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Non extendable holomorphic functions

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Abstract

We modify arguments used in the study of Universal Taylor Series and extend them from simply connected domains to general domains in $\mathbb{C}$ to give a new simple and natural proof of the fact that all holomorphic functions are generically non extendable.

1. *Introduction*

In 1896, Emile Borel claimed that in general the circle of convergence is a natural boundary (Comptes rendus de l'Académie des sciences 1896, Acta Mathematica 1897). Later in [**12**] Baire's Category Theorem was used to prove that generically all holomorphic functions in the open unit disc have the unit circle as a natural boundary and therefore, they are non extendable. In [**8**] it is stated that the above generic result can be extended to any domain in $\mathbb{C}$. Constructive methods without use of Baire's Theorem have been developed in [**13**–**15**], in order to establish existence of holomorphic functions with chaotic behavior near the boundary, which implies non extendability. In [**6**] the generic character of such results has been pointed out. In all the above arguments, when we approach a boundary point from the interior, the limit of the function in question does not exist. Furthermore in [**16**] a holomorphic function which is smooth up to the boundary and non-extendable is constructed. The method of proof is different and uses Ostrowski gaps.

In this paper we first analyze the notion of non-extendability in the sense of Riemann surfaces. A holomorphic function f in a domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, is exactly holomorphic in Ω, if the complete Riemann surface of f has exactly one sheet and its projection on $\mathbb{C}$ is Ω. We also say that f is non-extendable. It turns out, that this is equivalent to the following fact: for every ζ in Ω, the radius of convergence R_ζ of the Taylor development of f with center ζ is exactly $R_\zeta = \operatorname{dist}(\zeta, \partial\Omega)$. In fact it suffices to have this equality for all ζ in a denumerable dense subset J of Ω.

We prove that for any $\zeta \in \Omega$ we have $R_\zeta = \operatorname{dist}(\zeta, \partial\Omega)$ generically for all functions in the corresponding Fréchet Algebra of all holomorphic functions, or in the

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corresponding Fréchet Algebra of holomorphic functions smooth up to the boundary. Then Baire's Theorem allows us to consider a denumerable intersection for all $\zeta \in J$.

The main argument we are using is a modification and extension of the standard argument used in the case of Universal Taylor Series in simply connected domains ([**9**, **10**, **19**–**21**, **24**, **25**]). In particular this paper is closely related to a conjecture of Jean–Pierre Kahane, which has been answered in the affirmative in [**19**] using elements from [**5**] (see also [**8**]): Every Universal Taylor Series in the open unit disc has the unit circle as a natural boundary. The proof in this paper does not use Ostrowski gaps, it is simple, natural and different from all previous proofs; in fact we prove that the set of non-extendable functions is G_δ and dense in natural Fréchet Algebras of holomorphic functions, in any domain Ω in $\mathbb{C}$.

Finally we mention that it is possible to obtain some further results with the same methods. More precisely, if Ω is a domain in $\mathbb{C}$, bounded by a finite number of Jordan curves, generically in some convenient spaces of functions, we can obtain functions which are holomorphic exactly in Ω and which are smooth only on a part of the boundary; near another disjoint part of the boundary they may have a chaotic behavior from the interior. The smoothness up to a part of the boundary may be interpreted as continuous extension of the function only, or continuous extension of all order's derivatives. We do not include the proofs here. For details see [**16**].

Our results are based on the strongest and natural definition of non-extendability using Riemann surfaces. This is in contrast to a usual definition, saying that f is non-extendable in Ω, if it can not be homomorphically extended to any larger domain G, $\Omega \subsetneqq G \subset \mathbb{C}$. For instance a branch of $\sqrt{z}$ in $\Omega = \mathbb{C} - \{x \in \mathbb{R}: x \leqslant 0\}$ is non-extendable according to the latter definition, while it is not non-extendable according to our definition. We give complete arguments using the strong notion of non-extendability and we obtain results in this strong sense.

For the use of Baire's Theorem in various branches of analysis we refer to [**7**, **8**]. This paper is dedicated to the memory of Nikolaos Danikas.

2. *Preliminaries*

Definition 2·1. Let $\Omega \subset \mathbb{C}$ be a domain and $f: \Omega \to \mathbb{C}$ a holomorphic function. We say that f is non-extendable (in the sense of Riemann surfaces), or that f is holomorphic exactly in Ω, if the following holds:

The complete Riemann surface of f has exactly one sheet and its projection onto $\mathbb{C}$ is equal to Ω.

Equivalently, if $(D_0, f_0), \ldots, (D_N, f_N)$, $N = 1, 2, \ldots$ are such that D_j are open discs in $\mathbb{C}$ with $D_j \bigcap D_{j+1} \neq \varnothing$ for all $j = 0, \ldots, N-1$, f_j is holomorphic in D_j, $j = 0, \ldots, N$, $D_0 \subset \Omega$, $f_0 = f|_{D_0}$ and $f_j|_{D_j \cap D_{j+1}} = f_{j+1}|_{D_j \cap D_{j+1}}$, $j = 0, \ldots, N-1$, then all D_j are contained in Ω and $f_j = f|_{D_j}$, $j = 0, \ldots, N$.

