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The first moment of primitive cubic L-functions with fixed genus

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Abstract. We study the first moment of primitive cubic L-functions over $\mathbb{F}_q(T)$ in the non-Kummer setting $q \equiv 2 \bmod 3$. More precisely, we study the sum

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \operatorname{genus}(\chi)=g}} L_q\left(\frac{1}{2}, \chi\right),$$

where $L_q(s,\chi)$ denotes the L-function associated with primitive cubic character χ. Using a double Dirichlet series, we obtain an asymptotic formula with an error term of size $q^{\left(\frac{7}{8}+\varepsilon\right)g}$. The argument avoids an explicit residue computation and clarifies why the poles at $u^3 = 1$ do not contribute, thereby resolving the question left in Remark 4.6 of David et al. (2019, *algebra number theory*).

1 Introduction

In this article, we study the first moment of primitive cubic Dirichlet L-functions $L_q(s,\chi)$ over $\mathbb{F}_q(T)$ in the non-Kummer setting $q \equiv 2 \bmod 3$. More precisely, we average $L_q(s,\chi)$ over primitive cubic characters χ of fixed genus g. Our main result confirms the asymptotic formula established by David et al. [6] with a streamlined argument based on a double Dirichlet series.

The first moment of cubic L-functions has a substantial literature over number fields. Luo [9] analyzed the cubic twists over $\mathbb{Q}(\xi_3)$. Güloğlu and Yesilyurt [8] proved a mollified first-moment asymptotic over the Eisenstein field and David et al. [5] obtained a power-saving error term for the mollified second moment. Over $\mathbb{Q}$, Baier and Young [1] established the asymptotic formula

$$\sum_{(q,3)=1} \sum_{\substack{\chi \text{ primitive mod } q \\ \chi \text{ cubic}}} L\left(\frac{1}{2}, \chi\right) w\left(\frac{q}{Q}\right) = c\hat{w}(0)Q + O\left(Q^{\frac{37}{38}+\varepsilon}\right), \tag{1.1}$$

with an explicit constant c. This result was later refined by Gao and Zhao [7], who employed double Dirichlet series to improve the error term to

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$$\sum_{(q,3)=1}\sum_{\substack{\chi \text{ primitive mod } q\\ \chi \text{ cubic}}} L\left(\frac{1}{2},\chi\right)\Phi\left(\frac{q}{Q}\right) = C_3\hat{\Phi}(0)Q + O\left(Q^{\frac{7}{8}+\varepsilon}\right). \tag{1.2}$$

The key input in (1.2) is that one can continue the underlying double Dirichlet series to a larger region while preserving its functional equations. This continuation yields finer control of the generating series and ultimately produces the exponent $\frac{7}{8}$ in the error term. They established the functional equation for cubic L-function in terms of double Dirichlet series, whose coefficients are generalized Gauss sums. This method allows them to meromorphically continue the underlying double Dirichlet series to a larger region without altering it. This approach was first used by Soundararajan and Young [12], and later by Čech [4] to study the ratios conjecture for real Dirichlet characters.

In the function field setting, the mean value of cubic L-functions was first explored by Rosen [11], who averaged over all monic polynomials of a given degree. A significant advancement was made by David et al. [6], who investigated the first moment of cubic L-functions $L_q(s,\chi)$ over primitive cubic characters of fixed genus when $q \equiv 2 \bmod 3$, establishing the asymptotic formula

$$\sum_{\substack{\chi \text{ primitive cubic}\\ \text{genus}(\chi)=g}} L_q\left(\frac{1}{2},\chi\right) = \frac{\zeta_q(\frac{3}{2})}{\zeta_q(3)}\mathcal{A}_{nK}\left(\frac{1}{q^2},\frac{1}{q^{3/2}}\right)q^{g+2} + O\left(q^{\frac{7}{8}g+\varepsilon g}\right). \tag{1.3}$$

Here, $\zeta_q(s) = \frac{1}{1-q^{1-s}}$ and $\mathcal{A}_{nK}$ is an explicit function. Motivated by the work of Gao and Zhao [7], we adapt their approach using double Dirichlet series to compute the first moment of cubic L-functions in the function field setting. Our main result is stated as follows.

Theorem 1.1 *Let q be an odd prime power such that $q \equiv 2 \bmod 3$. Then*

$$\sum_{\substack{\chi \text{ primitive cubic}\\ \text{genus}(\chi)=g}} L_q\left(\frac{1}{2},\chi\right) = q^g(1-q^2)P(q^{-2})Z(q^{-2},q^{-\frac{1}{2}}) + O\left(q^{(\frac{7}{8}+\varepsilon)g}\right), \tag{1.4}$$

with $P(u)$ and $Z(u,v)$ given in Section 3.3.

This result confirms, and in fact refines, the asymptotic formula of David et al. [6], while substantially reducing the computational complexity. In particular, our method resolves the question left open in [6] concerning the putative contribution from the poles at $u^3 = 1$. We show that these poles lie outside the region of absolute convergence, and therefore the associated residue does not contribute to the asymptotic.

In [6], a secondary term of size $q^{5g/6}$ was conjectured to arise from the residue of the generating function at $u^3 = 1$, and considerable effort was devoted to its estimation. However, the dominant error originates from integration along the contour $|u| = q^{-(7/8+\varepsilon)}$, which already contributes an error term of order $q^{(\frac{7}{8}+\varepsilon)g}$; consequently, any such secondary term is necessarily absorbed by the error.

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Our approach predicts this in advance by exhibiting explicitly the convex hull of the convergence domain. Of course, if one could develop new techniques to enlarge the domain of convergence, this residue would reappear in the main term; in this sense, the work of [6] remains highly significant.

