Annals of Actuarial Science, Vol. 13, part 2, pp. 241–267. © Institute and Faculty of Actuaries 2018
doi:10.1017/S1748499518000209
First published online 13 July 2018

Beta transform and discounted aggregate claims under dependency

Zhehao Zhang* and Shuanming Li

Department of Economics, Centre for Actuarial Studies, The University of Melbourne, Parkville, VIC 3010, Australia

Abstract

This paper starts with the Beta transform and discusses the stochastic ordering properties of this transform under different parameter settings. Later, the distribution of discounted aggregate claims in a compound renewal risk model with dependence between inter-claim times and claim sizes is studied. Recursive formulas for moments and joint moments are expressed in terms of the Beta transform of the inter-claim times and claim severities. Particularly, our moments formula is more explicit and computation-friendly than earlier ones in the references. Lastly, numerical examples are provided to illustrate our results.

Keywords

Discounted aggregate claims; Renewal risk process; Copula; Beta distributional transform; Stochastic ordering

1. Introduction

Consider a continuous time compound renewal process representing aggregate claims of an insurance portfolio and let

$$Z_t = \sum_{i=1}^{N_t} \mathrm{e}^{-\delta T_i} X_i, \quad t \geq 0$$

be the aggregate discounted value at time 0 of all claims recorded over $[0, t]$, where $\delta \geq 0$ is the force of interest and $T_n = \sum_{i=1}^{n} W_i$ is the occurrence time of the nth claim. The X_i's are i.i.d. random variables with common distribution function F_X (such that $F_X(0) = 0$) and density f_X. We denote its moment generating function by $M_X(s) = \int_0^\infty \mathrm{e}^{sx} f_X(x)\, dx$. The counting process $\{N_t; t \geq 0\}$ is a renewal process representing the number of claims up to time t and is defined as $N_t = \max\{n \in \mathbb{N}^+ : W_1 + W_2 + \ldots + W_n \leq t\}$, where inter-claim times $W_i's$ are assumed i.i.d. with common distribution function F_W (and corresponding density function f_W). Define

$$V_n(t) = \mathbb{E}[Z_t^n], \quad t \geq 0, \quad n \in \mathbb{N}^+$$

to be the nth moment of Z_t. Note that $V_n(0) = 0$ and $V_0(t) = 1$.

Analysis on the moments and distributions of the discounted aggregate claims in the compound Poisson process can be found in Taylor (1979), Delbaen & Haezendonck (1987) and Willmot (1989). Léveillé & Garrido (2001*a*) give explicit expressions for the first two moments of the discounted aggregate claims in the compound renewal risk process by using renewal theory arguments. Further Léveillé &

*Correspondence to: Zhehao Zhang, Department of Economics, Centre for Actuarial Studies, The University of Melbourne, Parkville, VIC 3010, Australia. Tel: +61 3 83445616. E-mail: zhehaoz1@student.unimelb.edu.au

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Garrido (2001*b*) give a recursive formula for the moments of a compound renewal present value process. Jang (2004) obtains the Laplace transform of the distribution of the discounted aggregate claims using a shot noise process. Kim & Kim (2007) give the first two moments of the discounted aggregate claims when the rates of claim arrivals and the claim sizes change according to the states of an underlying Markov process. Ren (2008) obtains some matrix expressions for the Laplace transform and the first two moments of the aggregate discounted claims, when inter-claim times follow a Markovian arrival process (MAP). Léveillé *et al.* (2010) study the moment generating functions of the discounted aggregate claims process in finite time and infinite time. Covariance and joint moments of Z_t and Z_{t+s} are discussed in Léveillé & Adékambi (2011, 2012). Wang *et al.* (2016) give more explicit results on this topic and use them to calculate risk measures such as Value-at-Risk and Conditional Tail Expectation.

Attention has also been drawn to the introduction of the dependency within the model. Albrecher & Boxma (2004) propose dependence between X_n and W_{n+1}, i.e., claim size has an impact on the next inter-arrival time. This structure is also investigated in Cheung (2012). Conversely, Boudreault *et al.* (2006) assume inter-arrival time has effects on the next claim amount. Copula is introduced in Albrecher & Teugels (2006), and then discussed in Boudreault *et al.* (2006) and Cossette *et al.* (2008, 2010).

From literature, it shows that the distribution of Z_t for $t < \infty$ is hard to obtain even under independence assumption, let alone under dependent structure. Therefore, the study on the moments of Z_t, at least, provides some information for its distribution. Further, if an explicit formula on moments is provided, various moments matching methods can be applied for density approximation purpose.

Barges *et al.* (2011) investigate the moments of Z_t under a compound Poisson process via dependence being measured by the classical Farlie–Gumbel–Morgenstern (FGM) copula which is a special case of the copula defined in the next section and studied in our paper. The results in Barges *et al.* (2011) are much more complicated and involve multiple summations and significant choice problems for index. Woo & Cheung (2013) proposes a dependent risk model in which the conditional pdf of X given $W = t$ can be expressed as a mixture of densities and then the generalised FGM copula is a special case of their model. However, their method requires a large amount of work in identification and calculation of the coefficients, even just under the simplest case, i.e., the classical FGM. Thus, the motivation of this paper is to provide explicit formulas for moments and joint moments of Z_t for the generalised FGM copula, with the help of some interesting properties of the Beta transform.

In this paper, section 2 specifies the copula we are going to use, which is the generalisation of Barges *et al.* (2011). Section 3 revisits a distributional transform, the Beta transform, which is actually a generalisation of the Proportional Hazard (PH) transform in Wang (1995). Later, examples and stochastic ordering properties are presented. Section 4 gives the integral equation for moments and joint moments under this framework. Section 5 gives a few specific results, theoretically and numerically, when the arrival process is a Poisson process, which are more computationed friendly. Section 6 discusses the moments of this discounted aggregated sum when the inter-arrival times follow an Erlang(2) distribution. Section 7 gives some concluding remarks.

2. Dependence Between Inter-Claim Times and Claim Sizes

In the renewal risk model, inter-claim times and claim sizes are assumed to be independent. There exist real-world situations for which such an assumption is inappropriate. Recently, risk models with dependence between inter-claim times and claim sizes have been studied extensively by researchers.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Albrecher & Boxma (2004) propose an extension to the classical risk model with dependence between inter-claim times and claim sizes in which the distribution of the time between two successive claims depends on the previous claim size and study the Gerber–Shiu function for this dependent risk model. Such a dependent risk model can be transformed to a semi-Markovian risk model which was first introduced in Janssen & Reinhard (1985) and further studied by Albrecher & Boxma (2005). This semi-Markovian risk model is a special case of a MAP risk model, see Badescu *et al.* (2005). The moments of the aggregate discounted claims in a MAP risk model have been studied in Ren (2008).

Boudreault *et al.* (2006) propose an extension to the classical risk model in which a reverse dependent structure was introduced, the distribution of the next claim amount depends on the time elapsed since the last claim, i.e., the conditional distribution of X_i given $W_i = t$ is given by

$$f_{X_i|W_i=t}(x) = \mathrm{e}^{-\beta t} f_1(x) + \left(1-\mathrm{e}^{-\beta t}\right) f_2(x), \quad i = 1, 2, \ldots, \beta > 0$$

where f_1 and f_2 are two density functions.

Albrecher & Teugels (2006) use copulas to define the joint distribution of the inter-claim times and claim sizes. Rodriguez-Lallena & Ãbeda-Flores (2004) propose the following bivariate copula:

$$C(u, v) = uv + \theta h(u) g(v), \quad u, v \in [0, 1] \tag{2.1}$$

where h and g are non-zero real functions with support [0, 1] satisfying certain conditions and θ is constant. Cossette *et al.* (2008) studies the Gerber–Shiu function in the classical risk model with dependence introduced by the copula defined above with a special choice of h and g as follows:

$$h(u) = u^a (1-u)^b$$

$$g(v) = v^c (1-v)^d$$

where $a, b, c, d \geq 1$, and θ takes values from an interval depending on the values of a, b, c, d. This generalised FGM copula with the above choice of $h(u)$ and $g(v)$ has four parameters with more flexibility in modelling of dependence. In particular, if $a = b = c = d = 1$, then

$$C(u, v) = uv + \theta u(1-u)v(1-v), \quad u, v \in [0, 1]$$

with $-1 \leq \theta \leq 1$, is the classical FGM copula, which was discussed in details in Barges *et al.* (2011).

For h and g defined above, the joint distribution function of (X, W) is

$$\begin{aligned} F_{X,W}(x, t) &= C(F_X(x), F_W(t)) \\ &= F_X(x) F_W(t) + \theta (F_X(x))^a [1-F_X(x)]^b (F_W(t))^c (1-F_W(t))^d \end{aligned}$$

the joint density function of (X, W) is

$$f_{X,W}(x, t) = f_X(x) f_W(t) + \theta h'(F_X(x)) g'(F_W(t)) f_X(x) f_W(t) \tag{2.2}$$

and the conditional density of X given that $W = t$ is

$$f_{X|W=t}(x) = f_X(x) + \theta h'(F_X(x)) f_X(x) g'(F_W(t)) \tag{2.3}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

The covariance of X and W is given by

$$\begin{aligned}\mathrm{Cov}(X,\ W) &= \int_0^1\int_0^1 [C(u,\ v)-uv]dF_X^{-1}(u)dF_W^{-1}(v)\\ &= \theta\int_0^1 h(u)dF_X^{-1}(u)\int_0^1 g(v)dF_W^{-1}(v)\end{aligned} \tag{2.4}$$

This shows that when $\theta>0$ (<0), X and W are positively (negatively) correlated, while when $\theta=0$, X and W are independent.

3. Beta Distributional Transform

In this section, we revisit and study some properties of a distributional transform, the Beta transform. This transform was first proposed in Wirch & Hardy (1999) for distortion functions. Our main results can be expressed in terms of the Beta transforms of distributions of claim inter-arrivals and claim severities. Furthermore, upper and lower bounds for the moments of aggregate discounted claims can obtained by the ordering properties of the Beta transform.

Definition 1. Let X be a positive random variable with probability density function $f_X(x)$ and cumulative distribution function $F_X(x) = 1-\overline{F}_X(x)$. Define $X_{(\alpha,\ \beta)}$ to be a random variable with probability density function

$$f_{X_{(\alpha,\ \beta)}}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}[F_X(x)]^{\alpha-1}\left[\ \overline{F}_X(x)\right]^{\beta-1}f_X(x),\quad \alpha>0,\ \beta>0 \tag{3.1}$$

and probability distribution function

$$F_{X_{(\alpha,\ \beta)}}(x) = \frac{B(F_X(x);\alpha,\ \beta)}{B(\alpha,\ \beta)} \tag{3.2}$$

where $\Gamma(\alpha)$, $B(\alpha, \beta)$ and $B(z; \alpha, \beta)$ are the gamma function, the beta function and the incomplete beta function, respectively.