It is easy to see, that $\bigcup_{j=0}^{N} D_j \subset \Omega$ is a sufficient condition; because then, by analytic continuation, we automatically obtain $f_j = f|_{D_j}$. It follows that in the case $\Omega = \mathbb{C}$ every entire function is exactly holomorphic in $\mathbb{C}$.

Let Ω be a domain in $\mathbb{C}$, $\Omega \neq \mathbb{C}$, and $f: \Omega \to \mathbb{C}$ a holomorphic function, which does not extend to an entire function. Then, for every $\zeta \in \Omega$ the Taylor development of f with center ζ has a radius of convergence $R_\zeta \in (0, +\infty)$, which satisfies $R_\zeta \geqslant \operatorname{dist}(\zeta, \Omega^c) = \operatorname{dist}(\zeta, \partial\Omega)$.

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LEMMA 2·2. *Let Ω, f and R_ζ ($\zeta \in \Omega$) be as above. Then we have*:

(i) *If ζ, $w \in \Omega$ are such that the whole segment $[\zeta, w]$ is contained in Ω, then $|R_\zeta - R_w| \leqslant |\zeta - w|$;*

(ii) *The map $\Omega \ni \zeta \to R_\zeta \in (0, +\infty)$ is continuous.*

Proof. Suppose that $[\zeta, w] \subset \Omega$ and $R_w - R_\zeta > |\zeta - w|$ to arrive at a contradiction. We consider the open discs $D(w, R_w)$ with center w and radius R_w and $D(\zeta, R_\zeta)$ with center ζ and radius R_ζ. The closed disc $\overline{D(\zeta, R_\zeta)}$ is contained in the open disc $D(w, R_w)$. The Taylor development of f with center w defines a holomorphic function F in $D(w, R_w)$. The restriction of F to a small disc with center w coincides with that of f. By analytic continuation along $[\zeta, w]$, the restriction of F to a small disc with center ζ coincides with the restriction of f to that disc. Therefore, R_w and R_ζ coincides with the radii of convergence of the Taylor development of F with centers w and ζ, respectively. It follows that $R_\zeta \geqslant \operatorname{dist}(\zeta, \partial D(w, R_w)) = R_w - |w - \zeta| > R_\zeta$, which implies $R_\zeta > R_\zeta$; this is absurd and the proof of (i) is complete.

To prove (ii) we fix $\zeta \in \Omega$. Then for $w \in D(\zeta, \operatorname{dist}(\zeta, \Omega^c))$ we have $[\zeta, w] \subset \Omega$ and $|R_\zeta - R_w| \leqslant |\zeta - w|$ by (i). Therefore, the map $\Omega \ni z \to R_z \in (0, +\infty)$ is continuous at ζ, for all $\zeta \in \Omega$, and the proof is completed.

PROPOSITION 2·3. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be a domain and $f: \Omega \to \mathbb{C}$ be a holomorphic function. Then the following are equivalent.*

(i) *f is holomorphic exactly in Ω.*

(ii) *For every $\zeta \in \Omega$, we have $R_\zeta = \operatorname{dist}(\zeta, \partial\Omega)$.*

(iii) *$R_\zeta = \operatorname{dist}(\zeta, \partial\Omega)$, for all ζ in a dense subset of Ω.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $R_\zeta > \operatorname{dist}(\zeta, \partial\Omega)$ for some $\zeta \in \Omega$ to arrive at a contradiction.

Let F denote the holomorphic function in $D(\zeta, R_\zeta)$ defined by the Taylor development of f at ζ. We also consider the disc $D(\zeta, \operatorname{dist}(\zeta, \partial\Omega))$ with the restriction of f on it. Thus we have a continuation $(D(\zeta, \operatorname{dist}(\zeta, \partial\Omega)), f)$, $(D(\zeta, R_\zeta), F)$. This contradicts (i), because $D(\zeta, R_\zeta)$ is not contained in Ω.

(ii) $\Rightarrow$ (i) If (i) does not hold we easily arrive at the following situation: D_0, D_1 are open discs, $D_0 \subset \Omega \cap D_1$, $D_1 \cap \partial\Omega \neq \varnothing$, $F: D_1 \to \mathbb{C}$ is holomorphic in D_1 and $F|_{D_0} = f|_{D_0}$.

Let ζ be the center of D_0 and $w \in D_1 \cap \partial\Omega$. Then on the segment $[\zeta, w] \subset D_1$, there exists $z \in \partial\Omega \cap [\zeta, w]$, such that $[\zeta, z] \cap \partial\Omega = \{z\}$ and $[\zeta, z] \setminus \{z\} \subset \Omega$. Obviously $z \neq \zeta$ and $z \in D_1$. We set $r = \operatorname{dist}(z, \partial D_1) > 0$.