Methodologically, we rely solely on the functional equation, without invoking auxiliary approximation functions. While it is often assumed that the functional equation alone is insufficient, we prove that thanks to the equivalence of Fourier transform, functional equation, and approximational function equation in the function field setting, the functional equation by itself suffices. We reformulate the relevant sums in a way that reveals the region of convergence, apply Perron's formula directly to this expression, and bound the error by contour integration. This streamlined argument not only clarifies the absence of the $u^3 = 1$ contribution but also yields the main term with significantly less computational effort.

2 Preliminary

In this section, we recall the basic properties of cubic characters, Gauss sums, and the functional equation for the corresponding L-functions. These facts are standard and their proofs are readily available in the literature; we therefore record only the statements that will be used later.

Throughout, we work in the non-Kummer setting: q is an odd prime power satisfying $q \equiv 2 \bmod 3$. For simplicity, we do not discuss the Kummer case $q \equiv 1 \bmod 3$, where analogous methods yield a similar first-moment asymptotic for cubic L-functions.

2.1 Primitive cubic characters and L-functions

In the non-Kummer setting, primitive cubic characters over $\mathbb{F}_q[T]$ are most conveniently constructed by restricting cubic characters over $\mathbb{F}_{q^2}[T]$ (see [2]).

Assume $q \equiv 2 \bmod 3$ and fix an isomorphism Ω between the cubic roots of unity in $\mathbb{C}^*$ and those in $\mathbb{F}_{q^2}^*$. Let $\pi \in \mathbb{F}_{q^2}[T]$ be monic irreducible such that $\pi\pi^\sigma = P$ is irreducible in $\mathbb{F}_q[T]$, where σ denotes the nontrivial element of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. For $a \in \mathbb{F}_{q^2}[T]$, define the cubic residue symbol $\chi_\pi(a)$ by

$$\chi_\pi(a) = \begin{cases} 0, & \pi \mid a, \\ \alpha, & \pi \nmid a, \end{cases}$$

where α is the unique cubic root of unity in $\mathbb{C}$ such that

$$a^{\frac{q^{2\deg \pi}-1}{3}} \equiv \Omega(\alpha) \mod \pi.$$

Then $\chi_\pi|_{\mathbb{F}_q[T]}$ is a cubic character modulo P.

We extend this definition by multiplicativity. For a monic polynomial $F = \pi_1^{e_1} \dots \pi_s^{e_s} \in \mathbb{F}_{q^2}[T]$ with distinct π_i, set $\chi_F = \chi_{\pi_1}^{e_1} \dots \chi_{\pi_s}^{e_s}$. Restricting χ_F to $\mathbb{F}_q[T]$ yields a cubic character on $\mathbb{F}_q[T]$. Moreover, χ_F is primitive if and only if $e_i = 1$ for all i and F has no prime divisor in $\mathbb{F}_q[T]$; in this case, the conductor is FF^σ.

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Let $\mathcal{A}_q$ (resp. $\mathcal{A}_{q^2}$) denote the set of monic polynomials over $\mathbb{F}_q$ (resp. $\mathbb{F}_{q^2}$), and let $\mathcal{H}_q$ (resp. $\mathcal{H}_{q^2}$) denote the set of monic square-free polynomials in $\mathcal{A}_q$ (resp. $\mathcal{A}_{q^2}$). Recall that $\mathcal{H}_{q^2,g/2+1}$ denotes the set of monic square-free polynomials in $\mathcal{A}_{q^2}$ of degree $g/2+1$. We write $|a| = q^{\deg a}$ for $a \in \mathcal{A}_q$ and $|a|_2 = q^{2\deg a}$ for $a \in \mathcal{A}_{q^2}$. With this notation, David et al. [6] show the following lemma.

Lemma 2.1 (Lemma 2.11 in [6]) *Suppose $q \equiv 2 \bmod 3$. Then*

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L_q\left(\frac{1}{2}, \chi\right) = \sum_{\substack{F \in \mathcal{H}_{q^2,g/2+1} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} L_q\left(\frac{1}{2}, \chi_F\right). \tag{2.1}$$

Since

$$\chi_F(\alpha) = \Omega^{-1}\left(\alpha^{\frac{q^{2\deg F}-1}{3}}\right)$$

for $\alpha \in \mathbb{F}_q \subset \mathbb{F}_{q^2}$, and q is odd with $q \equiv 2 \bmod 3$, every cubic character over $\mathbb{F}_q[T]$ is even, that is, $\chi|_{\mathbb{F}_q^*} = 1$. Consequently, we have the following functional equation.

Lemma 2.2 (Functional equation for even L-functions) *Let $F \in \mathcal{H}_{q^2}$ and suppose that F has no prime divisor in $\mathcal{A}_q$. Then*

$$L_q(s, \chi_F) = \varepsilon(\chi_F) q^{2s-1} \frac{1-q^{-s}}{1-q^{s-1}} \frac{L_q(1-s, \overline{\chi_F})}{|F|_2^{s-\frac{1}{2}}},$$

where $\varepsilon(\chi_F) = q^{-\deg F} G(\chi_F)$ and $G(\chi_F)$ is the Gauss sum defined in Definition 2.1.

This is Lemma 2.3 in [6].

Next, we record bounds for cubic L-functions; this restates Lemmas 2.6 and 2.7 of [6].

