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A theorem of measurable rigidity in $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$

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Abstract. We consider measurable conjugacies between two non-solvable subgroups G_1, G_2 of the group of analytic diffeomorphisms of the circle which admit a finite generating set close to the identity. These measurable conjugacies are characterized by our main result which asserts that, up to finite coverings, they coincide almost everywhere with an analytic diffeomorphism of the circle.

1. *Introduction*

Consider the non-solvable subgroups of the group of orientation-preserving analytic diffeomorphisms of the circle, $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$, divided into two categories according to whether they are discrete or not. The main examples of discrete groups in $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ are the Fuchsian groups (or conjugates of them) whereas the main source of examples of non-discrete groups are those admitting a finite generating set close to the identity (cf. §2.2). Both types of groups have applications in different branches of mathematics (e.g. geometry, foliations, dynamical systems). Fuchsian groups have been intensively studied and their dynamical properties have been considered from the differentiable, topological and measure theoretic point of view.

However, for groups having a finite set of generators close to the identity, the best known features are related to topological behavior (e.g. presence of finite orbits, density of orbits). In a previous paper [**Reb**], we introduced an approach to questions involving 'differentiability'. In particular, we settled the question of the ergodicity of such groups.

In this work we extend the method of [**Reb**] in order to study the conjugacy classes of groups generated by a finite set of diffeomorphisms close to the identity from the measurable point of view. In this paper we shall be concerned with orientation-preserving diffeomorphisms. However, it is easy to generalize our methods to include orientation-reversing analytic diffeomorphisms of $\mathbb{S}^1$.

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Recall that a measurable transformation is a measurable mapping, having a measurable inverse, which preserves the class of the Lebesgue measure (cf. §2.1). Our main result is as follows.

RIGIDITY THEOREM. *Let $\mathcal{G}$ be an abstract non-solvable group generated by $\{g_1, \ldots, g_l\}$. Consider two faithful representations ρ_1, ρ_2 of $\mathcal{G}$ in* $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$. *Then there exists a neighborhood $\mathcal{U}$ of the identity in the real analytic topology (cf. §2.2) such that, if $\rho_1(g_i), \rho_2(g_i) \in \mathcal{U}$ for $i = 1, \ldots, l$, then the following are equivalent:*

(i) *$G_1 = \rho_1(\mathcal{G})$ and $G_2 = \rho_2(\mathcal{G})$ are C^{ω}-conjugate (i.e. $\mathfrak{f} \circ \rho_1(g_i) \circ \mathfrak{f}^{-1} = \rho_2(g_i)$, $i = 1, \ldots, l$, for some $\mathfrak{f} \in \mathrm{Diff}^{\omega}(\mathbb{S}^1)$);*

(ii) *$G_1 = \rho_1(\mathcal{G})$ and $G_2 = \rho_2(\mathcal{G})$ are topologically conjugate (i.e. $\mathfrak{h} \circ \rho_1(g_i) \circ \mathfrak{h}^{-1} = \rho_2(g_i)$, $i = 1, \ldots, l$, for some homeomorphism $\mathfrak{h} : \mathbb{S}^1 \to \mathbb{S}^1$);*

(iii) *$G_1 = \rho_1(\mathcal{G})$ and $G_2 = \rho_2(\mathcal{G})$ are measurably conjugate (i.e. $\varphi \circ \rho_1(g_i) \circ \varphi^{-1}(x) = \rho_2(g_i)(x)$, a.e. for $i = 1, \ldots, l$ and some measurable transformation φ).*

The equivalence between (i) and (ii) (called topological rigidity) was established in our work mentioned above so that the new contribution of the Rigidity theorem is the equivalence between (i) and (iii) (called measurable rigidity). Nevertheless, the part of the Rigidity theorem concerning measurable rigidity is much harder than the one related to topological rigidity. We also point out that, despite the fact that topological conjugacies are often singular with respect to the Lebesgue measure, the argument presented here can equally well be used to give an independent proof of topological rigidity. In any case our first proof is much simpler than this new one.

We also know that any topological conjugacy between G_1, G_2 is necessarily an analytic diffeomorphism (cf. [**Reb**]). It is then natural to ask whether the part of the Rigidity theorem concerning measurable conjugacies can similarly be improved. In other words, we want to decide whether a measurable transformation realizing a measurable conjugacy between groups G_1, G_2 as before must coincide almost everywhere with an analytic diffeomorphism. However counterexamples for the last claim can easily be produced by considering a group G_1 commuting with a suitable finite group of rotations (I thank the referee for pointing out this remark). Indeed it is enough to consider the lift to a finite cover of $\mathbb{S}^1$ of groups having suitable fixed points. Nonetheless, it turns out that this is the only source of counterexamples. This is the content of the following.

COMPLEMENT TO THE RIGIDITY THEOREM. *Suppose that groups G_1, G_2 as in the Rigidity theorem are measurably conjugate by a measurable transformation φ. Then there is a covering map $\pi : \mathbb{S}^1 \to \mathbb{S}^1$, of finite degree, such that G_1 (respectively G_2) is a lift with respect to π of another subgroup of* $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ *denoted by $\underline{G_1}$ (respectively $\underline{G_2}$). Furthermore, φ induces through π a measurable conjugacy $\underline{\varphi}$ between $\underline{G_1}, \underline{G_2}$. Finally $\underline{\varphi}$ coincides almost everywhere with an element of* $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$.

Remark. As we have said, for topological conjugacies, the improvement of the Rigidity theorem is immediate since the conjugating homeomorphism is in fact an analytic diffeomorphism. Note also that the covering map $\pi : \mathbb{S}^1 \to \mathbb{S}^1$ is generically the identity. Indeed for rather generic groups G_1, G_2 as before, any measurable conjugacy must coincide a.e. with an element of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$.

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It is interesting to compare the Rigidity theorem with the analogous question concerning Fuchsian groups. First of all topological rigidity does not hold in that case. Indeed, if $\pi_1(\Sigma)$ denotes the fundamental group of the oriented surface of genus greater than 1, then any two cocompact embeddings of $\pi_1(\Sigma)$ into $PSL(2,\mathbb{R})$ are topologically conjugate. However they are C^1-conjugate only if they define the same point in Teichmüller space. We observe that these homeomorphisms are singular with respect to the Lebesgue measure thanks to a result of Bowen and Series.

On the other hand, the result about measurable rigidity asserted by our theorem is in line with the corresponding problem for Fuchsian groups. If G_1 is a Fuchsian group acting ergodically on $\mathbb{S}^1 \times \mathbb{S}^1$, Kuusalo established a result similar to the Rigidity theorem under the additional assumption that the measurable transformation φ is a homeomorphism. More general results regarding measurable rigidity were obtained by Tukia under the assumption that G_1 acts ergodically on a convenient subset of $\mathbb{S}^1 \times \mathbb{S}^1$ (references for the various facts mentioned above can be obtained in [**Tuk**]).

At this point it seems important to stress two features of our proof of measurable rigidity. Firstly *the proof can be extended to show that our groups always act in an 'essentially' ergodic way on* $\mathbb{S}^1 \times \mathbb{S}^1$ ('essentially' means in particular, that the number of ergodic components is finite, cf. Theorem 3.5 and Proposition 4.1). Since this fact plays an important role for Fuchsian groups, one may expect it to have interesting applications in our setting as well. Actually a complete study of all these 'diagonal actions' will appear in [**R-S**]. Another fact that deserves to be noted is that our method of proving measurable rigidity may provide rigidity theorems even under hypotheses weaker than conjugation by a 'measurable transformation'. We have not investigated this possibility but we believe that it might provide results under hypotheses of regularity similar to those considered by Tukia in [**Tuk**].

Finally we want to say some words about our assumption concerning the non-solvability of our groups. This is of course a very weak assumption, but some assumption of this type is necessary since the same statement does not hold for $\mathbb{Z}$-actions. In fact, Arnold has constructed a real analytic diffeomorphism of $\mathbb{S}^1$ topologically conjugate to an irrational rotation but such that no conjugating homeomorphism is absolutely continuous; furthermore, this diffeomorphism can be made arbitrarily close to the identity (cf. [**KaHa**]). In a slightly different context, Furstenberg's example (see [**Fur**]) shows in particular that measurable equivalence also does not imply the existence of a topological conjugacy.

In what follows we shall outline the main ideas of this paper and describe its structure.

One of the most important analytic tools used is the notion of vector field in the 'closure of a pseudogroup'. In order to get an idea of the significance and applicability of this notion, let us suppose that $G_1 = \rho_1(\mathcal{G})$ and $G_2 = \rho_2(\mathcal{G})$ are contained in $PSL(2,\mathbb{R})$ acting on $\mathbb{S}^1$ viewed as the projective line. Note that this action induces a representation of $\mathfrak{psl}(2,\mathbb{R})$ (the Lie algebra of $PSL(2,\mathbb{R})$) in $\mathfrak{X}^{\omega}(\mathbb{S}^1)$ (the algebra of analytic vector fields on $\mathbb{S}^1$).

Next observe that G_1, G_2 are not discrete as subgroups of $PSL(2,\mathbb{R})$. In fact they are non-solvable groups (in particular, non-nilpotent) admitting a finite generating set close to the identity so that the assertion results from Zassenhaus's lemma. Now since G_1, G_2 are not discrete, their respective closures $\overline{G}_1, \overline{G}_2$ in $PSL(2,\mathbb{R})$ are Lie groups possessing

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non-trivial Lie algebrae. These Lie algebrae define, via the representation of $\mathfrak{psl}(2,\mathbb{R})$ in $\mathfrak{X}^{\omega}(\mathbb{S}^1)$ mentioned above, analytic vector fields on $\mathbb{S}^1$ which are said to be in *the closure of* G_1 (respectively G_2). Vector fields as above are the prototype of what we understand by vector fields in the closure of groups or pseudogroups. More precisely they possess the following property: let X^1 (respectively X^2) be a vector field in the closure of G_1 (respectively G_2), then for $t \in \mathbb{R}$ fixed, the diffeomorphism $\phi^t_{X^1}$ (respectively $\phi^t_{X^2}$) induced by the corresponding flow can be C^{∞} (or even analytically) approximated by elements in G_1 (respectively G_2). It is the possibility of approximating differentiably $\phi^t_{X^1}$ by elements of the corresponding group (respectively pseudogroup) that makes these vector fields useful for studying the dynamics of the group itself (respectively pseudogroup).

We continue to assume that G_1, G_2 are contained in $PSL(2,\mathbb{R})$ so that we have a non-trivial Lie algebra $\mathfrak{G}_1 \subset \mathfrak{psl}(2,\mathbb{R})$ (respectively $\mathfrak{G}_2 \subset \mathfrak{psl}(2,\mathbb{R})$) formed by vector fields in the closure of G_1 (respectively G_2). We have pointed out that these vector fields permit us to keep a very good control on the dynamics of the corresponding groups. Nonetheless it turns out that this control is still better if one considers the diagonal action of $\mathcal{G}$ on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ obtained by means of $\rho_1 \times \rho_2$ (by abuse of notation we refer to this action as the action of $G_1 \times G_2$ on $\mathbb{S}^1 \times \mathbb{S}^1$). The diagonal action is defined through the representation $\overline{\rho} : \mathcal{G} \to \mathrm{Diff}^{\omega}(\mathbb{S}^1 \times \mathbb{S}^1)$ by setting $\overline{\rho}(g)(x,y) = (\rho_1(g)(x), \rho_2(g)(y))$ ($g \in \mathcal{G}$). Just as before this new action possesses non-trivial vector fields in its closure (note that $G_1 \times G_2$ is contained in the Lie group $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ so that we can repeat the preceding arguments). Denote by $\mathfrak{G}_{G_1\times G_2}$ the algebra formed by vector fields in the closure of $G_1 \times G_2$. If $X \in \mathfrak{G}_{G_1\times G_2}$ then $X(x,y)$ can be written as $(X^1(x), X^2(y))$ with $X^1 \in \mathfrak{G}_1$, $X^2 \in \mathfrak{G}_2$. Furthermore $G_1 \times G_2$ acts naturally on $\mathfrak{G}_{G_1\times G_2}$, hence it is easy to show that $\mathfrak{G}_{G_1\times G_2}$ contains vector fields $X, Y, \ldots$ which are linearly independent at generic points.

We are now ready to explain how the proof of the Rigidity theorem runs and also to explain the difference between *measurable rigidity* and *topological rigidity*. First suppose that G_1, G_2 are topologically conjugate and denote by $\mathfrak{h}$ the conjugating homeomorphism. After the definition of the diagonal action of $G_1 \times G_2$ on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ it follows that the *graph of* $\mathfrak{h}$, Graph($\mathfrak{h}$), is invariant under $G_1 \times G_2$. Since $\mathfrak{h}$ is continuous it follows that Graph($\mathfrak{h}$) is also invariant under the vector fields in the closure of $G_1 \times G_2$. Hence Graph($\mathfrak{h}$) is contained in the union of the common trajectories of these vector fields which is a smooth analytic curve (since X, Y belong to this algebra of vector fields). The last claim will imply that $\mathfrak{h}$ belongs to $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$.

Suppose now that G_1, G_2 are (measurably) conjugate by the measurable transformation φ. Notice that all the arguments below should be understood 'modulo sets of measure zero'. The graph of φ, Graph(φ), is a measurable curve invariant under $G_1 \times G_2$. However we cannot ensure that Graph(φ) is invariant under the vector fields in $\mathfrak{G}_{G_1\times G_2}$ since φ might be highly discontinuous. We then refine the approach employed in the topological case by considering the set of points $p = (p^1, p^2) \in \mathbb{T}^2$ where there does not exist a vector field $Z = (Z^1, Z^2) \in \mathfrak{G}_{G_1\times G_2}$ satisfying $Z^1(p^1) = 0$ and $Z^2(p^2) \neq 0$ (i.e. Z is vertical at p). Roughly speaking we shall prove that these points form an analytic curve. Thus it is reasonable to expect that Graph(φ) be contained in this curve otherwise Graph(φ) could be displaced 'slightly upwards' which should be impossible (this idea is detailed in §4).

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The fact that $\mathrm{Graph}(\varphi)$ is contained in such a curve will imply the result on measurable rigidity.

Consider now general subgroups G_1, G_2 as in the statement of the Rigidity theorem. The first idea one has to use to prove the theorem for these groups is to imitate the constructions outlined before. However I have not been able to construct global vector fields in the closure of $G_1 \times G_2$. Nonetheless such vector fields do exist 'locally'. In other words, there are locally defined vector fields which behave as expected in relation to some pseudogroups. Naturally the pseudogroups in question are generated by suitable restrictions of elements in $G_1 \times G_2$ to certain open sets so that they reflect part of the dynamics associated to $G_1 \times G_2$. Using these local vector fields, we shall succeed in adapting the argument sketched above for general subgroups G_1, G_2.

In §2 we set up the notion of vector fields in the closure of pseudogroups. The section begins with §2.1 which contains the precise definition of measurable transformations. The reader will also find a simple lemma which will be used in §3. In §2.2 the analytic topology is introduced. It plays a role in the statement of the Rigidity theorem and in an important preliminary result due to Ghys. We then revisit Nakai's work [**Na1**] in §2.3 since some subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ contain pseudogroups as those considered in [**Na1**] (in fact it is enough to consider a subgroup of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ having a finite orbit). Here we also define vector fields in the closure of a pseudogroup. Finally in §2.4 another result (Proposition 2.7) about the existence of such vector fields for a suitable class of pseudogroups is presented. Pseudogroups in this class always appear in connection with subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ which do not contain (in a natural sense) pseudogroups verifying Nakai's assumptions (clearly only subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ satisfying the assumption of the Rigidity theorem are considered). However, a complete proof of Proposition 2.7 is deferred to Appendix A since it uses some ideas already appearing in [**Reb**].

§3 is devoted to 'globalizing' the vector fields constructed in §§2.3 and 2.4 to the whole of (or most of) $\mathbb{T}^2$. Notice that 'globalize' means only to (try to) obtain such a local vector field around every point $p \in \mathbb{T}^2$. In general one cannot patch these local vector fields together in order to construct a global vector field on $\mathbb{T}^2$. The fact that our vector fields are not global leads to further analytical difficulties.

In §4 we take for granted Proposition 4.1 which provides a lot of information on the structure of the points that admit neither a vertical nor a horizontal vector field around them. Using this proposition we settle the proof of the Rigidity theorem based on the argument of 'pushing up' the graph of the measurable transformation in question. Finally in §5 Proposition 4.1 is proved.

Main notation. First $\mathcal{U}$ always denotes an intrinsic neighborhood of the identity in the analytic topology. We denote by $\mathcal{G}$ an abstract group and by g an element of $\mathcal{G}$. We reserve the letter G for groups of diffeomorphisms (globally defined); pseudogroups will be denoted by Γ. The letters f, h stand for diffeomorphisms regardless of whether they are local or global (more generally they refer to 'mappings'). The corresponding capitals F, H also denote mappings, however these are mappings possessing two coordinates, i.e. $F = (f^1, f^2)$ (the coordinates are written with superscripts to distinguish them from a set of ordinary mappings $f_1, f_2, \dots$). We also add superscripts to 'mappings' f^1, f^2

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(respectively h^1, h^2) when they are associated to each other in some way (often this association is given by a conjugacy). Note that the context will make clear the distinction between two mappings with superscripts f^1, f^2 and iterates of f. In the case where we have to use both types of notation at the same time, we shall write $(f^i)^n$ (e.g. $(f^2)^4$ is the 4th iterate of the 'coordinate' f^2). Conjugating diffeomorphisms (respectively homeomorphisms) are denoted by $\mathfrak{f}$ (respectively $\mathfrak{h}$). Vector fields (local or global) on the circle, torus or open sets of $\mathbb{C}^n$ in general are denoted by capitals X, Y, Z. If it is necessary to specify their coordinates we shall write $X = (X^1, X^2)$. Ordinary points are usually denoted by x, y, however when we want to 'distinguish' or 'fix' a point it may be denoted by p, q (in fact x, y are usually 'variables' while p, q are given points). The Greek letters ϕ and φ stand respectively for a flow of a vector field and a measurable transformation. Finally ρ and $\overline{\rho}$ always denote representations of a group $\mathcal{G}$.

2. *Preliminaries*

In this section we collect some background material which will be useful in other parts of this work.

2.1. *Isomorphisms of measure.*

In this section we recall the notion of *isomorphism of measure* (sometimes also called a *measurable isomorphism*) which plays an important role in several aspects of ergodic theory. We consider $\mathbb{S}^1$ endowed with the normalized Lebesgue measure μ and with the σ-algebra of Borelian sets. Most of the concepts discussed here must be understood 'modulo sets of measure *zero*'.

By a measurable mapping $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ we mean a measurable mapping whose domain is a *full measure subset of* $\mathbb{S}^1$ (and not necessarily the whole $\mathbb{S}^1$). Such a mapping will be called a *measurable transformation* if it satisfies the further conditions:

- there is another measurable mapping $\varphi^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\varphi \circ \varphi^{-1}(x) = \varphi^{-1} \circ \varphi(x) = x$ for μ-a.e. $x \in \mathbb{S}^1$;
- φ preserves the *class of* μ, i.e. μ and $\varphi^*\mu$ have the same null sets.

We now consider an abstract group $\mathcal{G}$ and two faithful representations ρ_1, ρ_2 of $\mathcal{G}$ in $\mathrm{Diff}^\omega(\mathbb{S}^1)$. We put $G_1 = \rho_1(\mathcal{G})$ and $G_2 = \rho_2(\mathcal{G})$. A measurable transformation $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ is a *isomorphism of measure* (or a *measurable conjugacy* or a *measurable isomorphism*) between G_1 and G_2 if one has $\varphi \circ \rho_1(g)(x) = \rho_2(g) \circ \varphi(x)$ for any $g \in \mathcal{G}$ and μ-a.e. $x \in \mathbb{S}^1$. The groups G_1, G_2 are said to be *measurably conjugate* or *measurably equivalent*.

Since a measurable isomorphism takes into account only the features that may be described in terms of 'measure theory', it becomes a natural notion of equivalence between dynamical systems from the ergodic point of view. Indeed dynamical properties which are linked to the measurable structure of the dynamics are naturally invariant under this relation of equivalence. The most simple example of these properties is the ergodicity, however there are a good number of other important (and more subtle) examples (see [**CSV**]). We finish these comments by giving a simple lemma that will be needed later.

LEMMA 2.1. *Let f^1 and f^2 be two real analytic diffeomorphisms of $\mathbb{S}^1$ such that $\varphi \circ f^1 \circ \varphi^{-1}(x) = f^2(x)$ for μ-a.e. $x \in \mathbb{S}^1$ and a suitable measurable transformation*

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$\varphi : \mathbb{S}^1 \to \mathbb{S}^1$. *Assume that* $f^1 \neq \mathrm{id}$ *has a fixed point. Then, for an appropriate integer* $k \in \mathbb{N} \setminus \{0\}$, $(f^2)^k$ *is different from the identity and has a fixed point (where* $(f^2)^k$ *stands for the* k*th iterate of* f^2*).*

Proof. Since f^1 has a fixed point and $f^1 \neq \mathrm{id}$, $(f^1)^k$ is different from the identity for any $k \in \mathbb{N} \setminus \{0\}$. We claim that, for every $k \in \mathbb{N} \setminus \{0\}$, $(f^2)^k$ is different from the identity as well. In fact, if $(f^2)^k$ were the identity for some $k \in \mathbb{N} \setminus \{0\}$, then $(f^1)^k$ would coincide with the identity in a set of total measure. Thus $(f^1)^k$ would be the identity since it is analytic.