Let $t \in [\zeta, z]$, $t \neq z$ with $|t - z| < \frac{r}{2}$. Then $[\zeta, t] \subset \Omega$ and by analytic continuation along $[\zeta, t]$, the functions f and F coincide on a small disc with center t. Therefore, R_t coincides with the radius of convergence of F with center t. Since F is holomorphic in D_1 it follows that $R_t \geqslant \operatorname{dist}(t, \partial D_1) \geqslant \operatorname{dist}(z, \partial D_1) - |z - t| > r - \frac{r}{2} = \frac{r}{2}$. On the other hand $R_t = \operatorname{dist}(t, \partial\Omega) \leqslant |t - z| < \frac{r}{2}$, which implies $\frac{r}{2} < \frac{r}{2}$ and we arrived at a contradiction. This completes the proof of (ii) $\Rightarrow$ (i).

(ii)$\Leftrightarrow$(iii) This follows from the continuity of the maps $\Omega \ni \zeta \to \operatorname{dist}(\zeta, \partial\Omega) \in (0, +\infty)$ and $\Omega \ni \zeta \to R_\zeta \in (0, +\infty)$ (Lemma 2·2). The proof is completed.

Remark. Conditions (i), (ii) and (iii) of Proposition 2·3 are equivalent to the following.

(iv) There exists $\delta > 0$, such that $R_\zeta = \operatorname{dist}(\zeta, \partial\Omega)$ for all ζ in Ω satisfying $\operatorname{dist}(\zeta, \partial\Omega) < \delta$.

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PROPOSITION 2·4. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be a domain and $f: \Omega \to \mathbb{C}$ a holomorphic function. For $\zeta \in \Omega$ we denote by $R_\zeta \in (0, +\infty)$ the radius of convergence of the Taylor development of f with center ζ.*

We assume that $\operatorname{dist}(\zeta, \partial\Omega) \leqslant R_\zeta \leqslant \operatorname{dist}(\zeta, \partial\overline{\Omega})$ for every $\zeta \in \Omega$. Then there is no domain G, $\Omega \subset G$, $G \bigcap (\overline{\Omega})^c \neq \varnothing$ and holomorphic function $F: G \to \mathbb{C}$, such that $F|_\Omega = f$.

Proof. Suppose that there exist such G and F, to arrive at a contradiction.

G meets $(\overline{\Omega})^0$ and $(\overline{\Omega})^c$. Since G is a domain, it follows that there exist $w \in G \bigcap \partial(\overline{\Omega})$ and $r > 0$ so that $D(w, r) \subset G$. As $\partial(\overline{\Omega}) \subset \partial\Omega$, there exists $\zeta \in \Omega$ with $|\zeta - w| < \frac{r}{2}$. It follows that F is holomorphic in $D(\zeta, \frac{r}{2}) \subset D(w, r) \subset G$. Thus $R_\zeta \geqslant \frac{r}{2}$. On the other hand $R_\zeta \leqslant \operatorname{dist}(\zeta, \partial(\overline{\Omega})) \leqslant |\zeta - w| < \frac{r}{2}$; this implies $\frac{r}{2} < \frac{r}{2}$, which is absurd. The proof is completed.

One might think, that the assumption $\operatorname{dist}(\zeta, \partial\Omega) \leqslant R_\zeta \leqslant \operatorname{dist}(\zeta, \partial\overline{\Omega})$ for all $\zeta \in \Omega$ would imply the existence of a domain V, $\Omega \subset V \subset \overline{\Omega}$, so that f is holomorphic exactly in V. This is false, as one can see by the following example.

Example 2·5. Let u be a holomorphic function in $D = \{z : |z| < 1\}$ with the unit circle ∂D as a natural boundary. Let $\Omega = D \setminus [0, 1)$ and $\sqrt{z}$ a branch of the square root in Ω. Let $f(z) = u(z) + \sqrt{z}$. Then $R_\zeta = |\zeta|$, for $\zeta \in \Omega$, $|\zeta| \leqslant \frac{1}{2}$ and $R_\zeta = 1 - |\zeta|$, for all $\zeta \in \Omega$, $|\zeta| \geqslant \frac{1}{2}$. It follows that $\operatorname{dist}(\zeta, \partial\Omega) \leqslant R_\zeta \leqslant \operatorname{dist}(\zeta, \partial\overline{\Omega}) = 1 - |\zeta|$, for all $\zeta \in \Omega$. However, the Riemann surface of f has two sheets and therefore, it is not holomorphic exactly in some domain V in $\mathbb{C}$.

