{F}^{-1}[\widetilde{U}(x, v_1, v_2, r, \tau)] \\ &= \mathcal{F}^{-1}[e^{C(\omega,\tau)+D(\omega,\tau)v_1+E(\omega,\tau)v_2+F(\omega,\tau)r}\mathcal{F}[H(e^x)]]. \end{aligned}$$

■

REMARK 3.2: *The Fourier transform variable ω plays the role of a parameter in the function, $C(\omega,\tau)$, $D(\omega,\tau)$, $E(\omega,\tau)$, and $F(\omega,\tau)$.*

We review the definition of generalized Fourier transformation and the generalized inverse Fourier transformation. The generalized Fourier transform $\hat{f}$ of a function f is given by

$$\hat{f}(\omega) = \mathcal{F}[f(x)] = \int_{-\infty}^{+\infty} f(x)\, e^{-j\omega x}\, dx.$$

The generalized inverse Fourier transformation of the function $\hat{f}$ is defined as

$$f(x) = \mathcal{F}^{-1}[\hat{f}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)\, e^{j\omega x}\, d\omega.$$

The Fourier transform of the function $e^{j\xi x}$ is

$$\mathcal{F}[e^{j\xi x}] = 2\pi\delta_\xi(\omega),$$

where ξ is a fixed complex number and $\delta_\xi(\omega)$ is the generalized delta function which satisfies the selection property

$$\int_{-\infty}^{+\infty} \delta_\xi(x)\Phi(x)\, dx = \Phi(\xi).$$

Do the replace

$$\begin{cases} I = S(t_{i-1}), \\ S = e^x, \end{cases}$$

the payoff function Formula (3.1) with respect to x is converted to the following function:

$$\mathcal{F}\left[\left(\frac{e^x}{I} - 1\right)^2\right] = 2\pi\left(\frac{\delta_{-2j}(\omega)}{I^2} - 2\frac{\delta_{-j}(\omega)}{I} + \delta_0(\omega)\right). \tag{3.11}$$

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Substitute Formula (3.11) into Formula (3.4),

$$\begin{aligned} U_i(S, v_1, v_2, r, t_i) &= \mathcal{F}^{-1}\left[e^{C(\omega,\tau)+D(\omega,\tau)v_1+E(\omega,\tau)v_2+F(\omega,\tau)r}2\pi\right. \\ &\quad \left.\times\left[\frac{\delta_{-2j}(\omega)}{I^2}-2\frac{\delta_{-j}(\omega)}{I}+\delta_0(\omega)\right]\right] \\ &= \frac{1}{I^2}\,e^{C(\omega,\tau)+D(\omega,\tau)v_1+E(\omega,\tau)v_2+F(\omega,\tau)r}\,e^{j\omega x}|_{\omega=-2j} \\ &\quad -\frac{2}{I}\,e^{C(\omega,\tau)+D(\omega,\tau)v_1+E(\omega,\tau)v_2+F(\omega,\tau)r}\,e^{j\omega x}|_{\omega=-j} \\ &\quad +e^{C(\omega,\tau)+D(\omega,\tau)v_1+E(\omega,\tau)v_2+F(\omega,\tau)r}\,e^{j\omega x}|_{\omega=0} \\ &= \frac{e^{2x}}{I^2}e^{\widetilde{C}(\tau)+\widetilde{D}(\tau)v_1+\widetilde{E}(\tau)v_2+\widetilde{F}(\tau)r}-2\frac{e^x}{I}e^{\widehat{C}(\tau)+\widehat{F}(\tau)r}+1, \end{aligned} \tag{3.12}$$

where $t_{i-1} \le t \le t_i$ and $\tau = t_i - t$. The notation of

$$\begin{cases} \widetilde{C}(\tau) = C(-2j,\tau), \widetilde{D}(\tau) = D(-2j,\tau), \widetilde{E}(\tau) = E(-2j,\tau), \widetilde{F}(\tau) = F(-2j,\tau), \\ \widehat{C}(\tau) = C(-j,\tau), \widehat{F}(\tau) = F(-j,\tau). \end{cases} \tag{3.13}$$

where

$$\widehat{D}(\tau) = D(-j,\tau) = \widehat{E}(\tau) = E(-j,\tau) = 0.$$

By noting that $\lim_{t\downarrow t_{i-1}} \ln S(t) = \ln I$, we can get

$$\lim_{t\downarrow t_{i-1}} U_i(S, v_1, v_2, r, \tau) = e^{\widetilde{C}(\triangle t)+\widetilde{D}(\triangle t)v_1+\widetilde{E}(\triangle t)v_2+\widetilde{F}(\triangle t)r} - 2\,e^{\widehat{C}(\triangle t)+\widehat{F}(\triangle t)r} + 1. \tag{3.14}$$

