Teaching proofs without words using dynamic geometry

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1. *Introduction*

A *Proof Without Words* (PWW) contains data, the claim that is to be proved, and one or more diagrams, sometimes without anything else and in other cases with a few mathematical expressions, without any verbal justifications [1]. It is assumed that students and researchers who possess the related appropriate mathematical knowledge will view the drawings and the expressions, will be able to follow and justify each step in the proof and develop their own visual proof abilities. PWW is very much like a cartoon which contains a drawing with sometimes only a few words or sometimes no words at all and the observers are expected to understand the context or the projected message.

A PWW constitutes images or diagrams that help the viewer see and understand why a certain theorem is true and how to begin proving it. In some cases, certain clues or an equation or two are presented to guide the viewer in the proof processes. The emphasis is clearly on the provision of visual clues to the viewer in order to promote and develop visual mathematical thought. In fact, a PWW is an elegant visual proof (usually simple and brief) of a task or theorem whose formal proof is longer, more complicated, and uses more involved mathematical arguments.

PWW started to appear in the 1970s in *The College Mathematics Journal* and *Mathematics Magazine* and, to a lesser extent, in other journals. A variety of PWW cases can also be found in three books written by Nelsen [2, 3, 4] published by the Mathematical Association of America.

Other versions of PWW from China, Greece, and India, many of which are dated back to the 12th century, can be found in [5, 6]. The PWW equivalent in Chinese is ‘to see is to understand’, which is a testimony that PWW is a beautiful geometrical analogy in which the truthfulness of a theorem's proof in geometry is seen in a single glance, instead of the traditional long and tedious proof.

At various levels of learning geometry in grade school education, the requirements for justification change or are adjusted according to the students' abilities to comprehend the processes required to complete a proof. For example, in the early stages of learning geometry, visual justification suffices, while, in the later stages and levels of investigation, detailed justifications are required at each stage of the formulation of the proof as well as full statements of the theorems or the properties used in the proof. As the study progresses and deepens, the required justifications are not as detailed and therefore shorter formulations become sufficient. If the proof is presented properly, it can become the main part of proving the task (see for example [7, 8]).

https://doi.org/10.1017/mag.2019.51 Published online by Cambridge University Press

Most PWW are instrumental in teaching and learning geometry in all levels of education from K-16 (Kindergarten through college) (see for example [4]).

In the following, first we state a few examples of PWW that can be used in a classroom setting and then we elaborate on the effectiveness of geometrical semi-proof using dynamic geometry environment (DGE).

Throughout the article, we let S and P, respectively, denote the area and perimeter of a given geometrical shape and R denote the radius of the circumcircle of a triangle.

2. *Geometrical Problems and PWW*

Problem 1

$ABCD$ is a convex quadrilateral other than a trapezium.

MN is the bisector of the sides AD and BC (Figure 1).

Prove that $\dfrac{AB + DC}{2} > MN$.

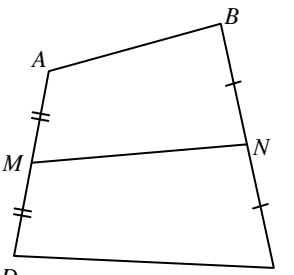


FIGURE 1

Proofs:

Method 1

The point K is the midpoint of the diagonal AC, that is $AK = KC$ (Figure 2a). Then

$$\begin{aligned} MN &< MK + KN \\ &= \tfrac{1}{2}DC + \tfrac{1}{2}AB \\ &= \tfrac{1}{2}(AB + DC). \end{aligned}$$

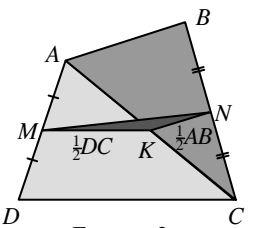


FIGURE 2a

Method 2

By area rotation of the quadrilateral $MNCD$ (Figure 2b). $AMM'D'$ is a parallelogram, so $AB + C'D' > 2MN$.

Equality exists when $ABCD$ is a trapezium.