Remark 2·6. If $\partial\Omega = \partial\overline{\Omega}$ and $\operatorname{dist}(\zeta, \partial\Omega) \leqslant R_\zeta \leqslant \operatorname{dist}(\zeta, \partial\overline{\Omega})$, for all $\zeta \in \Omega$, then we have the equality $R_\zeta = \operatorname{dist}(\zeta, \partial\Omega)$. It follows that f is holomorphic exactly in Ω (Proposition 2·3); in particular this holds, if Ω is a Jordan domain and $\operatorname{dist}(\zeta, \partial\Omega) \leqslant R_\zeta \leqslant \operatorname{dist}(\zeta, \partial\overline{\Omega})$, for all $\zeta \in \Omega$.

LEMMA 2·7 ([**4**, chapter XIV, p. 479 I]). *Let g denote a rational function and assume that $\zeta \in \mathbb{C}$ is not a pole of g. Let $S_N(g, \zeta)(z) = \sum_{n=0}^{N} \frac{g^{(n)}(\zeta)}{n!}(z - \zeta)^n$, $N = 0, 1, 2, \ldots$ be the sequence of partial sums of the Taylor development of g with center ζ.*

Consider the poles of g with minimum distance from ζ and let w be one of them which has maximal multiplicity. Then $\lim_{N\to\infty} S_N(f, \zeta)(w) = \infty$.

The above does not hold for all poles; such an example can be found in [**24**, page 1303].

3. *Universal Taylor series*

Overconvergence of power series is a generic property of holomorphic functions in simply connected domains. For $\zeta \in \mathbb{C}$ and f a holomorphic function in an open set containing ζ we denote by $S_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z - \zeta)^n$ the Nth partial sum of the Taylor expansion of f with center ζ.

Definition 3·1 ([**24**, **25**]). Let Ω be a simply connected domain in $\mathbb{C}$ and let $\zeta \in \Omega$ be fixed. A holomorphic function $f: \Omega \to \mathbb{C}$ belongs to the class $U(\Omega, \zeta)$, if, for every compact set $K \subset \mathbb{C}$, $K \bigcap \Omega = \varnothing$, with K^c connected, and every function $h: K \to \mathbb{C}$

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which is continuous on K and holomorphic in K^0, there exists a sequence $\lambda_n \in \{0, 1, 2, \ldots\}$, $n = 1, 2, \ldots$, such that

$$\sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \longrightarrow 0, \quad as \quad n \longrightarrow +\infty.$$

Definition 3·2 ([**25**]). Let Ω be a simply connected domain in $\mathbb{C}$. A holomorphic function $f: \Omega \to \mathbb{C}$ belongs to the class $U(\Omega)$, if, for every compact set $K \subset \mathbb{C}$, $K \bigcap \Omega = \varnothing$, with K^c connected and every function $h: K \to \mathbb{C}$ which is continuous on K and holomorphic in K^0, there exists a sequence $\lambda_n \in \{0, 1, 2, \ldots\}$, $n = 1, 2, \ldots$, such that, for every compact set $L \subset \Omega$ we have

$$\sup_{\zeta \in L} \sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \longrightarrow 0, \quad as \quad n \longrightarrow +\infty.$$

Definition 3·3 ([**1**, **17**, **18**]). Let Ω be a simply connected domain in $\mathbb{C}$ and let $\zeta \in \Omega$ be fixed. A holomorphic function $f: \Omega \to \mathbb{C}$ belongs to the class $U_1(\Omega, \zeta)$, if for every compact set $K \subset \mathbb{C}$, $K \bigcap \overline{\Omega} = \varnothing$, with K^c connected, and every function $h: K \to \mathbb{C}$ which is continuous on K and holomorphic in K^0, there exists a sequence $\lambda_n \in \{0, 1, 2, \ldots\}$, $n = 1, 2, \ldots$ such that

$$\sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \longrightarrow 0, \quad as \quad n \longrightarrow +\infty.$$

Definition 3·4. Let $\Omega \subset \mathbb{C}$ be a simply connected domain. A holomorphic function $f: \Omega \to \mathbb{C}$ belongs to the class $U_1(\Omega)$, if, for every compact set $K \subset \mathbb{C}$, $K \bigcap \overline{\Omega} = \varnothing$, with K^c connected, and every function $h: K \to \mathbb{C}$ which is continuous on K and holomorphic in K^0, there exists a sequence $\lambda_n \in \{0, 1, 2, \ldots\}$ $n = 1, 2, \ldots$, such that, for every compact set $L \subset \Omega$ we have

$$\sup_{\zeta \in L} \sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \longrightarrow 0, \quad as \quad n \longrightarrow +\infty.$$