We now consider $\mathrm{rot}(f^2)$, the rotation number of f^2. It is enough to prove that $\mathrm{rot}(f^2) \in \mathbb{Q}$. Assume by contradiction that one has $\mathrm{rot}(f^2) \in \mathbb{R} \setminus \mathbb{Q}$.

Let us choose an open interval I ($I \neq \emptyset$) strictly contained in a fundamental domain $[p, f^1(p)]$ for some $p \in \mathbb{S}^1$. Because f^1 has fixed points, the orbit of the interval I is a set of measure strictly comprised between 0 and 1. So we conclude that the orbit by f^2 of the set $\varphi(I)$ has positive measure strictly smaller than 1. However a diffeomorphism with irrational rotation number acts ergodically on $\mathbb{S}^1$ (cf. [**KaHa**]). Therefore f^2 cannot have an irrational rotation number which gives us the desired contradiction. The lemma is proved. □

2.2. *The analytic topology.* We want to equip $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ with a natural topology called the real analytic topology (also referred to as the analytic topology or the C^{ω}-topology) which has some remarkable features. This topology will be *intrinsic* so that the statement of the Rigidity theorem makes sense (i.e. one can consider neighborhoods of the identity from an intrinsic point of view without specifying any particular embedding of $\mathbb{S}^1$ in $\mathbb{R}^n$ or $\mathbb{C}^n$). However, in practice, the main use we shall make of this topology is summarized by a theorem of Ghys (Theorem 2.2) which can be stated without explicit reference to the analytic topology. This form of stating Ghys's result has the advantage of enabling computations and allowing explicit constants. Moreover, it may also appear 'more concrete' to the reader.

Thus we begin by introducing Theorem 2.2 and then we shall define the analytic topology in $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$.

Assume that G is a group contained in $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ and generated by the finite set $S = \{f_1, \ldots, f_l\}$ (notation: $G = \langle f_1, \ldots, f_l \rangle$ or $G = \langle S \rangle$). Following [**Gh**], let us associate to S a sequence of sets $S(\mathfrak{m}) \subset G$ ($\mathfrak{m} = 0, 1, \ldots$) by the inductive rule below:

(i) $S(0) = S$,

(ii) $S(\mathfrak{m}+1)$ is the set whose elements are written under the form $[g^{\pm 1}, h^{\pm 1}]$ where $g \in S(\mathfrak{m})$ and $h \in S(\mathfrak{m}) \cup S(\mathfrak{m}-1)$ ($h \in S(0)$ if $\mathfrak{m} = 0$).

Consider $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$. For a given τ verifying $0 < \tau < 1/4$, let $A_{2\tau}$ denote the annulus defined by $A_{2\tau} = \{x \in \mathbb{C} : 1 - 2\tau < \|x\| < 1 + 2\tau\}$. Now observe that any f belonging to $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ admits a holomorphic extension, denoted $\tilde{f}$, to some neighborhood of $\mathbb{S}^1$ in $\mathbb{C}$. Suppose we are given $\varepsilon > 0$ and $0 < \tau < 1/4$ (as above), we then define the set $U^{\varepsilon}_{2\tau} \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ by saying that $f \in U^{\varepsilon}_{2\tau}$ if and only if:

(1) f admits a holomorphic extension $\tilde{f}$ to $A_{2\tau}$;

(2) $\sup_{x \in A_{2\tau}} \|\tilde{f}(x) - x\| < \varepsilon$.

The following theorem due to Ghys will be needed in §3.

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THEOREM 2.2. [**Gh**] *Let us regard* $\mathbb{S}^1$ *as the unit circle contained in* $\mathbb{C}$. *For any given* τ *with* $0 < \tau < 1/4$, *there exists*† $\varepsilon > 0$ *such that, as long as the set* $S = \{f_1, \ldots, f_l\}$ *is contained in* $U^{\varepsilon}_{2\tau}$ *and* $G = \langle S \rangle$ *is not solvable, the following holds:*

(1) $S(\mathfrak{m})$ *does not degenerate into* $\{\mathrm{id}\}$ *for any* $\mathfrak{m} \geq 0$;

(2) *any* $h_{\mathfrak{m}} \in S(\mathfrak{m})$ *has a holomorphic extension to the annulus* $A_\tau = \{x \in \mathbb{C} : 1 - \tau < |x| < 1 + \tau\}$;

(3) *if* $h_{\mathfrak{m}} \in S(\mathfrak{m})$ *then* $\sup_{x \in A_\tau} \|\tilde{h}_{\mathfrak{m}}(x) - x\| < 2^{-\mathfrak{m}}\varepsilon$ *(where* $\tilde{h}_{\mathfrak{m}}$ *stands for the holomorphic extension of* $h_{\mathfrak{m}}$*).*

Let us now introduce the analytic topology (or C^ω-topology). As we have observed, for the rest of this paper we only need Theorem 2.2 so that the role of the analytic topology is just to provide a coherent and intrinsic set up to our work.

Fix hereafter a real analytic embedding of $\mathbb{S}^1$ in some Euclidean space $\mathbb{R}^N$. We identify $\mathbb{S}^1$ with the image of the fixed embedding so that we can say that $\mathbb{S}^1$ is contained in $\mathbb{R}^N$. An open Riemann surface $\widetilde{\mathbb{S}^1} \subset \mathbb{C}^N$ containing $\mathbb{S}^1 \subset \mathbb{R}^N$ is called a *complexification* of $\mathbb{S}^1$ if any real analytic function defined on $\mathbb{S}^1$ possesses a unique holomorphic extension to some neighborhood of $\mathbb{S}^1$ in $\widetilde{\mathbb{S}^1}$. Two complexifications of $\mathbb{S}^1$ (for the fixed analytic embedding) coincide in a neighborhood of $\mathbb{S}^1$ and we choose one of them.

For $\tau > 0$, we denote $\widetilde{\mathbb{S}^1_\tau}$ as the set of points in $\widetilde{\mathbb{S}^1}$ whose distance (the Euclidean distance of $\mathbb{C}^N$) to $\mathbb{S}^1$ is smaller than τ. Let u be a real analytic function from $\mathbb{S}^1$ to an Euclidean space $\mathbb{R}^n$. Assume that there is a holomorphic extension $\tilde{u}$ of u to $\widetilde{\mathbb{S}^1_\tau}$ (taking values in $\mathbb{C}^n$). Under this assumption, the symbol $\|u\|_\tau$ will denote the supremum of the norm $\|\tilde{u}(x)\|$ where x belongs to $\widetilde{\mathbb{S}^1_\tau}$. If there is no holomorphic extension of u to $\widetilde{\mathbb{S}^1_\tau}$, we put $\|u\|_\tau = \infty$.

Let $\tau > 0$ and $\varepsilon > 0$ be given numbers. The set defined by the diffeomorphisms $f : \mathbb{S}^1 \to \mathbb{S}^1 \subset \mathbb{R}^N$ which satisfy $\|f - \mathrm{id}\|_\tau < \varepsilon$ is going to be denoted by $\mathcal{U}^{\varepsilon}_{\tau}$. We are now ready to introduce a topology on $\mathrm{Diff}^\omega(\mathbb{S}^1)$ simply by declaring that:

(i) the sets $\mathcal{U}^{\varepsilon}_{\tau}$, $\tau > 0$ and $\varepsilon > 0$, define a base of neighborhoods for the identity;

(ii) given $f \in \mathrm{Diff}^\omega(\mathbb{S}^1)$, a base of neighborhoods for f is obtained by translation of the one for the identity.

The topology introduced just above is the so-called *analytic topology* (or C^ω*-topology*). Despite its 'extrinsic' definition (in which one has made use of an embedding), this topology is *intrinsic* since it does not depend on the embedding chosen at the beginning. When endowed with the C^ω-topology, $\mathrm{Diff}^\omega(\mathbb{S}^1)$ supports a unique structure of topological group. However there is no compatible structure of Lie groups on $\mathrm{Diff}^\omega(\mathbb{S}^1)$.

Finally let us point out that, strictly speaking, the embedding of $\mathbb{S}^1$ in $\mathbb{C}$ as the unit circle does not provide a complexification of $\mathbb{S}^1$. In fact, we should begin with an embedding of $\mathbb{S}^1$ in some $\mathbb{R}^N$ (and obviously the unit circle of $\mathbb{C}$ is not contained in $\mathbb{R} \subset \mathbb{C}$). To obtain a complexification of $\mathbb{S}^1$ through this embedding we have to consider $\mathbb{C}$ as $\mathbb{R}^2$ (so that one has an embedding of $\mathbb{S}^1$ in $\mathbb{R}^2$). The resulting complexification is therefore a Riemann surface contained in $\mathbb{C}^2$. Nevertheless the reader can easily check that the set $U^{\varepsilon}_{2\tau}$ used in the statement of Theorem 2.2 does contain an open neighborhood $\mathcal{U}$ of the identity for the C^ω-topology. Hence Theorem 2.2 can be rephrased as follows: there is a neighborhood of

† ε can be explicitly estimated in terms of τ.

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the identity in the C^{ω}-topology such that, if a generating set S of a non-solvable group G is contained in $\mathcal{U}$, then conclusions (1), (2) and (3) of the theorem in question do hold.

2.3. *Germs of holomorphic diffeomorphims in* $(\mathbb{C}, 0)$. In this section we shall recall a theorem due to Nakai ([**Na1**], see also [**Sch**]) concerning the local dynamics of non-solvable subgroups of $\mathrm{Diff}(\mathbb{C}, 0)$ (where $\mathrm{Diff}(\mathbb{C}, 0)$ stands for the group of germs of holomorphic diffeomorphisms fixing $0 \in \mathbb{C}$). In fact, for our purposes, it is sufficient to deal with the germs of holomorphic diffeomorphisms having real coefficients.

However, instead of working with germs of diffeomorphisms (with real coefficients), we prefer to consider *the pseudogroup* generated by a collection $\{f_1, \ldots, f_l\}$ of local holomorphic diffeomorphisms fixing $0 \in \mathbb{C}$. Let $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$ be the group formed by the *germs* of holomorphic diffeomorphisms with real coefficients. We say that the pseudogroup generated by $\{f_1, \ldots, f_l\}$ is non-solvable if the subgroup of $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$ generated by the germs of the f_i's is non-solvable.

Assume that the pseudogroup generated by $\{f_1, \ldots, f_l\}$ is not solvable and preserves the real line. Then there are local diffeomorphisms (fixing $0 \in \mathbb{C}$) $\mathbf{f}_1, \mathbf{f}_2$ in this pseudogroup, for which the respective Taylor's expansions centered at $0 \in \mathbb{C}$ are written

$$\begin{aligned} &\mathbf{f}_1(x) = x + c_1 x^{r_1+1} + \cdots; \\ &\mathbf{f}_2(x) = x + c_2 x^{r_2+1} + \cdots (c_1 c_2 \neq 0 \text{ and } 0 < r_1 < r_2). \end{aligned} \tag{1}$$

Furthermore, $\mathbf{f}_1, \mathbf{f}_2$ do not generate a solvable group (which in this case is equivalent to saying that $\mathbf{f}_1, \mathbf{f}_2$ do not commute). Therefore it is sufficient to deal with non-solvable pseudogroups Γ generated by elements $\mathbf{f}_1, \mathbf{f}_2$ as above. Since no misunderstanding can arise, in what follows we simply write f_1, f_2 for denoting $\mathbf{f}_1, \mathbf{f}_2$.

Given a local holomorphic diffeomorphism f satisfying $f(0) = 0, f'(0) = 1$ and $f \neq \mathrm{id}$, one denotes by $\mathrm{Bas}(f)$ the *basin* of f, namely the set of points $x \in \mathbb{C}$ for which $\{f^k(x)\}$ (is defined for all $k \in \mathbb{N}$ and) converges to $0 \in \mathbb{C}$ when k goes to infinity. It is a well known fact that $\mathrm{Bas}(f) \cup \mathrm{Bas}(f^{-1})$ always contains a neighborhood of $0 \in \mathbb{C}$.

The following theorem is due to Nakai and it is going to be used in §§3 and 5.

THEOREM 2.3. [**Na1**] *Assume that f_1, f_2 are as in (1) (in particular, they generate a non-solvable pseudogroup Γ). There exists a real analytic vector field X (without singularities) defined on* $\mathrm{Bas}(f_1) \setminus \{0\}$ *which has the following properties.*

(1) *$X(x)$ is defined as the uniform limit of the sequence of vector fields $\{X_k(x)\}$ given by*

$$X_k(x) = n_k D_{f_1^k(x)} f_1^{-1}.(f_2 \circ f_1^k(x) - f_1^k(x)),$$

for an appropriate sequence $\{n_k\} \in \mathbb{N}$ (here X_k is the vector field obtained by applying the linear transformation $n_k D_{f_1^k(x)} f_1^{-1}$ to the vector whose extremities are $f_1^k(x)$ as origin and $f_2 \circ f_1^k(x)$ as endpoint).

(2) *X induces a local flow such that, for t fixed, the corresponding mapping ϕ_X^t is holomorphic.*

(3) *Let V be a relatively compact open subset of* $\mathrm{Bas}(f_1) \setminus \{0\}$ *and $t_0 > 0$ so small that ϕ_X^t is defined on V whenever $0 \leq t \leq t_0$. Then there is a sequence of mappings $\{h_m\} \subset \Gamma$ converging to $\phi_X^{t_0}$ uniformly on V (and in fact each h_m is defined on V).*

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(4) *The sequence* $\{h_m\}$ *mentioned above is explicitly given by* $h_m = f_1^{-m} \circ f_2^{v(m)} \circ f_1^m$ *(for* m *large enough) and a suitable sequence* $v(m) \subset \mathbb{N}$. *Indeed it suffices to take* $v(m)$ *so that* $v(m).m^{(r_1-r_2)/r_1} \to t_0$ *when* m *goes to infinity.*

Vector fields as above will be referred to as Nakai's vector fields. None of Nakai's vector fields is defined in a neighborhood of $0 \in \mathbb{C}$, nevertheless the union of the domains of all these vector fields (for Γ fixed) contains a punctured neighborhood of $0 \in \mathbb{C}$. The last claim can easily be checked after Theorem 2.3. In fact it is sufficient to replace f_1 by f_1^{-1} and to recall that $\mathrm{Bas}(f_1) \cup \mathrm{Bas}(f_1^{-1})$ contains such a neighborhood.

The property (4) of Nakai's vector fields is one of the motivations for Definition 2.4.

Definition 2.4. Let U be an open set of $\mathbb{C}^n$ and Γ a pseudogroup of holomorphic mappings generated by a collection of holomorphic mappings from U to $\mathbb{C}^n$ (i.e. the generators of Γ are defined on the whole U). Suppose that U is equipped with a real analytic vector field X and denote by ϕ_X^t the local flow of X. One says that X is in the closure of Γ relative to U if for any relatively compact set $V \subset U$ and $t_0 > 0$, such that $\phi_X^t(V)$ is defined whenever $0 \leq t \leq t_0$, the induced mapping $\phi_X^{t_0} : V \to \mathbb{C}^n$ fulfils the following two conditions:

- $\phi_X^{t_0} : V \to \mathbb{C}^n$ is holomorphic;
- there is a sequence of mappings $\{h_m\} \subset \Gamma$ (h_m defined on V for every m) converging to $\phi_X^{t_0}$ uniformly on V.

According to Definition 2.4, the Nakai's vector field X defined on $\mathrm{Bas}(f_1) \setminus \{0\}$ is in the closure of Γ relative to $\mathrm{Bas}(f_1) \setminus \{0\}$. When there is no possibility of misunderstanding about the domain U considered, we simply say that X is in the closure of Γ.

Before finishing this paragraph, we have to introduce the n-fold version of Nakai's theorem. For the sake of notation, we shall state only the two-fold version. Actually the generalization to an arbitrary $n \in \mathbb{N}$ is obvious.

Let us consider four local holomorphic diffeomorphisms (preserving the real line) f_i^j $(i, j = 1, 2)$ fixing $0 \in \mathbb{C}$. In terms of Taylor series we set:

$$f_1^1(x) = x + c_1^1 x^{r_1+1} + \cdots ; \quad f_2^1(x) = x + c_2^1 x^{r_2+1} + \cdots ; \tag{2}$$

$$f_1^2(y) = y + c_1^2 y^{s_1+1} + \cdots ; \quad f_2^2(y) = y + c_2^2 y^{s_2+1} + \cdots , \tag{3}$$

where x, y are complex variables, $c_1^1 c_2^1 c_1^2 c_2^2 \neq 0$, $0 < r_1 < r_2$ and $0 < s_1 < s_2$ (we prefer to write s_1, s_2 instead of r_1^2, r_2^2 to reduce the number of indices).

Using the coordinates $(x, y) \in \mathbb{C} \times \mathbb{C} = \mathbb{C}^2$, the local diffeomorphisms (f_1^1, f_1^2) and (f_2^1, f_2^2) defined around $(0, 0) \in \mathbb{C}^2$ generate a pseudogroup of holomorphic mappings between open sets of $\mathbb{C}^2$. This pseudogroup will be denoted by $\Gamma_{\Gamma_1 \times \Gamma_2}$. Notice that the projection of $\Gamma_{\Gamma_1 \times \Gamma_2}$ on the first (respectively second) coordinate defines a pseudogroup Γ_1 (respectively Γ_2) of holomorphic mappings between open sets of $\mathbb{C}$. Moreover, Γ_1, Γ_2 are generated respectively by f_1^1, f_2^1 and f_1^2, f_2^2. In particular, they preserve the real line.

We are finally ready to prove the existence of 'Nakai's vector fields' associated to $\Gamma_{\Gamma_1 \times \Gamma_2}$. Of course the result asserting the existence of such vector fields corresponds to the so-called two-fold version of Nakai's theorem. Actually this result was not stated in his work [**Na1**] but it can immediately be deduced from the discussion carried out there complemented by a fairly simple remark.

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THEOREM 2.5. *Let* f_1^1, f_2^1 *and* f_1^2, f_2^2 *be as above. There exists a nowhere singular real analytic vector field* X *defined on* $(\mathrm{Bas}(f_1^1) \setminus \{0\}) \times (\mathrm{Bas}(f_1^2) \setminus \{0\})$ *having the following features.*

(1) $X(x, y) = (X^1(x), X^2(y))$ *where* X^1 *(respectively* X^2*) is a real analytic vector field on* $\mathrm{Bas}(f_1^1) \setminus \{0\}$ *(respectively* $\mathrm{Bas}(f_1^2) \setminus \{0\}$*).*

(2) X^1 *(respectively* X^2*) is in the closure of* Γ_1 *(respectively* Γ_2*) relative to* $\mathrm{Bas}(f_1^1) \setminus \{0\}$ *(respectively* $\mathrm{Bas}(f_1^2) \setminus \{0\}$*).*

(3) $X = (X^1, X^2)$ *is in the closure of* $\Gamma_{\Gamma_1 \times \Gamma_2}$ *relative to* $(\mathrm{Bas}(f_1^1) \setminus \{0\}) \times (\mathrm{Bas}(f_1^2) \setminus \{0\})$.

(4) *Assume that* ϕ_X^t *denotes the local flow of* X*. Let* V *be a relatively compact subset of* $(\mathrm{Bas}(f_1^1) \setminus \{0\}) \times (\mathrm{Bas}(f_1^2) \setminus \{0\})$ *and* $t_0 > 0$ *be such that* $\phi_X^t(V)$ *is defined for all* $0 \leq t \leq t_0$*. Then the restriction of* ϕ_X^t *to* V *is the uniform limit (on* V*) of a sequence* $H_m = (h_m^1, h_m^2)$ *where* $h_m^j = (f_1^j)^{-m} \circ (f_2^j)^{v(m)} \circ (f_1^j)^m$ *(*$j = 1, 2$*) for an appropriate sequence of integers* $v(m)$.

Proof. Let us consider the Nakai's vector field X^1 (respectively X^2) on $\mathrm{Bas}(f_1^1)$ (respectively $\mathrm{Bas}(f_1^2)$) associated to f_1^1, f_2^1 (respectively f_1^2, f_2^2). Clearly one can suppose that $(s_2 - s_1)/s_1 \geq (r_2 - r_1)/r_1$. Hence the proof of the theorem is reduced to the claim below.

CLAIM. *If* $(s_2 - s_1)/s_1 > (r_2 - r_1)/r_1$ *then the vector field* $(X^1, 0)$ *fulfils the required conditions. On the other hand, if* $(s_2 - s_1)/s_1 = (r_2 - r_1)/r_1$ *then it is enough to take* $X = (X^1, X^2)$.