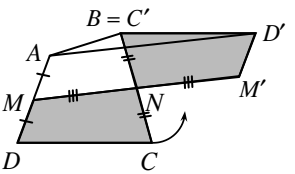


FIGURE 2b

GeoGebra has an applet [9] that demonstrates the inequality. It is possible to drag any of the vertices A, B, C and D and observe the changing inequality as the dimensions of the quadrilateral change. In any situation, the inequality

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exists, except where the quadrilateral becomes a trapezium.

Link 1: An equality between the half sum of two opposite side lengths of quadrilateral https://www.geogebra.org/m/g6uTHHuR

Problem 2

ABCD is a given arbitrary convex quadrilateral (Figure 3). *MN* is the bisector of the sides *AD* and *BC*. *PQ* is the bisector of the sides *AB* and *DC*.

Prove that $PQ \times MN > S_{ABCD}$.

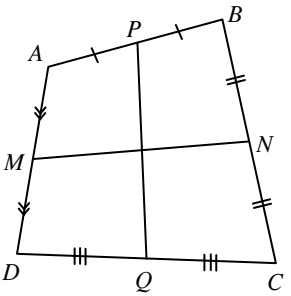


FIGURE 3

Proof

Method 1

$$S_{MPQN} = \frac{PQ \times MN \times \sin\alpha}{2}$$

(see Figure 4a). Then, by Varignon's Theorem,

$$\frac{PQ \times MN}{2} \geqslant S_{MPQN} = \tfrac{1}{2}S_{ABCD}.$$

Equality exists only if $PQ \perp MN$ or, equivalently, $DB = AC$.

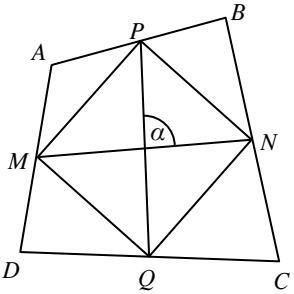


FIGURE 4a

Method 2

If the four regions in the quadrilateral *ABCD* are rearranged to form a parallelogram (Figures 4b, 4c) then (without resorting to Varignon's Theorem) we have

$$S_{MPNQ} = PQ \times MN \times \sin\alpha \leqslant PQ \times MN.$$

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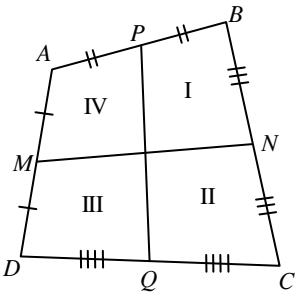