Therefore, the solution of the first phase is

$$\begin{aligned} H_i(v_1(t_{i-1}), v_2(t_{i-1}), r(t_{i-1})) &= U_i(S, v_1, v_2, r, \triangle t) \\ &= e^{\widetilde{C}(\triangle t)+\widetilde{D}(\triangle t)v_1(t_{i-1})+\widetilde{E}(\triangle t)v_2(t_{i-1})+\widetilde{F}(\triangle t)r(t_{i-1})} \\ &\quad -2\,e^{\widehat{C}(\triangle t)+\widehat{F}(\triangle t)r(t_{i-1})}+1. \end{aligned} \tag{3.15}$$

3.2. Computation of the Second Phase

In this section, we derive the solution of the second phase

$$E_0^T[H_i(v_1(t_{i-1}), v_2(t_{i-1}), r(t_{i-1}))].$$

In the first phase, we obtain the solution of $H_i(v_1(t_{i-1}), v_2(t_{i-1}), r(t_{i-1}))$. Then, the expectation is represented by

$$\begin{aligned} H_i(v_1(0), v_2(0), r(0)) &= E_0^T[H_i(v_1(t_{i-1}), v_2(t_{i-1}), r(t_{i-1}))] \\ &= E_0^T[e^{\widetilde{C}(\triangle t)+\widetilde{D}(\triangle t)v_1(t_{i-1})+\widetilde{E}(\triangle t)v_2(t_{i-1})+\widetilde{F}(\triangle t)r(t_{i-1})} \\ &\quad -2e^{\widehat{C}(\triangle t)+\widehat{F}(\triangle t)r(t_{i-1})}+1] \\ &= e^{\widetilde{C}(\triangle t)}E_0^T[e^{\widetilde{D}(\triangle t)v_1(t_{i-1})+\widetilde{E}(\triangle t)v_2(t_{i-1})+\widetilde{F}(\triangle t)r(t_{i-1})}] \\ &\quad -2e^{\widehat{C}(\triangle t)}E_0^T[e^{\widehat{F}(\triangle t)r(t_{i-1})}]+1 \end{aligned} \tag{3.16}$$

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In order to derive approximate solutions for $E_0^T[e^{\widetilde{D}(\triangle t)v_1(t_{i-1})+\widetilde{E}(\triangle t)v_2(t_{i-1})+\widetilde{F}(\triangle t)r(t_{i-1})}]$ and $E_0^T[e^{\widetilde{F}(\triangle t)r(t_{i-1})}]$, the variables $v_1(t_{i-1})$, $v_2(t_{i-1})$, and $r(t_{i-1})$ can be approximated by normally distributed random variable.

LEMMA 3.3: *If the underlying asset follows the dynamic process (2.1), we can assume the notations:*

$$\begin{cases} m_1(t) = v_1(0)\, e^{-k_1^* t} + \theta_1^*(1 - e^{k_1^* t}), \\ var_1(t) = \dfrac{(\sigma_1^*)^2 v_1(0)}{k_1^*}(e^{-k_1^* t} - e^{-2k_1^* t}) + \dfrac{\theta_1^*(\sigma_1^*)^2}{2k_1^*}(1 - 2e^{-k_1^* t} + e^{-2k_1^* t}), \\ m_2(t) = v_2(0)\, e^{-k_2^* t} + \theta_2^*(1 - e^{k_2^* t}), \\ var_2(t) = \dfrac{(\sigma_2^*)^2 v_2(0)}{k_2^*}(e^{-k_2^* t} - e^{-2k_2^* t}) + \dfrac{\theta_2^*(\sigma_2^*)^2}{2k_2^*}(1 - 2e^{-k_2^* t} + e^{-2k_2^* t}), \\ m_3(t) = r(0)\, e^{-\alpha t} + \eta(1 - e^{\alpha t}), \\ var_3(t) = \dfrac{\eta^2 r(0)}{\alpha}(e^{-\alpha t} - e^{-2\alpha t}) + \dfrac{\beta\eta^2}{2\alpha}(1 - 2e^{-\alpha t} + e^{-2\alpha t}). \end{cases} \tag{3.17}$$