Proof of the claim. Let $V \subseteq \mathrm{Bas}(f_1^1) \setminus \{0\}$ and $t_0 > 0$ be such that the local flow of X^1, $\phi_{X^1}^t$, is defined on V whenever $0 \leq t \leq t_0$. We choose a sequence of positive integers $v(m)$ such that $v(m).m^{(r_1 - r_2)/r_1} \to t_0$. According to Nakai's theorem, the sequence of mappings $\{(f_1^1)^{-m} \circ (f_2^1)^{v(m)} \circ (f_1^1)^m\}$ converges uniformly on V towards $\phi_{X^1}^{t_0}$. On the other hand, $\{v(m).m^{(s_1 - s_2)/s_1}\}$ converges either to zero (if $(s_2 - s_1)/s_1 > (r_2 - r_1)/r_1$) or to t_0 (if $(s_2 - s_1)/s_1 = (r_2 - r_1)/r_1$). Employing Nakai's argument once again, the sequence $\{(f_1^2)^{-m} \circ (f_2^2)^{v(m)} \circ (f_1^2)^m\}$ must converge uniformly on compact sets of $\mathrm{Bas}(f_1^2) \setminus \{0\}$ either to the identity or to $\phi_{X^2}^{t_0}$ (the local flow of X^2). The proof of the claim is over. □

Therefore so is the proof of the theorem. □

Remark 2.6. Recall that our diffeomorphisms preserve the real line. Suppose that for $x, y \in \mathbb{R}$, $x, y > 0$ small, $(f_1^1)^k(x)$ and $(f_1^2)^k(y)$ converge to zero when k increases. In this case it is easy to see that $(\mathrm{Bas}(f_1^1) \setminus \{0\}) \times (\mathrm{Bas}(f_1^2) \setminus \{0\})$ contains a rectangle of the form $\{(x, y) \in \mathbb{R}^2 : 0 < x < \epsilon \text{ and } 0 < y < \epsilon\}$ (for some small $\epsilon > 0$). Thus such a rectangle can be endowed with a Nakai's vector field X. This remark will often be used in §5.

2.4. *Vector fields in the closure of pseudogroups.* The purpose of this paragraph is to state a generalized version of the basic result concerning the existence of vector fields 'in the closure of certain pseudogroups' established in [**Reb**]. The desired generalization will be obtained by Proposition 2.7 which is a type of 'two-fold version' of the corresponding proposition in [**Reb**] (just as Theorem 2.5 is the two-fold version of Nakai's theorem).

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Nonetheless, in the proof of this new result we make use of a number of ideas which already appear in my previous work. Therefore, I have decided to include the corresponding proofs only in Appendix A.

We call B the unit ball of $\mathbb{C}$ and we write B^2 to denote the product $B \times B \subset \mathbb{C}^2$. In the following we will be concerned with a very special type of pseudogroup of holomorphic mappings between open subsets of $\mathbb{C}^2$. In fact these pseudogroups split into coordinates $\mathbb{C} \times \mathbb{C}$. Since the result we shall discuss is similar to Theorem 2.5, we keep some notation of the preceding paragraph, namely we shall denote by Γ_1, Γ_2 and $\Gamma_{\Gamma_1 \times \Gamma_2}$ some pseudogroups to be introduced below.

Denote by $\Gamma_{\Gamma_1 \times \Gamma_2}$ a pseudogroup of holomorphic mappings between open sets of $\mathbb{C}^2$ which verifies the following conditions.

(1) Each $H \in \Gamma_{\Gamma_1 \times \Gamma_2}$ has the form $H(x, y) = (h^1(x), h^2(y))$ where h^j $(j = 1, 2)$ is a holomorphic mapping from an open set $V_j \subset \mathbb{C}$ to $\mathbb{C}$. Furthermore, each h^j preserves the real line (i.e. $h^j(V_j \cap \mathbb{R}) \subset \mathbb{R}$).

(2) There exists $F = (f^1, f^2)$ in $\Gamma_{\Gamma_1 \times \Gamma_2}$ fulfilling the following hypotheses:
   (a) F is defined in the entire B^2;
   (b) $F(0, 0) = (0, 0)$, $f^j \neq \mathrm{id}$ and $f^j \circ f^j \neq \mathrm{id}$ $(j = 1, 2)$;
   (c) $\{(f^1)^k(x)\}$ (respectively $\{(f^2)^k(y)\}$) converge to $0 \in \mathbb{C}$ when k increases to infinity as long as $x \in \mathbb{R}$ (respectively $y \in \mathbb{R}$) is positive and sufficiently near to $0 \in \mathbb{C}$.

(3) There exists a sequence† $\mathcal{H} = \{H_d\}_{d \in \mathbb{N}}$ of elements in $\Gamma_{\Gamma_1 \times \Gamma_2}$ ($H_d \neq \mathrm{id}$ for all j) such that each H_d is defined in the whole B^2. Moreover, the sequence $\mathcal{H}$ converges uniformly to the identity on B^2.

(4) For any $H_d = (h_d^1, h_d^2) \in \mathcal{H}$, one has $h_d^1(0) > 0$ or $h_d^2(0) > 0$.

As in §2.3, the projection of $\Gamma_{\Gamma_1 \times \Gamma_2}$ on the first (respectively second) coordinate naturally defines a pseudogroup Γ_1 (respectively Γ_2) of holomorphic mappings between open sets of $\mathbb{C}$.

Suppose we are given two positive real numbers $\varepsilon_1, \varepsilon_2$ and a point $p \in \mathbb{R}$. Let us denote by $R_{\varepsilon_{12}, p} \subset \mathbb{C}$ the rectangle whose vertices are $(p - \varepsilon_1, \varepsilon_2)$, $(p + \varepsilon_1, \varepsilon_2)$, $(p + \varepsilon_1, -\varepsilon_2)$, $(p - \varepsilon_1, -\varepsilon_2)$. Let R_p^2 denote the product $R_{\varepsilon_{12}, p} \times R_{\varepsilon_{12}, p} \subset \mathbb{C}^2$ (we are going to omit the dependence of R_p^2 on $\varepsilon_1, \varepsilon_2$ since no misunderstanding is possible). We are finally ready to give the statement of Proposition 2.7.

PROPOSITION 2.7. *Let $\Gamma_{\Gamma_1 \times \Gamma_2}$ be a pseudogroup satisfying all the above conditions. Then there are positive real numbers $\varepsilon_1, \varepsilon_2$ and a point $p \in \mathbb{R}$ (in fact $p > 0$) such that R_p^2 is endowed with a nowhere zero real analytic vector field $X(x, y) = (X^1(x), X^2(y))$ which is in the closure of $\Gamma_{\Gamma_1 \times \Gamma_2}$ relative to R_p^2. Moreover, X^1 (respectively X^2) is in the closure of Γ_1 (respectively Γ_2) relative to $R_{\varepsilon_{12}, p} \subset \mathbb{C}$.*

3. *The diagonal action*

Let us begin by fixing some notation that are going to be used throughout the section. Firstly $\mathcal{G}$ denotes an abstract non-solvable group admitting a finite generating set

† The proposition in question will be used in the proof of Theorem 3.2. In that theorem the sequence $\{H_d\}$ appears as restrictions of mappings associated to the sets $S(\mathfrak{m})$ of §2.2.

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$\{g_1, \ldots, g_l\}$. Assume that we are given two faithful representations ρ_1, ρ_2 of $\mathcal{G}$ in $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$. In the following $G_1 \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ (respectively $G_2 \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$) is the image of $\mathcal{G}$ by ρ_1 (respectively ρ_2) endowed with the naturally induced group law. Finally one sets $f_i^1 = \rho_1(g_i)$ and $f_i^2 = \rho_2(g_i)$ ($i \in \{1, \ldots, l\}$).

Under the assumptions above, the group $\mathcal{G}$ has a natural action on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ which is called the diagonal action. More precisely, one obtains a faithful representation $\overline{\rho} : \mathcal{G} \to \mathrm{Diff}^{\omega}(\mathbb{T}^2)$ by

$$\begin{aligned} \overline{\rho}(g) : \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 &\longrightarrow \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \\ (x, y) &\longmapsto (\rho_1(g)(x), \rho_2(g)(y)). \end{aligned}$$

The homomorphism $\overline{\rho}$ is called the diagonal representation of ρ_1 and ρ_2 and the corresponding action of $\mathcal{G}$ on $\mathbb{T}^2$ is the *diagonal action*. This action, sometimes also referred to as the two-fold diagonal action, is the object of this section.

Denote by $\mathbb{G} \subset \mathrm{Diff}^{\omega}(\mathbb{T}^2)$ the image of $\overline{\rho}$. Given $p = (p^1, p^2) \in \mathbb{T}^2$, $\mathrm{Stab}_{\mathbb{G}}(p)$ stands for the *stabilizer* of p in $\mathbb{G}$, namely it is the subgroup consisting of those elements in $\mathbb{G}$ leaving p fixed. In what follows, we shall be concerned with vector fields defined on open sets of $\mathbb{T}^2$ having the form $X(x, y) = (X^1(x), X^2(y))$. These vector fields are associated to certain pseudogroups generated by restrictions of some elements in $\mathbb{G}$ to suitable open sets. Recall that G_1 (respectively G_2) acts on $\mathbb{S}^1$ which can be viewed as the unit circle in $\mathbb{C}$. In this setting, the vector field X^1 (respectively X^2) is in fact defined on an open set of $\mathbb{C}$ which intersects $\mathbb{S}^1$. It turns out that X^1 (respectively X^2) is in the closure of a certain pseudogroup generated by restrictions of some elements in G_1 (respectively G_2) to suitable open sets of $\mathbb{C}$ (in fact the elements in G_1 (respectively G_2) in question possess a holomorphic extension to a uniform open set of $\mathbb{C}$).

On the other hand, the sketch of the proof of the Rigidity theorem given at the introduction shows that only the orbits of X^1 (respectively X^2) contained in $\mathbb{S}^1$ are relevant to our discussion (since we are interested only in the dynamics of X on open sets of $\mathbb{T}^2$). Therefore we shall adapt Definition 2.4 to our context. Since X^1, X^2 will be considered as vector fields defined only on open intervals of $\mathbb{S}^1$ (i.e. their holomorphic extensions to open sets of $\mathbb{C}$ will not enter into the discussion), it is natural to work with the C^{∞}-topology.

Definition 3.1. Let $\mathbb{G}$ be a group of C^{ω} (or C^{∞}) diffeomorphisms acting on $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n times). Suppose that $U \subseteq \mathbb{T}^n$ is an open set endowed with a C^{ω} (or just $\mathbb{C}^{\infty}$) vector field X whose corresponding local flow is denoted ϕ_X^t. We say that X is in the C^{∞}-closure of $\mathbb{G}$ relative to U if and only if it has the following property: for every relatively compact open subset V of U and $t_0 \in \mathbb{R}$, $t_0 > 0$, so small that $\phi_X^t(V)$ is defined whenever $0 \leq t \leq t_0$, there is a sequence of diffeomorphisms $\{H_m\} \in \mathbb{G}$ such that the restriction of them to V converges C^{∞} to $\phi_X^{t_0} : V \to \phi_X^{t_0}(V)$.

In the rest of this section we are going to study the (two-fold) diagonal action generated by two non-solvable groups G_1, G_2 which are measurable equivalent and have a generating set close to the identity. Indeed we intend to proving Theorem 3.2.

THEOREM 3.2. *Let $\mathcal{U}$ be the neighborhood of the identity in* $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ *furnished by Theorem 2.2 (in its 'intrinsic' version). We suppose that for all* $i \in \{1, \ldots, l\}$, f_i^1, f_i^2

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belong to $\mathcal{U}$. We also suppose the existence of an isomorphism of measure φ between G_1 and G_2 (so that $\varphi \circ f_i^1(x) = f_i^2 \circ \varphi(x)$ for a.e. $x \in \mathbb{S}^1$). Then, given a point $p = (p^1, p^2) \in \mathbb{T}^2$ such that neither p^1 is periodic for G_1 nor p^2 is periodic for G_2, there is an open neighborhood $U \subseteq \mathbb{T}^2$ of p endowed with a nowhere zero C^ω-vector field $X(x, y) = (X^1(x), X^2(y))$ which is in the C^∞-closure of $\mathbb{G}$ relative to U.

The proof of Theorem 3.2 needs a good knowledge of the topological dynamics associated to a group like G_1 (namely a non-solvable group generated by a finite set of diffeomorphisms near to the identity). In particular, a result due to Hölder (cf. [**HH**]) asserts that such a group always contains non-trivial diffeomorphisms displaying fixed points.

On the other hand, it is also known that a non-solvable subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^1)$ generated by diffeomorphisms belonging to $\mathcal{U}$ is such that the union of all periodic points (i.e. points having finite orbit), $\mathrm{Per}(G)$, is a finite set (maybe empty). Furthermore, restricting the action to each connected component of $\mathbb{S}^1 \setminus \mathrm{Per}(G)$, all the points have G-orbits everywhere dense. In particular, if $\mathrm{Per}(G)$ is empty then G is minimal, i.e. every point has dense orbit (cf. [**Gh**]).

We recall that G_1 (respectively G_2) is generated by $\{f_1^1, \ldots, f_l^1\}$ (respectively $\{f_1^2, \ldots, f_l^2\}$) where f_i^j belongs to $\mathcal{U}$. By assumption there is an isomorphism of measure $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\varphi \circ f_i^1 \circ \varphi^{-1}(x) = f_i^2(x)$ a.e. $x \in \mathbb{S}^1$. Let us assign to G_1 (respectively G_2) the sequence of sets $S_1(\mathfrak{m}) \subset G_1$ (respectively $S_2(\mathfrak{m}) \subset G_2$) defined by means of the generating set $\{f_1^1, \ldots, f_l^1\}$ (respectively $\{f_1^2, \ldots, f_l^2\}$) as in §2.2.

For each $\mathfrak{m} \in \mathbb{N}$, $G_j(\mathfrak{m})$ $(j = 1, 2)$ denotes the subgroup of G_j generated by $S_j(\mathfrak{m}) \cup S_j(\mathfrak{m} - 1) \cup S_j(\mathfrak{m} - 2)$. Notice that $G_j(\mathfrak{m})$ is not solvable for every $\mathfrak{m} \in \mathbb{N}$. Indeed a solvable subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^1)$ is *metabelian* (i.e. its second derived group is trivial). However the second derived group of $G_j(\mathfrak{m})$ contains $S_j(\mathfrak{m} + 2)$ which is distinct from {id} after Ghys's result. One concludes that $G_j(\mathfrak{m})$ is not solvable.

Considering $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$, we observe that any diffeomorphism in $S_j(\mathfrak{m})$ admits a holomorphic extension to some annulus A_τ ($\tau > 0$ fixed). Furthermore, if $\{h_\mathfrak{m}^j\}_{\mathfrak{m}\in\mathbb{N}}$ is a sequence such that $h_\mathfrak{m}^j \in S_j(\mathfrak{m})$, then $\{h_\mathfrak{m}^j\}_{\mathfrak{m}\in\mathbb{N}}$ converges uniformly to the identity on A_τ.

We need the following lemma.

LEMMA 3.3. *Assume that $p = (p^1, p^2) \in \mathbb{T}^2$ is such that neither p^1 is periodic for G_1 nor p^2 is periodic for G_2. Then there exists $F = (f^1, f^2) \in \mathbb{G}$, such that f^1, f^2 have both fixed points and also verify $f^1(\tilde{p}^1) \neq \tilde{p}^1$ and $f^2(\tilde{p}^2) \neq \tilde{p}^2$ for some $\tilde{p} = (\tilde{p}^1, \tilde{p}^2)$ in the $\mathbb{G}$-orbit of p.*

Before proving the Lemma (3.3), let us deduce Theorem 3.2.

Proof of Theorem 3.2. Let $p = (p^1, p^2)$ as in the statement of the theorem. We want to show the existence of a (nowhere *zero*) vector field X in the closure of $\mathbb{G}$ and defined in a neighborhood of p. Obviously it suffices to find a vector field in the closure of $\mathbb{G}$ which is defined in a neighborhood of an arbitrary point of the $\mathbb{G}$-orbit of p. Actually if $\tilde{p} = H(p)$ for some $H \in \mathbb{G}$ and X is a (nowhere *zero*) vector field in the closure of $\mathbb{G}$ defined around

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$\tilde{p}$, then H^*X is a vector field defined in a neighborhood of p. Moreover, H^*X is a nowhere *zero* vector field in the closure of $\mathbb{G}$.

Combining Lemma 3.3 with the discussion above, we see that there is no loss of generality in assuming the existence of $F = (f^1, f^2) \in \mathbb{G}$ such that p^1 (respectively p^2) is not fixed by f^1 (respectively f^2). In fact we can also assume that both f^1, f^2 have fixed points. Therefore, it results immediately that the sequence $F^k(p) = ((f^1)^k(p^1), (f^2)^k(p^2))$ converges to a point $q = (q^1, q^2)$ such that $f^1(q^1) = q^1$ and $f^2(q^2) = q^2$ (recall that our diffeomorphisms preserve the orientation).

One identifies (q^1, q^2) with $(0, 0) \in \mathbb{R}^2 \subset \mathbb{C}^2$. We fix a bidisc $B_{\epsilon\epsilon} \subset \mathbb{C}^2$ with center at $(0, 0)$ and radius $\epsilon > 0$, on which F has a holomorphic extension. Furthermore, using Ghys's theorem, ϵ can be chosen such that any element $H_{\mathfrak{m}} = (h^1_{\mathfrak{m}}, h^2_{\mathfrak{m}})$ of $(S_1(\mathfrak{m}) \times S_2(\mathfrak{m})) \cap \mathbb{G}$ also admits such an extension to $B_{\epsilon\epsilon}$. In fact, any non-trivial sequence $\{H_{\mathfrak{m}}\}$ of elements in $(S_1(\mathfrak{m}) \times S_2(\mathfrak{m})) \cap \mathbb{G}$ converges uniformly to the identity on this bidisc.

Assume first that, for $\mathfrak{m}_0$ large enough, every element $H_{\mathfrak{m}_0}$ of $(S_1(\mathfrak{m}_0) \times S_2(\mathfrak{m}_0)) \cap \mathbb{G}$ fixes the origin $(0, 0)$ $(\simeq (q^1, q^2))$. Then the stabilizer of $(0, 0)$ $(\simeq (q^1, q^2))$ in $\mathbb{G}$ is not solvable since its 'first coordinate' contains $G_1(\mathfrak{m}_0 + 2)$. Therefore (by iterating commutators) we can find an element $H = (h^1, h^2)$ in $\mathbb{G}$, fixing $(0, 0)$ $(\simeq (q^1, q^2))$, and such that 'the flatness order' of h^1 (respectively h^2) at 0 is strictly greater than the corresponding order of f^1 (respectively f^2). Hence $\mathrm{Bas}(f^1_1) \cup \mathrm{Bas}(f^2_1)$ is equipped with a nowhere *zero* 'Nakai's vector field' (cf. Theorem 2.5). Since $F^k(p)$ converges to $q \simeq (0, 0)$, the theorem follows in this first case.

Suppose now that, for every $\mathfrak{m} \in \mathbb{N}$ fixed, there is $H_{\mathfrak{m}} \in (S_1(\mathfrak{m}) \times S_2(\mathfrak{m})) \cap \mathbb{G}$ such that $H_{\mathfrak{m}}(0, 0) \neq (0, 0)$ (i.e. $H_{\mathfrak{m}}(q^1, q^2) \neq (q^1, q^2)$). Maybe by replacing $H_{\mathfrak{m}}$ by $H^{-1}_{\mathfrak{m}}$ it is possible to choose a non-trivial sequence $\{H_{\mathfrak{m}} = (h^1_{\mathfrak{m}}, h^2_{\mathfrak{m}})\}$, $H_{\mathfrak{m}} \in (S_1(\mathfrak{m}) \times S_2(\mathfrak{m})) \cap \mathbb{G}$, for which at least one between $h^1_{\mathfrak{m}}(0), h^2_{\mathfrak{m}}(0)$ is positive.

Call $\mathcal{H} = \{H_d\}$ the sequence of mappings from $B_{\epsilon\epsilon}$ to $\mathbb{C}^2$ obtained by restriction of (the holomorphic extension of) these $H_{\mathfrak{m}}$ to $B_{\epsilon\epsilon}$. Finally let $\Gamma_{\Gamma_1 \times \Gamma_2}$ be the pseudogroup generated by:

(1) the sequence $\mathcal{H}$;

(1) the restriction to $B_{\epsilon\epsilon}$ of (the holomorphic extension of) F (note that the holomorphic extension of F is still one-to-one on $B_{\epsilon\epsilon}$ provided that $\epsilon > 0$ is small).