Remark 3·5. In all the above definitions it is equivalent to require $\lambda_n < \lambda_{n+1}$. Furthermore, all the above classes are dense and G_δ subsets of $H(\Omega)$, the space of holomorphic functions in Ω endowed with the topology of uniform convergence on compact subsets of Ω ([**6**, **19**, **20**, **24**, **25**]). Properties of the universal functions belonging to the above classes can be found in [**3**, **5**, **11**, **19**, **20**, **21**, **24**, **25**]. In [**19**] it has been proved that $U(\Omega, \zeta) = U(\Omega)$, provided that Ω is contained in the complement of an angle with strictly positive opening (see also [**5**]). In a recent paper [**23**] it has been proved that $U(\Omega, \zeta) = U(\Omega)$, for all simply connected domains Ω in $\mathbb{C}$. However we have $U_1(\Omega, \zeta) = U_1(\Omega)$, if Ω is contained in the complement of an angle with strictly positive opening, but not always ([**23**]). We have the inclusions $U(\Omega, \zeta) = U(\Omega) \subset U_1(\Omega) \subset U_1(\Omega, \tilde{\zeta})$, for all $\zeta, \tilde{\zeta}$ in the simply connected domain Ω.

For $\Omega = D = \{z \in \mathbb{C}: |z| < 1\}$ we have $U(D, 0) \neq U_1(D, 0)$ ([**20**]). The elements of the class $U(\Omega, \zeta) = U(\Omega)$ have wild properties; in particular they cannot be smooth on the boundary ([**19**–**21**]). On the contrary the classes $U_1(\Omega, \zeta)$ and $U_1(\Omega)$ contain also functions smooth up to the boundary, provided that $\{\infty\} \bigcup [\mathbb{C} \setminus \overline{\Omega}]$ is connected ([**10**, **20**, **21**]).

If $f \in U(\Omega)$ and $\zeta \in \Omega$, then the radius of convergence of the Taylor expansion of f with center ζ is $R_\zeta = \operatorname{dist}(\zeta, \partial\Omega)$. Therefore, according to Proposition 2·3, f is holomorphic exactly in Ω; this relates to a conjecture of J.-P. Kahane, which has been answered in the affirmative in [**19**].

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If $f \in U_1(\Omega)$ and $\zeta \in \Omega$, then the radius of convergence R_ζ of the Taylor expansion of f with center ζ satisfies $\operatorname{dist}(\zeta, \partial\Omega) \leqslant R_\zeta \leqslant \operatorname{dist}(\zeta, \partial\overline{\Omega})$. Taking $\Omega = D \setminus [0,1)$, where D denotes the open unit disc and considering $f \in U(D) \subset U_1(\Omega)$ we see that f may be holomorphically extendable outside Ω. If in addition $\partial\Omega = \partial\overline{\Omega}$, then every $f \in U_1(\Omega)$ is holomorphic exactly in Ω, according to Proposition 2·3; in particular this holds if Ω is a Jordan domain.

The above results show that generically in $H(\Omega)$, for any simply connected domain $\Omega \subset \mathbb{C}$, every holomorphic function $f \in H(\Omega)$ is holomorphic exactly in Ω. In the case $\Omega = D$ this result had been established in 1933 using different arguments ([**12**]). In [**8**] it is stated that this argument can be extended to any domain in $\mathbb{C}$, not necessarily simply connected; see also [**13**]. In all these papers the method of proof is so that, when we approach a boundary point from the interior, the limit of the function does not exist.

In the present paper we modify the arguments concerning universal Taylor series and we extend them from simply connected domains to general domains in $\mathbb{C}$, to show that generically in $H(\Omega)$ all functions are holomorphic exactly in Ω. When we approach a boundary point from the interior, the limits of the function may or may not exist. Thus we are also able to obtain a generic result, that quasi all functions are holomorphic exactly in Ω, in spaces of smooth functions as $A(\Omega)$ or subspaces of $A^\infty(\Omega)$, to be defined later, provided that Ω is a domain bounded by a finite number of Jordan curves.

4. *Main result*

Let Ω be a domain in $\mathbb{C}$, $\Omega \neq \mathbb{C}$. Let $H(\Omega)$ denote the space of holomorphic functions in Ω endowed with the topology of uniform convergence on compacta. It is well known that $H(\Omega)$ is mertizable complete, in fact a Fréchet space. Thus, Baire's Theorem is at our disposal.

Let ζ be a point in Ω and $w \in \partial\Omega$ such that $|\zeta - w| = \operatorname{dist}(\zeta, \partial\Omega)$. We fix ζ and w as above. For $n \in \{0, 1, 2, \ldots\}$ and $k \in \{1, 2, \ldots\}$ we consider the set $E(n,k) = \{g \in H(\Omega) : |S_n(g,\zeta)(w)| > k\}$, where

$$S_n(g,\zeta)(w) = \sum_{\ell=0}^{n} \frac{g^{(\ell)}(\zeta)}{\ell!}(w-\zeta)^\ell.$$

LEMMA 4·1. *For $n \in \{0, 1, 2, \ldots\}$ and $k \in \{1, 2, \ldots\}$ the set $E(n,k)$ is open in $H(\Omega)$.*

Proof. Let $f_\lambda \in H(\Omega)$ be a sequence converging in the topology of $H(\Omega)$ towards an element $f \in E(n,k)$. We have $\lim_{\lambda \to +\infty} f_\lambda^{(\ell)}(\zeta) = f^{(\ell)}(\zeta)$, by Weierstrass Theorem; this implies $S_n(f_\lambda, \zeta)(w) \to S_n(f,\zeta)(w)$, as $\lambda \to +\infty$.