The approximations of the variables $v_1(t_{i-1})$, $v_2(t_{i-1})$, and $r(t_{i-1})$ are given by:

$$\begin{cases} v_1(t) \approx \mathcal{N}(m_1(t), \mathrm{var}_1(t)), \\ v_2(t) \approx \mathcal{N}(m_2(t), \mathrm{var}_2(t)), \\ r(t) \approx \mathcal{N}(m_3(t), \mathrm{var}_3(t)). \end{cases} \tag{3.18}$$

PROOF: Consider a CIR-type process,

$$dv(t) = k(\theta - v(t))\, dt + \sigma\sqrt{v(t)}\, dW(t),$$

where $W(t)$ is a standard Brownian motion.

Applying Itô's lamma on $d(e^{-k(t-s)}v(s))$ with respect to s and integrating both side from 0 to t, we have

$$v(t) = v(0)\, e^{-kt} + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{k(s-t)}\sqrt{v(s)}\, dW(s).$$

This formula shows that $v(t)$ is normally distributed. The expectation $E(v(t))$ and the variance $\mathrm{Var}(v(t))$ can be get easily.

$$\begin{aligned} E(v(t)) &= v(0)e^{-kt} + \theta(1 - e^{-kt}), \\ \mathrm{Var}(v(t)) &= \mathrm{Var}\left(\sigma \int_0^t e^{k(s-t)}\sqrt{v(s)}\, dW(s)\right) \\ &= \frac{\sigma^2 v(0)}{k}(e^{-kt} - e^{-2kt}) + \frac{\theta\sigma^2}{2k}(1 - 2e^{-kt} + e^{-2kt}). \end{aligned}$$

Let $L(0, t_{i-1}) = \widetilde{D}(\triangle t)v_1(t_{i-1}) + \widetilde{E}(\triangle t)v_2(t_{i-1}) + \widetilde{F}(\triangle t)r(t_{i-1})$, the approximation of $E_0^T[e^{L(0,t_{i-1})}]$ can be got

$$E_0^T[e^{L(0,t_{i-1})}] \approx e^{E_0^T[L(0,t_{i-1})] + \frac{1}{2}\mathbb{V}ar^T[e^{L(0,t_{i-1})}]}. \tag{3.19}$$

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Since the characteristic function of the sum of $v_1(t)$, $v_2(t)$, and $r(t)$ is normally distributed, the approximation of $E_0^T[e^{L(0,t_{i-1})}]$ can be get

$$E_0^T[e^{L(0,t_{i-1})}] \approx \exp\{\widetilde{D}(\triangle t)[m_1(t_{i-1})] + \widetilde{E}(\triangle t)[m_2(t_{i-1})] + \widetilde{F}(\triangle t)[m_3(t_{i-1})] + \frac{1}{2}[\widetilde{D}^2 \operatorname{var}_1(t_{i-1}) + \widetilde{E}^2 \operatorname{var}_2(t_{i-1}) + \widetilde{F}^2 \operatorname{var}_3(t_{i-1})]\}. \tag{3.20}$$

In the same way,

$$E_0^T[e^{\widehat{F}(\triangle t)r(t_{i-1})}] \approx \exp\left\{\widehat{F}[m_3(t_{i-1})] + \frac{\widehat{F}^2}{2}\operatorname{var}_3(t_{i-1})\right\}. \tag{3.21}$$