Remarks

1. Clearly $f_{X(\alpha,\ \beta)}(x)$ and $F_{X(\alpha,\ \beta)}(x)$ are probability density function and distribution function, respectively. This distributional transform is called the Beta transform.
2. Random variable X can be generalised on $\mathbb{R}$ and the distribution function is still the same. However we restrict X to be positive as it would be claim payment in the following sections.
3. $X_{(1,1)}$ follows the same distribution as the original random variable X.
4. Using a property of the incomplete beta function, we have

$$\overline{F}_{X_{(\alpha,\ \beta)}}(x) = \frac{B(\overline{F}_X(x);\ \beta,\ \alpha)}{B(\alpha,\ \beta)} \tag{3.3}$$

5. If $\alpha=1$, then

$$\overline{F}_{X_{(1,\ \beta)}}(x) = \left[\overline{F}_X(x)\right]^{\beta},\quad \beta>0$$

The Beta transform is the PH transform. See Wang (1995) for the properties of this transform and its applications.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

6. If $\beta = 1$, then

$$F_{X_{(\alpha,\,1)}}(x) = [F_X(x)]^\alpha, \quad \alpha > 0$$

This is the dual power transform, see Wirch & Hardy (1999).

The following Lemma shows how to connect $F_{X(\alpha,\,\beta)}$ with F_X.

Lemma 1. The distribution and the survival function of $X_{(\alpha,\,\beta)}$ can be expressed in terms of the hypergeometric function as follows:

$$F_{X_{(\alpha,\beta)}}(x) = \frac{1}{\alpha B(\alpha,\,\beta)}[F_X(x)]^\alpha\, {}_2F_1(1-\beta,\,\alpha;\,\alpha+1;\,F_X(x)) \tag{3.4}$$

$$= \frac{1}{B(\alpha,\,\beta)}\sum_{n=0}^{\infty}\frac{(1-\beta)_n}{n!}\frac{[F_X(x)]^{n+\alpha}}{n+\alpha} \tag{3.5}$$

$$\overline{F}_{X_{(\alpha,\beta)}}(x) = \frac{1}{\beta B(\alpha,\,\beta)}\left[\overline{F}_X(x)\right]^\beta {}_2F_1\left(1-\alpha,\,\beta;\,\beta+1;\,\overline{F}_X(x)\right) \tag{3.6}$$

$$= \frac{1}{B(\alpha,\,\beta)}\sum_{n=0}^{\infty}\frac{(1-\alpha)_n}{n!}\frac{\left[\overline{F}_X(x)\right]^{n+\beta}}{n+\beta} \tag{3.7}$$

where ${}_2F_1(a_1, a_2; b_1)$ is a generalised hypergeometric function with definition

$${}_pF_q(a_1,\,a_2,\,\ldots\,,a_p;\,b_1,\,b_2,\,\ldots\,,b_q;\,z) = \sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m\ldots\,(a_p)_m}{(b_1)_m(b_2)_m\ldots\,(b_q)_m}\frac{z^m}{m!}$$

and $(a)_m$ is the Pochhammer symbol. The moments of $X_{(\alpha,\,\beta)}$ can be expressed in terms of the moments of $X_{(1,\,\beta+n)}$ as follows:

$$\mathbb{E}\left[X^j_{(\alpha,\,\beta)}\right] = \frac{1}{B(\alpha,\,\beta)}\sum_{n=0}^{\infty}\frac{(1-\alpha)_n}{n!}\frac{\mathbb{E}\left[X^j_{(1,\,\beta+n)}\right]}{\beta+n} \tag{3.8}$$

Proof. Applying the definition of the incomplete beta function and expanding $(1-x)^{(\beta-1)}$ into the MacLaurin series, we have

$$B(x;\,\alpha,\,\beta) = \frac{x^\alpha}{\alpha}\,{}_2F_1(1-\beta,\,\alpha;\,\alpha+1;\,x)$$

Using equation (3.2) and the equation above, we obtain equation (3.4). Equation (3.5) is obtained by the definition of ${}_2F_1$ function. Applying similar argument to equation (3.3), we obtain equations (3.6) and (3.7). Equation (3.8) is obtained by noting $\mathbb{E}\left[X^j_{(\alpha,\,\beta)}\right] = j\int_0^\infty x^{j-1}\,\overline{F}_{X_{(\alpha,\,\beta)}}(x)dx$. This completes the proof. □

Remarks

1. $\overline{F}_{X_{(\alpha,\,\beta)}}(x)$ can be expressed as the infinite mixture of $\overline{F}_{X_{(1,\,\beta+n)}}(x)$, the tail of the PH transformed distribution of X (Remark 5 of Definition 1), for $n = 0, 1, \ldots$.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

2. If β takes a positive integer value, say $\beta = k$, then

$$F_{X_{(\alpha,\beta)}}(x) = \frac{1}{B(\alpha,\beta)} \sum_{n=0}^{k-1} \frac{(1-\beta)_n}{n!} \frac{[F_X(x)]^{n+\alpha}}{n+\alpha}$$

as $(1-\beta)_n = (1-k)_n = 0$ when $n \geq k$.

3. If α takes a positive integer value, say $\alpha = m$, then

$$\overline{F}_{X_{(\alpha,\beta)}}(x) = \frac{1}{B(\alpha,\beta)} \sum_{n=0}^{m-1} \frac{(1-\alpha)_n}{n!} \frac{[\overline{F}_X(x)]^{n+\beta}}{n+\beta}$$

as $(1-\alpha)_n = (1-m)_n = 0$ when $n \geq m$.

3.1 Some examples

Example 1. If $X \sim U(0, 1)$, then

$$f_{X_{(\alpha,\beta)}}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1,\ \alpha > 0,\ \beta > 0$$

i.e., $X_{(\alpha,\beta)}$ follows the $Beta(\alpha, \beta)$ distribution.

Example 2. If $X \sim Exp(\eta)$, then $\overline{F}_X(x) = e^{-\eta x}$, and it follows from equation (3.7) that

$$\overline{F}_{X_{(\alpha,\beta)}}(x) = \sum_{k=0}^{\infty} \xi_k e^{-\eta(k+\beta)x} \tag{3.9}$$

where

$$\xi_k = \frac{1}{B(\alpha,\beta)} \frac{(1-\alpha)_k}{k!(k+\beta)}, \quad k = 0, 1, 2, \ldots \tag{3.10}$$

This is to say that $X_{(\alpha,\beta)}$ follows an infinite mixture of exponential distributions with parameter $\eta(k+\beta)$, for $k = 0, 1, 2, \ldots$. The moment generating function

$$M_{X_{(\alpha,\beta)}}(s) = \frac{\Gamma(\alpha+\beta)\Gamma\left(\beta - \frac{s}{\eta}\right)}{\Gamma(\beta)\Gamma\left(\alpha+\beta-\frac{s}{\eta}\right)}, \quad s < \eta\beta$$

The jth moment of $X_{(\alpha,\beta)}$ is given by

$$\begin{aligned}\mathbb{E}\left[X^j_{(\alpha,\beta)}\right] &= \frac{\eta}{B(\alpha,\beta)} \sum_{k=0}^{\infty} \frac{(1-\alpha)_k}{k!} \frac{\Gamma(1+j)}{(\eta(k+\beta))^{1+j}} \\ &= \frac{1}{B(\alpha,\beta)} \frac{j!}{\beta^{j+1}\eta^j} {}_{j+2}F_{j+1}(1-\alpha, \beta, \beta, \ldots, \beta; \beta+1, \beta+1, \ldots, \beta+1; 1)\end{aligned}$$

Example 3. If $X \sim Gamma(2, \lambda)$, with

$$f_X(x) = \lambda^2 x\, e^{-\lambda x}, \quad \overline{F}_X(x) = (1+\lambda x)e^{-\lambda x}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

then it follows from (3.7) that

$$\overline{F}_{X_{(\alpha,\beta)}}(x) = \frac{1}{B(\alpha,\beta)}\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{k!(k+\beta)}(1+\lambda x)^{(k+\beta)}e^{-\lambda(k+\beta)x}$$

$$= \sum_{k=0}^{\infty}\xi_k(1+\lambda x)^{(k+\beta)}e^{-\lambda(k+\beta)x} \tag{3.11}$$

where ξ_k is given in equation (3.10), and

$$M_{X_{(\alpha,\beta)}}(s) = \frac{1}{B(\alpha,\beta)}\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{k!}U\left(2, \beta+k+2, \beta+k-\frac{s}{\lambda}\right), \quad s<\lambda(\beta+k)$$

where $U(\alpha, \gamma, z)$ is the confluent hypergeometric function of the second kind with definition

$$U(\alpha,\gamma;z) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty}e^{-zt}t^{\alpha-1}(1+t)^{\gamma-\alpha-1}, \quad \alpha>0, \quad z>0$$

Example 4. If $X \sim Pareto(\gamma, \eta)$, with

$$f_X(x) = \frac{\gamma\eta^{\gamma}}{(\eta+x)^{\gamma+1}}, \quad \overline{F}_X(x) = \left(\frac{\eta}{\eta+x}\right)^{\gamma}, \quad x>0, \gamma>0, \eta>0$$

then it follows from (3.7) that

$$\overline{F}_{X_{(\alpha,\beta)}}(x) = \frac{1}{B(\alpha,\beta)}\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{k!(k+\beta)}\left(\frac{\eta}{\eta+x}\right)^{\gamma(k+\beta)}$$

$$= \sum_{k=0}^{\infty}\xi_k\left(\frac{\eta}{\eta+x}\right)^{\gamma(k+\beta)} \tag{3.12}$$

where ξ_k is given in equation (3.10). This is to say, $X_{(\alpha,\beta)}$, follows an infinite mixture of Pareto distributions with parameters $\gamma(k+\beta)$ and η for $k=0, 1, 2\ldots$. In particular, when $\alpha=1$, then $X_{(1,\beta)} \sim Pareto(\gamma\beta, \eta)$.

Example 5. If $X \sim Geometric(p)$, with

$$f_X(x) = pq^x, \quad \overline{F}_X(x) = q^{x+1}, \quad 0<p<1, p+q=1, x=0, 1, 2, \ldots$$

then it follows from (3.7) that

$$\overline{F}_{X_{(\alpha,\beta)}}(x) = \sum_{k=0}^{\infty}\xi_k q_k^{x+1}$$

where $q_k = q^{k+\beta}$, for $k=0, 1, 2, \ldots$. That is to say, that $X_{(\alpha,\beta)}$ follows an infinite mixture of geometric distributions with parameter $p_k = 1 - q_k$.

3.2 Stochastic ordering

Next we use the above equations to derive conditions for α and β under which $X_{(\alpha,\beta)}$ and X are ordered under stochastic ordering and the stop loss ordering. We start with the definitions of these two orderings.

Definition 2. Let X and Y be two non-negative random variables. X, is said to be smaller than Y in the stochastic ordering, denoted as $X \preceq_{st} Y$, if $\overline{F}_X(x) \le \overline{F}_Y(x)$, $\forall x$.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Definition 3. Let X and Y be two non-negative random variables. X, is said to be smaller than Y, in the convex ordering, denoted as $X \preccurlyeq_{\mathrm{sl},=} Y$, or $X \preccurlyeq_{\mathrm{cx}} Y$, if $\mathbb{E}[X] = \mathbb{E}[Y]$, and $\mathbb{E}[\max(X-x, 0)] \le \mathbb{E}[\max(Y-x, 0)]$, $\forall x \in \mathbb{R}^+$ or equivalently, if $\mathbb{E}[v(X)] \le \mathbb{E}[v(Y)]$ for all convex functions v for which the expectations exist.

Theorem 1. We have the following stochastic ordering between $X_{(\alpha,\beta)}$ and X.