Since $|S_n(f,\zeta)(w)| > k$, it follows that there exists λ_0, so that $|S_n(f_\lambda,\zeta)(w)| > k$, for all $\lambda \geqslant \lambda_0$. Therefore, $f_\lambda \in E(n,k)$, for all $\lambda \geqslant \lambda_0$. This shows that $E(n,k)$ is open.

LEMMA 4·2. *The set $\bigcup_{n=0}^{\infty} E(n,k)$ is open and dense in $H(\Omega)$, for all $k = 1, 2, \ldots$*

Proof. According to Lemma 4·1 the sets $E(n,k)$ are open. Therefore, the same is true for their union. We shall also show that $\bigcup_{n=0}^{\infty} E(n,k)$ is dense in $H(\Omega)$.

Let $L \subset \Omega$ be a compact set. According to Theorem 13·3 in [**27**], we may assume that each component of $[\mathbb{C} \cup \{\infty\}] \setminus L$ contains a component of $[\mathbb{C} \cup \{\infty\}] \setminus \Omega$. Let A be a set consisting of exactly one point from each component of $[\mathbb{C} \cup \{\infty\}] \setminus \Omega$.

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We also consider a function $f \in H(\Omega)$ and $\varepsilon > 0$. We have to find $n \in \{0, 1, 2, \ldots\}$ and $g \in E(n, k)$ so that $\sup_{z \in L} |f(z) - g(z)| < \varepsilon$.

By Runge's Approximation Theorem ([**27**]), there exists a rational function φ with poles in A, such that $\sup_{z \in L} |f(z) - \varphi(z)| < \frac{\varepsilon}{2}$.

Let q be a positive integer strictly greater than all multiplicities of poles of φ, which are finite in number, since φ is a rational function. We choose $\delta > 0$ small enough, such that $\sup_{z \in L} |\delta \frac{1}{(z-w)^q}| < \frac{\varepsilon}{2}$ and we set $g(z) = \varphi(z) + \delta \frac{1}{(z-w)^q}$. Since g is a rational function with poles off Ω, we have $g \in H(\Omega)$. We also have $\sup_{z \in L} |f(z) - g(z)| \leqslant \sup_{z \in L} |f(z) - \varphi(z)| + \sup_{z \in L} |\delta \frac{1}{(z-w)^q}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Finally, since w is the nearest pole of g to ζ with greatest multiplicity, we have $\lim_{N \to \infty} S_N(g, \zeta)(w) = \infty$ (Lemma 2·7). It follows that there exists $n \in \{0, 1, 2, \ldots\}$, such that $|S_n(g, \zeta)(w)| > k$. Therefore $g \in E(n, k)$ and $\bigcup_{n=0}^{\infty} E(n, k)$ is dense in $H(\Omega)$. The proof is completed.

Definition 4·3. Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be a domain and $\zeta \in \Omega$ be fixed. A function $f \in H(\Omega)$ belongs to the class $F(\Omega, \zeta)$ if the radius of convergence of the Taylor expansion of f with center ζ is exactly equal to $\operatorname{dist}(\zeta, \partial\Omega)$.

Proposition 4·4. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be a domain and $\zeta \in \Omega$. Then $F(\Omega, \zeta)$ is dense and G_δ in $H(\Omega)$.*

Proof. We obviously have $F(\Omega, \zeta) \supset \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{\infty} E(n, k)$, where the sets $E(n, k)$ were previously defined with respect to the point ζ in Ω. Combining Baire's Theorem with Lemma 4·2 we conclude that $F(\Omega, \zeta)$ contains a dense and G_δ subset in $H(\Omega)$. It remains to prove that $F(\Omega, \zeta)$ is itself a G_δ set.

Let $d = \operatorname{dist}(\zeta, \partial\Omega) > 0$. The radius of convergence

$$R_\zeta(f) = \frac{1}{\limsup\limits_n \sqrt[n]{\frac{|f^{(n)}(\zeta)|}{n!}}}$$

satisfies $R_\zeta(f) \geqslant d$ for all $f \in H(\Omega)$. We easily see that

$$F(\Omega, \zeta) = \bigcap_{s=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \left\{ f \in H(\Omega) \colon \frac{1}{d} - \frac{1}{s} < \sqrt[n]{\frac{|f^{(n)}(\zeta)|}{n!}} \right\}.$$

Since the maps $H(\Omega) \ni f \to f^{(n)}(\zeta) \in \mathbb{C}$, $n = 0, 1, \ldots$ are continuous, it easily follows that $F(\Omega, \zeta)$ is G_δ and the proof is completed.