Lemma 2.3 (Lindelöf Hypothesis) *Let χ be a primitive cubic character of conductor $h \neq 1$ defined over $\mathbb{F}_q[T]$. Then, for $\Re(s) \geq \frac{1}{2}$ and all $\varepsilon > 0$,*

$$|L_q(s, \chi)| \ll q^{\varepsilon \deg h}; \tag{2.2}$$

and for $\Re(s) \geq 1$ and all $\varepsilon > 0$,

$$|L_q(s, \chi)| \gg q^{-\varepsilon \deg h}. \tag{2.3}$$

2.2 Gauss sums

We recall standard properties of polynomial Gauss sums (see, for instance, [13]).

Definition 2.1 For χ a primitive character of modulus h on $\mathbb{F}_q[T]$, define

$$G(\chi) = \sum_{a \bmod h} \chi(a) e_q\left(\frac{a}{h}\right).$$

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Here,

$$e_q(a/h) = e^{2\pi i \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_1)/p},$$

where a_1 is the coefficient of $1/T$ in the Laurent expansion of $\frac{a}{h}$ in $\mathbb{F}_q((1/T))$.

We also define generalized Gauss sums as follows.

Definition 2.2 Let χ_f be a cubic character of $\mathbb{F}_q[T]$ defined as above for some $f \in \mathbb{F}_q[T]$. We define

$$G_q(V, f) = \sum_{u \bmod f} \chi_f(u) e_q\left(\frac{uV}{f}\right). \tag{2.4}$$

Lemma 2.4 *Suppose that $q \equiv 1 \bmod 3$ and let $f_1, f_2 \in \mathcal{A}_q$.*

(1) *If $(f_1, f_2) = 1$, then*

$$\begin{aligned} G_q(V, f_1 f_2) &= \chi_{f_1}(f_2)^2 G_q(V, f_1) G_q(V, f_2) \\ &= G_q(V f_2, f_1) G_q(V, f_2). \end{aligned}$$

(2) *If $V = V_1 P^\alpha$ with $P \nmid V_1$, then*

$$G_q(V, P^i) = \begin{cases} 0, & \text{if } i \le \alpha \text{ and } i \not\equiv 0 \mod 3; \\ \phi(P^i), & \text{if } i \le \alpha \text{ and } i \equiv 0 \mod 3; \\ -|P|^{i-1}, & \text{if } i = \alpha + 1 \text{ and } i \equiv 0 \mod 3; \\ \varepsilon(\chi_{P^i}) \chi_{P^i}(V_1^{-1}) |P|^{i-\frac{1}{2}}, & \text{if } i = \alpha + 1 \text{ and } i \not\equiv 0 \mod 3; \\ 0, & \text{if } i \ge \alpha + 2, \end{cases} \tag{2.5}$$

where ϕ is the Euler function for polynomials. We recall that $\varepsilon(\chi) = 1$ when χ is even. For χ_{P^i}, this holds when $3 \mid \deg P^i$.

This is Lemma 2.12 of [6]. For the primitive character χ_F defined above, we have the following lemma.

Lemma 2.5 *For $F \in \mathbb{F}_{q^2}[T]$, we have*

$$G(\chi_F) = G_{q^2}(1, F).$$

Moreover, if F is square-free, then

$$|G_{q^2}(1, F)| = q^{\deg F}. \tag{2.6}$$

2.3 Distribution of Gauss sums

The generating series for Gauss sums is defined by

$$\Psi_q(f, u) = \sum_{F \in \mathcal{A}_q} G_q(f, F) u^{\deg F}, \tag{2.7}$$

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and

$$\widetilde{\Psi}_q(f,u) = \sum_{\substack{F \in \mathcal{A}_q \\ (F,f)=1}} G_q(f,F) u^{\deg F}. \tag{2.8}$$

We now discuss the distribution of Gauss sums. Patterson established the distribution of cubic Gauss sums for S-integers (see [10]); with minor adjustments, we obtain the following estimate.

Lemma 2.6 (Lemma 3.11 in [6]) *If $q^{-\frac{1}{2}} \geq |u| \geq q^{-\frac{3}{2}}$ and both $|u^3 - q^{-4}| > \delta$ and $|u^3 - q^{-2}| > \delta$, then*

$$\widetilde{\Psi}_q(f,u) \ll_\delta |f|^{\frac{3}{4}+\varepsilon} |u^{\frac{1}{2}}|^{\deg f}.$$

2.4 Multivariable complex analysis lemmas

Our approach uses two standard results from multivariable complex analysis. We begin with the notion of a tube domain.

Definition 2.3 An open set $T \subset \mathbb{C}^n$ is a tube if there exists an open set $U \subset \mathbb{R}^n$ such that

$$T = \{z \in \mathbb{C}^n : \Re(z) \in U\}.$$

For any set $U \subset \mathbb{C}^n$, we define $T(U) = U + i\mathbb{R}^n \subset \mathbb{C}^n$. We quote Bochner's Tube Theorem [3].

Theorem 2.1 *Let $U \subset \mathbb{R}^n$ be a connected open set and let $f(z)$ be holomorphic on $T(U)$. Then $f(z)$ admits a holomorphic continuation to the convex hull of $T(U)$.*

We denote the convex hull of an open set $T \subset \mathbb{C}^n$ by $\widehat{T}$. Our next result is [4, Proposition C.5] on the preservation of modulus inequalities under holomorphic continuation.