FIGURE 4a

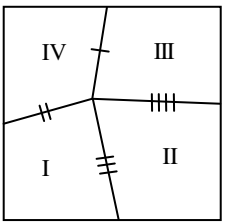


FIGURE 4b

Problem 3

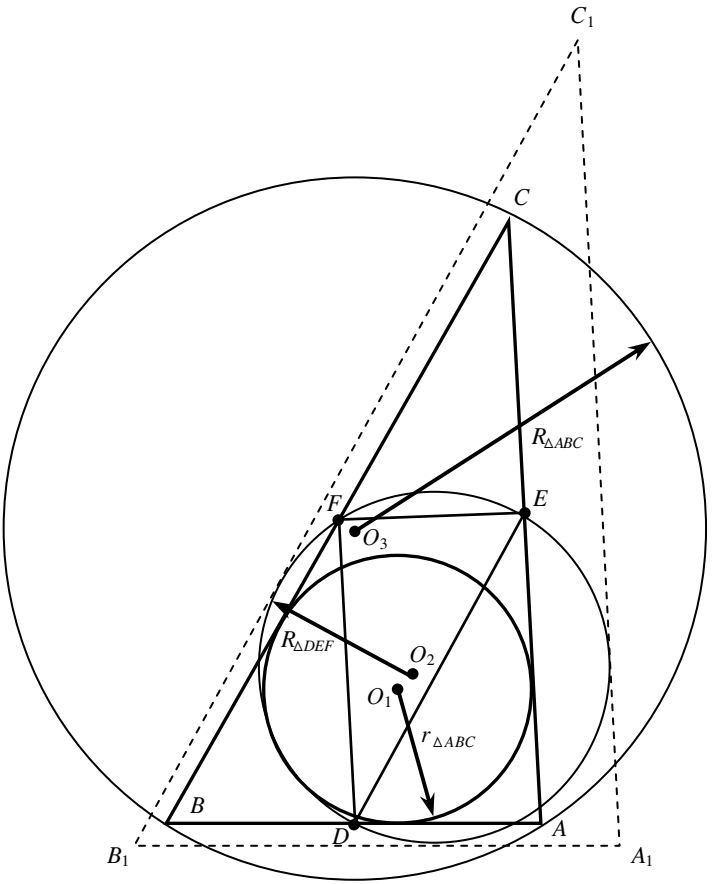


FIGURE 5

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Given the triangle ABC and the inscribed and circumscribed circles, the relation between the radius of the inscribed circle and the radius of the circumcircle of a triangle is

$$R_{\triangle ABC} > 2 \times r_{\triangle ABC}$$

A PWW is given in Figure 5.

An alternative way instead of using the Euler formula for the diameter O_2O_3 is

$$\left.\begin{aligned} AD &= DB \\ BF &= FC \\ AE &= EC \end{aligned}\right\} \Rightarrow \frac{AD}{AB} = \frac{BF}{BC} = \frac{AE}{AC} = \frac{1}{2} \Rightarrow \triangle ABC \sim \triangle DEF$$

$$\Rightarrow R_{\triangle ABC} > 2 \times R_{\triangle DEF}.$$

$$\left.\begin{aligned} A_1B_1 &\parallel AB \\ A_1C_1 &\parallel AC \\ B_1C_1 &\parallel BC \\ r_{\triangle A_1B_1C_1} &= R_{\triangle DEF} \end{aligned}\right\} \Rightarrow \triangle A_1B_1C_1 \sim \triangle ABC \Rightarrow r_{\triangle A_1B_1C_1} > r_{\triangle ABC}$$

$$\Rightarrow R_{\triangle DEF} > r_{\triangle ABC}.$$

$$\left.\begin{aligned} R_{\triangle ABC} &= 2 \times R_{\triangle DEF} \\ R_{\triangle DEF} &> r_{\triangle ABC} \end{aligned}\right\} \Rightarrow R_{\triangle ABC} = 2 \times r_{\triangle ABC}.$$

We remark that the circumcircle of DEF is the nine-point circle, and for an alternative proof of Problem 3 one can refer to [10].

Problem 4

The triangle DEF is inscribed in an acute-angled triangle ABC (Figure 6). Then

$$P_{\triangle DEF} \geqslant \frac{2 \times S_{\triangle ABC}}{R_{\triangle ABC}}.$$

The problem is investigated dynamically using the link to the applet [11].

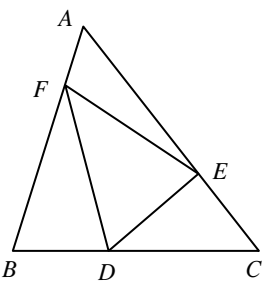


FIGURE 6

Using the applet in [11], one can drag each of the vertices of the triangles ABC and DEF to view the interactions between the components of the two triangles ABC and DEF including the interrelations between the perimeter $P_{\triangle DEF}$ and $\dfrac{2 \times S_{\triangle ABC}}{R_{\triangle ABC}}$.

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More interestingly, the screen shows the changing angles between the lines *AD*, *BE* and *FC* and the sides of the triangles.

The dynamic investigation that equality is obtained only when the straight lines *AD*, *BE* and *FC* are altitudes in the triangle. In this case, the triangle *DEF* is the orthic triangle – the triangle whose vertices are the feet of the altitudes of a given triangle.

The proof is presented as a PWW. Figure 7 shows the triangle *ABC*, the circle that circumscribes *ABC* and the triangle *DEF* inscribed in *ABC*.