- If $\alpha \ge 1$, $0 < \beta < 1$, then $X \preccurlyeq_{\mathrm{st}} X_{(\alpha,\beta)}$.
- If $0 < \alpha < 1$, $\beta \ge 1$, then $X_{(\alpha,\beta)} \preccurlyeq_{\mathrm{st}} X$.
- If $\alpha = 1$, $\beta = 1$, then $X_{(\alpha,\beta)} \overset{d}{=} X$.

Proof. It follows from Lebedev (1972, Section 9) that

$$ {}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \quad \Re(\gamma-\alpha-\beta) > 0 \tag{3.13} $$

When $\alpha \ge 1$, $0 < \beta < 1$ it follows from equation (3.4) that

$$ \begin{aligned} F_{X_{(\alpha,\beta)}}(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} (F_X(x))^{\alpha} {}_2F_1(1-\beta, \alpha, \alpha+1, F_X(x)) \\ &\le \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} (F_X(x))^{\alpha} {}_2F_1(1-\beta, \alpha, \alpha+1, 1) \\ &= (F_X(x))^{\alpha} \le F_X(x) \qquad (\alpha \ge 1) \end{aligned} $$

When $0 < \alpha < 1$, $\beta \ge 1$, equation (3.6) gives

$$ \begin{aligned} \overline{F}_{X_{(\alpha,\beta)}}(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)} \left(\overline{F}_X(x)\right)^{\beta} {}_2F_1\left(1-\alpha, \beta, \beta+1, \overline{F}_X(x)\right) \\ &\le \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)} \left(\overline{F}_X(x)\right)^{\beta} {}_2F_1(1-\alpha, \beta, \beta+1, 1) \\ &= \left(\overline{F}_X(x)\right)^{\beta} \le \overline{F}_X(x) \qquad (\beta \ge 1) \end{aligned} $$

In summary, we have

$$ \overline{F}_{X_{(\alpha,\beta)}}(x) \begin{cases} \ge \overline{F}_X(x), & \text{if} \quad \alpha \ge 1, 0 < \beta < 1 \\ = \overline{F}_X(x), & \text{if} \quad \alpha = \beta = 1 \\ \le \overline{F}_X(x), & \text{if} \quad 0 < \alpha < 1, \beta \ge 1 \end{cases} $$

This proves the stochastic ordering stated in Theorem 1. □

Remark. For $j \in \mathbb{N}$, we have the following inequalities for moments by using the results in Theorem 1

$$ \mathbb{E}\left[X^j_{(\alpha,\beta)}\right] \begin{cases} \le \mathbb{E}[X^j], & \text{if} \quad 0 < \alpha < 1, \beta \ge 1 \\ \ge \mathbb{E}[X^j], & \text{if} \quad \alpha \ge 1, 0 < \beta < 1 \\ = \mathbb{E}[X^j], & \text{if} \quad \alpha = \beta = 1 \end{cases} $$

Theorem 2. Given $\mathbb{E}[X] = \mathbb{E}\left[X_{(\alpha,\beta)}\right]$ we have the following stochastic ordering between $X_{(\alpha,\beta)}$ and X.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

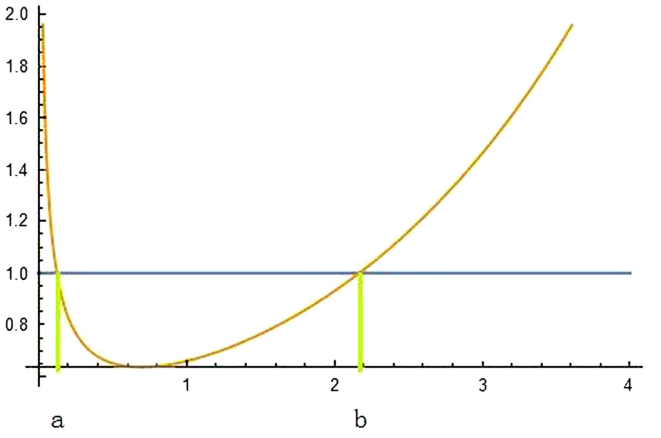


Figure 1. $\xi(x)$ for $0<\alpha<1$ and $0<\beta<1$.

- If $0<\alpha<1$, $0<\beta<1$, then $X \leqslant_{\text{cx}} X_{(\alpha,\beta)}$.
- If $\alpha>1$, $\beta>1$, then $X_{(\alpha,\beta)} \leqslant_{\text{cx}} X$.

Proof. Define

$$\xi(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}[F_X(x)]^{\alpha-1}[1-F_X(x)]^{\beta-1}, \qquad 0<x<\infty,\ \alpha>0,\ \beta>0$$

and then $f_{X_{(\alpha,\beta)}}(x)$ is a weighted version of $f_X(x)$, i.e.

$$f_{X_{(\alpha,\beta)}}(x) = \xi(x) f_X(x)$$

We discuss the plots for $\xi(x)$ under two cases.

- When $0<\alpha<1$ and $0<\beta<1$, the plot for $\xi(x)$ is given in Figure 1.
  It follows from Figure 1 that $\xi(a) = \xi(b) = 1$ and

$$\begin{cases} \xi(x)>1, & x\in(0,a)\cup(b,\infty) \\ \xi(x)<1, & x\in(a,b) \end{cases}$$

That is to say

$$\begin{cases} f_{X_{(\alpha,\beta)}}(x) > f_X(x), & x\in(0,a)\cup(b,\infty) \\ f_{X_{(\alpha,\beta)}}(x) < f_X(x), & x\in(a,b) \end{cases}$$

It now follows from theorem 3.A.44 in Shaked & Shanthikumar (2007) that $X \leqslant_{\text{cx}} X_{(\alpha,\beta)}$ when $0<\alpha<1$ and $0<\beta<1$.

- When $\alpha>1$ and $\beta>1$, the plot for $\xi(x)$ is given in Figure 2.
  It follows from Figure 2 that $\xi(c)=\xi(d)=1$ and

$$\begin{cases} \xi(x)<1, & x\in(0,c)\cup(d,\infty) \\ \xi(x)>1, & x\in(c,d) \end{cases}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

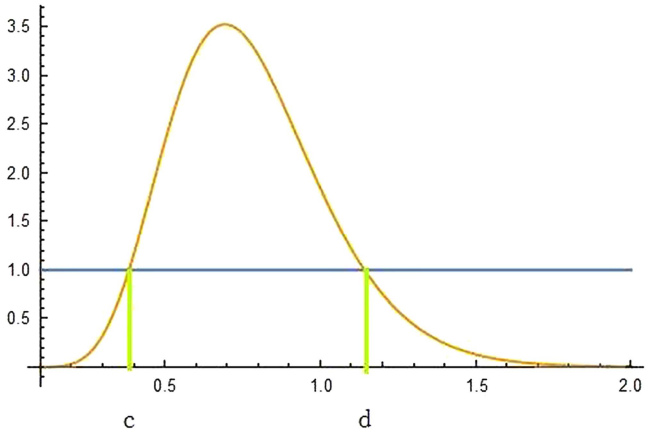


Figure 2. $\xi(x)$ for $\alpha > 1$ and $\beta > 1$.

That is to say

$$\begin{cases} f_{X_{(\alpha,\beta)}}(x) < f_X(x), & x \in (0, c) \cup (c, \infty) \\ f_{X_{(\alpha,\beta)}}(x) > f_X(x), & x \in (c, d) \end{cases}$$

It now follows from theorem 3.A.44 in Shaked & Shanthikumar (2007) that $X_{(\alpha,\beta)} \leqslant_{\text{cx}} X$ when $\alpha > 1$ and $\beta > 1$.

This completes the proof. □

Remarks.

1. Theorems 1 and 2 can be interpreted in the following way:
   - If $\alpha \geq 1$ and $0 < \beta < 1$, the weighted function $\xi(x)$ increases monotonically. This implies that $f_{X_{(\alpha,\beta)}}$ and f_X cross once and $f_{X_{(\alpha,\beta)}}$ is larger than f_X at the tail, therefore $\overline{F}_X(x) \leq \overline{F}_{X_{(\alpha,\beta)}}(x)$ for all x, i.e., $X \leqslant_{\text{st}} X_{(\alpha,\beta)}$.
   - If $0 < \alpha < 1$ and $0 < \beta < 1$, $\xi(x)$ decreases first and then increases, as illustrated in Figure 1. This implies that $f_{X_{(\alpha,\beta)}}$ and f_X cross twice and $\overline{F}_X(x)$ and $\overline{F}_{X_{(\alpha,\beta)}}(x)$ cross once, therefore $\overline{F}_X(x) \leq \overline{F}_{X_{(\alpha,\beta)}}(x)$ for large x, i.e., $X \leqslant_{\text{cx}} X_{(\alpha,\beta)}$.
   - Similar reasoning for the other two cases in Theorems 1 and 2.
2. By analysing the monotonicity property of the weight functions between densities, we have

$$X_{(\alpha_1,\beta)} \leqslant_{\text{st}} X_{(\alpha_2,\beta)}, \quad 0 < \alpha_1 < \alpha_2, \beta > 0$$

   and

$$X_{(\alpha,\beta_2)} \leqslant_{\text{st}} X_{(\alpha,\beta_1)}, \quad 0 < \beta_1 < \beta_2, \alpha > 0$$

   In particular

$$X_{(\alpha,\beta+1)} \leqslant_{\text{st}} X_{(\alpha,\beta)} \leqslant_{\text{st}} X_{(\alpha+1,\beta)}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

3. Further, if $\mathbb{E}\left[X_{(\alpha+1,\beta+1)}\right] = \mathbb{E}\left[X_{(\alpha,\beta)}\right]$, then

$$X_{(\alpha+1,\beta+1)} \lesssim_{\text{cx}} X_{(\alpha,\beta)}$$

This can be obtained by analysing the monotonicity property of $\zeta(x)$ where $f_{X_{(\alpha+1,\beta+1)}}(x) = \zeta(x) f_{X_{(\alpha,\beta)}}(x)$.

Finally, when $\alpha \geq 1$ and $0 < \beta \leq 1$, the expectation of Beta transform of survival function defines a coherent risk measure, i.e., the mean of the risk adjusted loss of $X_{(\alpha,\beta)}$. Details could be found in Wirch & Hardy (1999). Related definitions and theorems can be found in Denuit *et al.* (2005, section 2) and Tse (2009, section 4).

4. Integral Equations for Moments and Joint Moments

In this section, we aim to provide general formulas for the moments and joint moments of Z_t when the joint distribution of (X, W) is defined via the copula defined in section 2. To our knowledge, only few special cases have been done in the literature up to now. When $a=b=c=d=1$, recursive formulas for moments under Poisson arrival process are discussed in Barges *et al.* (2011, equation (19)). Later, Woo & Cheung (2013, example 3.2) propose a systematic way for this type of dependence structure and give numerical results under $a=b=c=d=1$ and $a=b=c=d=2$ cases. However, their method requires a large amount of calculations for coefficients and is therefore not computation friendly.

