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A Graph Integral Formulation of the Circuit Partition Polynomial

CRISTOPHER MOORE[1] and ALEXANDER RUSSELL[2]

[1] Computer Science Department, University of New Mexico, Albuquerque, NM 87131, USA
and
The Santa Fe Institute, Santa Fe, NM 87501, USA
(e-mail: moore@santafe.edu)

[2] Department of Computer Science & Engineering, University of Connecticut, Storrs, CT 06269, USA
(e-mail: acr@cse.uconn.edu)

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We present a simple graph integral equivalent to a multiple of the circuit partition polynomial. Let G be a directed graph, and let k be a positive integer. Associate with each vertex v of G an independent, uniformly random k-dimensional complex vector x_v of unit length. We define $q(G;k)$ to be the expected value of the product, over all edges (u,v), of the inner product $\langle x_u, x_v\rangle$. We show that $q(G;k)$ is proportional to G's cycle partition polynomial, and therefore that computing $q(G;k)$ is $\#P$-complete for any $k > 1$. We also study the natural variants that arise when the x_v are real or drawn from the Gaussian distribution.

1. Introduction

Let $x, y \in \mathbb{C}^k$ be k-dimensional complex-valued vectors. We denote their inner product as

$$\langle x, y\rangle = \sum_{i=1}^{k} x_i^* y_i.$$

Now suppose we have a directed graph $G = (V, E)$. Associating a vector $x_v \in \mathbb{C}^k$ with each vertex v, consider the product, over all edges (u,v), of the inner products of the corresponding vectors:

$$\prod_{(u,v)\in E} \langle x_u, x_v\rangle. \tag{1.1}$$

For example, for the graph in Figure 1 this product is

$$\prod_{(u,v)\in E} \langle x_u, x_v\rangle = \langle x_1, x_2\rangle \langle x_2, x_3\rangle \langle x_3, x_1\rangle |\langle x_3, x_4\rangle|^2. \tag{1.2}$$

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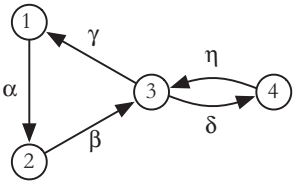


Figure 1. A little directed graph. We use the edge labels in the proof of Theorem 1.1.

The expectation of this product, where each x_v is chosen independently and uniformly from the set of vectors in $\mathbb{C}^k$ of norm 1, is a type of moment where each x_v appears with order $d_v^{\text{in}} + d_v^{\text{out}}$. It is a function of the graph G and the dimension k, which we denote by

$$q(G;k) \triangleq \mathop{\mathbb{E}}_{\{x_v\}} \prod_{(u,v)\in E} \langle x_u, x_v \rangle.$$

A simple observation is that $q(G;k)$ is zero unless G is Eulerian – that is, unless $d_v^{\text{in}} = d_v^{\text{out}}$ for each vertex v. Since x_v appears in the product d_v^{in} times unconjugated and d_v^{out} times conjugated, multiplying x_v by $e^{i\theta}$ multiplies $q(G;k)$ by $e^{i\theta(d_v^{\text{in}} - d_v^{\text{out}})}$. As multiplying by a phase preserves the uniform measure, the expectation is zero if $d_v^{\text{in}} \neq d_v^{\text{out}}$ for any v. We remark that the discussion above, as well as the results developed later in the article, apply to graphs with multiple edges and self-loops.

So, let us suppose that G is Eulerian. In that case, what is $q(G;k)$? Does it have a combinatorial interpretation? And how difficult is it to calculate?

Our main result asserts that $q(G;k)$ is directly related to $j(G;k)$, the *circuit partition polynomial*; we recall the definition of $j(G;k)$ in Section 2 below. Our main result is the following.

Theorem 1.1. *For any Eulerian directed graph $G = (V, E)$,*

$$q(G;k) = \left(\prod_{v \in V} \frac{(k-1)!}{(k+d_v-1)!} \right) j(G;k), \tag{1.3}$$

where d_v denotes $d_v^{\text{in}} = d_v^{\text{out}}$ and $j(G;k)$ denotes the circuit partition polynomial.

As computing the circuit partition polynomial is $\#P$-hard, we have the following corollary.

Corollary 1.2. *For any $k \geqslant 2$, computing $q(G;k)$, given G as input, is $\#P$-hard under Turing reductions.*

If we extend $\#P$ to rational functions in the natural way, then we can replace $\#P$-hardness in this theorem with $\#P$-completeness.

The rest of the article is devoted to establishing Theorem 1.1 and exploring, in addition, the behaviour of $q(G;k)$ when the vectors v_k are chosen from $\mathbb{R}^k$ or according to the

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Gaussian measure. Along the way, we will meet some nice combinatorics and diagrammatic methods, and touch on the representation theory of the unitary and orthogonal groups.

2. Representing the circuit partition polynomial

A *circuit partition* of G is a partition of G's edges into circuits. Let r_t denote the number of circuit partitions containing t circuits; for instance, r_1 is the number of Eulerian circuits. The *circuit partition polynomial* $j(G;z)$ is the generating function

$$j(G;z) = \sum_{t=1}^{\infty} r_t z^t. \tag{2.1}$$

For instance, for the graph in Figure 1 we have $j(G;z) = z + z^2$. This polynomial was first studied by Martin [11], with a slightly different parametrization; see also [1, 3, 4, 7, 10, 12, 13].