Theorem 2.2 *Assume that $T \subset \mathbb{C}^n$ is a tube domain and that $g, h : T \to \mathbb{C}$ are holomorphic. Let $\tilde{g}, \tilde{h}$ be their holomorphic continuations to $\widehat{T}$. If $|g(z)| \leq |h(z)|$ for all $z \in T$ and $h(z)$ is nonzero on T, then $|\tilde{g}(z)| \leq |\tilde{h}(z)|$ for all $z \in \widehat{T}$.*

3 Analytical behavior of $A_3(u,v)$

3.1 The generating function

Now, we construct a double Dirichlet series to rewrite the sum in equation (2.1). From now on, we will always use P, P_1, or P_2 to denote irreducible polynomial in corresponding polynomial ring. For $\Re(s)$ and $\Re(w)$ sufficiently large, we define

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$$A_3(s,w) = \sum_{\substack{F\in\mathcal{H}_{q^2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[T]}} \frac{L_q(w,\chi_F)}{|F|_2^s}. \tag{3.1}$$

Upon taking $u = q^{-2s}$, we rewrite the sum as

$$\begin{aligned}\sum_{\substack{F\in\mathcal{H}_{q^2,g/2+1}\\ P|F\Rightarrow P\notin\mathbb{F}_q[T]}} L_q\left(\frac{1}{2},\chi_F\right) &= \frac{1}{2\pi i}\oint_{|u|=r}\sum_{\substack{F\in\mathcal{H}_{q^2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[T]}} L_q\left(\frac{1}{2},\chi_F\right)u^{\deg F}\frac{du}{u^{g/2+2}}\\ &= \frac{1}{2\pi i}\oint_{|u|=r} A_3\left(u,\frac{1}{2}\right)\frac{du}{u^{g/2+2}}, \end{aligned}\tag{3.2}$$

by applying Perron's formula for function fields

$$\sum_{n\le N} a(n) = \frac{1}{2\pi i}\oint_{|u|=r}\left(\sum_{n=0}^{\infty} a(n)u^n\right)\frac{du}{(1-u)u^{N+1}}. \tag{3.3}$$

Here, $A_3(u,w) = A_3(s,w)$. Hereafter, we will use $A_3(s,w)$ and $A_3(u,w)$ interchangeably when there is no ambiguity.

Next, we consider the different expressions of $A_3(s,w)$ on different convergence regions and find out the convex hull of convergence regions.

3.2 The first convergence region

To analyze the behavior of $A_3(u,v)$, we begin by using Möbius inversion

$$\sum_{\substack{D\in\mathcal{A}_q\\ D|F}}\mu(D) = \begin{cases}1 & F \text{ has no prime divisor in } \mathbb{F}_q[T];\\ 0 & \text{otherwise.}\end{cases} \tag{3.4}$$

This allows us to remove the divisor condition. Applying this, $A_3(s,w)$ can be rewritten as

$$\begin{aligned} A_3(s,w) &= \sum_{\substack{F\in\mathcal{H}_{q^2}\\ P|F\to P\notin\mathbb{F}_q[T]}} \frac{L_q(w,\chi_F)}{|F|_2^s}\\ &= \sum_{D\in\mathcal{A}_q}\mu(D)\sum_{\substack{F\in\mathcal{H}_{q^2}\\ (D,F)=1}}\frac{L_q(w,\chi_{DF})}{|DF|_2^s}\\ &= \sum_{D\in\mathcal{A}_q}\mu(D)\sum_{\substack{F\in\mathcal{H}_{q^2}\\ (D,F)=1}}\sum_{N\in\mathcal{A}_q}\frac{\chi_{DF}(N)}{|N|^w|DF|_2^s}\\ &= \sum_{N\in\mathcal{A}_q}\frac{1}{|N|^w}\sum_{D\in\mathcal{A}_q}\frac{\mu(D)\chi_D(N)}{|D|_2^s}\sum_{\substack{F\in\mathcal{H}_{q^2}\\ (D,F)=1}}\frac{\chi_F(N)}{|F|_2^s}\\ &= \sum_{N\in\mathcal{A}_q}\frac{1}{|N|^w}\frac{L_{q^2}(s,\chi^{(N)})}{L_{q^2}(2s,\bar{\chi}^{(N)})}\sum_{D\in\mathcal{A}_q}\frac{\mu(D)\chi_D(N)}{|D|_2^s}\prod_{\substack{P_2|D\\ P_2\in\mathcal{A}_{q^2}}}\left(1+\frac{\chi_{P_2}(N)}{|P_2|_2^s}\right)^{-1}. \end{aligned}\tag{3.5}$$

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The last equality uses a trick

$$\sum_{\substack{F\in\mathcal{H}_{q^2}\\(D,F)=1}}\frac{\chi_F(N)}{|F|_2^s}=\prod_{\substack{P_2\in\mathcal{A}_{q^2}\\(P_2,D)=1}}\left(1+\frac{\chi_{P_2}(N)}{|P_2|_2^s}\right)=\frac{L_{q^2}(s,\chi^{(N)})}{L_{q^2}(2s,\overline{\chi}^{(N)})}\prod_{\substack{P_2\in\mathcal{A}_{q^2}\\P_2|D}}\left(1+\frac{\chi_{P_2}(N)}{|P_2|_2^s}\right)^{-1}.$$

Here, $\chi^{(N)}(F)=\chi_F(N)$ is the Hecke character induced by χ_F. $L_{q^2}(s,\chi^{(N)})$ denotes the L-function over $\mathbb{F}_{q^2}(T)$. Meanwhile, we recall that $|F|=q^{\deg F}$ and $|F|_2=q^{2\deg F}$.

Using the lemma below, we simplify $A_3(s,w)$ further.