Clearly $\Gamma_{\Gamma_1 \times \Gamma_2}$ satisfies conditions (1), (2), (3) and (4) of Proposition 2.7. So this proposition ensures the existence of a non-trivial vector field X whose restriction to $\mathbb{R}^2$ produces the required vector field. Theorem 3.2 is proved. □

It remains to prove Lemma 3.3. For the convenience of the reader we divide the proof into two cases depending on whether or not $G_1(\mathfrak{m})$ has finite orbits (for some $\mathfrak{m}$).

Case 1. Suppose that $G_1(\mathfrak{m})$ has no finite orbits for all $\mathfrak{m} \in \mathbb{N}$. In this case one has the following.

LEMMA 3.4. *The set of finite orbits of* $G_2(\mathfrak{m})$ *is either empty or consists of exactly one fixed point.*

Proof. Suppose for a contradiction that the statement were false. We consider a $\mathfrak{m}_0 \in \mathbb{N}$ such that the set of periodic points (i.e. points having finite orbits) of $G_2(\mathfrak{m}_0)$,

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$\mathrm{Per}(G_2(\mathfrak{m}_0))$, contains at least two points. Because the groups $G_2(\mathfrak{m})$ are generated by diffeomorphisms which become closer and closer to the identity, there is $\mathfrak{m}_1 \in \mathbb{N}$ such that all points in $\mathrm{Per}(G_2(\mathfrak{m}_0))$ are fixed by $G_2(\mathfrak{m}_1)$.

On the other hand, φ gives a natural correspondence between $G_1(\mathfrak{m}_1)$ and $G_2(\mathfrak{m}_1)$. However $G_1(\mathfrak{m}_1)$ acts ergodically on $\mathbb{S}^1$ with respect to the Lebesgue measure since it has no periodic points (cf. [**Reb**]). But since $G_2(\mathfrak{m}_1)$ leaves invariant the connected components of $\mathbb{S}^1 \setminus \mathrm{Per}(G_2(\mathfrak{m}_0))$, it does not act ergodically on $\mathbb{S}^1$. The resulting contradiction establishes the lemma. □

Proof of Lemma 3.3 in Case 1. Consider a point $p = (p^1, p^2) \in \mathbb{T}^2$ (if $\mathrm{Per}(G_2(\mathfrak{m}_0))$ is not empty for $\mathfrak{m}$ large, assume also that p^2 is not the corresponding fixed point, cf. Lemma 3.4). We claim the existence of $F = (f^1, f^2) \in \mathbb{G}$ such that $(f^1)^k(p^1)$ (respectively $(f^2)^k(p^2)$) converge to q^1 (respectively q^2), $q^1 \neq p^1$ (respectively $q^2 \neq p^2$), where $f^1(q^1) = q^1$ (respectively $f^2(q^2) = q^2$). In order to settle the claim, recall that Hölder's theorem enables us to find a diffeomorphism $f^1 \in G_1$ ($f^1 \neq \mathrm{id}$) displaying a fixed point. Applying Lemma 2.1, we see that we can assume the corresponding diffeomorphism $f^2 \in G_2$ ($f^2 = \varphi \circ f^1 \circ \varphi^{-1}$ a.e.) has fixed points as well. Thus, if p^1, p^2 are simultaneously not fixed by f^1, f^2 respectively, the diffeomorphism $F = (f^1, f^2)$ satisfies the required conditions. On the other hand, assume that p^1 is fixed by f^1 (respectively p^2 is fixed by f^2). Because $G_1(\mathfrak{m})$ has no fixed points for all $\mathfrak{m} \in \mathbb{N}$ (respectively the unique possible fixed point for $G_2(\mathfrak{m})$ is distinct from p^2), there is $H = (h^1, h^2)$ in $\mathbb{G}$, close to the identity, such that h^1 (respectively h^2) takes p^1 (respectively p^2) to a point which cannot be fixed by f^1 (respectively f^2) because it is 'very close' to p^1 (respectively p^2). Hence the diffeomorphism $H^{-1} \circ F \circ H$ satisfies the required conditions.

Finally, if (p^1, p^2) is such that p^2 is fixed by $G_2(\mathfrak{m})$ (cf. the assumptions of Theorem 3.2), with $\mathfrak{m}$ large, but is not fixed by G_2, then there is a point $(\tilde{p}^1, \tilde{p}^2)$ in the $\mathbb{G}$-orbit of (p^1, p^2) verifying $\tilde{p}^2 \neq p^2$. This establishes the lemma in the first case. □

Case 2. $G_1(\mathfrak{m})$ has finite orbits for $\mathfrak{m}$ large. If $G_2(\mathfrak{m})$ has no finite orbits for all $\mathfrak{m}$, then we use the arguments of Case 1. Thus it suffices to discuss the case in which both $G_1(\mathfrak{m})$, $G_2(\mathfrak{m})$ have finite orbits. Indeed we can assume that they have *fixed points*. Denote by $\mathrm{Per}(G_1(\mathfrak{m}))$ (respectively $\mathrm{Per}(G_2(\mathfrak{m}))$) the set of periodic points (i.e. points having finite orbit) for G_1 (respectively G_2).

Proof of Lemma 3.3 in Case 2. Choose $\mathfrak{m}_0 \in \mathbb{N}$ so large that $G_1(\mathfrak{m}_0)$, $G_2(\mathfrak{m}_0)$ have fixed points and leave fixed all respective points in $\mathrm{Per}(G_1)$, $\mathrm{Per}(G_2)$ (these latest sets may be empty). Note however that, since every element in $G_1(\mathfrak{m}_0)$ (respectively $G_2(\mathfrak{m}_0)$) has fixed points, a point in $\mathbb{S}^1$ is periodic (i.e. has finite orbit) for $G_1(\mathfrak{m}_0)$ (respectively $G_2(\mathfrak{m}_0)$) only if it is fixed by $G_1(\mathfrak{m}_0)$ (respectively $G_2(\mathfrak{m}_0)$).

Fix (p^1, p^2) as in the statement and suppose first that p^1 (respectively p^2) is not periodic for $G_1(\mathfrak{m}_0)$ (respectively $G_2(\mathfrak{m}_0)$). Take $F = (f^1, f^2) \neq (\mathrm{id}, \mathrm{id})$ in $(S_1(\mathfrak{m}_0) \times S_2(\mathfrak{m}_0)) \cap \mathbb{G}$. In particular, f^1 (respectively f^2) fixes all points in $\mathrm{Per}(G_1(\mathfrak{m}_0)) = \mathrm{Fix}(G_1(\mathfrak{m}_0))$ (where $\mathrm{Fix}(G_1(\mathfrak{m}_0))$ stands for the set of fixed points of $G_1(\mathfrak{m}_0)$) (respectively $\mathrm{Per}(G_2(\mathfrak{m}_0)) = \mathrm{Fix}(G_2(\mathfrak{m}_0))$). Because $G_1(\mathfrak{m}_0)$ possesses all orbits dense in the connected components of $\mathbb{S}^1 \setminus \mathrm{Per}(G_1(\mathfrak{m}_0))$, there is a point $(\tilde{p}^1, \tilde{p}^2)$ in the orbit of (p^1, p^2) such that $f^1(\tilde{p}^1) \neq \tilde{p}^1$ and $(f^1)^k(\tilde{p}^1)$ converges to

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$q^1 \in \mathrm{Per}(G_1(\mathfrak{m}_0)) = \mathrm{Fix}(G_1(\mathfrak{m}_0))$. Therefore, if $\tilde{p}^2$ is not fixed by f^2 the proof of the lemma is over.

Hence let us suppose that $f^2(\tilde{p}^2) = \tilde{p}^2$. First notice that given $h^1 \in G_1(\mathfrak{m}_0)$, the point $h^1 \circ (f^1)^k(p^1)$ is not fixed by f^1 provided that $k \in \mathbb{N}$ is large enough. In fact, we have seen that $(f^1)^k(p^1)$ converges to q^1 which belongs to $\mathrm{Per}(G_1(\mathfrak{m}_0)) = \mathrm{Fix}(G_1(\mathfrak{m}_0))$. Thus q^1 is fixed by h^1 and therefore $h^1 \circ (f^1)^k(p^1)$ cannot be fixed by f^1. On the other hand, $\tilde{p}^2$ lies on the complement of $\mathrm{Per}(G_2(\mathfrak{m}_0))$ since it belongs to the $G_2(\mathfrak{m}_0)$-orbit of p^2. Thus there is $H = (h^1, h^2)$ (where $h^1 \in G_1(\mathfrak{m}_0)$, $h^2 \in G_2(\mathfrak{m}_0)$) such that $h^2(\tilde{p}^2)$ is not fixed by f^2. Therefore, for k large enough the point $H \circ F^k(\tilde{p}^1, \tilde{p}^2)$ in the $\mathbb{G}$-orbit of (p^1, p^2) verifies the statement for the initial F (note that $(f^2)^k(\tilde{p}^2) = \tilde{p}^2$).

Now consider the general case. Obviously we can suppose that p^1 lies away from $\mathrm{Per}(G_1(\mathfrak{m}_0))$ but p^2 belongs to $\mathrm{Per}(G_2(\mathfrak{m}_0))$. Clearly there is a finite set of diffeomorphisms $H_1 = (h_1^1, h_1^2), \ldots, H_e = (h_e^1, h_e^2)$ in $\mathbb{G}$ satisfying the following: for any point $q \in \mathrm{Per}(G_2(\mathfrak{m}_0)) \setminus \mathrm{Per}(G_2)$ there is h_i^2 ($i \in \{1, \ldots, e\}$) such that $h_i^2(q)$ lies in the complement of $\mathrm{Per}(G_2(\mathfrak{m}_0))$. Next choose $H_{\mathfrak{m}_1} = (h_{\mathfrak{m}_1}^1, h_{\mathfrak{m}_1}^2)$ with $\mathfrak{m}_1$ very large. Using the local density of the orbits of G_1, we can assume that $h_{\mathfrak{m}_1}^1(p^1) \neq p^1$ (in fact we 'displace' (p^1, p^2) by elements in $\mathbb{G}$ to obtain this condition, obviously we can suppose that during this procedure the 'corresponding' p^2 remains in $\mathrm{Per}(G_2(\mathfrak{m}_0))$). Finally let $H_i = (h_i^1, h_i^2)$ be such that $h_i^2(p^2) \notin \mathrm{Per}(G_2(\mathfrak{m}_0))$. Note that the same holds for $h_i^2 \circ h_{\mathfrak{m}_1}^2(p^2)$ for $\mathfrak{m}_1$ very large (since $h_{\mathfrak{m}_1}^2$ must fix points in $\mathrm{Per}(G_2(\mathfrak{m}_0))$). However, $h_i^1(p^1)$ and $h_i^1 \circ h_{\mathfrak{m}_1}^1(p^1)$ cannot simultaneously belong to $\mathrm{Per}(G_1(\mathfrak{m}_0))$. The lemma is proved. □

We leave the following straightforward version of Theorem 3.2 which is interesting in its own right to the reader (naturally this version depends on the n-fold version of Proposition (2.7), cf. Remark A.8). Let $G \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ be a non-solvable group admitting a finite generating set which is contained in the neighborhood $\mathcal{U}$ of the identity furnished by Ghys's result. Given $f \in G$, we define a diffeomorphism $(f, \ldots, f)$ (n-times) of the torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n-times) by letting

$$(f, \ldots, f)(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n)).$$

This type of component-wise iteration raises a natural action of G on $\mathbb{T}^n$. This action is named the n-fold diagonal action of G on $\mathbb{T}^n$ and is going to be denoted by $G^{(n)}$.

THEOREM 3.5. *Let $G = \langle f_1, \ldots f_l \rangle \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ be a non-solvable group and assume that $f_i \in \mathcal{U}$ ($i = 1, 2, \ldots, l$). Assume in addition that $p = (p_1, \ldots, p_n) \in \mathbb{T}^n$ is such that none of its coordinate represents a periodic point for G. Then there is an open neighborhood $U \subset \mathbb{T}^n$ of p, equipped with a nowhere zero C^{ω} vector field $(X(x_1), \ldots, X(x_n))$ (where X is a vector field on an open interval of $\mathbb{S}^1$) which is in the C^{∞}-closure of $G^{(n)}$ relative to U.*

4. *Measurable rigidity*

We keep the notation of the preceding section. Here the Rigidity theorem will be proved by means of the diagonal action arising from G_1 and G_2. The proof we are going to give is based on Proposition 4.1 which in turn depends on Theorem 3.2. Nevertheless we will

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postpone the proof of this proposition to the next section in order to motivate its statement which may sound a bit technical.

So let $G_1 = \rho_1(\mathcal{G}), G_2 = \rho_2(\mathcal{G})$ be as in §3 (G_1 and G_2 measurably conjugate). Recall that we have the diagonal representation $\overline{\rho} : \mathcal{G} \to \mathrm{Diff}^{\omega}(\mathbb{T}^2)$ and we have called $\mathbb{G} = \overline{\rho}(\mathcal{G})$. Assume that there is a vector field of the form $X(x, y) = (X^1(x), X^2(y))$ in the closure of $\mathbb{G}$ relative to some (non-empty) open set $V \subset \mathbb{T}^2$ (cf. §2.3 and §2.4). Given a point $p = (p^1, p^2)$ of V, one says that X is *vertical at* p if $X(p) = (X^1(p^1), X^2(p^2))$ is such that $X^1(p^1) = 0$ and $X^2(p^2) \neq 0$. Similarly we define *horizontal* vector fields.

From now on $P(\mathbb{G})$ will denote the set of points $p = (p^1, p^2) \in \mathbb{T}^2$ such that p^1 is periodic (i.e. has finite orbit) for G_1 or p^2 is periodic for G_2. Clearly $P(\mathbb{G})$ consists of a finite union of vertical and horizontal paths. Finally, $\mathrm{Per}(\mathbb{G}) \subset \mathbb{T}^2$ is the set of periodic points for $\mathbb{G}$. This means that $p = (p^1, p^2)$ belongs to $\mathrm{Per}(\mathbb{G})$ if and only if p^1 is periodic for G_1 and p^2 is periodic for G_2. We are now able to state Proposition 4.1.

PROPOSITION 4.1. *There exists an analytic curve $\mathcal{C} \subset \mathbb{T}^2$ which is invariant under the action of $\mathbb{G}$ and has the following properties.*

(1) *If $p \in \mathbb{T}^2 \setminus (\mathcal{C} \cup P(\mathbb{G}))$ then there is a (nowhere zero) vector field X defined in a neighborhood of p (X in the closure of $\mathbb{G}$ relative to this neighborhood) which is vertical or horizontal at p.*

(2) *The curve $\mathcal{C}$ is smooth (without isolated points). Moreover, the tangent to $\mathcal{C}$ at a point $q \in \mathcal{C} \setminus \mathrm{Per}(\mathbb{G})$ is neither vertical nor horizontal.*

(3) *If $(p^1, p^2) \in \mathrm{Per}(\mathbb{G}) \cap \mathcal{C}$, then there is a neighborhood V of (p^1, p^2) such that $V \cap \mathcal{C}$ is either the graph of a strictly monotone function or the union of the graphs of two strictly monotone functions (one increasing and the other decreasing).*

The rest of the present section is devoted to setting up the proof of the Rigidity theorem. Since G_1 and G_2 are measurably conjugate by the measure isomorphism φ, we can consider the *graph* of φ defined by

$$\mathrm{Graph}(\varphi) = \{(x, \varphi(x)) \subset \mathbb{T}^2 : \text{ for } \mu\text{-a.e. } x \in \mathbb{S}^1\}.$$

Obviously $\mathrm{Graph}(\varphi)$ is a measurable curve in $\mathbb{T}^2$.

Modulo a set of measure zero, $\mathrm{Graph}(\varphi)$ is invariant under the action of $\mathbb{G}$. In fact, for any $g \in \mathcal{G}$, we 'forget' the set of the points $x \in \mathbb{S}^1$ such that $\varphi \circ \rho_1(g) \circ \varphi^{-1}(x) \neq \rho_2(g)(x)$. By assumption this set has null measure for each $g \in \mathcal{G}$. As $\mathcal{G}$ is a countable group, it follows that there is a full measure set of $\mathbb{S}^1$ on which φ is defined and such that $\varphi \circ \rho_1(g) \circ \varphi^{-1}(x) = \rho_2(g)(x)$ for any $g \in \mathcal{G}$ and x in this set. Therefore, restricting φ to the set in question, one concludes that $\mathrm{Graph}(\varphi)$ is a measurable curve invariant by $\mathbb{G}$ as desired.

In what follows many assertions should be understood 'modulo sets of measure zero'. We want to estimate the size of $\mathrm{Graph}(\varphi)$ which lies away from $\mathcal{C}$. Hence one introduces the set $D_\varphi = \{x \in \mathbb{S}^1 : (x, \varphi(x)) \notin \mathcal{C} \cup P(\mathbb{G})\}$. Obviously D_φ is a measurable set (i.e. a Borelian). The next proposition has crucial importance for the proof of the Rigidity theorem.

PROPOSITION 4.2. *D_φ has null measure (i.e. $\mu(D_\varphi) = 0$).*

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The most convenient way of proving this proposition is reasoning by contradiction. So we start our approach by supposing that $\mu(D_\varphi) \geq \varepsilon > 0$. By means of Lusin's theorem, we can find a Borel set $\mathcal{B} \subset \mathbb{S}^1$ and a continuous function $\overline{\varphi} : \mathbb{S}^1 \to \mathbb{S}^1$ such that:

(1) $\mu(\mathcal{B}) \geq 1 - \varepsilon/2$;

(2) $\overline{\varphi}$ coincides with φ on $\mathcal{B}$.

As $\mu(\mathcal{B} \cap D_\varphi) \geq \varepsilon/2 > 0$, we can pick a point $p^1 \in \mathcal{B} \cap D_\varphi$ which is a Lebesgue density point for this set. We now choose a sequence $\{I_s\}_{s\in\mathbb{N}}$ of intervals $I_s \subset \mathbb{S}^1$ centered at p^1 whose diameters converge to *zero*. Thus one obtains

$$\lim_{s\to\infty} \frac{\mu((\mathcal{B} \cap D_\varphi) \cap I_s)}{\mu(I_s)} = 1. \tag{4}$$

Let $V \subset \mathbb{T}^2$ be a neighborhood of $(p^1, \varphi(p^1)) = (p^1, \overline{\varphi}(p^1))$ equipped with a nowhere zero vector field X, vertical or horizontal at $(p^1, \overline{\varphi}(p^1))$, contained in the closure of $\mathbb{G}$ relative to V (see Proposition 4.1). Replacing φ by φ^{-1} and regarding the graph of φ^{-1} 'from the second coordinate', we see that it suffices to consider the case in which X is vertical at $(p^1, \overline{\varphi}(p^1))$. Next choose $t_0 > 0$ so small that $\phi_X^{t_0}(p^1, \overline{\varphi}(p^1)) \neq (p^1, \overline{\varphi}(p^1))$ belongs to V (where ϕ_X^t stands for the local flow arising from X). Finally let $U \subset V$ be a small ball centered at $(p^1, \varphi(p^1))$ and such that $\phi_X^{t_0}(U) \subset V$ does not intersect U (i.e. $\phi_X^{t_0}(U) \cap U = \emptyset$).

Since the diameter of I_s goes to zero, we can assume that the point $(x, \overline{\varphi}(x))$ belongs to U for any $x \in I_s$ (thanks to the continuity of $\overline{\varphi}$). Let us write I_s under the form $I_s = [p^1 - \mathfrak{c}_s, p^1 + \mathfrak{c}_s]$.

One denotes by $\pi_1 : \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ the projection onto the first coordinate.

Recall that the vector field X has the form $(X^1(x), X^2(y))$, so that $\phi_X^{t_0}(x, y) = (\phi_{X^1}^{t_0}(x), \phi_{X^2}^{t_0}(y))$. Because X is vertical at $(p^1, \overline{\varphi}(p^1))$, one concludes that $\phi_{X^1}^{t_0}(p^1) = p^1$. In particular, there is a positive constant c_1 $(1 \geq c_1 > 0)$ such that the interval $[p^1 - c_1\mathfrak{c}_s, p^1 + c_1\mathfrak{c}_s]$ is contained in $I_s \cap \pi_1[\phi_X^{t_0}(\mathrm{Graph}(\overline{\varphi})|_{I_s})]$ (actually c_1 is 'essentially' the derivative of $\phi_{X^1}^{t_0}$ at p^1 so that it does not depend on s). Therefore we can also define an interval $I_{sc}^{(-1)} \subset I_s$ by letting $I_{sc}^{(-1)} = (\phi_{X^1}^{t_0})^{-1}([p^1 - c_1\mathfrak{c}_s, p^1 + c_1\mathfrak{c}_s])$. Clearly $\mu(I_s) \geq \mu(I_{sc}^{(-1)}) \geq C_2\mu(I_s)$ for an appropriate constant $C_2 > 0$ and sufficiently large $s \in \mathbb{N}$.