Define $M_{Z_t}(s) = \mathbb{E}\left[e^{sZ_t}\right]$ to be the moment generating function of Z_t. Using the renewal technique, we have

$$M_{Z_t}(s) = \mathbb{P}(W_1 > t) + \int_0^t M_{X_1 \mid W_1 = v}(se^{-\delta v}) M_{Z_{t-v}}(se^{-\delta v}) f_W(v)\,dv \tag{4.1}$$

The nth derivative of $M_{Z_t}(s)$ at s is

$$M_{Z_t}^{(n)}(s) = \sum_{j=0}^{n} \binom{n}{j} \int_0^t e^{-n\delta v} M_{X_1 \mid W_1 = v}^{(j)}(se^{-\delta v}) M_{Z_{t-v}}^{(n-j)}(se^{-\delta v}) f_W(v)\,dv$$

In order to construct a recursive formula, the term with $M_{Z_{t-v}}^{(n)}(se^{-\delta v})$ needs to be eliminated and that can be done through the use of convolution. First, we start with rearranging $M_{Z_t}^{(n)}(s)$ so that

$$M_{Z_t}^{(n)}(s) = \sum_{j=1}^{n} \binom{n}{j} \int_0^t e^{-n\delta v} M_{X_1 \mid W_1 = v}^{(j)}(se^{-\delta v}) M_{Z_{t-v}}^{(n-j)}(se^{-\delta v}) f_W(v)\,dv$$

$$+ \int_0^t e^{-n\delta v} M_{Z_{t-v}}^{(n)}(se^{-\delta v}) f_W(v)\,dv \tag{4.2}$$

Evaluating equation (4.2) at $s=0$ gives the following result in terms of the moments of Z_t:

$$V_n(t) = \sum_{j=1}^{n} \binom{n}{j} \int_0^t e^{-n\delta v} \mathbb{E}\left[X_1^j \mid W_1 = v\right] V_{n-j}(t-v) f_W(v)\,dv$$

$$+ \int_0^t e^{-n\delta v} V_n(t-v) f_W(v)\,dv \tag{4.3}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Under the dependence assumption in equation (2.3):

$$\mathbb{E}\left[X_1^j \mid W_1 = v\right] = \int_0^\infty x^j f_{X_1 \mid W_1 = v}(x)dx$$

$$= \mathbb{E}\left[X^j\right] + \theta ac[F_W(v)]^{c-1}[1-F_W(v)]^d \int_0^\infty x^j [F_X(x)]^{a-1}[1-F_X(x)]^b f_X(x)dx$$

$$-\theta bc[F_W(v)]^{c-1}[1-F_W(v)]^d \int_0^\infty x^j [F_X(x)]^a [1-F_X(x)]^{b-1} f_X(x)dx$$

$$-\theta ad[F_W(v)]^c[1-F_W(v)]^{d-1} \int_0^\infty x^j [F_X(x)]^{a-1}[1-F_X(x)]^b f_X(x)dx$$

$$+\theta bd[F_W(v)]^c[1-F_W(v)]^{d-1} \int_0^\infty x^j [F_X(x)]^a [1-F_X(x)]^{b-1} f_X(x)dx$$

Applying Definition 1 and using notation $X_{(a,\ b+1)}$ and $X_{(a,\ b)}$, we have

$$\mathbb{E}\left[X_1^j \mid W_1 = v\right] = \mathbb{E}[X^j] + \theta ac\frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)}[F_W(v)]^{c-1}[1-F_W(v)]^d \mathbb{E}[X^j_{(a,b+1)}]$$

$$-\theta bc[F_W(v)]^{c-1}[1-F_W(v)]^d \int_0^\infty x^j [F_X(x)]^a [1-F_X(x)]^{b-1} f_X(x)dx$$

$$-\theta ad\frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)}[F_W(v)]^c[1-F_W(v)]^{d-1}\mathbb{E}\left[X^j_{(a,b+1)}\right]$$

$$+\theta bd[F_W(v)]^c[1-F_W(v)]^{d-1} \int_0^\infty x^j [F_X(x)]^a [1-F_X(x)]^{b-1} f_X(x)dx$$

$$= \mathbb{E}[X^j] + \theta c\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}[F_W(v)]^{c-1}[1-F_W(v)]^d \mathbb{E}[X^j_{(a,b+1)}]$$

$$-\theta d\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}[F_W(v)]^c[1-F_W(v)]^{d-1}\mathbb{E}[X^j_{(a,b+1)}]$$

$$+\theta bc[F_W(v)]^{c-1}[1-F_W(v)]^d \int_0^\infty x^j [F_X(x)]^{a-1}(1-F_X(x))^b f_X(x)dx$$

$$-\theta bd[F_W(v)]^c[1-F_W(v)]^{d-1} \int_0^\infty x^j [F_X(x)]^{a-1}(1-F_X(x))^b f_X(x)dx$$

$$-\theta bc\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}[F_W(v)]^{c-1}[1-F_W(v)]^d \mathbb{E}\left[X^j_{(a,b)}\right]$$

$$+\theta bd\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}[F_W(v)]^c[1-F_W(v)]^{d-1}\mathbb{E}[X^j_{(a,b)}]$$

$$= \mathbb{E}\left[X^j\right] + \theta c\frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b)}[F_W(v)]^{c-1}[1-F_W(v)]^d \left(\mathbb{E}\left[X^j_{(a,b+1)}\right] - \mathbb{E}\left[X^j_{(a,b)}\right]\right)$$

$$-\theta d\frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b)}[F_W(v)]^c[1-F_W(v)]^{d-1}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right] - \mathbb{E}\left[X^j_{(a,b)}\right]\right) \quad (4.4)$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

When $a=b=c=d=1$, the equation above is much neater than the expression in Barges *et al.* (2011) and Woo & Cheung (2013). Substituting $\mathbb{E}[X_1^j \mid W_1 = v]$ into equation (4.3), we have the following equation:

$$\begin{aligned}
V_n(t) = &\sum_{j=1}^{n}\binom{n}{j}\mathbb{E}\left[X^j\right]\int_0^t \mathrm{e}^{-n\delta v}V_{n-j}(t-v)dF_W(v)\\
&+\theta c\frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)\\
&\times\int_0^t \mathrm{e}^{-n\delta v}[F_W(v)]^{c-1}[1-F_W(v)]^d V_{n-j}(t-v)dF_W(v)\\
&-\theta d\frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}[X^j_{(a,b)}]\right)\\
&\times\int_0^t \mathrm{e}^{-n\delta v}[F_W(v)]^{c}[1-F_W(v)]^{d-1} V_{n-j}(t-v)dF_W(v)\\
&+\int_0^t \mathrm{e}^{-n\delta v}V_n(t-v)dF_W(v)
\end{aligned}$$

After identifying $W_{(c,\ d)}$, we have the following result:

Theorem 3. For any $\delta \geq 0$ and $n \in \mathbb{N}$, the nth moment of Z_t satisfies

$$\begin{aligned}
V_n(t) = &\sum_{j=0}^{n}\binom{n}{j}\mathbb{E}\left[X^j\right]\int_0^t \mathrm{e}^{-n\delta v}V_{n-j}(t-v)dF_W(v)\\
&+\theta bcB(a,b)B(c,d+1)\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,\,b+1)}\right]-\mathbb{E}\left[X^j_{(a,\,b)}\right]\right)\\
&\times\int_0^t \mathrm{e}^{-n\delta v}V_{n-j}(t-v)dF_{W_{(c,d+1)}}(v)\\
&-\theta bdB(a,b)B(c+1,d)\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,\,b+1)}\right]-\mathbb{E}\left[X^j_{(a,\,b)}\right]\right)\\
&\times\int_0^t \mathrm{e}^{-n\delta v}V_{n-j}(t-v)dF_{W_{(c+1,d)}}(v) \qquad (4.5)
\end{aligned}$$

Remarks.

1. As the expressions for the Beta transform of f_W. and for the moments of the Beta transform of f_X have been given in previous section, the recursion formula given in equation (4.5) can be evaluated efficiently by using numerical integration method and using the starting value $V_0(t)=1$.
2. When the distribution of W is specified, equation (4.5) can be further simplified. In sections 5 and 6, we deal with the cases where $W \sim Exp(\lambda)$ and $W \sim \Gamma(2,\lambda)$. Explicit expressions for the moments in particular the lower moments can be derived.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Theorem 3 suggests that, because of Beta transform, the computational capacity of $V_n(t)$ depends on $f_{W_{(c,d)}}$. In particular, we have seen in equation (3.6) and will see in the following sections, whether c is an integer or not makes a significant difference in calculation of $f_{W_{(c,d)}}$. With the aid of stochastic ordering properties obtained in section 3.2, lower and upper bounds for $V_n(t)$ can be derived when c is not an integer. The following lemma is developed for this purpose.

Lemma 2. Let ϕ be a non-negative and non-decreasing function defined on $(-\infty, \infty)$ with $\phi(\infty) < \infty$. Let Y_1 and Y_2 be two positive random variables with corresponding density functions f_{Y_1} and f_{Y_2}. If there exists a positive number m, such that $f_{Y_2}(y) < f_{Y_1}(y), y < m,$ and $f_{Y_1}(y) < f_{Y_2}(y), y > m,$ i.e., $Y_1 \leq_{st} Y_2$, then

$$\int_0^t \mathrm{e}^{-rv}\phi(t-v)f_{Y_1}(v)dv \geq \int_0^t \mathrm{e}^{-rv}\phi(t-v)f_{Y_2}(v)dv, \quad t \geq 0, r > 0 \tag{4.6}$$

Proof. Clearly, inequality (4.6) is true for $t \leq m$. For $t > m$, the following inequality is true:

$$\int_t^\infty \mathrm{e}^{-rv}\phi(t-v)f_{Y_1}(v)dv \leq \int_t^\infty \mathrm{e}^{-rv}\phi(t-v)f_{Y_2}(v)dv \tag{4.7}$$

Consider function $\psi(v) = \mathrm{e}^{-rv}\phi(t-v)$, $0 < v < \infty$. As $\psi'(v) = -re^{-rv}\phi(t-v) - \mathrm{e}^{-rv}\phi'(t-v) \leq 0$, then $\psi(v)$ is a non-increasing function. It follows from theorem 1.A.3(a) in Shaked & Shanthikumar (2007) that $\psi(Y_2) \leq_{\mathrm{st}} \psi(Y_1)$, therefore $\mathbb{E}[\psi(Y_2)] \leq \mathbb{E}[\psi(Y_1)]$ and this gives

$$\int_0^\infty \mathrm{e}^{-rv}\phi(t-v)f_{Y_2}(v)dv \leq \int_0^\infty \mathrm{e}^{-rv}\phi(t-v)f_{Y_1}(v)dv \tag{4.8}$$

Combining (4.7) and (4.8) shows that inequality (4.6) is also true for $t > m$ and the proof is completed. □