Lemma 3.1 *If $D,F\in\mathcal{A}_q$ and $(D,F)=1$, then we have $\chi_D(F)=1$.*

Proof Let $\sigma:\mathbb{F}_{q^2}\to\mathbb{F}_{q^2}$ be the nontrivial Frobenius automorphism. It's sufficient to treat the case $D=P\in\mathcal{A}_q$ is irreducible. Assume first that P does not split in extension $\mathbb{F}_{q^2}(T)/\mathbb{F}_q(T)$. By definition, we have

$$F^{\frac{q^{2\deg P}-1}{3}}\equiv\Omega(\alpha)\mod P,$$

which is equivalent to say $F^{\frac{q^{2\deg P}-1}{3}}=f(T)P+\Omega(\alpha)$. Applying σ and using $\sigma(F)=F$, we have $F^{\frac{q^{2\deg P}-1}{3}}=f^\sigma(T)P+\sigma(\Omega(\alpha))\equiv\sigma(\Omega(\alpha))\mod P$. Since two constants congruent modulo P must coincide, $\sigma(\Omega(\alpha))=\Omega(\alpha)$, hence $\Omega(\alpha)=1$. If instead $P=\pi\pi^\sigma$ splits, then $\chi_P(F)=\chi_\pi\overline{\chi_\pi}(F)=1$. For the general case, $D=P_1^{e_1}\dots P_k^{e_k}$ follows by multiplicativity, since $\chi_D=\chi_{P_1}^{e_1}\dots\chi_{P_k}^{e_k}$. ∎

Using the lemma, we obtain

$$\begin{aligned}A_3(s,w)&=\sum_{N\in\mathcal{A}_q}\frac{1}{|N|^w}\frac{L_{q^2}(s,\chi^{(N)})}{L_{q^2}(2s,\bar{\chi}^{(N)})}\sum_{\substack{D\in\mathcal{A}_q\\(D,N)=1}}\frac{\mu(D)}{|D|_2^s}\prod_{\substack{P_2\in\mathcal{A}_{q^2}\\P_2|D}}\left(1+\frac{\chi_{P_2}(N)}{|P_2|_2^s}\right)^{-1}\\&=\sum_{N\in\mathcal{A}_q}\frac{1}{|N|^w}\frac{L_{q^2}(s,\chi^{(N)})}{L_{q^2}(2s,\bar{\chi}^{(N)})}\prod_{\substack{P_1\in\mathcal{A}_q\\(P_1,N)=1}}\left(1-\frac{1}{|P_1|_2^s}\prod_{\substack{P_2|P_1\\P_2\in\mathcal{A}_{q^2}}}\left(1+\frac{\chi_{P_2}(N)}{|P_2|_2^s}\right)^{-1}\right)\\&=\sum_{N\in\mathcal{A}_q}\frac{1}{|N|^w}\frac{L_{q^2}(s,\chi^{(N)})}{L_{q^2}(2s,\bar{\chi}^{(N)})}P\left(s,\chi^{(N)}\right)\prod_{\substack{P_1|N\\P_1\in\mathcal{A}_q}}\left(1-\frac{1}{|P_1|_2^s}\right)^{-1},\end{aligned}$$

where

$$P(s,\chi^{(N)})=\prod_{P_1\in\mathcal{A}_q}\left(1-\frac{1}{|P_1|_2^s}\prod_{\substack{P_2|P_1\\P_2\in\mathcal{A}_{q^2}}}\left(1+\frac{\chi_{P_2}(N)}{|P_2|_2^s}\right)^{-1}\right).\tag{3.6}$$

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For $\Re(s) > \frac{1}{2}$, it follows that

$$P(s, \chi^{(N)}) = \prod_{P_1 \in \mathcal{A}_q} \left(1 - \frac{1}{|P_1|_2^s} + O\left(\frac{1}{|P_1|_2^{3s/2}}\right)\right)$$

$$= \zeta_q^{-1}(2s) \prod_{P \in \mathcal{A}_q} \left(1 + O\left(\frac{1}{|P|_2^{3s/2}}\right)\right). \tag{3.7}$$

As $(1 - \frac{1}{|P_1|_2^s})^{-1} = 1 + \frac{1}{|P_1|_2^s - 1} \ll |P_1|_2^{\max\{0, -\Re(s)\}}$, we will obtain

$$\prod_{\substack{P_1 | N \\ P_1 \in \mathcal{A}_q}} \left(1 - \frac{1}{|P_1|_2^s}\right)^{-1} \ll |N|_2^{\max\{0, -\Re(s)\} + \varepsilon}. \tag{3.8}$$

Taking $u = q^{-2s}$ and $v = q^{-w}$, we have

$$A_3(u, v) = \sum_{N \in \mathcal{A}_q} v^{\deg N} \frac{L_{q^2}(u, \chi^{(N)})}{L_{q^2}(u^2, \bar{\chi}^{(N)})} P\left(u, \chi^{(N)}\right) \prod_{\substack{P_1 | N \\ P_1 \in \mathcal{A}_q}} \left(1 - u^{\deg P_1}\right)^{-1}, \tag{3.9}$$

where

$$L_{q^2}\left(u, \chi^{(N)}\right) = \sum_{F \in \mathcal{A}_{q^2}} \chi_F(N) u^{\deg F}$$

and

$$P\left(u, \chi^{(N)}\right) = \prod_{P_1 \in \mathcal{A}_q} \left(1 - u^{\deg P_1} \prod_{\substack{P_2 \in \mathcal{A}_{q^2} \\ P_2 | P_1}} \left(1 + \chi_{P_2}(N) u^{\deg P_2}\right)^{-1}\right).$$

It is easy to see that $P(u, \chi^{(N)})$ is analytic for $|u| < q^{-1}$.