Suppose we are given (a small) $\delta > 0$. For s large enough we can assume that

$$1 - \delta \leq \min\left\{ \frac{\mu(\mathcal{B} \cap D_\varphi \cap I_{sc}^{(-1)})}{\mu(I_{sc}^{(-1)})}, \frac{\mu(\mathcal{B} \cap D_\varphi \cap I_s)}{\mu(I_s)}, \frac{\mu(\mathcal{B} \cap D_\varphi \cap [p^1 - c_1\mathfrak{c}_s, p^1 + c_1\mathfrak{c}_s])}{\mu([p^1 - c_1\mathfrak{c}_s, p^1 + c_1\mathfrak{c}_s])} \right\}. \tag{5}$$

LEMMA 4.3. *Fixed* $\delta > 0$, *for* s *large enough we have*

$$\frac{\mu(\phi_{X^1}^{t_0}(\mathcal{B} \cap D_\varphi \cap I_{sc}^{(-1)}))}{\mu(\phi_{X^1}^{t_0}(I_{sc}^{(-1)}))} = \frac{\mu(\phi_{X^1}^{t_0}(\mathcal{B} \cap D_\varphi \cap I_{sc}^{(-1)}))}{\mu([p^1 - c_1\mathfrak{c}_s, p^1 + c_1\mathfrak{c}_s])} \geq (1 - \delta)^2 C_2 > 0. \tag{6}$$

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Proof. Recall that $I_{sc}^{(-1)} \subset I_s$ is such that $\mu(I_s) \geq \mu(I_{sc}^{(-1)}) \geq C_2\mu(I_s)$. Moreover, choosing s very large, it results that $\mu(\mathcal{B}\cap D_\varphi \cap I_{sc}^{(-1)}) \geq (1-\delta)\mu(I_{sc}^{(-1)}) \geq (1-\delta)C_2\mu(I_s)$ (cf. (5)).

We now remind the reader that $\phi_{X^1}^{t_0}$ is a C^1 (C^∞) local diffeomorphism (since so is $\phi_X^{t_0}$). Thus the variation of its derivative between points x_1, x_2 is bounded by a constant (the supremum of its derivative) times the distance between x_1, x_2. Therefore, for s large, one has

$$\frac{\mu(\phi_{X^1}^{t_0}(\mathcal{B} \cap D_\varphi \cap I_{sc}^{(-1)}))}{\mu(\phi_{X^1}^{t_0}(I_{sc}^{(-1)}))} \geq (1-\delta)\frac{\mu(\mathcal{B} \cap D_\varphi \cap I_{sc}^{(-1)})}{\mu(I_{sc}^{(-1)})}.$$

Putting together the estimates above, one gets

$$\begin{aligned}\frac{\mu(\phi_{X^1}^{t_0}(\mathcal{B} \cap D_\varphi \cap I_{sc}^{(-1)}))}{\mu(\phi_{X^1}^{t_0}(I_{sc}^{(-1)}))} &= \frac{\mu(\phi_{X^1}^{t_0}(\mathcal{B} \cap D_\varphi \cap I_{sc}^{(-1)}))}{\mu([p^1 - c_1\mathfrak{c}_s, p^1 + c_1\mathfrak{c}_s])} \geq (1-\delta)\frac{\mu(\mathcal{B} \cap D_\varphi \cap I_{sc}^{(-1)})}{\mu(I_{sc}^{(-1)})} \\ &\geq \frac{(1-\delta)^2 C_2\mu(I_s)}{\mu(I_s)} = (1-\delta)^2 C_2 > 0.\end{aligned}$$

The lemma is proved. □

Proof of Proposition 4.2. We fix $\delta > 0$ very small. Choose s large enough such that estimates (5) and (6) hold.

Recall that $\phi_X^{t_0}$ is induced by the local flow of X which is in the closure of $\mathbb{G}$ relative to V (recall that $U \subset V$ and $\phi_{X^1}^{t_0}(U) \subset V$). Hence we can find a sequence of elements $\{H_m\} \in \mathbb{G}$ which converges C^1 to $\phi_X^{t_0}$ on U ($H_m = (h_m^1, h_m^2)$). Because s is fixed, for a sufficiently large m, estimate (6) 'essentially' holds replacing $\phi_{X^1}^{t_0}$ by h_m^1. Precisely one has

$$\mu(h_m^1(\mathcal{B} \cap D_\varphi \cap I_{sc}^{(-1)})) > (1-2\delta)^2 C_2\mu(h_m^1(I_{sc}^{(-1)})).$$

Furthermore, taking m still larger if needed, we can also assume that:

(1) $h_m^1(I_{sc}^{(-1)}) \subset I_s$ and $H_m(U) \cap U = \emptyset$;
(2) $\mu(\mathcal{B} \cap D_\varphi \cap h_m^1(I_{sc}^{(-1)})) \geq (1-2\delta)\mu(h_m^1(I_{sc}^{(-1)}))$.

In fact, assumption (1) is obvious because $\{h_m^1\}$ converges to $\phi_{X^1}^{t_0}$ on U (and H_m converges to $\phi_X^{t_0}$). To show assumption (2), let us observe that estimate (5) implies that

$$\frac{\mu(\mathcal{B} \cap D_\varphi \cap \phi_{X^1}^{t_0}(I_{sc}^{(-1)}))}{\mu(\phi_{X^1}^{t_0}(I_{sc}^{(-1)}))} = \frac{\mu(\mathcal{B} \cap D_\varphi \cap [p^1 - c_1\mathfrak{c}_s, p^1 + c_1\mathfrak{c}_s])}{\mu([p^1 - c_1\mathfrak{c}_s, p^1 + c_1\mathfrak{c}_s])} \geq 1 - \delta.$$

Therefore, assumption (2) follows from the C^1-convergence of $\{h_m^1\}$ to $\phi_{X^1}^{t_0}$.

We now consider $\mathrm{Graph}(\overline{\varphi})|_{\mathcal{B}\cap D_\varphi \cap I_{sc}^{(-1)}}$ and $\mathrm{Graph}(\overline{\varphi})|_{\mathcal{B}\cap D_\varphi \cap h_m^1(I_{sc}^{(-1)})}$, the restrictions of the graph of $\overline{\varphi}$ to the indicated sets. To accomplish our proof, we first observe that

$$[H_m(\mathrm{Graph}(\overline{\varphi})|_{I_{sc}^{(-1)}})] \cap \mathrm{Graph}(\overline{\varphi})|_{h_m^1(I_{sc}^{(-1)})} = \emptyset$$

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(since $I_{sc}^{(-1)} \subset I_s$, $h_m^1(I_{sc}^{(-1)}) \subset I_s$ and $\mathrm{Graph}(\overline{\varphi})|_{I_s}$ is enclosed in the ball U). However

$$\begin{aligned}\mu[\pi_1(H_m(\mathrm{Graph}(\overline{\varphi})|_{\mathcal{B}\cap D_\varphi \cap I_{sc}^{(-1)}})) \cap \pi_1(\mathrm{Graph}(\overline{\varphi})|_{\mathcal{B}\cap D_\varphi \cap h_m^1(I_{sc}^{(-1)})})] \\ \geq \mu(h_m^1(\mathcal{B}\cap D_\varphi \cap I_{sc}^{(-1)})) + \mu(\mathcal{B}\cap D_\varphi \cap h_m^1(I_{sc}^{(-1)})) - \mu(h_m^1(I_{sc}^{(-1)})) \\ \geq [(1-2\delta)^2 C_2 - 2\delta]\mu(h_m^1(I_{sc}^{(-1)})) > 0. \qquad (7)\end{aligned}$$

As $\overline{\varphi}$ coincides with φ on $\mathcal{B}$, the same holds when replacing the graph of $\overline{\varphi}$ by that of φ. Nonetheless, $\mathrm{Graph}(\varphi)$ is invariant under H_m so that $H_m(\mathrm{Graph}(\varphi)|_{\mathcal{B}\cap D_\varphi \cap I_{sc}^{(-1)}})$ has to be contained in $\mathrm{Graph}(\varphi)|_{h_m^1(I_{sc}^{(-1)})}$. On the other hand, $H_m(\mathrm{Graph}(\varphi)|_{\mathcal{B}\cap D_\varphi \cap I_{sc}^{(-1)}}) \subset \mathrm{Graph}(\varphi)|_{h_m^1(I_{sc}^{(-1)})}$ does not intersect (in the measure sense) $\mathrm{Graph}(\varphi)|_{\mathcal{B}\cap D_\varphi \cap h_m^1(I_{sc}^{(-1)})} \subset \mathrm{Graph}(\varphi)|_{h_m^1(I_{sc}^{(-1)})}$. By definition of the graph of φ, it follows that the measure of the projection on the first coordinate of the intersection of these two sets must be zero. It results in a contradiction with inequality (7) which proves the proposition. □

The proof of the Rigidity theorem requires a few additional lemmas.

LEMMA 4.4. *Let $G \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ be a non-solvable group admitting a finite generating set close to the identity. Denote by $G^c \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ the group consisting of those diffeomorphisms in $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ which commute with all elements in G. Then the group G^c is C^{ω}-conjugate to a finite group of rotations.*

Proof. Denote by $\mathrm{Per}(G)$ the finite set of periodic points for G. Choose a diffeomorphism $f \in G$ ($f \neq \mathrm{id}$) possessing at least one fixed point in $\mathbb{S}^1 \setminus \mathrm{Per}(G)$ (such an f always exists; if $\mathrm{Per}(G) = \emptyset$ its existence follows from Hölder's theorem and if $\mathrm{Per}(G) \neq \emptyset$ then, for instance, it is an easy consequence of Nakai's result). Observe that a diffeomorphism $h^c \in G^c$ displaying a fixed point must coincide with the identity. Indeed, suppose that $h^c \neq \mathrm{id}$ has a fixed point and commutes with G. In view of the preceding discussion, there is $f \in G$ ($f \neq \mathrm{id}$) having a fixed point p such that $h^c(p), (h^c)^2(p), \ldots$ form an infinite set (i.e. p is not fixed by h^c). However, since f commutes with h^c, all these points are fixed by f. Because f is analytic, it implies $f = \mathrm{id}$ which is impossible.

Next denote by $\mathrm{Fix}(f) \neq \emptyset$ the set of fixed points of f. The diffeomorphisms in G^c permutes the points in $\mathrm{Fix}(f)$, therefore G^c possesses a subgroup of finite index whose elements admit a common fixed point. This subgroup is therefore reduced to $\{\mathrm{id}\}$. It follows that G^c is a finite group acting properly discontinuously on $\mathbb{S}^1$ (since no element in G^c other than the identity has fixed points). Clearly such a group is C^{ω}-conjugate to a finite group of rotations. □

We fix a group G as in Lemma 4.4. Consider the (two-fold) diagonal action $G^{(2)}$ of G on $\mathbb{S}^1 \times \mathbb{S}^1$ (as explained at the end of §3, i.e. $G = G_1 = G_2$). In the following let us identify $G^{(2)}$ with the corresponding group of diffeomorphisms of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and write $P(G^{(2)})$ for denoting the set of points $p = (p^1, p^2) \in \mathbb{T}^2$ such that p^1 or p^2 is periodic for G.

Notice that Proposition 4.1 can be used to study the action of $G^{(2)}$ (just let $\varphi = \mathrm{id}$). Hence there is an analytic curve $\mathcal{C}_{G^{(2)}}$, possessing all the properties stated in Proposition 4.1, such that any point p in the complement of $\mathcal{C}_{G^{(2)}} \cup P(G^{(2)})$ has a neighborhood equipped with a vector field in the closure of $G^{(2)}$ and vertical (or horizontal)

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at p. The curve $\mathcal{C}_{G^{(2)}}$ may not be connected, however the *diagonal* of the torus is necessarily one of its connected components. Furthermore, the diagonal of $\mathbb{T}^2$ is obviously invariant under $G^{(2)}$. We conclude from the mentioned properties of $\mathcal{C}_{G^{(2)}}$ that *all the connected components* of $\mathcal{C}_{G^{(2)}}$ are invariant under $G^{(2)}$(recall that G preserves orientation). This remark will enable us to prove Lemma 4.5.

After Lemma 4.4, it is known that a group of diffeomorphisms commuting with G is conjugate to a finite group of rotations. It is therefore cyclic and 'changing coordinates' we can suppose that the rational rotation R_α generates this group.

LEMMA 4.5. *Let G and R_α be as above. Suppose that $f \in \mathrm{Diff}^\omega(\mathbb{S}^1)$ is such that R_α commutes with all diffeomorphisms in $f^{-1} \circ G \circ f$. Then f verifies the functional equation*

$$(R_\alpha)^M \circ f(x) = f(x + \alpha),$$

for some $M \in \mathbb{N}$.

Proof. Given $x \in \mathbb{S}^1$, we consider the point $(f(x), f(x+\alpha)) \in \mathbb{T}^2$. Assume that, for some $x \in \mathbb{S}^1$, the point $(f(x), f(x+\alpha))$ lies away from $\mathcal{C}_{G^{(2)}}$. In this case, we can suppose without loss of generality the existence of a vector field $X = (X^1, X^2)$ vertical at $(f(x), f(x+\alpha))$ and in the closure of $G^{(2)}$ (the case X horizontal is analogous). This results in the existence of a sequence $\{H_m = (h_m, h_m)\}_{m\in\mathbb{N}} \subset G^{(2)}$ such that:

(1) $\{h_m \circ f(x)\}$ converges to $f(x)$;

(2) $|h_m \circ f(x+\alpha) - f(x+\alpha)|$ converges towards some $\epsilon > 0$.

It follows from (1) that $\{f^{-1} \circ h_m \circ f(x)\}$ tends to x. Since $f^{-1} \circ h_m \circ f$ commutes with R_α, we see that $\{f^{-1} \circ h_m \circ f(x+\alpha)\}$ should converge to $x + \alpha$. However this last convergence is impossible in view of (2). The resulting contradiction proves that $(f(x), f(x+\alpha))$ belongs to $\mathcal{C}_{G^{(2)}}$ for every $x \in \mathbb{S}^1$.

The preceding result shows that the set $\{(x, f \circ R_\alpha \circ f^{-1}(x)) : x \in \mathbb{S}^1\}$ coincides with some connected component of $\mathcal{C}_{G^{(2)}}$. It is then invariant under $G^{(2)}$ (cf. the discussion above). The last claim implies that $f \circ R_\alpha \circ f^{-1}$ commutes with G. Thus $f \circ R_\alpha \circ f^{-1}$ is of the form $(R_\alpha)^M$ for some $M \in \mathbb{N}$. The proof of the lemma follows at once. □

Proof of the Rigidity theorem. First of all we notice that the set of points $x \in \mathbb{S}^1$ such that $(x, \varphi(x))$ belongs to $P(\mathbb{G})$ has measure zero. Otherwise φ would take a set having positive measure onto a set of measure zero which is impossible. Thus one concludes from Proposition 4.2 that the set $\{x \in \mathbb{S}^1 : (x, \varphi(x)) \in \mathcal{C}\}$ contained in $\mathbb{S}^1$ has total Lebesgue measure.

Assume firstly that $\mathcal{C}$ is connected. In order to prove our theorem it is sufficient to prove that $\mathcal{C}$ is a graph. In fact we want to show the existence of a homeomorphism $\mathfrak{h} : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\mathcal{C} = \mathrm{Graph}(\mathfrak{h})$. If such an $\mathfrak{h}$ exists then it must be a real analytic diffeomorphism. Furthermore, as it conjugates G_1, G_2 in a set with positive measure it has to conjugate these groups in the entire $\mathbb{S}^1$. Finally, as $\mathrm{Graph}(\varphi) \subseteq \mathcal{C}$, one sees that φ coincides a.e. with $\mathfrak{h}$, therefore the proof of the theorem will be complete.

We now proceed with the construction of the required homeomorphism $\mathfrak{h}$. Pick a point $p = (p^1, p^2)$ in $\mathcal{C} \setminus \mathrm{Per}(\mathbb{G})$ such that $p^2 = \varphi(p^1)$ and $p^1 = e^{\theta_0}$ is a Lebesgue density point for the set $\{x \in \mathbb{S}^1 : (x, \varphi(x)) \in \mathcal{C}\}$. Recall that $\mathcal{C}$ is smooth and has 'oblique tangents' around p (i.e. it is known that the tangent to $\mathcal{C}$ in a point which lies

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away from $\mathrm{Per}(\mathbb{G})$ is neither vertical nor horizontal). Hence for small $\theta \in \mathbb{R}$, $\theta > 0$, we can define a local homeomorphism $\mathfrak{h}$ through the condition $(e^{\theta_0+\theta}, \mathfrak{h}(e^{\theta_0+\theta})) \in \mathcal{C}$. If $\mathfrak{h}$ is defined at $e^{\theta_0+\theta_1}$ ($e^{\theta_0+\theta_1} \notin \mathrm{Per}(\mathbb{G})$) then one can extend $\mathfrak{h}$ to a neighborhood of $e^{\theta_0+\theta_1}$ in the same way. When $e^{\theta_0+\theta_1}$ is a periodic point for G_1, we extend $\mathfrak{h}$ to a neighborhood of $e^{\theta_0+\theta_1}$ using item (3) of Proposition 4.1 (or item (2), if $(e^{\theta_0+\theta_1}, \mathfrak{h}(e^{\theta_0+\theta_1}))$ does not belong to $\mathrm{Per}(\mathbb{G})$). Proceeding in this way, it is possible to define $\mathfrak{h}(e^{\theta_0+\theta})$ for any $\theta \in [0, 2\pi)$. The resulting function $\mathfrak{h}$ is a C^{ω}-mapping which is monotone (increasing or decreasing) thanks to items (2) and (3) ('oblique tangents') of Proposition 4.1.

Therefore to show that $\mathfrak{h}$ is a homeomorphism (and thus a C^{ω}-diffeomorphism) it is enough to verify the assertions below:

(a) $\mathfrak{h}$ is one-to-one;

(b) when $\theta \to 2\pi$, $\mathfrak{h}(e^{\theta_0+\theta})$ converges to $\mathfrak{h}(e^{\theta_0}) = p_2$.

Let us prove assertion (a). Suppose for a contradiction that $\mathfrak{h}$ is not one-to-one. Since $\mathfrak{h}$ is monotone, the existence of $\theta_1 \in (0, 2\pi)$ follows such that the restriction of $\mathfrak{h}$ to $[e^{\theta_0}, e^{\theta_0+\theta_1})$ satisfies the following conditions: $\mathfrak{h}|_{[e^{\theta_0}, e^{\theta_0+\theta_1})}$ is one-to-one and onto $\mathbb{S}^1$, in addition $\mathfrak{h}(e^{\theta_0}) = \mathfrak{h}(e^{\theta_0+\theta_1}) = p^2$. Hence we can define the inverse $\mathfrak{h}^{-1} : \mathbb{S}^1 \to [e^{\theta_0}, e^{\theta_0+\theta_1})$. We observe that $\mathfrak{h}$ coincides (around p) with φ in a subset of $[e^{\theta_0}, e^{\theta_0+\theta_1})$ which has positive measure. Note however that the equation $\mathfrak{h} \circ h^1 = h^2 \circ \mathfrak{h}$, for arbitrary $h^1 = \rho_1(g)$, $h^2 = \rho_2(g)$, does not need to hold in a set with positive measure (even though $\mathfrak{h}$ coincides with φ in such a set). In fact, h^1 may take the points x of the set $\{x \in \mathbb{S}^1 : \varphi(x) = \mathfrak{h}(x)\}$ to its complement, so that one cannot ensure the equality above. On the other hand, if h^1 is sufficiently C^1-close to the identity, then $h^1(\{x \in \mathbb{S}^1 : \varphi(x) = \mathfrak{h}(x)\})$ must have non-trivial intersection (in the measure sense) with $\{x \in \mathbb{S}^1 : \varphi(x) = \mathfrak{h}(x)\}$. Since the mappings involved are analytic, we deduce that the equation $\mathfrak{h} \circ h^1 \mathfrak{m}_0 = h^2_{\mathfrak{m}_0} \circ \mathfrak{h}$ holds for every $x \in [e^{\theta_0}, e^{\theta_0+\theta_1}]$ provided that the diffeomorphism $h^1_{\mathfrak{m}_0}$ (respectively $h^2_{\mathfrak{m}_0}$) belongs to $S_1(\mathfrak{m}_0)$ (respectively $S_2(\mathfrak{m}_0)$) for $\mathfrak{m}_0 \in \mathbb{N}$ large enough. Hence the equation $h^1_{\mathfrak{m}_0} \circ \mathfrak{h}^{-1} = \mathfrak{h}^{-1} \circ h^2_{\mathfrak{m}_0}$ holds for $x \in \mathbb{S}^1$. This implies that the image of $\mathfrak{h}^{-1}$, namely the set $[e^{\theta_0}, e^{\theta_0+\theta_1}) = [p^1, e^{\theta_0+\theta_1}) \subset \mathbb{S}^1$, is invariant under elements in $S_1(\mathfrak{m}_0)$ and thus invariant under $G_1(\mathfrak{m}_0)$ (using the notation of §3). Choosing p^1 'generic', we obtain a contradiction which proves that $\mathfrak{h}$ is one-to-one.