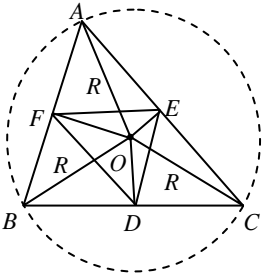


FIGURE 7

$$S_{FOEA} \leqslant \frac{AO \times FE}{2} = \frac{R \times FE}{2}$$

$$S_{EODC} \leqslant \frac{CO \times ED}{2} = \frac{R \times ED}{2}$$

$$S_{BFOD} \leqslant \frac{BO \times FD}{2} = \frac{R \times FD}{2}$$

$$\Rightarrow S_{\Delta ABC} = S_{FOEA} + S_{EODC} + S_{BFOD} \leqslant \frac{R(FE + ED + FD)}{2}$$

$$\Rightarrow P_{\Delta DEF} \geqslant \frac{2\,(S_{\Delta ABC})}{R_{\Delta ABC}}.$$

Note: using a dynamic investigation, it can be seen that for an obtuse-angled triangle *ABC*, the perimeter of the triangle *DEF* is always greater than the ratio $\frac{2 \times S_{\Delta ABC}}{R_{\Delta ABC}}$.

Problem 5

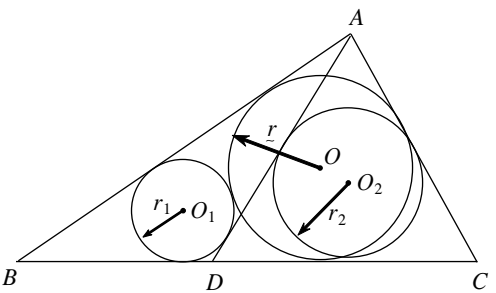


FIGURE 8

The triangle *ABC* is given. Choose a point *D* (other than *B* and *C*) on the base *BC*. Inscribe the circles, (O_1, r_1), (O_2, r_2) and (O, r) in the triangles *ABD*, *ADC* and *ABC*, respectively, as shown in the Figure 8.

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Prove that $r_1 + r_2 > r$.

See [11].

Proof:

A very well-known formula:

$$r_1 = \frac{S_{\triangle ABD}}{\frac{1}{2}P_{\triangle ABD}} > \frac{S_{\triangle ABD}}{\frac{1}{2}P_{\triangle ABC}}$$

$$r_2 = \frac{S_{\triangle ADC}}{\frac{1}{2}P_{\triangle ADC}} > \frac{S_{\triangle ADC}}{\frac{1}{2}P_{\triangle ABC}}$$

$$r_1 + r_2 > \frac{S_{\triangle ABD} + S_{\triangle ADC}}{\frac{1}{2}P_{\triangle ABC}} > \frac{S_{\triangle ABC}}{\frac{1}{2}P_{\triangle ABC}} = r.$$

In the following, [12] is a GeoGebra applet that demonstrates the interdependence between the radii of the inscribed circles in Figure 8. We remark that when dragging the point D to the left, the size of the circle inscribed in the triangle ABD decreases and the size of the circle inscribed in the triangle ADC increases, if the point D is dragged to the right, the size of the circle inscribed in the triangle ABD increases and the size of the circle inscribed in the triangle ADC decreases, but in either case the sum of the radii of the two smaller circles is greater than the radius of the larger circle.

3. *Semi-proof using Dynamic Geometry Environment* (DGE)

With over nine decades of teaching experience among the three of us, we noticed that when the illustration for a certain task is presented to our students (see for example [9, 11, 12]), the level of frustration to prove is diminished and instead the difficulties of the students (mostly) are shifted and/or are expressed in preparing the drawings in accordance to the given and/or required data. This latter difficulty can be helped or modified by the introduction of the dynamic geometry software (DGS) such as GeoGebra or Geometers' Sketchpad. This introduction of DGS (in many cases) helps students and learners to experiment (through different dragging modalities) with the geometrical objects that they are asked to construct and consequently to infer properties and generalities, and to make conjectures. It enables the students to experiment with a variety of scenarios and comprehend a whole class of cases and situations which can eventually assure them of the high possibility of the correctness of their conjectures. Nevertheless, because of the inductive nature of the DGE, we call this process a semi-proof.

These non-deductive methods of investigation, which rely on experimental, intuitive and inductive reasoning are seen to provide more meaningful contexts for teaching and learning geometry in a dynamic geometry environment than the classical approach of proof as a way of obtaining certainty. We emphasise that students must be aware that they still need to prove a result rather than rely on the virtual experiment.

https://doi.org/10.1017/mag.2019.51 Published online by Cambridge University Press

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10.1017/mag.2019.51

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