The function $L_{q^2}(u, \chi^{(N)})$ is a polynomial in u and convergent for $|u| < q^{-1}$ in general. If N is a cube, then $L_{q^2}(u, \chi^{(N)})$ has a simple pole at $u = q^{-2}$ and $L_{q^2}(u^2, \chi^{(N)})^{-1}$ is a polynomial in u. If N is not a cube, then $L_{q^2}(u, \chi^{(N)})$ is analytic and $L_{q^2}(u^2, \chi^{(N)})$ has zeros on the circle $|u| = q^{-1/2}$. Hence, $(u - q^{-2}) \frac{L_{q^2}(u, \chi^{(N)})}{L_{q^2}(u^2, \bar{\chi}^{(N)})}$ is analytic for $|u| < q^{-\frac{1}{2}}$.

When N is not a cube, $\frac{L_{q^2}(u, \chi^{(N)})}{L_{q^2}(u^2, \bar{\chi}^{(N)})}$ is a rational function in u with possible simple poles on the line $|u| = q^{-\frac{1}{2}}$. The Lindelöf Hypothesis gives, for $|u| < q^{-1}$ and arbitrary $\varepsilon > 0$,

$$\frac{L_{q^2}(u, \chi^{(N)})}{L_{q^2}(u^2, \bar{\chi}^{(N)})} \ll |N|_2^{\varepsilon}. \tag{3.10}$$

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There is a possible simple pole at $u = q^{-2}$ when N is a cube in $A_3(u, v)$. In fact, combining (3.7)–(3.10), for $|u| < q^{-1}$ and any $\varepsilon > 0$, we have the bound

(3.11)
$$|(u - q^{-2})A_3(u, v)| \ll |(u - q^{-2})(1 - qu)| \sum_{N \in \mathcal{A}_q} |q^{\varepsilon} v|^{\deg N} \ll |(u - q^{-2})(1 - qu)| \zeta_q(vq^{\varepsilon}).$$

Meanwhile, $(u - q^{-2})A_3(u, v)$ is holomorphic in the convergence region of the right-hand side of the above inequality:

$$S_1 = \{(u, v) || u| < q^{-1}, |v| < q^{-1}\}. \tag{3.12}$$

Remark 3.1 In (s, w)-coordinates, the region S_1 becomes

$$S_1 = \{(s, w) | \Re(s) > \frac{1}{2}, \Re(w) > 1\}. \tag{3.13}$$

We will always describe all convergence regions in the (s, w)-coordinate, since Bochner's Tube Theorem is stated for tube domains.

3.3 Residue at $u = q^{-2}$

We now analyze the possible poles at $u = q^{-2}$. Let $\zeta_{q^2}(u) = \frac{1}{1-q^2u}$ be the zeta function for the field $\mathbb{F}_{q^2}(T)$. The possible pole comes from the term N being a cube. We have

$$\begin{aligned}
&\operatorname*{Res}_{u=q^{-2}} A_3(u, v) \\
&= \operatorname*{Res}_{u=q^{-2}} \sum_{N \in \mathcal{A}_q} v^{3 \deg N} \frac{\zeta_{q^2}(u)}{\zeta_{q^2}(u^2)} \prod_{\substack{P_2 \in \mathcal{A}_{q^2} \\ P_2 | N}} \frac{1 - u^{\deg P_2}}{1 - u^{2 \deg P_2}} \prod_{\substack{P_1 \in \mathcal{A}_q \\ P_1 \nmid N}} \left(1 - u^{\deg P_1} \prod_{\substack{P_2 \in \mathcal{A}_{q^2} \\ P_2 | P_1}} (1 + u^{\deg P_2})^{-1}\right) \\
&= P(q^{-2}) Z(q^{-2}, v) \operatorname*{Res}_{u=q^{-2}} \frac{\zeta_{q^2}(u)}{\zeta_{q^2}(u^2)},
\end{aligned} \tag{3.14}$$

where

$$P(u) = \prod_{P_1 \in \mathcal{A}_q} \left(1 - u^{\deg P_1} \prod_{\substack{P_2 \in \mathcal{A}_{q^2} \\ P_2 | P_1}} (1 + u^{\deg P_2})^{-1}\right) \tag{3.15}$$

and

(3.16)
$$Z(u, v) = \sum_{N \in \mathcal{A}_q} v^{3 \deg N} \prod_{\substack{P_2 \in \mathcal{A}_{q^2} \\ P_2 | N}} (1 + u^{\deg P_2})^{-1} \prod_{\substack{P_1 \in \mathcal{A}_q \\ P_1 | N}} \left(1 - u^{\deg P_1} \prod_{\substack{P_2 \in \mathcal{A}_{q^2} \\ P_2 | P_1}} (1 + u^{\deg P_2})^{-1}\right)^{-1}.$$

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Taking $v = q^{-\frac{1}{2}}$, we have

$$\operatorname*{Res}_{u=q^{-2}} A_3(u, q^{-\frac{1}{2}}) = (q^{-4} - q^{-2})P(q^{-2})Z(q^{-2}, q^{-\frac{1}{2}}).$$

3.4 The second region

The Lindelöf Hypothesis gives, for $|v| \le q^{-\frac{1}{2}}$ and any $\varepsilon > 0$,

$$|L_q(v, \chi_F)| \ll |F|^{\varepsilon}$$

and hence

$$\sum_{\substack{F \in \mathcal{H}_{q^2} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T] \\ F \neq 1}} L_q(v, \chi_F) u^{\deg F} \ll \sum_{F \in \mathcal{H}_{q^2}} |F|^{\varepsilon} |u|^{\deg F} \ll \frac{\zeta_{q^2}(uq^{\varepsilon})}{\zeta_{q^2}((uq^{\varepsilon})^2)}.$$