Let $J = (Q + iQ) \cap \Omega$, which is a denumerable dense subset of Ω. Baire's Theorem implies that $\bigcap_{\zeta \in J} F(\Omega, \zeta)$ is G_δ and dense in $H(\Omega)$. According to Lemma 2·3, this intersection coincides with the set of $f \in H(\Omega)$ which are holomorphic exactly in Ω. Thus we have proved the following.

Theorem 4·5. *Let Ω be a domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$. There exist functions $f \in H(\Omega)$ which are holomorphic exactly in Ω. The set of such functions is dense and G_δ in $H(\Omega)$, endowed with the topology of uniform convergence on compacta.*

5. *Smooth non extendable functions*

In this section Ω denotes a domain in $\mathbb{C}$, which is bounded by a finite number of disjoint Jordan curves. In particular Ω is bounded.

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A holomorphic function $f \in H(\Omega)$ belongs to $A(\Omega)$, if f extends continuously on $\overline{\Omega}$. $A(\Omega)$ becomes a Banach Algebra with the norm $\|f\| = \sup_{z\in\overline{\Omega}} |f(z)|$. Therefore, Baire's Theorem can be applied.

Let A be a set which contains exactly one point from each component of $[\mathbb{C} \cup \{\infty\}] \setminus \overline{\Omega}$. According to an extension of Mergelyan's Theorem ([**27**, ch. 20, p. 394 ex. 1]) rational functions with poles in A are dense in $A(\Omega)$, because $[\mathbb{C} \cup \{\infty\}] \setminus \overline{\Omega}$ has a finite number of components. We fix such a set A.

A holomorphic function $f \in H(\Omega)$ belongs to $A^\infty(\Omega)$, if every derivative $f^{(\ell)}$, $\ell = 0, 1, 2, \ldots$ extends continuously on $\overline{\Omega}$. $A^\infty(\Omega)$ becomes a Fréchet Algebra with the seminorms $\sup_{z\in\overline{\Omega}} |f^{(\ell)}(z)|$, $\ell = 0, 1, 2, \ldots$. We do not know if rational functions with poles off Ω are dense in $A^\infty(\Omega)$. For this reason, we consider $X^\infty(\Omega)$ to be the closure in $A^\infty(\Omega)$ of rational functions with poles off $\overline{\Omega}$ (see [**10**]). Because of the choice of the set A we easily see that rational functions with poles in A are dense in $X^\infty(\Omega)$ in the topology of $A^\infty(\Omega)$. The space $X^\infty(\Omega)$ with the relative topology from $A^\infty(\Omega)$ is a Fréchet Algebra and therefore Baire's Theorem is at our disposal.

Let $\zeta \in \Omega$ and $w \in \mathbb{C} \setminus \overline{\Omega}$ be fixed such that $|\zeta - w| < \operatorname{dist}(\zeta, A)$. We consider the set

$$E(\Omega, \zeta, w, n, k) = \{f \in H(\Omega): \mid S_n(f, \zeta)(w) \mid > k\}$$

for $n \in \{0, 1, 2, \ldots\}$ and $k \in \{1, 2, \ldots\}$.

LEMMA 5·1. *For Ω, ζ, w, n, k as above we have*:
(i) *$E(\Omega, \zeta, w, n, k) \bigcap A(\Omega)$ is open in $A(\Omega)$*;
(ii) *$E(\Omega, \zeta, w, n, k) \bigcap X^\infty(\Omega)$ is open in $X^\infty(\Omega)$.*

The proof is similar to that of Lemma 4·1 and is omitted.

LEMMA 5·2. *Under the above assumptions and notation we have*:
(i) *The set $\bigcup_{n=0}^\infty [E(\Omega, \zeta, w, n, k) \bigcap A(\Omega)]$ is dense in $A(\Omega)$*;
(ii) *The set $\bigcup_{n=0}^\infty [E(\Omega, \zeta, w, n, k) \bigcap X^\infty(\Omega)]$ is dense in $X^\infty(\Omega)$, provided that $|\zeta - w| < \operatorname{dist}(\zeta, A)$.*

Proof. (i) Let $f \in A(\Omega)$ and $\varepsilon > 0$. We consider the function $F: \overline{\Omega} \cup \{w\} \to \mathbb{C}$ defined by $F(z) = f(z)$ on $\overline{\Omega}$ and $F(w) = k+1$. According to an extension of Mergelyan's theorem (**27**, ch. 20, p. 394 ex. 1]), there exists a rational function g with poles in A, so that $|F(z) - g(z)| < \min(\varepsilon, \frac{1}{2})$ on $\overline{\Omega} \cup \{w\}$. It follows that $|f(z) - g(z)| < \varepsilon$ on $\overline{\Omega}$ and $|g(w)| > k + \frac{1}{2}$. Since $|\zeta - w| < \operatorname{dist}(\zeta, A)$, the Taylor series of g with center ζ satisfies $\lim_n S_n(g, \zeta)(w) = g(w)$. Therefore, there exists $n \in \{0, 1, 2, \ldots\}$ so that $|S_n(g, \zeta)(w)| > k$. It follows that $g \in E(\Omega, \zeta, w, n, k) \bigcap A(\Omega)$ which proves (i).