The proof of (b) is now easy. We note that the argument employed above shows in particular that $\lim_{\theta \to 2\pi} \mathfrak{h}(e^{\theta_0+\theta}) \leq p^2$. The reciprocal argument (replacing φ by φ^{-1}) will imply that $\lim_{\theta \to 2\pi} \mathfrak{h}(e^{\theta_0+\theta}) \geq p^2$. Hence $\lim_{\theta \to 2\pi} \mathfrak{h}(e^{\theta_0+\theta}) = p^2$ as required. The theorem is proved for $\mathcal{C}$ connected.

Consider now the general case in which $\mathcal{C}$ may have distinct connected components $\mathcal{C}_1, \ldots, \mathcal{C}_r$. Modulo erasing some of these components, we can assume that the set $\mathfrak{C}_i = \{x \in \mathbb{S}^1 : (x, \varphi(x) \in \mathcal{C}_i\}$ has positive measure for every $i = 1, \ldots, r$. Furthermore, these sets are such that their union has total Lebesgue measure in $\mathbb{S}^1$. Denote by $\{f^1_1, \ldots, f^1_l\}$ (respectively $\{f^2_1, \ldots, f^2_l\}$) a generating set (close to the identity) for G_1 (respectively G_2). Proceeding as before, to each $\mathcal{C}_i$ we can associate a diffeomorphism $\mathfrak{h}_i$ such that (for some $\mathfrak{m}_i$ large enough):

(1) $\mathfrak{h}_i$ is a C^{ω}-conjugacy between $G_1(\mathfrak{m}_i)$ and $G_2(\mathfrak{m}_i)$;

(2) $\mathfrak{h}_i$ is a C^{ω}-conjugacy between $f^1_j \circ G_1(\mathfrak{m}_i) \circ (f^1_j)^{-1}$ and $f^2_j \circ G_2(\mathfrak{m}_i) \circ (f^2_j)^{-1}$ for every $j = 1, \ldots, l$.

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In fact for concluding (2) it suffices to observe that the generators of $f_j^1 \circ G_1(\mathfrak{m}_i) \circ (f_j^1)^{-1}$ (respectively $f_j^2 \circ G_2(\mathfrak{m}_i) \circ (f_j^2)^{-1}$) become arbitrarily close to the identity when $\mathfrak{m}_i$ increases.

Taking $\mathfrak{m}_0 = \max\{\mathfrak{m}_i\}$, we see in particular that any $\mathfrak{h}_i$ provides a C^ω-conjugacy between $G_1(\mathfrak{m}_0)$ and $G_2(\mathfrak{m}_0)$. Consider the group generated by the diffeomorphisms $\{\mathfrak{h}_{i_1}^{-1} \circ \mathfrak{h}_{i_2}\}$ and note that any $\mathfrak{h}_{i_1}^{-1} \circ \mathfrak{h}_{i_2}$ commutes with $G_1(\mathfrak{m}_0)$. It follows that these diffeomorphisms preserve the orientation (otherwise they would have fixed points which in turn would imply that they coincide with the identity, cf. the proof of Lemma 4.4). Hence Lemma 4.4 permits us to assume that $\{\mathfrak{h}_{i_1}^{-1} \circ \mathfrak{h}_{i_2}\}$ generate a finite group of rotations (similar considerations hold for $\{\mathfrak{h}_{i_1} \circ \mathfrak{h}_{i_2}^{-1}\}$).

Denote by $\pi_1 : \mathbb{S}^1 \to \mathbb{S}^1$ the covering map associated to the rotation group generated by $\{\mathfrak{h}_{i_1}^{-1} \circ \mathfrak{h}_{i_2}\}$. The group $G_1(\mathfrak{m}_0)$ naturally induces another group $\underline{G_1}(\mathfrak{m}_0)$ of diffeomorphisms of $\mathbb{S}^1$ whose (one) lift by π_1 is exactly $G_1(\mathfrak{m}_0)$. In fact it is enough to recall that $G_1(\mathfrak{m}_0)$ commutes with the mentioned rotations. Similarly, if $\pi_2 : \mathbb{S}^1 \to \mathbb{S}^1$ is the covering map corresponding to the group generated by $\{\mathfrak{h}_{i_1} \circ \mathfrak{h}_{i_2}^{-1}\}$, $G_2(\mathfrak{m}_0)$ also induces a group $\underline{G_2}(\mathfrak{m}_0) \subset \mathrm{Diff}^\omega(\mathbb{S}^1)$ whose (one) lift by π_2 is $G_2(\mathfrak{m}_0)$.

We claim that φ naturally induces a measurable conjugacy $\underline{\varphi}$ between $\underline{G_1}(\mathfrak{m}_0)$ and $\underline{G_2}(\mathfrak{m}_0)$. Indeed, the union of the sets $\mathfrak{C}_i$ has total measure so that the set $\{x \in \mathbb{S}^1 : \varphi(x) = \mathfrak{h}_i(x) \text{ for some } i = 1, \ldots, r\}$ has total measure as well. Thus φ can be 'pushed down' by π_1 (and π_2) to define the required measurable conjugacy $\underline{\varphi}$. Actually $\mathfrak{h}_{i_1}, \mathfrak{h}_{i_2}$ differ by an element of the group associated to these coverings; more precisely if π_1 identifies points $x_1, x_2 \in \mathbb{S}^1$ then $\mathfrak{h}_{i_1}(x_1), \mathfrak{h}_{i_2}(x_2)$ are identified by π_2 as well. Moreover, after the case treated firstly (i.e. $\mathcal{C}$ connected), it results that $\underline{\varphi}$ coincides a.e. with an analytic diffeomorphism.

To accomplish the proof of the Rigidity theorem (along with its 'complement') it remains to verify that the initial groups G_1, G_2 are C^ω-conjugate. Note that $G_1(\mathfrak{m}_0), G_2(\mathfrak{m}_0)$ are so after the above discussion. In fact the groups $\underline{G_1}(\mathfrak{m}_0), \underline{G_2}(\mathfrak{m}_0)$ are C^ω-conjugate by a diffeomorphism coinciding a.e. with $\underline{\varphi}$. To show that G_1, G_2 are C^ω-conjugate as well, we are going to verify that they can be 'pushed down' by π_1, π_2. In other words, we shall see that G_1, G_2 naturally induce through π_1, π_2 groups of diffeomorphisms $\underline{G_1}, \underline{G_2}$ which are therefore measurably conjugated by $\underline{\varphi}$.

So fix a generator f_j^1 of G_1 and let us check f_j^1 can be pushed down by π. Recall that $\mathfrak{h}_i$ conjugates $f_j^1 \circ G_1(\mathfrak{m}_0) \circ (f_j^1)^{-1}$ to $f_j^2 \circ G_2(\mathfrak{m}_0) \circ (f_j^2)^{-1}$. Thus the diffeomorphisms in $f_j^1 \circ G_1(\mathfrak{m}_0) \circ (f_j^1)^{-1}$ commute with the rotations $\mathfrak{h}_{i_1}^{-1} \circ \mathfrak{h}_{i_2}$. Hence Lemma 4.5 shows that f_j^1 satisfies a suitable functional equation which in turn proves that f_j^1 can be 'pushed down' as desired (i.e. two points $x_1, x_2 \in \mathbb{S}^1$ are identified by π_1 if and only if $f_j^1(x_1), f_j^1(x_2)$ are identified by π_2). The theorem is proved. □

5. *Vertical vector fields*

In this last section we shall provide the proof of Proposition 4.1 accomplishing, in this way, the demonstration of the Rigidity theorem.

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According to Theorem 3.2, any point $p \in \mathbb{T}^2 \setminus P(\mathbb{G})$ is contained in a neighborhood equipped with a (nowhere zero) vector field X which is in the closure of $\mathbb{G}$. Of course it may happen that a point $p \in \mathbb{T}^2 \setminus P(\mathbb{G})$ possesses a neighborhood with vector fields X, Y as above which are also linearly independent at p. In this case one obtains a vector field Z vertical at p and contained in the closure of $\mathbb{G}$ by simply considering a suitable linear combination of X, Y.

Next *we define* $\mathcal{C}' \subset \mathbb{T}^2 \setminus P(\mathbb{G})$ to be the set formed by those points $p \in \mathbb{T}^2 \setminus P(\mathbb{G})$ which does not admit any neighborhood equipped with a vector field X, in the closure of $\mathbb{G}$, which is vertical or horizontal at p. Clearly any point $p \in \mathcal{C}'$ is such that all the vector fields in the closure of $\mathbb{G}$ defined around p are parallel at p. We now define $\mathcal{C} \subset \mathbb{T}^2$ as the closure of $\mathcal{C}'$ in $\mathbb{T}^2$. It is immediate to check that $\mathcal{C}$ is invariant under $\mathbb{G}$. The rest of the paper is devoted to showing that $\mathcal{C}$ is an analytic curve with the properties announced in Proposition 4.1.

Our strategy for proving that $\mathcal{C}$ is such a curve consists of showing that $\mathcal{C}$ is *locally* so, that is $\mathcal{C}$ is an analytic curve in a neighborhood of any point $p \in \mathbb{T}^2$. Note that $\mathcal{C}'$ is invariant under the local orbits of vector fields like X because it is invariant under $\mathbb{G}$ (in particular, $\mathcal{C}$ has no isolated points). Hence locally $\mathcal{C}'$ looks like a union of orbits of X. There are two distinct cases depending on whether or not p belongs to $P(\mathbb{G})$. Let us consider first the case $p \in \mathbb{T}^2 \setminus P(\mathbb{G})$.

We fix a point $p \in \mathbb{T}^2 \setminus P(\mathbb{G})$, a neighborhood U of p endowed with a (nowhere zero) vector field X which is in the closure of $\mathbb{G}$. The first part of Proposition 4.1 is established by the following.

PROPOSITION 5.1. *If $p \in \mathbb{T}^2 \setminus P(\mathbb{G})$ then there is a neighborhood V of p such that $\mathcal{C} \cap V$ is either empty or a smooth analytic curve.*

We begin by showing the following lemma.

LEMMA 5.2. *If $p \in \mathbb{T}^2 \setminus \mathcal{C}' \cup P(\mathbb{G})$, then there is a neighborhood V of p such that $V \cap \mathcal{C} = \emptyset$.*

Proof. Assume first that around p we have vector fields X, Y (in the closure of $\mathbb{G}$) linearly independent at p. Thus X, Y remain linearly independent at any point of a small neighborhood V of p. So every point of V possesses a vertical vector field in the closure of $\mathbb{G}$. It follows that $V \cap \mathcal{C} = \emptyset$.

We now assume that all vector fields in the closure of $\mathbb{G}$ and defined around p are parallel at p. After Lemma 3.3, we can assume the existence of $F = (f^1, f^2)$ in $\mathbb{G}$ ($f^1(p^1) \neq p^1$, $f^2(p^2) \neq p^2$) such that both f^1, f^2 have fixed points. Employing once again the argument used in the proof of Theorem 3.2, we see that the sequence $\{F^k(p)\}$ is converging and, for k large enough, contained in the domain of a (nowhere zero) vector field Y in the closure of $\mathbb{G}$. Furthermore, Y must be vertical (or horizontal) at every point $F^k(p)$. Therefore the first (or second) coordinate of Y has to be trivial† and a suitable pull-back $(F^k)^*Y$ provides a vector field in the closure of $\mathbb{G}$ which is vertical (or horizontal)

† Maybe $F^k(p)$ converges to a point at which Y is not defined. In this case Y is a 'Nakai's vector field' in the sense of Theorem 2.5. Even in this case the argument works because if such a Y is vertical (or horizontal) at one point of its domain then it must be vertical (or horizontal) in the whole domain.

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at any point of a neighborhood of p. Therefore, $\mathcal{C}$ does not intersect this neighborhood. The lemma is proved. □

Lemma 5.2 says that $\mathcal{C}'$ is a closed set in $\mathbb{T}^2 \setminus P(\mathbb{G})$), in particular $\mathcal{C}' = \mathcal{C} \cap (\mathbb{T}^2 \setminus P(\mathbb{G}))$. So we just need to prove Proposition 5.1 for the points p in $\mathcal{C}' = \mathcal{C} \setminus P(\mathbb{G})$.

Recall that, for $\mathfrak{m} \in \mathbb{N}$, we have sets $S_1(\mathfrak{m}) \subset G_1$ (respectively $S_2(\mathfrak{m}) \subset G_2$) associated to the set $\{f_1^1, \ldots, f_l^1\}$ (respectively $\{f_1^2, \ldots, f_l^2\}$) as in §2.2. Define $\mathbb{S}(\mathfrak{m}) \subset \mathbb{G}$ in an analogous way by means of the set $\{(f_1^1, f_1^2), \ldots, (f_l^1, f_l^2)\}$. Finally $G_1(\mathfrak{m})$ denotes the group generated by $S_1(\mathfrak{m}) \cup S(\mathfrak{m}-1) \cup S(\mathfrak{m}-2)$. Similarly one defines $G_2(\mathfrak{m})$, $\mathbb{G}(\mathfrak{m})$.

Take $p = (p^1, p^2) \in \mathcal{C}' = \mathcal{C} \setminus P(\mathbb{G})$. As usual let $X = (X^1, X^2)$ be a vector field in the closure of $\mathbb{G}$ which is defined in a neighborhood V of p. For fixed $\mathfrak{m}$, denote by $H_{\mathfrak{m},i} = (h^1_{\mathfrak{m},i}, h^2_{\mathfrak{m},i})$ the diffeomorphisms in $\mathbb{S}(\mathfrak{m}) \cup \mathbb{S}(\mathfrak{m}-1) \cup \mathbb{S}(\mathfrak{m}-2)$ ($i = 1, \ldots, r(\mathfrak{m})$). If $\mathfrak{m}$ is large enough, then $H_{\mathfrak{m},i}(p) = (h^1_{\mathfrak{m},i}(p^1), h^2_{\mathfrak{m},i}(p^2))$ belongs to V.

Now let ϕ_X^t be the local flow of X. Fix $\mathfrak{m}$ as above and, for each $H_{\mathfrak{m},i}$, let t_i be such that the local diffeomorphism $H^{(\phi)}_{\mathfrak{m},i} = \phi_X^{t_i} \circ H_{\mathfrak{m},i} = (\phi_{X^1}^{t_i} \circ h^1_{\mathfrak{m},i}, \phi_{X^2}^{t_i} \circ h^2_{\mathfrak{m},i})$ satisfies $\phi_{X^1}^{t_i} \circ h^1_{\mathfrak{m},i}(p^1) = p^1$. Let us consider the neighborhood of p given by $V \cap \bigcap_{i=1}^{r(\mathfrak{m})} H^{(\phi)}_{\mathfrak{m},i}(V)$.

Proof of Proposition 5.1. Because $p \in H^{(\phi)}_{\mathfrak{m},i}(V)$, it is enough to check that, for some $H^{(\phi)}_{\mathfrak{m},i}$ as before, the vector fields X and $(H^{(\phi)}_{\mathfrak{m},i})^* X$ are linearly independent at generic points. In fact, if X, $(H^{(\phi)}_{\mathfrak{m},i})^* X$ are linearly independent, then $\mathcal{C} \cap V \cap \bigcap_{i=1}^{r(\mathfrak{m})} H^{(\phi)}_{\mathfrak{m},i}(V)$ is contained in a local analytic curve along which these vector fields are parallel. Furthermore, $\mathcal{C}$ is invariant under the corresponding flows of X, $(H^{(\phi)}_{\mathfrak{m},i})^* X$ so that $\mathcal{C}$ is a union of common trajectories for these vector fields (in particular, it is smooth). Since it is also an analytic curve passing through p, the statement follows.

In view of the preceding discussion, let us suppose for a contradiction that all the vector fields in question are parallel at every point of $V \cap \bigcap_{i=1}^{r(\mathfrak{m})} H^{(\phi)}_{\mathfrak{m},i}(V)$. First observe that, for some $i \in \{1, \ldots, r(\mathfrak{m})\}$, $H^{(\phi)}_{\mathfrak{m},i}$ does not coincide with the identity, otherwise all the $H^{(\phi)}_{\mathfrak{m},i}$ would be embedded in a commutative group (locally given by ϕ_X^t). Hence $\mathbb{S}(\mathfrak{m}+1)$ would degenerate into {id} which is impossible. Now let y be such that (p^1, y) belongs to $V \cap \bigcap_{i=1}^{r(\mathfrak{m})} H^{(\phi)}_{\mathfrak{m},i}(V)$. Since X, $(H^{(\phi)}_{\mathfrak{m},i})^* X$ are parallel, one gets

$$(\phi_{X^2}^{t_i} \circ h^2_{\mathfrak{m},i})'(y) X^2(y) = (\phi_{X^1}^{t_i} \circ h^1_{\mathfrak{m},i})'(p^1) X^2(\phi_{X^2}^{t_i} \circ h^2_{\mathfrak{m},i}(y))$$

(where the prime $'$ stands for the derivative). Thus the action of $\phi_{X^2}^{t_i} \circ h^2_{\mathfrak{m},i}$ on X^2 consists of multiplying X^2 by a constant (namely $(\phi_{X^1}^{t_i} \circ h^1_{\mathfrak{m},i})'(p^1)$). However $\phi_{X^2}^{t_i}$ preserves X^2. It follows that $h^2_{\mathfrak{m},i}$ also acts on X^2, multiplying it by the same constant. Parametrizing (locally) the vertical line through p by the flow of X^2, one sees that the group generated by $h^2_{\mathfrak{m},i}$ ($i = 1, \ldots, r(\mathfrak{m})$) is conjugate to a subgroup of the linear affine group. Therefore $G_2(\mathfrak{m})$ is solvable which gives us a contradiction. Proposition 5.1 is proved. □

After Proposition 5.1, the proof of Proposition 4.1 is essentially reduced to study the behavior of $\mathcal{C}$ around points in $P(\mathbb{G})$. In fact we shall see that it is sufficient to analyze the case in which $p \in \mathrm{Per}(\mathbb{G})$, which in turn is settled by the next proposition.

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PROPOSITION 5.3. *Let* $p = (p_1, p_2)$ *be a point belonging to* $\mathrm{Per}(\mathbb{G}) \cap \mathcal{C}$. *There is a neighborhood* $V \subset \mathbb{T}^2$ *of* p *such that either* $V \cap \mathcal{C}$ *is the graph of a strictly monotone* C^{ω}*-function or it is the union of the graphs of two strictly monotone* C^{ω}*-functions (one increasing and the other decreasing).*

Before proving Proposition 5.3 we want to show how Proposition 4.1 can be deduced from it.

Proof of Proposition 4.1. It is already known that $\mathcal{C}$ is invariant under $\mathbb{G}$. We have to prove that $\mathcal{C}$ is an analytic curve. Because $\mathbb{T}^2$ is compact, it suffices to see that any point $p = (p^1, p^2) \in \mathbb{T}^2$ is contained in a neighborhood U such that $U \cap \mathcal{C}$ is such a curve. For a $p \in \mathbb{T}^2 \setminus P(\mathbb{G})$, the assertion follows from Proposition 5.1. If p belongs to $\mathrm{Per}(\mathbb{G})$ then the existence of such a neighborhood results from Proposition 5.3. Hence we just need to prove that, if $p \in P(\mathbb{G}) \setminus \mathrm{Per}(\mathbb{G})$, then there is U so that $U \cap \mathcal{C} = \emptyset$.

Fix $p = (p^1, p^2) \in P(\mathbb{G}) \setminus \mathrm{Per}(\mathbb{G})$. There is no loss of generality in supposing that p^1 is periodic for G_1. Using the $\mathbb{G}$-invariance of $\mathcal{C}$, it is enough to check the existence of an arbitrary point $\tilde{p}$, in the $\mathbb{G}$-orbit of p, such that $\mathcal{C} \cap U = \emptyset$. Since $\mathrm{Per}(G_1)$ is invariant under G_1, one can also assume the existence of $F = (f^1, f^2) \in \mathbb{G}$ verifying $f^1(p^1) = p^1$ and such that $\{(f^2)^k(p^2)\}$ converges to a point q^2 in $\mathrm{Per}(G_2)$ (in particular, p^2 is not fixed by f^2). Notice that (p^1, q^2) belongs to $\mathrm{Per}(\mathbb{G})$, hence there exists a neighborhood V of (p^1, q^2) as in Proposition 5.3.

Suppose by contradiction the existence of a sequence $\{(x_n, y_n)\}$, contained in $\mathcal{C}' \subset \mathbb{T}^2 \setminus P(\mathbb{G})$, converging to (p^1, p^2). Fix k so large that $F^k(p^1, p^2)$ lies on V. This implies that sequence $\{F^k(x_n, y_n)\}_{n\in\mathbb{N}}$ is contained in $\mathcal{C}' \cap V$ (for n large). Furthermore this sequence converges to a point of the form $(p^1, \tilde{q}^2) \in V$ with $\tilde{q}^2 \neq q^2$. It follows that $\mathcal{C} \cap V$ cannot be the graph of a regular function. The resulting contradiction shows that $\mathcal{C}$ is analytic.