For the term $F = 1$, we note that

$$(v - q^{-1})\zeta_q(v) \ll 1$$

for all v, where $\zeta_q(v) = \frac{1}{1-qv}$ is the zeta function of $\mathbb{F}_q(T)$. Therefore, $(v - q^{-1})$ $A_3(u, v)$ is holomorphic in the convergence region

$$S_{2,1} = \{(u, v) \mid |v| \le q^{-\frac{1}{2}}, |u| < q^{-2}\}. \tag{3.17}$$

Similarly, when $|v| \ge q^{-\frac{1}{2}}$, the Lindelöf Hypothesis with the functional equation yields, for any $\varepsilon > 0$,

$$|L_q(v, \chi_F)| \ll |v^2 q^{1+\varepsilon}|^{\deg F}.$$

Consequently,

$$A_3(u, v) = \zeta_q(v) + \sum_{\substack{F \in \mathcal{H}_{q^2} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T] \\ F \neq 1}} L_q(v, \chi_F) u^{\deg F} \ll \sum_{F \in \mathcal{H}_{q^2}} |v^2 u q^{1+\varepsilon}|^{\deg F} \ll \frac{\zeta_{q^2}(uv^2 q^{(1+\varepsilon)})}{\zeta_{q^2}((uv^2 q^{(1+\varepsilon)})^2)}$$

in the convergence region

$$S_{2,2} = \{(u, v) \mid |v| \ge q^{-\frac{1}{2}}, |u^{1/2} v| < q^{-\frac{3}{2}}\}. \tag{3.18}$$

Combining the two cases, we obtain

$$S_2 = S_{2,1} \cup S_{2,2} = \{(u, v) \mid |u^{1/2} v| < q^{-\frac{3}{2}}, |u| < q^{-2}\}. \tag{3.19}$$

Remark 3.2 In (s, w)-coordinates, this region becomes

$$S_2 = \{(s, w) | \Re(s + w) > \frac{3}{2}, \Re(s) > 1\}. \tag{3.20}$$

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4 The dual term

4.1 The third convergence region

We now derive the third convergence region. The functional equation (Lemma 2.2) gives

$$\begin{aligned} A_3(s,w) &= \sum_{\substack{F\in\mathcal{H}_{q^2}\\ P\mid F\Rightarrow P\notin\mathbb{F}_q[T]}} \frac{L_q(w,\chi_F)}{|F|_2^s} \\ &= q^{2w-1}\frac{1-q^{-w}}{1-q^{w-1}} \sum_{\substack{F\in\mathcal{H}_{q^2}\\ P\mid F\Rightarrow P\notin\mathbb{F}_q[T]}} \frac{L_q(1-w,\overline{\chi}_F)}{|F|_2^{s+w}} G_{q^2}(1,F) \\ &= q^{2w-1}\frac{1-q^{-w}}{1-q^{w-1}} \sum_{N\in\mathcal{A}_q}\frac{1}{|N|^{1-w}} \sum_{D\in\mathcal{A}_q}\frac{\mu(D)\overline{\chi}_D(N)G_{q^2}(1,D)}{|D|_2^{s+w}} \sum_{\substack{F\in\mathcal{H}_{q^2}\\ (F,DN)=1}} \frac{G_{q^2}(ND,F)}{|F|_2^{s+w}}. \end{aligned} \tag{4.1}$$

Replacing q by q^2, Lemma 2.4 shows $G_{q^2}(ND,F)=0$ whenever F is not square-free. Hence, we may rewrite $A_3(s,w)$ as

$$\begin{aligned} A_3(s,w) &= q^{2w-1}\frac{1-q^{-w}}{1-q^{w-1}} \sum_{N\in\mathcal{A}_q}\frac{1}{|N|^{1-w}} \sum_{\substack{D\in\mathcal{A}_q\\ (D,N)=1}}\frac{\mu(D)G_{q^2}(N,D)}{|D|_2^{s+w}} \sum_{\substack{F\in\mathcal{A}_{q^2}\\ (F,DN)=1}} \frac{G_{q^2}(ND,F)}{|F|_2^{s+w}} \\ &= q^{2w-1}\frac{1-q^{-w}}{1-q^{w-1}} C_3(s,w), \end{aligned}$$

where

$$C_3(s,w) = \sum_{N\in\mathcal{A}_q}\frac{1}{|N|^{1-w}} \sum_{\substack{D\in\mathcal{A}_q\\ (D,N)=1}}\frac{\mu(D)G_{q^2}(N,D)}{|D|_2^{s+w}} H(ND,s+w) \tag{4.2}$$

and

$$H(Q,s) = \sum_{\substack{F\in\mathcal{A}_{q^2}\\ (F,Q)=1}} \frac{G_{q^2}(Q,F)}{|F|_2^s}. \tag{4.3}$$

Taking $u=q^{-2s}$ and $v=q^{-w}$, $A_3(u,v)$ becomes

$$A_3(u,v) = \frac{1}{qv^2}\frac{1-v}{1-\frac{1}{qv}} C_3(u,v), \tag{4.4}$$

where

$$C_3(u,v) = \sum_{N\in\mathcal{A}_q}\frac{1}{(qv)^{\deg N}} \sum_{\substack{D\in\mathcal{A}_q\\ (D,N)=1}}\mu(D)G_{q^2}(N,D)H(ND,uv^2)(uv^2)^{\deg D} \tag{4.5}$$

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and

$$H(Q,u)=\sum_{\substack{F\in\mathcal{A}_{q^2}\\(F,Q)=1}}G_{q^2}(Q,F)u^{\deg F}. \tag{4.6}$$

As before, we do not distinguish $C_3(s,w)$ from $C_3(u,v)$ nor $H(Q,s)$ from $H(Q,u)$ when no confusion can arise.