Now we derive upper and lower bounds for $V_n(t)$ by using Lemma 2. By the second remark on Theorem 2, we have $X_{(a,b+1)} \leq_{\mathrm{st}} X_{(a,b)}$, therefore $\mathbb{E}[X^j_{(a,b+1)}] - \mathbb{E}[X^j_{(a,b)}] < 0$. If $\theta < 0 (>0)$, following Lemma 2, $V_n(t)$ has the following lower (upper) bound:

$$\begin{aligned} V_n(t) \geq (\leq) & \sum_{j=0}^{n} \binom{n}{j} \mathbb{E}[X^j] \int_0^t \mathrm{e}^{-n\delta v} V_{n-j}(t-v) dF_W(v) \\ & + \theta bcB(a,b)B(c,d+1) \sum_{j=1}^{n} \binom{n}{j} \left(\mathbb{E}\left[X^j_{(a,b+1)}\right] - \mathbb{E}\left[X^j_{(a,b)}\right] \right) \\ & \times \int_0^t \mathrm{e}^{-n\delta v} V_{n-j}(t-v) dF_{W_{([c]+1,d+1)}}(v) \\ & - \theta bdB(a,b)B(c+1,d) \sum_{j=1}^{n} \binom{n}{j} \left(\mathbb{E}\left[X^j_{(a,b+1)}\right] - \mathbb{E}\left[X^j_{(a,b)}\right] \right) \\ & \times \int_0^t \mathrm{e}^{-n\delta v} V_{n-j}(t-v) dF_{W_{([c]+1,d)}}(v) \end{aligned}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

and the upper (lower) bound

$$V_n(t) \le (\ge) \sum_{j=0}^{n} \binom{n}{j} \mathbb{E}[X^j] \int_0^t \mathrm{e}^{-n\delta v} V_{n-j}(t-v) dF_W(v)$$

$$+ \theta bcB(a,b)B(c,d+1) \sum_{j=1}^{n} \binom{n}{j} (\mathbb{E}[X^j_{(a,b+1)}] - \mathbb{E}[X^j_{(a,b)}])$$

$$\times \int_0^t \mathrm{e}^{-n\delta v} V_{n-j}(t-v) dF_{W_{([c],d+1)}}(v)$$

$$- \theta bdB(a,b)B(c+1,d) \sum_{j=1}^{n} \binom{n}{j} (\mathbb{E}[X^j_{(a,b+1)}] - \mathbb{E}[X^j_{(a,b)}])$$

$$\times \int_0^t \mathrm{e}^{-n\delta v} V_{n-j}(t-v) dF_{W_{([c]+2,d)}}(v)$$

where $[c]$ is the integer part of c.

Now we study the joint moments of Z_t and Z_{t+h} by starting with their joint moment generating function.

$$M_{Z_t,Z_{t+h}}(s,r) = \mathbb{E}\left[\mathrm{e}^{sZ_t + rZ_{t+h}}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{(s+r)\sum_{k=1}^{N_t} \mathrm{e}^{-\delta T_k} X_k + r \sum_{k=N_t+1}^{N_{t+h}} \mathrm{e}^{-\delta T_k} X_k} \mid W_1\right]\right]$$

$$= \int_0^t \mathbb{E}\left[\mathrm{e}^{(s+r)\sum_{k=1}^{N_t} \mathrm{e}^{-\delta T_k} X_k + r \sum_{k=N_t+1}^{N_{t+h}} \mathrm{e}^{-\delta T_k} X_k} \mid W_1\right] dF_W(v)$$

$$+ \int_t^{t+h} \mathbb{E}\left[\mathrm{e}^{r\sum_{k=1}^{N_{t+h}} \mathrm{e}^{-\delta T_k} X_k} \mid W_1\right] dF_W(v) + \int_{t+h}^{\infty} \mathbb{E}\left[\mathrm{e}^0 \mid W_1\right] dF_W(v)$$

$$= \int_0^t \mathbb{E}\left[\mathrm{e}^{(s+r)\mathrm{e}^{-\delta v} X_1} \mid W_1\right] \mathbb{E}\left[\mathrm{e}^{(s+r)\mathrm{e}^{-\delta v} \sum_{k=2}^{N_t} \mathrm{e}^{-\delta(T_k - v)} X_k + r\mathrm{e}^{-\delta v} \sum_{k=N_t+1}^{N_{t+h}} \mathrm{e}^{-\delta(T_k - v)} X_k}\right] dF_W(v)$$

$$+ \int_t^{t+h} \mathbb{E}\left[\mathrm{e}^{r\mathrm{e}^{-\delta v} X_1} \mid W_1\right] \mathbb{E}\left[\mathrm{e}^{r\mathrm{e}^{-\delta v} \sum_{k=2}^{N_{t+h}} \mathrm{e}^{-\delta(T_k - v)} X_k}\right] dF_W(v) + \int_{t+h}^{\infty} dF_W(v)$$

$$= \int_0^t \mathbb{E}\left[\mathrm{e}^{(s+r)\mathrm{e}^{-\delta v} X_1} \mid W_1\right] M_{Z_{t-v}, Z_{t-v+h}}\left(s\mathrm{e}^{-\delta v}, r\mathrm{e}^{-\delta v}\right) dF_W(v)$$

$$+ \int_t^{t+h} \mathbb{E}\left[\mathrm{e}^{r\mathrm{e}^{-\delta v} X_1} \mid W_1\right] M_{Z_{t-v+h}}\left(r\mathrm{e}^{-\delta v}\right) dF_W(v) + \int_{t+h}^{\infty} dF_W(v) \tag{4.9}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Similar to $\mathbb{E}\left[X^j \middle| W_1 = v\right]$, we have

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{(s+r)\mathrm{e}^{-\delta v}X_1} \mid W_1] &= M_X\left((s+r)\mathrm{e}^{-\delta v}\right) \\ &\quad + \theta\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\left(M_{X_{(a,b+1)}}\left((s+r)\mathrm{e}^{-\delta v}\right) - M_{X_{(a,b)}}\left((s+r)\mathrm{e}^{-\delta v}\right)\right) \\ &\quad \times \left[cF_W(v)^{c-1}(1-F_W(v))^d - dF_W(v)^c(1-F_W(v))^{d-1}\right] \end{aligned} \tag{4.10}$$

Same reasoning for $\mathbb{E}[\mathrm{e}^{re^{-\delta v}X_1} \mid W_1]$. A quick check that could be done for equation (4.9) is to let $\theta = 0$, i.e. X_k and W_k are independent, then it reduces to Léveillé & Adékambi (2012, equation (2.5)). Differentiating $M_{Z_t,Z_{t+h}}(s, r)$ with respect to s and r, n and m times, respectively, and then letting $s = r = 0$, we have the following result.

Theorem 4. For any $\delta \geq 0$, and $n, m \in \mathbb{N}$, the expression of joint moments between Z_t and Z_{t+h} is given by

$$\begin{aligned} V_{n,m}(t, t+h) &= \sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\mathbb{E}\left[X^{k+j}\right]\int_0^t \mathrm{e}^{-(n+m)\delta v}V_{n-k,m-j}(t-v, t-v+h)dF_W(v) \\ &\quad + \theta bcB(a,b)B(c,d+1)\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\left(\mathbb{E}\left[X_{(a,b+1)}^{k+j}\right] - \mathbb{E}\left[X_{(a,b)}^{k+j}\right]\right) \\ &\quad \times \int_0^t \mathrm{e}^{-(n+m)\delta v}V_{n-k,m-j}(t-v, t-v+h)dF_{W_{(c,d+1)}}(v) \\ &\quad - \theta bdB(a,b)B(c+1,d)\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\left(\mathbb{E}\left[X_{(a,b+1)}^{k+j}\right] - \mathbb{E}\left[X_{(a,b)}^{k+j}\right]\right) \\ &\quad \times \int_0^t \mathrm{e}^{-(n+m)\delta v}V_{n-k,m-j}(t-v, t-v+h)dF_{W_{(c+1,d)}}(v) \end{aligned} \tag{4.11}$$

Similar to equation (4.5), equation (4.11) can be evaluated recursively by using numerical integration and noting that $V_{0,0}(t, t+h) = 1$. In particular, if the counting process is a Poisson process, the joint moments can be further simplified and $V_{1,1}(t, t+h)$ can be found explicitly. Bounds issue of $V_{n,m}(t, t+h)$ can be discussed similarly to $V_n(t)$.

5. Explicit Results Under Poisson Process

In this section, we aim to simplify the results in section 4 to obtain moments and joint moments of the aggregate discounted claims if W follows an exponential distribution with parameter $\lambda > 0$.

5.1 Simplified expressions for moments and joint moments

We use $\mathcal{L}_r\{z(t)\} = \int_0^\infty \mathrm{e}^{-rt}z(t)dt$ to denote the Laplace transform of function $z(t)$.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Lemma 3. A recursive formula for the Laplace transform of the nth moment of Z_t with respect to t with parameter r is

$$\begin{aligned}\mathcal{L}_r\{V_n(t)\} = &\sum_{j=1}^{n}\binom{n}{j}\mathbb{E}[X^j]\frac{\lambda}{r+n\delta}\mathcal{L}_r\{V_{n-j}(t)\}\\ &+\theta c\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)\\ &\times\frac{r+n\delta+\lambda}{(r+n\delta)}B\left(c,1+d+\frac{r+n\delta}{\lambda}\right)\mathcal{L}_r\{V_{n-j}(t)\}\\ &-\theta d\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}(\mathbb{E}[X^j_{(a,b+1)}]-\mathbb{E}[X^j_{(a,b)}])\\ &\times\frac{r+n\delta+\lambda}{r+n\delta}B\left(1+c,d+\frac{r+n\delta}{\lambda}\right)\mathcal{L}_r\{V_{n-j}(t)\}\end{aligned}$$

Proof. The following integral is frequently used in this case

$$\int_0^\infty e^{-rv}e^{-(d-1)\lambda v}\left(1-e^{-\lambda v}\right)^c dv = \frac{1}{\lambda}\frac{\Gamma(c+1)\Gamma\left(\frac{r}{\lambda}+d-1\right)}{\Gamma\left(c+\frac{r}{\lambda}+d\right)} = \frac{1}{\lambda}B\left(c+1,\frac{r}{\lambda}+d-1\right)$$

which can be done by a change of variable $u=e^{-\lambda v}$. Then apply the Laplace transform in Theorem 3 and the result follows after direct calculations. □

Inverting the Laplace transform in Lemma 3, we obtain the following result:

Theorem 5. For any $\delta\geq 0$ and $n\in\mathbb{N}$, a recursive formula for the nth moment of Z_t is

$$\begin{aligned}V_n(t) = &\sum_{j=1}^{n}\binom{n}{j}\mathbb{E}[X^j]\int_0^t\lambda e^{-n\delta v}V_{n-j}(t-v)dv\\ &+\theta c\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)\\ &\times\sum_{k=0}^{\infty}\frac{(1-c)_k}{k!}\int_0^t\lambda e^{-n\delta v}\frac{1+e^{-(1+d+k)\lambda v}(d+k)}{1+d+k}V_{n-j}(t-v)dv\\ &-\theta d\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)\\ &\times\sum_{k=0}^{\infty}\frac{(-c)_k}{k!}\int_0^t\lambda e^{-n\delta v}\frac{1+e^{-(d+k)\lambda v}(-1+d+k)}{d+k}V_{n-j}(t-v)dv\end{aligned}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Proof. Inverting Lemma 3 gives

$$V_n(t) = \sum_{j=1}^{n} \binom{n}{j} \mathbb{E}\left[X^j\right] \int_0^t \lambda e^{-n\delta v} V_{n-j}(t-v)dv$$

$$+ \theta c \frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)} \sum_{j=1}^{n} \binom{n}{j} \left(\mathbb{E}\left[X^j_{(a,b+1)}\right] - \mathbb{E}\left[X^j_{(a,b)}\right]\right)$$

$$\times \left[\int_0^t \lambda e^{-n\delta v}\left(1-e^{-\lambda v}\right)^{c-1} e^{-d\lambda v} e^{-\lambda v} V_{n-j}(t-v)dv\right.$$

$$\left. + \int_0^t \lambda^2 e^{-n\delta v}\left(\int_0^v \left(1-e^{-\lambda u}\right)^{c-1} e^{-d\lambda u} e^{-\lambda u} du\right) V_{n-j}(t-v)dv\right]$$

$$- \theta d \frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)} \sum_{j=1}^{n} \binom{n}{j} \left(\mathbb{E}\left[X^j_{(a,b+1)}\right] - \mathbb{E}\left[X^j_{(a,b)}\right]\right)$$

$$\times \left[\int_0^t \lambda e^{-n\delta v}\left(1-e^{-\lambda v}\right)^{c} e^{-(d-1)\lambda v} e^{-\lambda v} V_{n-j}(t-v)dv\right.$$

$$\left. + \int_0^t \lambda^2 e^{-n\delta v}\left(\int_0^v \left(1-e^{-\lambda u}\right)^{c} e^{-(d-1)\lambda u} e^{-\lambda u} du\right) V_{n-j}(t-v)dv\right]$$