It remains to verify that $\mathcal{C}$ satisfies properties (2) and (3) of Proposition 4.1. Naturally property (3) is nothing but Proposition 5.3. As to property (2), we see that, if $p \in \mathcal{C} \setminus \mathrm{Per}(\mathbb{G})$, then $\mathcal{C}$ is smooth around p because it is tangent to a non-singular vector field which in addition is neither vertical nor horizontal. On the other hand, if $p \in \mathcal{C} \cap \mathrm{Per}(\mathbb{G})$, then the statement follows once again from Proposition 5.3. The proof of Proposition 4.1 is over. □

The problem is therefore reduced to proving Proposition 5.3. Since this proposition has local nature, we identify (p^1, p^2) with $(0, 0)$. Let $V^{(1)} \subset \mathbb{R}^2$ be given by $\{(x, y) \in \mathbb{R}^2 : 0 < x < \epsilon \text{ and } 0 < y < \epsilon\}$ for some small $\epsilon > 0$. We denote by $V^{(2)}$ (respectively $V^{(3)}$, $V^{(4)}$) the square obtained by reflexion of $V^{(1)}$ with respect to $\{x = 0\}$ (respectively $(0, 0)$, $\{y = 0\}$). Put $W = \overline{V^{(1)} \cup V^{(3)}}$ (where the bar stands for the topological closure). By 'symmetry' it is enough to prove that $\mathcal{C} \cap W$ is the graph of a monotone increasing C^{ω}-function (provided that it is neither empty nor reduced to $(0, 0)$).

We shall often write $G_1 \times G_2$ instead of $\mathbb{G}$ (similarly to as in §2.3) to emphasize the local nature of some arguments. It is enough to consider the case in which $(0, 0)$ is accumulated by points in $\mathcal{C} \cap V^{(1)}$. Unless otherwise stated, we shall suppose the existence of $(f^1, f^2) \in G_1 \times G_2$ ($\simeq \mathbb{G}$) such that $(f^1)^k(x), (f^2)^k(y)$ tend to zero when $k \to \infty$ (provided that x, y are positive and small). Under this assumption $V^{(1)}$ is endowed with a non-singular Nakai vector field $X = (X^1, X^2)$.

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If X is neither vertical nor horizontal (in the sense of Theorem 2.5, i.e. neither X^1 nor X^2 is trivial), one can employ the same reasoning used in Proposition 5.1 to find $H = (h^1, h^2) \in G_1 \times G_2$ for which H^*X and X are linearly independent at generic points of the intersection of their domains. Since H preserves vertical and horizontal lines, it can be supposed that this intersection contains $V^{(1)}$ (maybe reducing $\epsilon > 0$). It results that $\mathcal{C}' \cap V^{(1)}$ is an analytic curve given by a union of common trajectories of X and H^*X. Write $H^*X = ((h^1)^*X^1, (h^2)^*X^2)$.

The coordinates X^1, X^2 (respectively $(h^1)^*X^1$, $(h^2)^*X^2$) of X (respectively H^*X) can obviously be considered as C^ω-functions defined on $(0, \epsilon)$. In general, none of these functions admit a C^ω-extension to a neighborhood of $0 \in \mathbb{R}$. However all of them possess an *asymptotic expansion* at $0 \in \mathbb{R}$. Let us explain the last assertion. The formal series $\mathbf{s} = \sum_{i=0}^{\infty} c_i^1 x^i$ is called the *asymptotic expansion* of X^1 if there are constants Const_i such that, for every $n \in \mathbb{N}$, one has

$$\left\| X^1(x) - \sum_{i=0}^{n} c_i^1 x^i \right\| < \text{Const}_n \, \|x\|^{n+1}$$

for small $x \in \mathbb{R}$ ($x > 0$). The existence of an asymptotic expansion $\mathbf{s}^1$ for X^1 (respectively X^2, $(h^1)^*X^1$, $(h^2)^*X^2$) follows from the construction of the ('ordinary') Nakai's vector field X^1. Indeed X^1 is defined and holomorphic in a sector $\mathcal{S}_\theta$ having vertex at $0 \in \mathbb{C}$ and positive angle θ (and also 'positive radius'). Furthermore, keeping $x \in \mathcal{S}_\theta$, the limit $\lim_{x\to 0}(X^1)^{(r)}(x)$ exists for any $r \in \mathbb{N}$ (where $(X^1)^{(r)}$ stands for the rth derivative of X^1). This property guarantees the existence of the corresponding asymptotic expansion. Actually one has $c_r^1 = r! \lim_{x\to 0}(X^1)^{(r)}(x)$ (cf. [**Car**]). Naturally similar conclusions do hold for X^2, $(h^1)^*X^1$, $(h^2)^*X^2$.

LEMMA 5.4. *Suppose that $\mathcal{C}' \cap V^{(1)}$ accumulates $(0, 0)$ ($\simeq (p_1, p_2)$). Then it is a union of trajectories of X, all of them converging to $(0, 0)$.*

Proof. Obviously X is neither vertical nor horizontal (in the sense of Theorem 2.5), otherwise $\mathcal{C}'$ is disjoint from $V^{(1)}$. We parametrize $X = (X^1, X^2)$ so that, for small positive $x \in \mathbb{R}$, $\phi^t_{X^1}(x)$ goes to zero when t increases (where $\phi^t_{X^1}(x)$ is the local flow of X^1). Because $\mathcal{C}' \cap V^{(1)}$ is a union of trajectories of X, it suffices to check that any trajectory of X converges to $(0, 0)$. This will be obtained by proving that $\phi^t_{X^2}(y) \to 0$ when t increases, provided that y is small and positive ($\phi^t_{X^2}(y)$ stands for the local flow of X^2). Hence we suppose by contradiction that the orientation of X^2 is such that $\phi^t_{X^2}(y)$ increases with t for y positive. It results that X has a saddle-like behavior on $V^{(1)}$. Since $\mathcal{C}' \cap V^{(1)}$ accumulates $(0, 0)$, we conclude that, for any arbitrarily small $\delta > 0$, $\mathcal{C}' \cap V^{(1)}$ cuts the section $\{(\epsilon/2, y) : 0 < y < \delta\}$ infinitely many times. Therefore the equation

$$X^2(y) - \frac{X^1(\epsilon/2)}{(h^1)^*X^1(\epsilon/2)}(h^2)^*X^2(y) = 0 \tag{8}$$

has infinitely many solutions for $0 < y < \delta$. On the other hand, the formal equation which arises from replacing X^1, X^2 and $(h^1)^*X^1$, $(h^2)^*X^2$ by their asymptotic expansion in (8) is not trivial because X and H^*X are linearly independent at generic points (and we precisely know the asymptotic expansion of a Nakai vector field). Thus this formal equation shows

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that (8) can have only a finite number of solutions for small $y \in \mathbb{R}$ ($y > 0$). A contradiction that proves the lemma then follows. □

Clearly we are interested in the case in which $\mathcal{C}' \cap V^{(1)}$ accumulates $(0,0)$ ($\simeq(p_1, p_2)$). Therefore we assume that Nakai's vector field X satisfies $\lim_{t\to\infty} \phi^t_{X^1}(x) = \lim_{t\to\infty} \phi^t_{X^2}(y) = 0$. After Lemma 5.4, $\mathcal{C}' \cap V^{(1)}$ is the union of a number of trajectories of X, all of them ending at $(0,0) \in \mathbb{R}^2$. Let us call $\mathcal{T}$ one of these trajectories. It can immediately be checked that $\mathcal{T}$ can be given as the graph of a C^{ω}-function $\mathfrak{h} : (0, \epsilon) \to \mathbb{R}$ which is monotone increasing.

LEMMA 5.5. *Suppose that* $X, \mathcal{T}, \mathfrak{h}$ *are as above. Then* $\mathfrak{h}$ *admits a* C^{ω}*-extension* $\tilde{\mathfrak{h}}$ *to a neighborhood of zero which (locally) conjugates two non-solvable subgroups of* $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$.

Proof. Modulo choosing an appropriate subgroup of $G_1 \times G_2$, we see that $\mathcal{T}$ can be supposed invariant under $G_1 \times G_2$. Let $\mathfrak{h}^{(r)}$ be the rth derivative of $\mathfrak{h}$. The limit $\lim_{x\to 0^+} \mathfrak{h}^{(r)}(x)$ clearly exists for every $r \in \mathbb{N}$. According to Borel (cf. [**Car**]), there is a C^{∞}-function $\overline{\mathfrak{h}}$ defined in a neighborhood of $0 \in \mathbb{R}$ which coincides with $\mathfrak{h}$ on $(0, \epsilon)$. In particular, $\overline{\mathfrak{h}}^{(r)}(0) = \lim_{x\to 0^+} \mathfrak{h}^{(r)}(x)$.

Because $\mathcal{T}$ is invariant under $G_1 \times G_2$, one concludes that $\mathfrak{h}$ is a local conjugacy between the (local) actions of G_1 and G_2 restricted to $(0, \epsilon)$. We claim that the Taylor series of $\overline{\mathfrak{h}}$ at zero furnishes a formal conjugacy between G_1, G_2. Indeed assume by contradiction that $h^1 \in G_1$ and $h^2 \in G_2$ are not formally conjugated by the series of $\overline{\mathfrak{h}}$ (where $h^1 \times h^2$ belongs to $G_1 \times G_2$). Thus the C^{∞}-local diffeomorphism $\overline{\mathfrak{h}} \circ h^1 \circ \overline{\mathfrak{h}}^{-1} \circ (h^2)^{-1}$ is not infinitely tangent to the identity at $0 \in \mathbb{R}$. However it coincides with the identity on $(0, \epsilon)$. A contradiction that proves the preceding claim follows.

To finish the proof, we recall a result of Cerveau and Moussu establishing the convergence of a formal conjugacy between two non-solvable subgroups of $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$ [**CeMo**]. Therefore the Taylor series of $\overline{\mathfrak{h}}$ converges and defines a local analytic diffeomorphism $\tilde{\mathfrak{h}}$ around $0 \in \mathbb{R}$. Obviously $\tilde{\mathfrak{h}}$ is the desired extension of $\mathfrak{h}$. □

Recall that we have supposed the existence of $(f^1, f^2) \in G_1 \times G_2$ such that $\lim_{n\to\infty}(f^1)^k(x) = \lim_{n\to\infty}(f^2)^k(y) = 0$ (provided that x, y are small and positive). Let us prove Proposition 5.3 in this special case.

Proof of Proposition 5.3 in the special case. Let $p = (p^1, p^2) \in \mathrm{Per}(\mathbb{G})$ be a point accumulated by points in $\mathcal{C}'$. We want to show that $\mathcal{C} \cap W$ is the graph of a monotone analytic function (as long as it is neither empty nor reduced to one point). Naturally we are assuming that p is accumulated by points in $\mathcal{C}' \cap V^{(1)}$.

We equip $V^{(1)}$ with a Nakai's vector field $X = (X^1, X^2)$. Thanks to Lemma 5.4, we know that the corresponding local flows $\phi^t_{X^1}, \phi^t_{X^2}$ are such that $\phi^t_{X^1}(x), \phi^t_{X^2}(y)$ go to zero when t increases (where x, y are positive and small). Actually $\mathcal{C}' \cap V^{(1)}$ is a union of trajectories of X which end at $(0,0) \in \mathbb{R}^2$. Obviously there is at least one such trajectory. We shall prove that there is exactly one such trajectory.

Suppose there were distinct trajectories $\mathcal{T}_1, \mathcal{T}_2$ as above. Again passing to a suitable subgroup of $G_1 \times G_2$ we can assume that both $\mathcal{T}_1, \mathcal{T}_2$ are invariant under $G_1 \times G_2$. These

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curves are the graphs of C^ω-functions $\mathfrak{h}_1, \mathfrak{h}_2$ defined on $(0, \epsilon)$. According to Lemma 5.5, $\mathfrak{h}_1, \mathfrak{h}_2$ have C^ω-extensions $\tilde{\mathfrak{h}}_1, \tilde{\mathfrak{h}}_2$ to a neighborhood of $0 \in \mathbb{R}$. Furthermore, $\tilde{\mathfrak{h}}_1, \tilde{\mathfrak{h}}_2$ are conjugacies between G_1, G_2. Because G_1, G_2 are non-solvable groups, it is well known that such a conjugacy is uniquely determined. In other words, $\tilde{\mathfrak{h}}_1 = \tilde{\mathfrak{h}}_2$ which is a contradiction. One concludes that $\mathcal{C}' \cap V^{(1)}$ consists of one unique trajectory $\mathcal{T}$.

Now, again using Lemma 5.5, $\mathcal{T}$ is associated to a C^ω-conjugacy $\tilde{\mathfrak{h}}$ between G_1, G_2 which is monotone increasing in $[0, \epsilon)$ (and $\tilde{\mathfrak{h}}(0) = 0$). Since $\tilde{\mathfrak{h}} \circ f^1 \circ \tilde{\mathfrak{h}}^{-1} = f^2$ we see that f^1, f^2 allow us to define a Nakai vector field on $V^{(3)}$ which has properties analogous to those of X. Therefore one can deduce that $\tilde{\mathfrak{h}}$ is also monotone increasing in $V^{(3)}$. The proof of the special case is over. □

We now drop the assumption on the existence of $(f^1, f^2) \in G_1 \times G_2$ as above. In order to prove Proposition 5.3 in this general setting, we shall need the fact that $G_1 \times G_2$ has a global realization as $\mathbb{G}$.

Proof of Proposition 5.3 in the general case. Fix $p = (p^1, p^2) \in \mathrm{Per}(\mathbb{G})$. Again we shall prove that $\mathcal{C} \cap W$ is the graph of a monotone analytic function (provided that $\mathcal{C}' \cap V^{(1)}$ accumulates p). After the special case treated above, we just need to verify the existence of $(f^1, f^2) \in G_1 \times G_2$ such that $\lim_{k\to\infty}(f^1)^k(x) = \lim_{k\to\infty}(f^2)^k(y) = 0$. Replacing f^1 by $(f^1)^{-1}$, we can assume that f^1 satisfies the required condition. Therefore let us suppose by contradiction that $(f^2)^{-k}(y) \to 0$ when k increases (y small and positive).

We recall that (f^1, f^2) has a global realization. Following the positive orbit of a point $q = (q^1, q^2) \in \mathcal{C}' \cap V^{(1)}$ by (f^1, f^2), we see that this orbit converges to a point $(p^1, \tilde{p}^2)$. We can suppose that $\tilde{p}^2$ is fixed by a suitable subgroup of G_2, otherwise there is a vector field in the closure of $\mathbb{G}$ defined around $(p^1, \tilde{p}^2)$ (cf. Proposition 2.7). If X is such a vector field, we immediately see that $\mathcal{C}' \cap V^{(1)}$ cannot accumulate (p^1, p^2).

On the other hand, assuming that $(p^1, \tilde{p}^2)$ is fixed by a suitable subgroup of $G_1 \times G_2$ ($\simeq \mathbb{G}$), we can apply the proposition in the special case to $(p^1, \tilde{p}^2)$. It results that $\mathcal{C}$ is the graph of a monotone C^ω-function around $(p^1, \tilde{p}^2)$. As $\mathcal{C}'$ is invariant under $\mathbb{G}$, we see that it cannot accumulate (p^1, p^2). The resulting contradiction proves the statement. □

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A. *Appendix. Proof of Proposition 2.7*

I have tried my best in order to make this appendix as self-contained as possible. This attempt explains the presence of some overlaps with [**Reb**]. We keep the notation of §2.4.

Let us start our approach to Proposition 2.7 by recalling a few basic facts about local holomorphic diffeomorphisms fixing $0 \in \mathbb{C}$. Let $f(x)$ be such a local diffeomorphism. For our purposes, we can assume that $f \in \mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$. We remind the reader that, if $f'(0) \neq \pm 1$, then f is locally holomorphic conjugate to the homothety $x \mapsto f'(0)x$ (here x is a complex variable).

On the other hand, the case $f'(0) = \pm 1$ can also occur. However, since we are able to replace f by f^{-1} as well as to iterate f twice, it is sufficient to deal with the case

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$f'(0) = 1$ and $f(x) < x$ for very small real positive x. For fixed positive $\theta, r \in \mathbb{R}$, the sector $\mathcal{S}_{r,\theta}$ is defined by

$$\mathcal{S}_{r,\theta} = \{x \in \mathbb{C} : x = |x|e^{i\alpha} \text{ where } -\theta < \alpha < \theta \text{ and } 0 <| x |< r\}.$$

For an f normalized as above, the next lemma is fairly easy and well known.

LEMMA A.1. *There are positive real numbers* r, θ *so that* $\mathcal{S}_{r,\theta}$ *verifies the properties below:*

(1) *for any* x *in* $\mathcal{S}_{r,\theta}$, $f^k(x)$ *is defined for all* $k \in \mathbb{N}$ *and converges to* $0 \in \mathbb{C}$ *when* $k \to \infty$*;*

(2) *if* V *is an open (relatively compact) set of* $\mathcal{S}_{r,\theta}$, *then* $f^k : V \to f^k(V)$ *is a holomorphic diffeomorphism (for any* $k \in \mathbb{N}$*).*

Let us return to Proposition 2.7. Maybe performing a rescaling of coordinates, we can add one more assumption on $\Gamma_{\Gamma_1 \times \Gamma_2}$ without loss of generality: if $|(f^j)'(0)| \neq 1$ $(j = 1, 2)$, then f^j is analytically conjugate to the homothety $x \mapsto (f^j)'(0)x$ in the whole B. Furthermore, if $(f^j)'(0) = 1$, then there is a sector $\mathcal{S}_{1,\theta}$ $(r = 1, \theta > 0)$ which satisfies the conclusions of Lemma A.1. In the following these hypotheses will be assumed with no additional comment.

From now on, we fix some $p \in \mathbb{R}$ $(0 < p < 1)$ and denote by p^2 the point $(p, p) \in \mathbb{C}^2$. According to assumption (2c) about $\Gamma_{\Gamma_1 \times \Gamma_2}$, $(f^j)^k(p)$ converges to $0 \in \mathbb{C}$ when k goes to infinity. For $\varepsilon > 0$, we write $R_{1\varepsilon}$ to denote the rectangle of $\mathbb{C}$ whose vertices are $(f^1(p), \varepsilon), (p, \varepsilon), (p, -\varepsilon), (f^1(p), -\varepsilon)$. Analogously $R_{2\varepsilon}$ is the rectangle with vertices $(f^2(p), \varepsilon), (p, \varepsilon), (p, -\varepsilon), (f^2(p), -\varepsilon)$. Finally R_ε is $R_{1\varepsilon} \times R_{2\varepsilon}$.

LEMMA A.2. *If* $\varepsilon > 0$ *is sufficiently small then the following assertions hold for* $j = 1, 2$.

(1) $R_{j\varepsilon}$ *is contained in the basin of* f^j *and* $(f^j)^k : R_{j\varepsilon} \to (f^j)^k(R_{j\varepsilon})$ *is a holomorphic diffeomorphism.*

(2) *There is a positive constant* Const *such that*

$$\|(f^j)^k(x) - (f^j)^k(p)\| \leq \text{Const}\, \|(f^j)^k(p) - (f^j)^{k+1}(p)\| \tag{A.1}$$

for every $x \in R_{j\varepsilon}$ *(where* $\|(f^j)^k(x) - (f^j)^k(p)\|$ *is the norm of the vector whose extremities are* $(f^j)^k(p)$ *and* $(f^j)^k(x)$*).*

(3) *There are positive constants* $C_2 \geq C_1 > 0$ *such that*

$$0 < C_1 < \frac{\sup_{x \in R_{j\varepsilon}} \|D_{(f^j)^k(x)}((f^j)^{-k})\|}{\inf_{x \in R_{j\varepsilon}} \|D_{(f^j)^k(x)}((f^j)^{-k})\|} \leq C_2. \tag{A.2}$$

Proof. The case in which $0 \in \mathbb{C}$ is a hyperbolic fixed point of f^j is easy. Indeed, under this assumption and in appropriate coordinates, f^j is a contractive homothety defined in B. Since the requested features are obvious for such a homothety, the lemma is proved by noticing that these features are invariant under change of coordinates.

We now assume that $(f^j)'(0) = 1$. In this case claim (1) results immediately from Lemma A.1. Thus we just need to show claims (2) and (3). For a very small $\varepsilon > 0$, $R_{j\varepsilon}$ $(j = 1, 2)$ is enclosed in a strictly larger domain U still contained in $A_{1,\theta}$ so that it verifies

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the conclusions of Lemma A.1. We put $\tilde{p}_{(j)} = (p + (f^j)(p))/2$ and $\lambda_{jk} = 1/((f^j)^k)'(p)$. In order to prove claim (2), let us consider the mappings $T_{jk} : U \to \mathbb{C}$ given by

$$z \longmapsto \lambda_{jk}(f^j)^k(z) + (\tilde{p}_{(j)} - \lambda_{jk}(f^j)^k(\tilde{p}_{(j)})).$$

One sees immediately that T_{jk} is univalent (one-to-one) on U. Besides $T_{jk}(\tilde{p}_{(j)}) = \tilde{p}_{(j)}$ and $T'_{jk}(\tilde{p}_{(j)}) = 1$. Thanks to the homogeneity of (A.1), it is enough to prove the existence of a uniform constant Const for which one has $\|T_{jk}(z) - T_{jk}((f^j)(p))\| \leq \text{Const}\,\|T_{jk}(p) - T_{jk}((f^j)(p))\|$. Nevertheless, the latest estimate is a consequence of Köebe's bounded distortion theorem, since there is a Riemann mapping from U to the unit disc taking $\tilde{p}_{(j)}$ at the center of the disc (note also that T_{jk} is conveniently normalized (cf. [**Alf**])).