Substitute Formulas (3.20) and (3.21) into Formula (3.16), the approximation can be obtained:

$$\begin{aligned} H_i(v_1(0), v_2(0), r(0)) \approx{}& e^{\widetilde{C}(\triangle t)} \exp\{\widetilde{D}(\triangle t)[m_1(t_{i-1})] + \widetilde{E}(\triangle t)[m_2(t_{i-1})] + \widetilde{F}(\triangle t)[m_3(t_{i-1})] \\ &+ \frac{1}{2}[\widetilde{D}^2 \operatorname{var}_1(t_{i-1}) + \widetilde{E}^2 \operatorname{var}_2(t_{i-1}) + \widetilde{F}^2 \operatorname{var}_3(t_{i-1})]\} \\ &- 2e^{\widehat{C}(\triangle t)} \exp\left\{\widehat{F}[m_3(t_{i-1})] + \frac{\widehat{F}^2}{2}\operatorname{var}_3(t_{i-1})\right\} + 1 \end{aligned} \tag{3.22}$$

■

3.3. Pricing of a Variance Swap

Combining the results derived in the above two sections, an approximation for the optimal pricing of variance swaps is obtained. As mentioned in Remark 2.1, the fair delivery price of a variance swap is represented by the sum over the whole time period as

$$K = E_0^T[\mathrm{RV}] = \frac{100^2}{T}\left(H(v_1(0), v_2(0), r(0)) + \sum_{i=2}^{N} H_i(v_1(0), v_2(0), r(0))\right), \tag{3.23}$$

where the case $i > 1$, $H_i(v_1(0), v_2(0), r(0))$ is approximated by the Formula (3.22). In the case $i = 1$, the expression is given by

$$\begin{aligned} H(v_1(0), v_2(0), r(0)) &= E_0^T\left[\frac{S(t_1)}{S(t_0)} - 1\right]^2 \\ &= e^{\widetilde{C}(\triangle t)+\widetilde{D}(\triangle t)v_1(0)+\widetilde{E}(\triangle t)v_2(0)+\widetilde{F}(\triangle t)r(0)} - 2e^{\widehat{C}(\triangle t)+\widehat{F}(\triangle t)r(0)} + 1. \end{aligned} \tag{3.24}$$

4. NUMERICAL EXPERIMENTS

A numerical analysis of pricing variance swaps under our model can help analyze the performance of our Formula (3.24). Our formula results are compared with resulting a Monte Carlo(MC) simulation which reflects the real market.

Unless otherwise stated, the default parameters of the hybrid double Heston model in Table 1 will be used for all numerical experiments.

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TABLE 1. Model parameters of the double Heston–CIR hybrid model

S_0	ρ_1	ρ_2	T	$v_1(0)$	k_1^*	θ_1^*	σ_1^*
1	−0.4	−0.6	1	0.05	2.5	0.05	0.1
$v_2(0)$	k_2^*	θ_2^*	σ_0^*	$r(0)$	α	β	η
0.06	3	0.006	0.005	0.05	2	0.01	0.01

We perform the MC simulation using the Euler scheme with 200,000 sample paths. A simple Euler discretization for the double Heston hybrid model can be expressed as

$$\begin{cases} S(t) = S(t-1)\exp\left\{\left[r(t-1) - \dfrac{v_1(t-1)+v_2(t-1)}{2}\right]\Delta t \right. \\ \qquad \left. + \sqrt{v_1(t-1)}\sqrt{\Delta t}W_t^1 + \sqrt{v_2(t-1)}\sqrt{\Delta t}W_t^2\right\}, \\ v_1(t) = v_1(t-1) + k_1^*(\theta_1^* - v_1(t-1))\Delta t \\ \qquad + \sigma_1^*\sqrt{v_1(t-1)}\sqrt{\Delta t}(\rho_1 W_t^1 + \sqrt{1-\rho_1^2}W_t^3), \\ v_2(t) = v_2(t-1) + k_2^*(\theta_2^* - v_2(t-1))\Delta t \\ \qquad + \sigma_2^*\sqrt{v_2(t-1)}\sqrt{\Delta t}(\rho_2 W_t^2 + \sqrt{1-\rho_2^2}W_t^4), \\ r(t) = r(t-1) + \alpha(\beta - r(t-1))\Delta t + \eta\sqrt{\Delta t}\sqrt{r(t-1)}W_t^5, \end{cases} \tag{4.1}$$

where W_t^1, W_t^2, W_t^3, W_t^4 , and W_t^5 are five independent standard normal random variables.