$\lambda, u > 0$ implies $e^{-\lambda u} < 1$ and therefore we can apply the MacLaurin series for $(1 - e^{-\lambda u})^{c-1}$. This gives

$$\int_0^v \left(1-e^{-\lambda u}\right)^{c-1} e^{-d\lambda u} e^{-\lambda u} du = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{(1-c)_k}{k!} \frac{1-e^{-(1+d+k)\lambda v}}{1+d+k}$$

Bring this relation back and the result follows after direct calculations. □

Remarks

1. As $V_0(t) = 1$, $V_1(t)$ in Theorem 5 can be expressed explicitly as the integrals on the right-hand side of the expression for $V_1(t)$ can be integrated out. Once we have an explicit expression for $V_1(t)$, we can find an explicit expression for $V_2(t)$ and therefore for $V_n(t)$, after simplifying the integrals.
2. If c is an integer, then the summations of k terminate after finite number of terms. In particular, when $c = 1$, $\sum_{k=0}^{\infty} \frac{(0)_k}{k!}$ and $\sum_{k=0}^{\infty} \frac{(-1)_k}{k!}$ only count for the first term and the first two terms, respectively:

$$V_n(t) = \sum_{j=1}^{n} \binom{n}{j} \mathbb{E}\left[X^j\right] \int_0^t \lambda e^{-n\delta v} V_{n-j}(t-v)dv$$

$$+ \theta \frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)} \sum_{j=1}^{n} \binom{n}{j} \left(\mathbb{E}\left[X^j_{(a,b+1)}\right] - \mathbb{E}\left[X^j_{(a,b)}\right]\right)$$

$$\times \int_0^t \lambda e^{-n\delta v}\left[e^{-(1+d)\lambda v} d - e^{-d\lambda v}(d-1)\right] V_{n-j}(t-v)dv$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Therefore the first moment is

$$V_1(t) = \lambda\frac{1-\mathrm{e}^{-\delta t}}{\delta}\mathbb{E}[X] + \theta\lambda\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)$$
$$\times\frac{\delta+\lambda-d\mathrm{e}^{-t(\delta+\lambda+d\lambda)}(\delta+d\lambda)+(d-1)\mathrm{e}^{-(\delta+d\lambda)t}(\delta+\lambda+d\lambda)}{(\delta+d\lambda)(\delta+\lambda+d\lambda)}$$

3. Further, letting $a=b=c=d=1$ and noticing that $\mathbb{E}[X^j_{(1,1)}] = \mathbb{E}[X^j]$, we have

$$V_n(t) = \sum_{j=1}^{n}\binom{n}{j}\mathbb{E}\left[X^j\right]\int_0^t \lambda\mathrm{e}^{-n\delta v}V_{n-j}(t-v)dv$$
$$+\theta\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(1,2)}\right]-\mathbb{E}\left[X^j\right]\right)\int_0^t \lambda\mathrm{e}^{-(n\delta+2\lambda)v}V_{n-j}(t-v)dv$$

This is a more tractable result in comparison to the formulae in Barges *et al.* (2011), in a form of multiply summation, and in Woo & Cheung (2013), requiring identifying and calculating of the coefficients. This formula is more efficient in terms of numerical analysis, as we only need to input the formula into a software, e.g., Mathematica.

Corollary 1. When $t\to\infty$, the moments of Z_∞, for any $\delta>0$ and $n\in\mathbb{N}$, can be calculated recursively as follows:

$$V_n(\infty) = \sum_{j=1}^{n}\binom{n}{j}\frac{\lambda}{n\delta}\mathbb{E}\left[X^j\right]V_{n-j}(\infty) + \theta\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}$$
$$\times\sum_{j=1}^{n}\binom{n}{j}\frac{n\delta+\lambda}{\lambda}B\left(1+c,\ d+\frac{n\delta}{\lambda}\right)\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)V_{n-j}(\infty)$$

In particular,

$$V_1(\infty) = \tfrac{\lambda}{\delta}\mathbb{E}[X] + \theta\tfrac{\delta+\lambda}{\lambda}\tfrac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}B\left(1+c,\ d+\tfrac{\delta}{\lambda}\right)\left(\mathbb{E}[X_{(a,b+1)}]-\mathbb{E}[X_{(a,b)}]\right)$$

Proof. Letting $t\to\infty$ in Theorem 5, we have

$$V_n(\infty) = \sum_{j=1}^{n}\binom{n}{j}\frac{\lambda}{n\delta}\mathbb{E}[X^j]V_{n-j}(\infty)$$
$$+\theta c\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\frac{\lambda}{n\delta}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)V_{n-j}(\infty)\sum_{k=0}^{\infty}\frac{(1-c)_k}{k!}\frac{n\delta+\lambda}{n\delta+(1+d+k)\lambda}$$
$$-\theta d\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\frac{\lambda}{n\delta}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)V_{n-j}(\infty)\sum_{k=0}^{\infty}\frac{(-c)_k}{k!}\frac{n\delta+\lambda}{n\delta+(d+k)\lambda}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Apply equation (3.13), then we have

$$\sum_{k=0}^{\infty}\frac{(-c)_k}{k!}\frac{n\delta+\lambda}{n\delta+(d+k)\lambda}=\frac{n\delta+\lambda}{\lambda}\frac{\Gamma\left(\frac{n\delta}{\lambda}+d\right)}{\Gamma\left(1+\frac{n\delta}{\lambda}+d\right)}{}_2F_1\left(-c,\frac{n\delta}{\lambda}+d,1+\frac{n\delta}{\lambda}+d,1\right)$$

$$=\frac{n\delta+\lambda}{\lambda}\frac{\Gamma\left(\frac{n\delta}{\lambda}+d\right)\Gamma(1+c)}{\Gamma\left(1+\frac{n\delta}{\lambda}+d+c\right)}$$

Same idea also works for the other summation and then the result follows after direct calculation. □

Remark. The second remark on Theorem 2 suggests $\mathbb{E}[X^j_{(a,b+1)}]-\mathbb{E}[X^j_{(a,b)}]<0$ and $B(\alpha,\beta)$ decreases when α or β increases, therefore for $\theta>0(<0)$, $V_n(\infty)$ increases (decreases) when c or d increases.

Based on Theorem 4 and applying the same technique as in Lemma 3, as well as Theorem 5, we have the following result:

Theorem 6. For any $\delta, h\geq 0$ and $n, m\in\mathbb{N}$, the joint moment of Z_t and Z_{t+h} is given by

$$\begin{aligned}V_{n,m}(t,t+h)=&\sum_{i=1}^{n+m}\sum_{k=\max\{i-m,0\}}^{\min\{i,n\}}\binom{n}{k}\binom{m}{i-k}\mathbb{E}[X^i]\int_0^t\lambda e^{-(m+n)\delta v}V_{n-k,m-(i-k)}(t-v,t-v+h)dv\\&+\theta c\frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b)}\sum_{i=1}^{n+m}\sum_{k=\max\{i-m,0\}}^{\min\{i,n\}}\sum_{j=0}^{\infty}\binom{n}{k}\binom{m}{i-k}\frac{(1-c)_j}{j!}\left(\mathbb{E}\left[X^i_{(a,b+1)}\right]-\mathbb{E}\left[X^i_{(a,b)}\right]\right)\\&\times\int_0^t\lambda e^{-(m+n)\delta v}\frac{1+e^{-v(1+d+j)\lambda}(d+j)}{1+d+j}V_{n-k,m-(i-k)}(t-v,t-v+h)dv\\&-\theta d\frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b)}\sum_{i=1}^{n+m}\sum_{k=\max\{i-m,0\}}^{\min\{i,n\}}\sum_{j=0}^{\infty}\binom{n}{k}\binom{m}{i-k}\frac{(-c)_j}{j!}\left(\mathbb{E}\left[X^i_{(a,b+1)}\right]-\mathbb{E}\left[X^i_{(a,b)}\right]\right)\\&\times\int_0^t\lambda e^{-(m+n)\delta v}\frac{1+e^{-v(d+j)\lambda}(-1+d+j)}{d+j}V_{n-k,m-(i-k)}(t-v,t-v+h)dv\end{aligned}\tag{5.1}$$

Remarks

1. For $n=m=1$, we have

$$\begin{aligned}V_{1,1}(t,t+h)=&\mathbb{E}[X]\int_0^t\lambda e^{-2\delta v}[V_1(t-v)+V_1(t-v+h)]dv+\mathbb{E}[X^2]\frac{\lambda(1-e^{-2t\delta})}{2\delta}\\&+\theta c\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\left(\mathbb{E}[X_{(a,b+1)}]-\mathbb{E}[X_{(a,b)}]\right)\sum_{l=0}^{\infty}\frac{(1-c)_l}{l!}\\&\times\int_0^t\lambda e^{-2\delta v}\frac{1+e^{-v(1+d+l)\lambda}(d+l)}{1+d+l}[V_1(t-v)+V_1(t-v+h)]dv\end{aligned}$$

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

$$+\theta c\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\left(\mathbb{E}\left[X^2_{(a,b+1)}\right]-\mathbb{E}\left[X^2_{(a,b)}\right]\right)\sum_{l=0}^{\infty}\frac{(1-c)_l}{l!}$$

$$\times\int_0^t\lambda e^{-2\delta v}\frac{1+e^{-v(1+d+l)\lambda}(d+l)}{1+d+l}dv$$

$$-\theta d\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\left(\mathbb{E}\left[X_{(a,b+1)}\right]-\mathbb{E}\left[X_{(a,b)}\right]\right)\sum_{l=0}^{\infty}\frac{(-c)_l}{l!}$$

$$\times\int_0^t\lambda e^{-2\delta v}\frac{1+e^{-v(d+l)\lambda}(-1+d+l)}{d+l}[V_1(t-v)+V_1(t-v+h)]dv$$

$$-\theta d\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\left(\mathbb{E}\left[X^2_{(a,b+1)}\right]-\mathbb{E}\left[X^2_{(a,b)}\right]\right)\sum_{l=0}^{\infty}\frac{(-c)_l}{l!}$$

$$\times\int_0^t\lambda e^{-2\delta v}\frac{1+e^{-v(d+l)\lambda}(-1+d+l)}{d+l}dv$$

As $V_1(t)$ has an explicit expression, an explicit expression for $V_{1,1}(t, t+h)=1$ can be obtained by simplifying all integrals in the formula above.