It remains to prove claim (3) under the hypothesis that $(f^j)'(0) = 1$. The homogeneity of (A.2) permits us to multiply the function $(f^j)^{-k} : (f^j)^k(U) \to U$ by $D_{(f^j)^k(\tilde{p}_{(j)})}((f^j)^{-k})$ in order to get an univalent function with derivative 1 at $(f^j)^k(\tilde{p}_{(j)})$. By means of the Riemann mapping considered above, Köebe's bounded distortion theorem (version for derivatives) applies to provide the existence of constants C_1, C_2 as required. The proof of the lemma is over. □

Let us assume from now until the end of this section that such a $\varepsilon > 0$ is fixed. Hence so are the rectangles $R_{j\varepsilon}$ $(j = 1, 2)$.

An important part of our construction derives from the simple remark according to the which diffeomorphisms C^1-close to the identity behave in short distances like translations. This remark is going to be accurately stated by the next two lemmas which are elementary and borrowed from [**Reb**].

For an open set $U \subseteq \mathbb{C}^N$ $(\simeq R^{2N})$ and a C^1-mapping h from U to $\mathbb{C}^N$ $(\simeq \mathbb{R}^{2N})$, one denotes $\|h\|_{1,U} = \sup\{\|h(x)\| + \|D_x h\| : x \in U\}$, where $D_x h$ is the differential of h at x. Sometimes we denote by $d(.,.)$ the Euclidean distance of $\mathbb{C}^N$, in other cases we use the symbol $\|\cdot\|$ to denote the norm of a vector. At last, for mappings h_1, h_2 defined in a domain of $\mathbb{C}^N$ and taking values in $\mathbb{C}^N$, the notation $(h_1 - h_2)(x) = h_1(x) - h_2(x)$ stands for the vector whose origin is $h_2(x)$ and whose final extremity is $h_1(x)$.

LEMMA A.3. *Let $U \subseteq \mathbb{C}$ be an open set and $V \subseteq U$ be a relatively compact subset of U. Given arbitrary numbers $\mathbf{n}_0 \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, as long as $\|h - \mathrm{id}\,\|_{1,U} < \delta$, one has*

$$\|(h^n - \mathrm{id})(x) - n(h - \mathrm{id})(x)\| < \varepsilon\|(h - \mathrm{id})(x)\|,$$

for every $x \in V$ and $0 \leq n \leq 2\mathbf{n}_0$ (where h^n is the nth iterated $h \circ \cdots \circ h$).

LEMMA A.4. *Let V, U as in Lemma A.3. Given $\mathbf{n}_0 \in \mathbb{N}$ and fixed positive constants C_3, C_4, there is $\delta > 0$ such that*

$$\|(h - \mathrm{id})(y) - (h - \mathrm{id})(x)\| \leq \|(h - \mathrm{id})(x)\|/4C_4,$$

provided that $x, y \in V$,

$$\|(h - \mathrm{id})\|_{1,U} < \delta \text{ and } d(x, y) < C_3\|(h^{2\mathbf{n}_0} - \mathrm{id})(x)\|.$$

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The desired vector field X is going to be built up as the limit of a suitable sequence of vector fields. We start the construction of this sequence by choosing a monotone increasing sequence by of positive integers $\{\mathbf{n}_J\}$. To each $\mathbf{n}_J$ we shall associate a vector field X_J so that a subsequence $\{X_J\}$ will converge to a vector field X which has the required properties.

One defines $c = \max\{1, \sup_{z\in B}|(f^1)'|, \sup_{z\in B}|(f^1)^{-1'}|, \sup_{z\in B}|(f^2)'|, \sup_{z\in B}|(f^2)^{-1'}|\}$. We choose and fix $\mathbf{n}_J \in \mathbb{N}$. Because the diffeomorphisms $H_d \in \mathcal{H}$ converge uniformly to the identity on B^2, by applying Cauchy's integral formula, we can suppose that they converge in the C^1 (C^∞) topology (modulo restricting our domain). From Lemmas A.3 and A.4 it results that, for a fixed $\mathbf{n}_J$, there exists $\mathbf{n}_{0J} \in \mathbb{N}$ such that, as long as $J > \mathbf{n}_{0J}$, the diffeomorphisms h_J^1, h_J^2 ($(h_J^1, h_J^2) \in \mathcal{H}$) satisfy the estimates

$$\|((h_J^j)^n - \mathrm{id})(x) - n(h_J^j - \mathrm{id})(x)\| \le \frac{1}{2c\mathbf{n}_J \,\mathrm{Const}_1}\|(h_J^j - \mathrm{id}(x)\|, \tag{A.3}$$

for every $x \in B$ and $0 \le n < 2\mathbf{n}_J$. Furthermore,

$$\|(h_J^j - \mathrm{id})(y) - (h_J^j - \mathrm{id})(x)\| \le \frac{1}{4\mathbf{n}_J}\|(h_J^j - \mathrm{id})(x)\|, \tag{A.4}$$

provided that $d(y, x) < 2c\max\{1, \mathrm{Const}_1\}\|((h_J^j)^{2\mathbf{n}_J} - \mathrm{id})(x)\|$.

Let now us fix a $H_J \in \mathcal{H}$ which fulfils the preceding conditions and such that H_J^{-1} fulfils the same conditions as well. Thus we have already fixed $p, \varepsilon > 0, \mathbf{n}_J \in \mathbb{N}$ and $H_J \in \mathcal{H}$. Next we define $k_J \in \mathbb{N}$ as the *smallest* positive integer for which at least one of the inequalities below holds:

$$(f^1)^{k_J}(p) - (f^1)^{k_J+1}(p) < \mathbf{n}_J\|(h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\| \tag{A.5}$$

or

$$(f^2)^{k_J}(p) - (f^2)^{k_J+1}(p) < \mathbf{n}_J\|(h_J^2 - \mathrm{id})((f^2)^{k_J}(p))\|. \tag{A.6}$$

The reader should observe that such a k_J always exists because $f^1(0) = f^2(0) = 0$ and, at least for one $j \in \{1, 2\}$, $h_J^j(0) > 0$. On the other hand, we can have $k_J > 3$ since it can be supposed that $\mathbf{n}_J\|(h_J^j - \mathrm{id})(p)\|$ converges to zero (in fact when $\mathbf{n}_J$ is fixed the corresponding diffeomorphism H_J can be chosen arbitrarily C^1-close to the identity). Passing to an appropriate subsequence, there is no loss of generality in supposing also that, for every $\mathbf{n}_J$, k_J occurs for f^1. We shall work on this assumption in the following and we notice that it gives us the following converse inequality

$$(f^2)^{k_J-1}(p) - (f^2)^{k_J}(p) \ge \mathbf{n}_J\|(h_J^2 - \mathrm{id})((f^2)^{k_J}(p))\|. \tag{A.7}$$

The next lemma furnishes us with two estimates which are elementary but crucial for the construction of our vector field.

LEMMA A.5. *We still have the following estimates:*

$$(\mathbf{n}_j - 1)\|(h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\| \le (f^1)^{k_J-1}(p) - (f^1)^{k_J}(p); \tag{A.8}$$

$$\|(h_J^1 - \mathrm{id})(y) - (h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\| \le \|(h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\|/4, \tag{A.9}$$

whenever $d(y, (f^1)^{k_J}(p)) \le \mathrm{Const}\,\|(f^1)^{k_J-1}(p) - (f^1)^{k_J}(p)\|$ *(where* Const *is as in Lemma A.2).*

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Proof. We consider the points p, $f^1(p)$, $(f^1)^2(p)$, ... and the corresponding sequence of fundamental domains $[f^1(p), p]$, $[(f^1)^2(p), f^1(p)]$, ... Thanks to the definition of k_J, we have

$$(f^1)^{k_J}(p) - (f^1)^{k_J+1}(p) < \mathbf{n}_J \|(h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\|.$$

We may assume that $h_J^1 \circ (f^1)^{k_J}(p) > (f^1)^{k_J}(p)$ (since the other possibility is completely analogous). Hence estimates (A.3) and (A.4) give that

$$h_J^1 \circ (f^1)^{k_J+1}(p) > (f^1)^{k_J+1}(p)$$

and

$$h_J^1 \circ (f^1)^{k_J-1}(p) > (f^1)^{k_J-1}(p)$$

(where we have used the fact that the ratio

$$\|(f^1)^{k_J-1}(p) - (f^1)^{k_J}(p)\| / \|(f^1)^{k_J}(p) - (f^1)^{k_J+1}(p)\|$$

is less than c). We claim that estimate (A.8) holds. Indeed assume for a contradiction that one has the converse inequality. In this case inequality (A.4) gives us that

$$\|(h_J^1 - \mathrm{id})((f^1)^{k_J-1}(p)) - (h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\| \le \|(h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\|/4\mathbf{n}_J.$$

Therefore we conclude that

$$\begin{aligned}\mathbf{n}_J \|(h_J^1 - \mathrm{id})((f^1)^{k_J-1}(p))\| &\ge \mathbf{n}_j \|(h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\| - \|(h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\|/4 \\ &\ge (f^1)^{k_J-1}(p) - (f^1)^{k_J}(p),\end{aligned}$$

which contradicts the minimality of k_J. Thus the first estimate is verified.

It remains to check the second estimate. Consider $y \in B$ be such that

$$d(y, (f^1)^{k_J}(p)) < \mathrm{Const}\, \|(f^1)^{k_J-1}(p) - (f^1)^{k_J}(p)\|.$$

We get the sequence of estimates below:

$$\begin{aligned}d(y, (f^1)^{k_J}(p)) &< \mathrm{Const}\, \|(f^1)^{k_J-1}(p) - (f^1)^{k_J}(p)\| \\ &\le c\, \mathrm{Const}\, \|(f^1)^{k_J}(p) - (f^1)^{k_J+1}(p)\| \\ &\le c\mathbf{n}_J\, \mathrm{Const}\, \|(h_J^1 - \mathrm{id})((f^1)^{k_J}(p))\| \\ &\le 2c\, \mathrm{Const}\, \|((h_J^1)^{2\mathbf{n}_J} - \mathrm{id})((f^1)^{k_J}(p))\|,\end{aligned}$$

where in the last estimate we have used (A.3). Now the desired inequality follows from (A.4). The proof of the lemma is over. □

Let us define e_J as the *smallest* positive integer for which $(h_J^1)^{e_J} \circ (f^1)^{k_J}(p) > (f^1)^{k_J-1}(p)$ (we recall that we have assumed $h_J^1 \circ (f^1)^{k_J}(p) > (f^1)^{k_J}(p)$ so that the existence of such an e_J results from the discussion above). Applying the estimates we have obtained, the reader will easily conclude the next lemma.

LEMMA A.6. *There are uniform constants (independing on J)* c_5, c_6 *so that* $c_5\mathbf{n}_J < e_J \le c_6\mathbf{n}_J$. □

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We now proceed with the construction of the sequence of vector fields $\{X_J\}$. For fixed $\mathbf{n}_J \in \mathbb{N}$, we choose $H_J \in \mathcal{H}$, $k_J, e_J \in \mathbb{N}$ as above. The reader should also recall that $F = (f^1, f^2)$, $H_J = (h_J^1, h_J^2)$ and $R_\varepsilon = R_{1\varepsilon} \times R_{2\varepsilon}$. Let us define the vector field X_J on R_ε by the formula

$$X_J(x, y) = e_J . D_{F^{k_J}(x,y)}(F^{-k_J}).(H_J - \mathrm{id}) \circ F^{k_J}(x, y). \tag{A.10}$$

One observes that this formula enable us to write X_J under the form $X_J(x, y) = (X_J^1(x), X_J^2(y))$ where X_J^1 (respectively X_J^2) is a vector field on $R_{1\varepsilon}$ (respectively $R_{2\varepsilon}$) given by

$$X_J^1(x) = e_J . D_{(f^1)^{k_J}(x)}((f^1)^{-k_J}).(h_J^1 - \mathrm{id}) \circ ((f^1)^{k_J}(x))$$

(respectively $X_J^2(y) = e_J . D_{(f^2)^{k_J}(y)}((f^2)^{-k_J}).(h_J^2 - \mathrm{id}) \circ ((f^2)^{k_J}(y))$).

We still need one more lemma.

LEMMA A.7. *There are positive uniform constants* $C_8 \geq C_7 > 0$ *verifying*

$$C_7 \leq \inf_{R_\varepsilon} \|X_J(x, y)\| \leq \sup_{R_\varepsilon} \|X_J(x, y)\| \leq C_8.$$

Proof. Firstly we are going to find a positive lower bound for $\inf_{R_{1\varepsilon}} \|X_J^1\|$ which in turn will imply the left-hand side of the desired double inequality.

As f^1 preserves the real line, the mean value theorem enables us to conclude the existence of a point $q \in [(f^1)^{k_J+1}(p), (f^1)^{k_J}(p)]$ such that

$$D_q((f^1)^{-k_J}) = \|p - f^1(p)\| / \|(f^1)^{k_J}(p) - (f^1)^{k_J+1}(p)\|.$$

Now given $x \in R_{1\varepsilon}$, we can write

$$\begin{aligned}
&\|e_J D_{(f^1)^{k_J}(x}((f^1)^{-k_J})(h_J^1 - \mathrm{id}) \circ (f^1)^{k_J}(x)\| \\
&\quad \geq c_5 \mathbf{n}_J \|D_{(f^1)^{k_J}(x)}((f^1)^{-k_J}).(h_J^1 - \mathrm{id}) \circ (f^1)^{k_J}(x)\| \\
&\quad = c_5 \mathbf{n}_J \frac{\|D_{(f^1)^{k_J}(x)}((f^1)^{-k_J})\|}{\|D_q((f^1)^{-k_J})\|} \|D_q((f^1)^{-k_J})(h_J^1 - \mathrm{id}) \circ (f^1)^{k_J}(x)\| \\
&\quad \geq c_5 C_1 \mathbf{n}_J \|D_q((f^1)^{-k_J}).(h_J^1 - \mathrm{id}) \circ (f^1)^{k_J}(x)\|
\end{aligned}$$

(where in the first estimate we use Lemma A.6 and in the last one Lemma A.2).

Moreover, by applying Lemma A.2, we see that

$$d((f^1)^{k_J}(x), (f^1)^{k_J}(p)) \leq \mathrm{Const}\, \|(f^1)^{k_J}(p) - (f^1)^{k_J+1}(p)\|.$$

Hence Lemma A.5 shows that

$$\|(h_J^1 - \mathrm{id}) \circ (f^1)^{k_J}(x)\| \geq 3\|(h_J^1 - \mathrm{id}) \circ (f^1)^{k_J}(p)\|/4.$$

So the sequence of inequalities can be continued as follows:

$$\begin{aligned}
\|X_J^1(x)\| &\geq c_5 C_1 \mathbf{n}_J \|D_q((f^1)^{-k_J})\| \|(h_J^1 - \mathrm{id}) \circ (f^1)^{k_J}(x)\| \\
&\geq \frac{3}{4} c_5 C_1 \mathbf{n}_J \frac{\|p - (f^1)(p)\|}{\|(f^1)^{k_J}(p) - (f^1)^{k_J+1}(p)\|} \|(h_J^1 - \mathrm{id}) \circ (f^1)^{k_J}(p)\| \\
&\geq \frac{3}{4} c_5 C_1 \|p - f^1(p)\| = C_7 > 0 \quad \text{(by (A.6))},
\end{aligned}$$

proving the first part of the lemma.

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Now we have to prove that X^1_J (respectively X^2_J) is uniformly bounded on $R_{1\varepsilon}$ (respectively $R_{2\varepsilon}$). The case of X^1_J is left to the reader since the argumentation is completely analogous to that employed above (modulo using the other half of the same inequalities above). So we shall deal with X^2_J. Let $q_2 \in [(f^2)^{k_J+1}(p), (f^2)^{k_J}(p)]$ be such that one has

$$D_{q_2}((f^2)^{-k_J}) = \|p - f^2(p)\|/\|(f^2)^{k_J}(p) - (f^2)^{k_J+1}(p)\|.$$

Using again the same type of reasoning, we obtain

$$\|X^2_J(y)\| \leq c_6 C_2 \mathbf{n}_J \frac{\|p - f^2(p)\|}{\|(f^2)^{k_J}(p) - (f^2)^{k_J+1}(p)\|} \|(h^2_J - \mathrm{id}) \circ (f^2)^{k_J}(y)\|. \tag{A.11}$$

Therefore it is sufficient to show that

$$\mathbf{n}_J \|(h^2_J - \mathrm{id}) \circ (f^2)^{k_J}(y)\|/\|(f^2)^{k_J}(p) - (f_2)^{k_J+1}(p)\|$$

is less than $2c$. In fact if this were false, then one would have

$$\|((h^2_J)^{2\mathbf{n}_J} - \mathrm{id}) \circ (f^2)^{k_J}(y)\| > \|\,(f^2)^{k_J}(p) - (f^2)^{k_J+1}(p)\|$$

(thanks to inequalities (A.3) and (A.4)). Thus

$$\|(h^2_J - \mathrm{id}) \circ (f^2)^{k_J}(y) - (h^2_J - \mathrm{id}) \circ (f^2)^{k_J}(p)\| \leq \|(h^2_J - \mathrm{id}) \circ (f^2)^{k_J}(p)\|/4\mathbf{n}_J.$$

It follows that

$$\begin{aligned} \mathbf{n}_J \|(h^2_J - \mathrm{id}) \circ (f^2)^{k_J}(p)\| &> (\mathbf{n}_J - 1/4)\|\,(h^2_J - \mathrm{id}) \circ (f^2)^{k_J}(p)\| \\ &\geq c\|(f^2)^{k_J}(p) - (f^2)^{k_J+1}(p)\| \\ &> \|(f^2)^{k_J-1}(p) - (f^2)^{k_J}(p)\|, \end{aligned}$$

so that a contradiction with inequality (A.7) results. The lemma is proved. □

Proof of Proposition 2.7. Let us consider the sequence of vector fields $\{X_J\}$ defined on R_ε by (A.10). According to Lemma A.7 the sequence $\{X_J\}$ is uniformly bounded on R_ε. Furthermore each X_J is a holomorphic mapping from R_ε to $\mathbb{C}^2$; hence one concludes from Montel's theorem that the $\{X_J\}$ form a compact family on R_ε. Choosing a domain R^2_p strictly contained in R_ε, we can find a limit X (defined on R^2_p) for a suitable subsequence of $\{X_J\}$. Clearly X has no singularities on R^2_p since all the X_J have a positive uniform lower bound (Lemma A.7). Finally we have to verify the fact that X is in the closure of $\Gamma_{\Gamma_1 \times \Gamma_2}$. So we denote by ϕ^t_X the local flow associated to X. Consider a relatively compact open set $V \subset R^2_p$ and a certain $t_0 > 0$ such that $\phi^t_X(V) \subset R^2_p$ for any $0 \leq t \leq t_0$. In order to prove the proposition, it is enough to show that the sequence $\{((f^1)^{-k_J} \circ (h^1_J)^{[e_J t_0]} \circ (f^1)^{k_J}, (f^2)^{-k_J} \circ (h^2_J)^{[e_J t_0]} \circ (f^2)^{k_J})\}$ (where $[e_J t_0]$ stands for the integral part of $e_J t_0$) converges to $\phi^{t_0}_X$ uniformly on V. However, because $X_J(x, y) = (X^1_J(x), X^2_J(y))$ the resulting vector field X can also be written under the form (X^1, X^2) so that $\phi^t_X = (\phi^t_{X^1}, \phi^t_{X^2})$. Thus we can apply the argument of [**Reb**] (cf. also [**Na1**]) to see that the analogous convergence occurs on each coordinate. This means that $(f^1)^{-k_J} \circ (h^1_J)^{[e_J t_0]} \circ (f^1)^{k_J}$ (respectively $(f^2)^{-k_J} \circ (h^2_J)^{[e_J t_0]} \circ (f^2)^{k_J}$) converge to $\phi^{t_0}_{X^1}$ (respectively $\phi^{t_0}_{X^2}$) uniformly on a suitable domain. It follows that $\{((f^1)^{-k_J} \circ (h^1_J)^{[e_J t_0]} \circ (f^1)^{k_J}, (f^2)^{-k_J} \circ (h^2_J)^{[e_J t_0]} \circ (f^2)^{k_J})\}$ converges to $\phi^{t_0}_X$ as desired. The proof of the proposition is complete. □

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Remark A.8. The reader may easily deduce the statement of the higher-dimensional version of Proposition 2.7.

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