(ii) Let f be a rational function with poles in A, $\varepsilon > 0$ and $N \in \{0, 1, 2, \ldots\}$. We have to find $n \in \{0, 1, 2, \ldots\}$ and $g \in E(\Omega, \zeta, w, n, k) \bigcap X^\infty(\Omega)$ so that $\sup_{z\in\overline{\Omega}} |f^{(\ell)}(z) - g^{(\ell)}(z)| < \varepsilon$ for $\ell = 0, 1, \ldots, N$.

We can find a domain Ω_1 and an open disk $D(w, \delta)$ with center w and radius $\delta > 0$, such that the following holds: $\overline{\Omega} \subset \Omega_1$, $\Omega_1 \bigcap A = \varnothing$, $D(w, \delta) \bigcap \Omega_1 = \varnothing$, $D(w, \delta) \bigcap A = \varnothing$ and each component of $[\mathbb{C} \cup \{\infty\}] \setminus \Omega_1$ meets A (see [**2**, **6**] for a particular case). We consider the holomorphic function $F: \Omega_1 \cup D(w, \delta) \to \mathbb{C}$ defined by $F(z) = f(z)$ in Ω_1 and $F(z) = k + 1$ in $D(w, \delta)$. By Runge's Approximation Theorem F can be approximated uniformly on compacta by rational functions with

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poles in A. By Weierstrass theorem in the open set $\Omega_1 \cup D(w,\delta)$, the approximation is also valid for all order's derivatives. Hence, there exists a rational function g with poles in A, such that, $\sup_{z\in\overline{\Omega}} |f^{(\ell)}(z) - g^{(\ell)}(z)| < \varepsilon$ for $\ell = 0, 1, \ldots, N$ and $|g(w) - (k+1)| < \frac{1}{2}$. Since $|\zeta - w| < \operatorname{dist}(\zeta, A)$ and all poles of g are contained in A, we have $\lim_n S_n(g,\zeta)(w) = g(w)$. Therefore there exists $n \in \{0,1,\ldots\}$ so that $|S_n(g,\zeta)(w)| > k$. Obviously $g \in E(\Omega,\zeta,w,n,k) \bigcap X^\infty(\Omega)$ and g approximates f in the topology of $X^\infty(\Omega)$. The proof is completed.

For fixed $\zeta \in \Omega$ we can find a sequence w_s, $s = 1, 2, \ldots$ in $\mathbb{C} \setminus \overline{\Omega}$, such that $|\zeta - w_s| < \operatorname{dist}(\zeta, A)$ for all s and $\lim_s |\zeta - w_s| = \operatorname{dist}(\zeta, \partial\Omega)$. Combining the previous lemma with Baire's Theorem we conclude that

(i) $\bigcap_{s=1}^\infty \bigcap_{k=1}^\infty \bigcup_{n=0}^\infty [E(\Omega,\zeta,w_s,n,k) \bigcap A(\Omega)]$ is G_δ and dense in $A(\Omega)$

(ii) $\bigcap_{s=1}^\infty \bigcap_{k=1}^\infty \bigcup_{n=0}^\infty [E(\Omega,\zeta,w_s,n,k) \bigcap X^\infty(\Omega)]$ is G_δ and dense in $X^\infty(\Omega)$.

It is easy to see that, for every element f of these sets, the radius of convergence $R_\zeta(f)$ of the Taylor series of f with center ζ is exactly equal to $\operatorname{dist}(\zeta, \partial\Omega)$. Taking intersections of these sets for all ζ in the denumerable set $[Q + iQ] \bigcap \Omega$ we obtain again G_δ and dense sets, which contain only non-extendable holomorphic functions (Lemma 2·3). On the other hand, as in Proposition 4·4, we can see that the set of non-extendable elements of $A(\Omega)$ is itself a G_δ set in $A(\Omega)$. Similarly the set of non-extendable elements of $X^\infty(\Omega)$ is itself a G_δ set in $X^\infty(\Omega)$. Thus we have proved the following.

THEOREM 5·3. *Let Ω be a domain in $\mathbb{C}$ bounded by a finite number of disjoint Jordan curves. Then there exist holomorphic functions which are holomorphic exactly in Ω and which extend continuously on $\overline{\Omega}$. The set of such functions is a dense and G_δ subset of $A(\Omega)$.*

THEOREM 5·4. *Let Ω be a domain in $\mathbb{C}$ bounded by a finite number of disjoint Jordan curves. Then there exists a function which is holomorphic exactly in Ω and such that every order's derivative extends continuously on $\overline{\Omega}$.*

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