2. Similar to moments, when c is an integer, the summations with respect to j terminates after finite number of terms. In particular, when $c=1$, we have

$$V_{1,1}(t,\,t+h)=\mathbb{E}[X]\int_0^t\lambda e^{-2\delta v}[V_1(t-v)+V_1(t-v+h)]dv+\mathbb{E}\left[X^2\right]\frac{\lambda\left(1-e^{-2t\delta}\right)}{2\delta}$$

$$+\theta\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\left(\mathbb{E}\left[X_{(a,\,b+1)}\right]-\mathbb{E}\left[X_{(a,\,b)}\right]\right)$$

$$\times\int_0^t\lambda e^{-2\delta v}e^{-(1+d)v\lambda}\left(d-(d-1)e^{v\lambda}\right)[V_1(t-v)+V_1(t-v+h)]dv$$

$$+\theta\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\left(\mathbb{E}\left[X^2_{(a,\,b+1)}\right]-\mathbb{E}\left[X^2_{(a,\,b)}\right]\right)$$

$$\times\frac{e^{-t(2\delta+\lambda+d\lambda)}\lambda\left[2d\left(e^{t\lambda}-1\right)\delta+d^2\left(e^{t\lambda}-1\right)\lambda+e^{t\lambda}\left(e^{2t\delta+dt\lambda}-1\right)(2\delta+\lambda)\right]}{(2\delta+d\lambda)(2\delta+\lambda+d\lambda)}$$

Based on above the theorem, we may get a formula for $\mathrm{Cov}(Z_t,\,Z_{t+h})$. The formula itself is too long to present here, even for the case $c=1$. However, an interesting property of covariance will be discussed in the following section.

5.2 Numerical illustration

In this section, we are aiming to give some examples for the theoretical formulas derived above. Woo & Cheung (2013, tables 3–6) give the $\mathbb{E}[Z_t]$ and $SD[Z_t]$ for $\delta=0.02$, 0.04 and $\lambda=1$ under $a=b=c=d=1$ and $a=b=c=d=2$, respectively, although an explicit formula for the latter case is not

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

presented in the paper. It turns out when a, b, c, d are integers, calculations become more straightforward. We will show our moment formula can lead to various choices of a, b, c, d. Based on the first remark on Theorem 3, the complexity of our formula is dependent on the choice of c rather than any other parameters. Thus, for illustration purposes, we let $a=b=d=1.5$, $c=\lambda=1$ and $\delta=0.01$ considering current low interest rate environment. Finally, we assume claim severities follow an exponential distribution with mean 100 throughout the whole section.

Tables 1 and 2 give the expected values $\mathbb{E}[Z_t]$ and standard deviations $SD[Z_t]$ of discounted aggregate claims Z_t. The following two observations could be made.

1. For a fixed θ, $\mathbb{E}[Z_t]$ and $SD[Z_t]$ are increasing with time t. As the duration of policy gets longer, it is reasonable to expect more claims.
2. For a fixed t, $\mathbb{E}[Z_t]$ and $SD[Z_t]$ are decreasing functions of θ. This is intuitive for $SD[Z_t]$ but not for $\mathbb{E}[Z_t]$ as we expect, at least, $\mathbb{E}[Z_t]$ is increasing when the dependence structure between claim severity and claim frequency is changing from negative to positive. However, this view is supported in a few references and interested readers may refer to Woo & Cheung (2013), Cossette *et al.* (2008) and Chadjiconstantinidis & Vrontos (2014) for more details. It might be more explainable by writing down the equation and applying Theorem 3, from which we will end with

$$\mathbb{E}[Z_t] = 10000\left(1-e^{-0.01t}\right)-5.23240\theta+11.73404e^{-2.51t}\theta-6.50164e^{-1.51t}\theta \tag{5.2}$$

The exponential functions vanish when t tends to infinity and therefore we can see that positive θ leads to lower expectations.

For correlation, in order to see the relationship between Z_t and Z_{t+h}, we take time t and step h starting from 10 with gap 10 as shown in Tables 3 and 4.

Table 1. $\mathbb{E}[Z_t]$.

t \ θ	−1	−0.5	0	0.5	1
5	492.9	490.3	487.7	485.1	482.5
10	956.9	954.2	951.6	949.0	946.4
50	3,939.9	3,937.3	3,934.7	3,932.1	3,929.5
100	6,326.4	6,323.8	6,321.2	6,318.6	6,316.0
500	9,937.9	9,935.2	9,932.6	9,930.0	9,927.4
1,000	10,004.8	10,002.2	9,999.6	9,996.9	9,994.3
5,000	10,005.2	10,002.6	10,000.0	9,997.4	9,994.8

Table 2. $SD[Z_t]$.

t \ θ	−1	−0.5	0	0.5	1
5	318.3	313.4	308.5	303.5	298.5
10	438.1	432.0	425.8	419.5	413.1
50	816.3	805.8	795.1	784.2	773.2
100	954.5	942.3	929.9	917.3	904.6
500	1,026.4	1,013.3	1,000.0	986.5	972.9
1,000	1,026.4	1,013.3	1,000.0	986.5	972.9
5,000	1,026.4	1,013.3	1,000.0	986.5	972.9

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Table 3. $\rho(Z_t, Z_{t+h})$, $\theta = 0.5$.

t \ h	10	20	30	40	50	60	70	80	90	100
10	0.74092	0.63319	0.57309	0.53486	0.50867	0.48988	0.47596	0.46540	0.45726	0.45090
20	0.85457	0.77345	0.72185	0.68652	0.66116	0.64236	0.62811	0.61712	0.60855	0.60178
30	0.90505	0.84468	0.80333	0.77365	0.75166	0.73498	0.72213	0.71209	0.70418	0.69789
40	0.93328	0.88759	0.85480	0.83050	0.81208	0.79787	0.78678	0.77804	0.77109	0.76554
50	0.95104	0.91590	0.88987	0.87013	0.85491	0.84302	0.83365	0.82621	0.82027	0.81549
60	0.96305	0.93567	0.91491	0.89891	0.88642	0.87657	0.86874	0.86249	0.85747	0.85342
70	0.97157	0.95001	0.93340	0.92042	0.91019	0.90207	0.89558	0.89036	0.88616	0.88277
80	0.97781	0.96071	0.94736	0.93683	0.92846	0.92178	0.91642	0.91209	0.90860	0.90577
90	0.98251	0.96885	0.95808	0.94953	0.94269	0.93721	0.93279	0.92921	0.92631	0.92396
100	0.98610	0.97514	0.96643	0.95948	0.95389	0.94939	0.94575	0.94281	0.94041	0.93847

Table 4. $\rho(Z_t, Z_{t+h})$, $\theta = -0.5$.

t \ h	10	20	30	40	50	60	70	80	90	100
10	0.74208	0.63485	0.57438	0.53613	0.50993	0.49112	0.47718	0.46661	0.45846	0.45209
20	0.85503	0.77404	0.72250	0.68719	0.66184	0.64305	0.62881	0.61782	0.60925	0.60248
30	0.90530	0.84502	0.80372	0.77407	0.75210	0.73544	0.72259	0.71256	0.70465	0.69837
40	0.933428	0.88781	0.85506	0.83079	0.81238	0.79819	0.78711	0.77838	0.77144	0.76589
50	0.95114	0.91606	0.89005	0.87033	0.85513	0.84326	0.83390	0.82646	0.82052	0.81575
60	0.96312	0.93578	0.91505	0.89906	0.88658	0.87674	0.86893	0.86268	0.85766	0.85362
70	0.97162	0.95009	0.93350	0.92054	0.91032	0.90220	0.89572	0.89051	0.88631	0.88292
80	0.97785	0.96077	0.94743	0.93692	0.92856	0.92189	0.91653	0.91221	0.90872	0.90589
90	0.98254	0.96890	0.95814	0.94960	0.94277	0.93729	0.93287	0.92930	0.92641	0.92406
100	0.98612	0.97518	0.96648	0.95953	0.95395	0.94946	0.94582	0.94288	0.94049	0.93855

The following observations can be made from Tables 3 and 4.

1. For all time t, step h and θ, Z_t and Z_{t+h} are positively correlated as expected, because Z_t is part of Z_{t+h}.
2. For a fixed time t and a fixed θ, ρ decreases as step h increases. This is intuitively reasonable as we expect risk profiles of one policy to become more distinct when time shifts.
3. For a fixed θ, step h has greater impacts when time t is small than when t is large.

Up to now, under this topic, results are easier to be achieved under time t infinity cases rather than for any t as one parameter is gone under infinity cases. Therefore our intention for these correlations is to see whether it is possible to give insights to any time t if we know the time infinity cases. However, above tables show it is not an ideal choice as the correlation of Z_t is not good enough between small t and large t. In any case, ρ may not be a good choice as Z_t may not be correlated with Z_{t+h} in a linear form.

6. Explicit Results Under *Gamma*(2, λ) Process

This section aims to show that the Beta transform is also applicable for other arrival processes. We will focus on the case where the inter-arrival times follow the *Gamma*(2, λ) distribution. For the Beta transform of a *Gamma*(2, λ) variable, see Example 3.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

By Theorem 3 and taking the Laplace transform with respect to time t on both sides, we have

$$\begin{aligned}\mathcal{L}_r\{V_n(t)\} = &\sum_{j=1}^{n}\binom{n}{j}\mathbb{E}[X^j]\mathcal{L}_r\{V_{n-j}(t)\}\mathcal{L}_r\left\{e^{-n\delta t}\frac{\lambda}{2}(1-e^{-2\lambda t})\right\}\\ &+\theta c\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)\mathcal{L}_r\{V_{n-j}(t)\}\\ &\times\frac{\mathcal{L}_r\left\{e^{-n\delta t}F_W(t)^{c-1}(1-F_W(t))^d f_W(t)\right\}}{1-\mathcal{L}_r\{e^{-n\delta t}f_W(t)\}}\\ &-\theta d\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)\mathcal{L}_r\{V_{n-j}(t)\}\\ &\times\frac{\mathcal{L}_r\left\{e^{-n\delta t}F_W(t)^{c}(1-F_W(t))^{d-1} f_W(t)\right\}}{1-\mathcal{L}_r\{e^{-n\delta t}f_W(t)\}}\end{aligned}$$

The key point is whether we are able to invert the fractional part involving $f_W(t)$. The calculation of the denominator is straightforward and the numerator part can be done in the following way.