The continuous model is a wide utilized tool in the variance swaps field. Swishchuk [1] showed that the integral form of the RV was conducive to obtain the expectation of the continuous integral. For the double Heston model, the continuous model of the fair delivery price for the variance swaps is defined as

$$\begin{aligned} E_0^T[\mathrm{RV}] &= \frac{100^2}{T} \times \sum_{i=1}^{2}\left[\int_0^T v_i(t)\,dt\right] \\ &= 100^2 \times \sum_{i=1}^{2}\left[v_i(0)\frac{1-e^{-k_i^*T}}{k_i^*T} + \theta_i^*\left(1 - \frac{1-e^{-k_i^*T}}{k_i^*T}\right)\right]. \end{aligned} \tag{4.2}$$

4.1. MC Simulations

Firstly, the numerical results are compared with the MC simulations results. The stochastic process of the model is discretized using the Euler scheme (4.1).

Figure 1 shows three data sets which represent the numerical results of Formula (3.24), the MC simulations of the hybrid model (4.1), and the numerical results of Formula (4.2), respectively.

In Table 2, we show some numerical results of three data sets with different time periods. Five different sampling frequencies are adopted: quarterly, monthly, fortnightly, weekly, and daily. With the increasing of observed frequency, the results from the approximation Formula (3.23) match better than the results from the MC simulations.

Table 3 presents the difference between the results from our semi-closed form solution and those from the MC simulations by taking $N = 252$ which is the daily sampling frequency.

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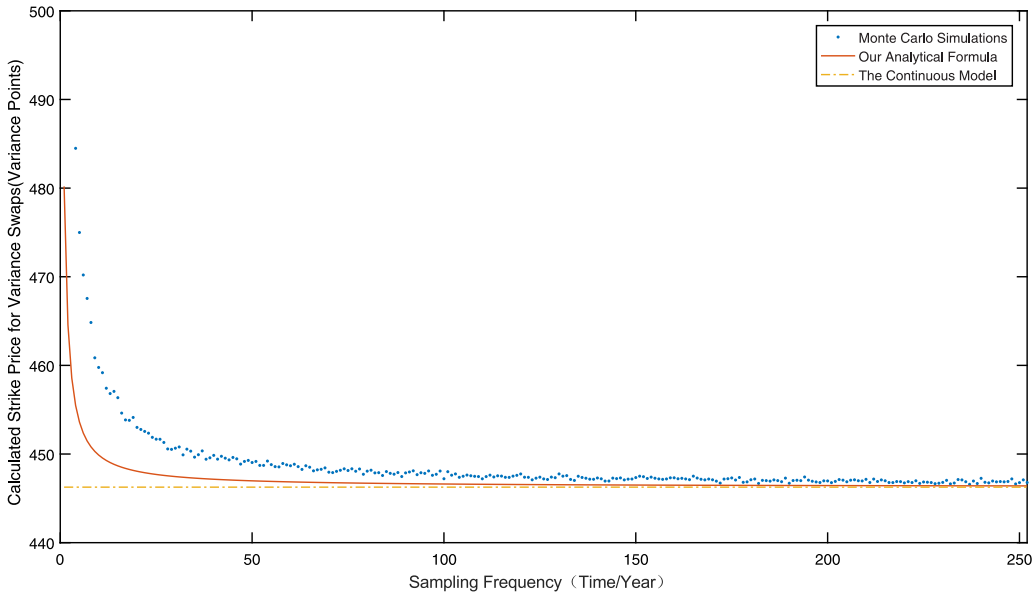


FIGURE **1.** The comparison of variance swap prices from our numerical formula, the MC simulations, and the continuous model.

TABLE 2. The comparison of variance swap prices from our numerical formula, the MC simulations, and the continuous model

Sampling frequency	Our numerical results	MC simulations	Continuous model
Quarterly ($N = 4$)	455.48	484.49	446.26
Monthly ($N = 12$)	449.27	457.42	446.26
Fortnightly ($N = 26$)	447.64	451.66	446.26
Weekly ($N = 52$)	446.95	448.72	446.26
Daily ($N = 252$)	446.40	446.79	446.26

TABLE 3. Relative errors and computational time of MC simulations

Path numbers of the MC simulations	Relative error (%)	Computational time (s)
10,000	0.1803	15.75
100,000	0.0991	103.44
200,000	0.0858	213.22
500,000	0.0792	402.05
1,000,000	0.0787	789.90

The calculation of the relative error uses our solution as the reference point. The reduction of relative errors with the increasing of path number demonstrates the convergence of the MC simulations toward our semi-closed solution. Furthermore, our analytical solution has a definite speed advantage that it takes 2.46 s, a much shorter time than the MC's operation time which takes 402.05 s when the daily sampling frequency $N = 252$ and path numbers is 500,000. These finding verify the accuracy and efficiency of our formula.