Replacing q by q^2 in Lemma 2.6, we obtain $H(ND,uv^2)\ll|u^{\frac{1}{2}}vq^{\frac{3}{2}+\varepsilon}|^{\deg ND}$ for $q^{-\frac{3}{2}}\le|u^{\frac{1}{2}}v|\le q^{-\frac{1}{2}}$. Moreover,

$$\begin{aligned}A_3(u,v)&\ll\frac{1}{qv^2}\frac{1-v}{1-\frac{1}{qv}}\sum_{N\in\mathcal{A}_q}\frac{1}{|qv|^{\deg N}}\sum_{\substack{D\in\mathcal{A}_q\\(D,N)=1}}|G_{q^2}(N,D)||uv^2|^{\deg D}|u^{\frac{1}{2}}vq^{\frac{3}{2}+\varepsilon}|^{\deg ND}\\&\ll\frac{1}{qv^2}\frac{1-v}{1-\frac{1}{qv}}\sum_{N\in\mathcal{A}_q}\frac{1}{|qv|^{\deg N}}\sum_{\substack{D\in\mathcal{A}_q\\(D,N)=1}}|D||uv^2|^{\deg D}|u^{\frac{1}{2}}vq^{\frac{3}{2}+\varepsilon}|^{\deg ND}\\&\ll\frac{1}{qv^2}\frac{1-v}{1-\frac{1}{qv}}\sum_{N\in\mathcal{A}_q}|u^{\frac{1}{2}}q^{\frac{1}{2}+\varepsilon}|^{\deg N}\sum_{\substack{D\in\mathcal{A}_q\\(D,N)=1}}|q^{\frac{5}{2}+\varepsilon}u^{\frac{3}{2}}v^3|^{\deg D}\\&\ll\frac{1}{qv^2}\frac{1-v}{1-\frac{1}{qv}}\zeta_q(u^{\frac{1}{2}}q^{\frac{1}{2}+\varepsilon})\zeta_q(u^{\frac{3}{2}}v^3q^{\frac{5}{2}+\varepsilon}).\end{aligned}$$

It follows that $A_3(u,v)$ is holomorphic in region

$$S_3=\{|u|<q^{-3},|u^{\frac{1}{2}}v|<q^{-\frac{7}{6}},|v|>1,q^{-\frac{3}{2}}\le|u^{\frac{1}{2}}v|\le q^{-\frac{1}{2}}\}.$$

Remark 4.1 In (s,w)-coordinates, S_3 becomes

$$S_3=\{(s,w)|\Re(s)>\frac{3}{2},\Re(s+w)>\frac{7}{6},\frac{1}{2}\le\Re(s+w)\le\frac{3}{2},\Re(w)<0\}. \tag{4.7}$$

We now combine the three domains obtained above. Recall that S_1 and S_2 were first obtained in (u,v)-coordinates and then translated into (s,w)-coordinates (see Remarks 3.1 and 3.2), while S_3 is already stated in (s,w)-coordinates. Since Bochner's Tube Theorem applies to tube domains in $\mathbb{C}^2$, we work in the (s,w)-system and compute the convex hull of $S_1\cup S_2\cup S_3$. The convex hull in (s,w)-coordinates can be determined by elementary geometric computation, and we obtain

$$S_4=\{(s,w)|\Re(s)>\frac{1}{2},\ \Re(s+w)>\frac{7}{6},\ \Re(4s+3w)>5\}. \tag{4.8}$$

By Bochner's Tube Theorem, $(u-q^{-2})(v-q^{-1})A_3(u,v)$ admits a holomorphic continuation to the corresponding region in (u,v)-coordinates, namely,

$$S_4=\{(u,v)||u|<q^{-1},\ |u^{\frac{1}{2}}v|<q^{-\frac{7}{6}},\ |u^2v^3|<q^{-5}\}. \tag{4.9}$$

Remark 4.2 The potential poles at $(uv^2)=q^{-8}$ do not lie in S_4.

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4.2 Complete the proof

We now complete the proof by evaluating the integral (3.2). Fix $v = q^{-\frac{1}{2}}$. We shift the contour from $|u| = r$ to $|u| = q^{-(\frac{7}{4}+\varepsilon)}$. The only pole crossed is the simple pole at $u = q^{-2}$. Therefore,

$$\frac{1}{2\pi i}\oint_{|u|=r} A_3(u,\frac{1}{2})\frac{du}{u^{g/2+2}} = \operatorname*{Res}_{u=q^{-2}} \frac{A_3(u,\frac{1}{2})}{u^{g/2+2}} + \frac{1}{2\pi i}\oint_{|u|=q^{-(\frac{7}{4}+\varepsilon)}} A_3(u,\frac{1}{2})\frac{du}{u^{g/2+2}}.$$

The residue at $u = q^{-2}$ produces the main term in (1.4). For the shifted contour integral, since $A_3(u,v)$ is holomorphic on the contour $|u| = q^{-(\frac{7}{4}+\varepsilon)}$, we bound it by

$$\ll q^{\frac{g}{2}(\frac{7}{4}+\varepsilon)} \oint_{|u|=q^{-(\frac{7}{4}+\varepsilon)}} |A_3(u,\frac{1}{2})|\,|du| \ll q^{(\frac{7}{8}+\varepsilon)g},$$

which yields the claimed error term in (1.4).

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