$$\begin{aligned}\mathcal{L}_r\left\{e^{-n\delta t}F_W(t)^{c-1}(1-F_W(t))^d f_W(t)\right\} &= \int_0^\infty e^{-rt}e^{-n\delta t}\left(1-e^{-t\lambda}(1+t\lambda)\right)^{c-1}\left(e^{-t\lambda}(1+t\lambda)\right)^d\frac{\lambda^2}{\Gamma(2)}te^{-\lambda t}dt\\ &= \sum_{k=0}^{\infty}\frac{(1-c)_k}{k!}\int_0^\infty e^{-rt}e^{-n\delta t}\left(e^{-t\lambda}(1+t\lambda)\right)^{d+k}\lambda^2te^{-\lambda t}dt\\ &= \sum_{k=0}^{\infty}\frac{(1-c)_k}{k!}\int_0^\infty e^{-t(r+n\delta+(d+k+1)\lambda)}t\lambda^2(1+t\lambda)^{d+k}dt\end{aligned}$$

where we apply Bromwich & MacRobert (1947: 151) to exchange the order of summation and integration. Further, when d is an integer, we can apply the Binomial theorem and see that

$$\begin{aligned}\mathcal{L}_r\{V_n(t)\} = &\sum_{j=1}^{n}\binom{n}{j}\mathbb{E}[X^j]\mathcal{L}_r\{V_{n-j}(t)\}\mathcal{L}_r\left\{e^{-n\delta t}\frac{\lambda}{2}(1-e^{-2\lambda t})\right\}\\ &+\theta c\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)\mathcal{L}_r\{V_{n-j}(t)\}\sum_{k=0}^{\infty}\frac{(1-c)_k}{k!}\\ &\times\left\{\sum_{i=0}^{d+k}\binom{d+k}{i}\Gamma(2+i)\frac{\lambda^{2+i}}{(r+n\delta+(1+d+k)\lambda)^{2+i}}\frac{(r+n\delta+\lambda)^2}{(r+n\delta)(r+n\delta+2\lambda)}\right\}\\ &-\theta d\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,b+1)}\right]-\mathbb{E}\left[X^j_{(a,b)}\right]\right)\mathcal{L}_r\{V_{n-j}(t)\}\sum_{k=0}^{\infty}\frac{(-c)_k}{k!}\\ &\times\left\{\sum_{i=0}^{d-1+k}\binom{d-1+k}{i}\Gamma(2+i)\frac{\lambda^{2+i}}{(r+n\delta+(d+k)\lambda)^{2+i}}\frac{(r+n\delta+\lambda)^2}{(r+n\delta)(r+n\delta+2\lambda)}\right\}\end{aligned}$$

There are different ways to invert this Laplace transform by manipulating with the expression.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Theorem 7. For any $\delta \geq 0$ and $n, d \in \mathbb{N}$, a recursive formula for the nth moment of Z_t is

$$\begin{aligned} V_n(t) = {} & \frac{\lambda}{2}\sum_{j=1}^{n}\binom{n}{j}\mathbb{E}\left[X^j\right]\int_0^t e^{-n\delta v}\left(1-e^{-2\lambda v}\right)V_{n-j}(t-v)dv \\ & + \theta c\lambda\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,\,b+1)}\right]-\mathbb{E}\left[X^j_{(a,\,b)}\right]\right)\sum_{k=0}^{\infty}\frac{(1-c)_k}{k!} \\ & \times\sum_{i=0}^{d+k}\binom{d+k}{i}\int_0^t e^{-n\delta v}g(v,d+1)V_{n-j}(t-v)dv \\ & - \theta d\lambda\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(a+b)}\sum_{j=1}^{n}\binom{n}{j}\left(\mathbb{E}\left[X^j_{(a,\,b+1)}\right]-\mathbb{E}\left[X^j_{(a,\,b)}\right]\right)\sum_{k=0}^{\infty}\frac{(-c)_k}{k!} \\ & \times\sum_{i=0}^{d-1+k}\binom{d-1+k}{i}\int_0^t e^{-n\delta v}g(v,d)V_{n-j}(t-v)dv \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} g(v,\,d) = {} & \int_0^v\left[(1+i)e^{-u(d+k)\lambda}u^i\lambda^{1+i}-(d+k)e^{-u(d+k)\lambda}u^{1+i}\lambda^{2+i}\right] \\ & \times\left[\frac{3}{4}+\frac{1}{4}e^{-(v-u)2\lambda}+\frac{1}{2}(v-u)\lambda\right]du \\ = {} & \frac{3}{4}e^{-(d+k)v\lambda}v^{1+i}\lambda^{1+i}-\frac{e^{-2v\lambda}}{4(d+k-2)^{2+i}}\left[2\Gamma(2+i)+(1+i)(d+k-2)\right. \\ & \times\Gamma((d+k-2)v\lambda,1+i)-(d+k)\Gamma((d+k-2)v\lambda,2+i)] \\ & +\frac{1}{2(d+k)^{2+i}}\left[\Gamma(2+i)-e^{-(d+k)v\lambda}((d+k)v\lambda)^{1+i}-(1+i)\Gamma((d+k)v\lambda,1+i)\right] \end{aligned}$$

and $\Gamma(z;\,\alpha) = \int_z^\infty x^{\alpha-1}e^{-x}dx$ is the upper incomplete gamma function.

Remarks

1. Equation (6.1) is computationally tractable and gives an explicit expression for $V_n(t)$, in particular for small n, can be obtained by further simplifying the integrals involved, as $g(v,\,d)$ is essentially a linear combination of functions of type $v^\gamma e^{-\xi v}$.
2. Due to page limit, we choose not to include a numerical example for equation (6.1).

7. Concluding Remarks

This paper studies the moments and covariance of the discounted aggregate claims under a renewal risk model with dependence between the inter-claim times and claim amounts. The dependence is given by a generalised FGM copula which has four parameters with more flexibilities in dependence modelling. According to the structure of this copula, it is natural to define the Beta transform so that

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

the main results can be expressed in terms of the Beta transforms of both claim amount distribution and inter-claim time distribution. This transform has many nice properties and some stochastic ordering properties can be used to derive upper and lower bounds for the moments of the discounted aggregate claims.

The Beta transform, with many nice properties, is demonstrated to be an efficient weapon in analysing quantities arising in risk models with the dependency being measured by the generalised FGM copula. It is believed that the Beta transform and its properties may have other applications in risk modelling and risk managements.

Acknowledgements

The authors would like to thank two anonymous referees for their valuable comments and suggestions which lead to significant improvement of the paper.

References

Albrecher, H. & Boxma, O.J. (2004). A ruin model with dependence between claim sizes and claim intervals. *Insurance: Mathematics and Economics*, **35**(2), 245–254.

Albrecher, H. & Boxma, O.J. (2005). On the discounted penalty function in a Markov-dependent risk model. *Insurance: Mathematics and Economics*, **37**(3), 650–672.

Albrecher, H. & Teugels, J.L. (2006). Exponential behavior in the presence of dependence in risk theory. *Journal of Applied Probability*, **43**(1), 257–273.

Badescu, A.L., Breuer, L., Drekic, S., Latouche, G. & Stanford, D.A. (2005). The surplus prior to ruin and the deficit at ruin for a correlated risk process. *Scandinavian Actuarial Journal*, **2005**(6), 433–445.

Barges, M., Cossette, H., Loisel, S. & Marceau, E. (2011). On the moments of aggregate discounted claims with dependence introduced by a FGM copula. *ASTIN Bulletin: The Journal of the IAA*, **41**(1), 215–238.

Boudreault, M., Cossette, H., Landriault, D. & Marceau, E. (2006). On a risk model with dependence between interclaim arrivals and claim sizes. *Scandinavian Actuarial Journal*, **2006**(5), 265–285.

Bromwich, T.J.I. & MacRobert, T.M. (1947). *An Introduction to the Theory of Infinite Series*. Macmillan, London.

Chadjiconstantinidis, S. & Vrontos, S. (2014). On a renewal risk process with dependence under a Farlie–Gumbel–Morgenstern copula. *Scandinavian Actuarial Journal*, **2014**(2), 125–158.

Cheung, E.C.K. (2012). A unifying approach to the analysis of business with random gains. *Scandinavian Actuarial Journal*, **2012**(3), 153–182.

Cossette, H., Marceau, E. & Marri, F. (2008). On the compound Poisson risk model with dependence based on a generalized Farlie–Gumbel–Morgenstern copula. *Insurance: Mathematics and Economics*, **43**(3), 444–455.

Cossette, H., Marceau, E. & Marri, F. (2010). Analysis of ruin measures for the classical compound Poisson risk model with dependence. *Scandinavian Actuarial Journal*, **2010**(3), 221–245.

Delbaen, F. & Haezendonck, J. (1987). Classical risk theory in an economic environment. *Insurance: Mathematics and Economics*, **6**(2), 85–116.

Denuit, M., Dhaene, J., Goovaerts, M. & Kaas, R. (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. Wiley, Hoboken, NJ and Chichester.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press

Jang, J.-W. (2004). Martingale approach for moments of discounted aggregate claims. *Journal of Risk and Insurance*, **71**(2), 201–211.

Janssen, J. & Reinhard, J.-M. (1985). Probabilités de ruine pour une classe de modeles de risque semi-Markoviens. *ASTIN Bulletin*, **15**(2), 123–133.

Kim, B. & Kim, H.-S. (2007). Moments of claims in a Markovian environment. *Insurance: Mathematics and Economics*, **40**(3), 485–497.

Lebedev, N.N. (1972). *Special Functions and Their Applications*. Dover Publications, New York.

Léveillé, G. & Adékambi, F. (2011). Covariance of discounted compound renewal sums with a stochastic interest rate. *Scandinavian Actuarial Journal*, **2011**(2), 138–153.

Léveillé, G. & Adékambi, F. (2012). Joint moments of discounted compound renewal sums. *Scandinavian Actuarial Journal*, **2012**(1), 40–55.

Léveillé, G. & Garrido, J. (2001*a*). Moments of compound renewal sums with discounted claims. *Insurance: Mathematics and Economics*, **28**(2), 217–231.

Léveillé, G. & Garrido, J. (2001*b*). Recursive moments of compound renewal sums with discounted claims. *Scandinavian Actuarial Journal*, **2001**(2), 98–110.

Léveillé, G., Garrido, J. & Wang, Y.F. (2010). Moment generating functions of compound renewal sums with discounted claims. *Scandinavian Actuarial Journal*, **2010**(3), 165–184.

Ren, J. (2008). On the Laplace transform of the aggregate discounted claims with Markovian arrivals. *North American Actuarial Journal*, **12**(2), 198–206.

Rodriguez-Lallena, J. & Ãbeda-Flores, M. (2004). A new class of bivariate copulas. *Statistics and Probability Letters*, **66**(3), 315–325.

Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic Orders*. Springer Science & Business Media, New York.

Taylor, G.C. (1979). Probability of ruin under inflationary conditions or under experience rating. *ASTIN Bulletin*, **10**(2), 149–162.

Tse, Y.-K. (2009). *Nonlife Actuarial Models: Theory, Methods and Evaluation*. Cambridge University Press, Cambridge and New York.

Wang, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics*, **17**(1), 43–54.

Wang, Y.F., Garrido, J. & Léveillé, G. (2016). The distribution of discounted compound PH–renewal processes. *Methodology and Computing in Applied Probability*, **20**(1), 1–28.

Willmot, G.E. (1989). The total claims distribution under inflationary conditions. *Scandinavian Actuarial Journal*, **1989**(1), 1–12.

Wirch, J.L. & Hardy, M.R. (1999). A synthesis of risk measures for capital adequacy. *Insurance: Mathematics and Economics*, **25**(3), 337–347.

Woo, J.-K. & Cheung, E.C. (2013). A note on discounted compound renewal sums under dependency. *Insurance: Mathematics and Economics*, **52**(2), 170–179.

https://doi.org/10.1017/S1748499518000209 Published online by Cambridge University Press