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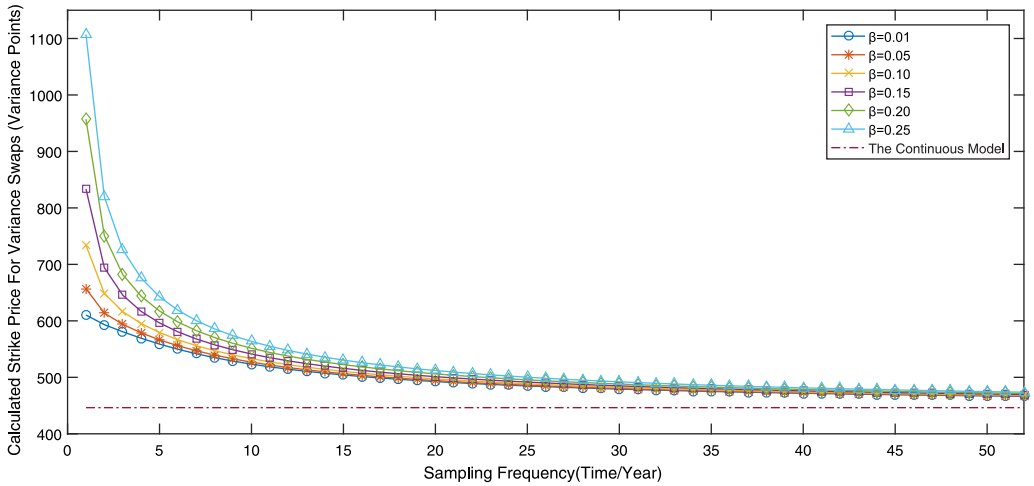


FIGURE **2.** The comparison of variance swap prices from our numerical formula with different β: $0.01, 0.05, 0.10, 0.15, 0.20, 0.25$. Model parameters are presented in Table 2, expect for β. The sampling frequency is $N = 52$.

4.2. Effect of Stochastic Interest Rate

Suppose that the ex-dividend stock price process is determined by two factors $v_1(t)$ and $v_2(t)$. If the parameters $k_2^* = \theta_2^* = \sigma_2^* = 0$, the Double Heston model will degrade into the one-factor Heston model. In this case, our Double Heston–CIR hybrid model degrades into the Heston–CIR hybrid model. And our pricing formula is consistent with the results in Cao *et al.* [5]. Meanwhile, if the CIR stochastic interest rate process degrades into a deterministic interest rate, our model will deduce into the model framework in Zhu and Lian [24]. Zhu and Lian [24] presented a closed formula for variance swaps with the one-factor Heston model which is different with our pricing formula and the results in Cao *et al.* [5].

Figure 2 illustrates the fair strike price data representing our numerical results with different β: $0.01, 0.05, 0.10, 0.15, 0.20, 0.25$. With the increasing of sampling frequency, the values of variance swaps are decreasing, converging to the continuous model. Furthermore, when β is increasing, the values of variance swaps are increasing correspondingly. These results show that the interest rate can impact and change the value of a variance swap, ignoring the effect of interest rate will result in mispricing.

5. CONCLUSION

This paper investigates the problem of pricing discrete variance swaps in the double Heston–CIR hybrid model. Our hybrid framework model is a combination with the double Heston stochastic volatility model and CIR stochastic interest rate model. The derivation of a semi-closed analytical formula via the dimension-reduction technique and the characteristic functions. We can only obtain an efficient semi-closed formula for variance swaps due to the application of the T-forward measure. Our numerical results are validated to be fast and accurate via comparisons with those from MC simulations. Our numerical results also discuss the impact of interest rate on the values of variance swaps. Our hybrid model is a more realistic model with practical importance for pricing and hedging.

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