With this definition in place, we turn to the proof of Theorem 1.1. Following Dirac, we use $|u\rangle$ and $\langle u|$ to denote a vector and its dual. Given two vectors $u, v \in \mathbb{C}^k$, we denote their outer product as $|v\rangle\langle u|$. If we view this as a linear operator on $\mathbb{C}^k$, its trace is their inner product:

$$\operatorname{tr} |v\rangle\langle u| = \langle u, v\rangle.$$

Given a vector $x \in \mathbb{C}^k$ and an integer d, the outer product of

$$x^{\otimes d} = \underbrace{x \otimes \cdots \otimes x}_{d \text{ times}}$$

with itself is a tensor of rank $2d$, or equivalently a linear operator on $(\mathbb{C}^k)^{\otimes d}$:

$$|x^{\otimes d}\rangle\langle x^{\otimes d}| = \big(|x\rangle\langle x|\big)^{\otimes d}.$$

In terms of indices $1 \leqslant \alpha_i, \beta_i \leqslant k$ where $1 \leqslant i \leqslant d$, we have

$$|x^{\otimes d}\rangle\langle x^{\otimes d}|^{\alpha_1\alpha_2\cdots\alpha_d}_{\beta_1\beta_2\cdots\beta_d} = \prod_{\ell=1}^{d} x_{\alpha_\ell} x^*_{\beta_\ell}.$$

The product $\prod_{(u,v)\in E}\langle x_u, x_v\rangle$ is a contraction of the tensor product of these tensors, where upper and lower indices correspond to incoming and outgoing edges respectively. For instance, for the graph in Figure 1 we can rewrite the product (1.2) as

$$\prod_{(u,v)\in E} \langle x_u, x_v\rangle = |x_1\rangle\langle x_1|^{\gamma}_{\alpha}\ |x_2\rangle\langle x_2|^{\alpha}_{\beta}\ |x_3 \otimes x_3\rangle\langle x_3 \otimes x_3|^{\beta\eta}_{\gamma\delta}\ |x_4\rangle\langle x_4|^{\delta}_{\eta}.$$

Here we use the Einstein summation convention, where any index that appears once above and once below is automatically summed from 1 to k. Now, since the x_v are independent for different v, we can compute $q(G;k)$ by taking the expectation over each x_v separately. At each vertex v this gives the tensor

$$X_{d_v} = \mathop{\mathbb{E}}_{x}\ |x^{\otimes d_v}\rangle\langle x^{\otimes d_v}|, \tag{2.2}$$

and $q(G;k)$ is a contraction of the tensor product $\bigotimes_v X_{d_v}$.

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In order to compute X_d, we introduce some notation. Let S_d denote the symmetric group on d elements. We identify a permutation $\pi \in S_d$ with the linear operator on $(\mathbb{C}^k)^{\otimes d}$ that permutes the d factors in the tensor product. That is,

$$\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_d) = x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(d)},$$

or, using indices,

$$\pi^{\alpha_1 \alpha_2 \cdots \alpha_d}_{\beta_1 \beta_2 \cdots \beta_d} = \prod_{\ell=1}^{d} \delta^{\alpha_{\pi(\ell)}}_{\beta_\ell},$$

where δ^i_j is the Kronecker delta operator, $\delta^i_j = 1$ if $i = j$ and 0 if $i \neq j$. Diagrammatically, π is a gadget with d incoming edges and d outgoing edges, wired to each other according to the permutation π.

Lemma 2.1. *With X_d defined as in* (2.2)*, if x is uniform in the set of vectors in $\mathbb{C}^k$ of norm* 1*, then*

$$X_d = \frac{(k-1)!}{(k+d-1)!} \sum_{\pi \in S_d} \pi. \tag{2.3}$$

Proof. First, X_d is a member of the commutant of the group $U(k)$ of $k \times k$ unitary matrices, since these preserve the uniform measure. That is, X_d commutes with $U^{\otimes d}$ for any $U \in U(k)$. By Schur duality, the commutant is a quotient of the group algebra $\mathbb{C}[S_d]$; namely, the image of $\mathbb{C}[S_d]$ under the identification above. Thus X_d is a superposition of permutations, $\sum_{\pi \in S_d} a_\pi \pi$.

We also have $X_d \pi = \pi X_d = X_d$ for any π. Thus X_d is proportional to the uniform superposition on S_d, or equivalently the projection operator $\Pi_{\text{sym}} = (1/d!) \sum_\pi \pi$ onto the totally symmetric subspace V_{sym} of $(\mathbb{C}^k)^{\otimes d}$. Since $\operatorname{tr} X_d = \mathbb{E}_x |x|^{2d} = 1$ while $\operatorname{tr} \Pi_{\text{sym}} = \dim V_{\text{sym}}$, we have $X_d = (1/\dim V_{\text{sym}}) \Pi_{\text{sym}}$.

Finally, $\dim V_{\text{sym}}$ is the number of ways to label the d factors of the tensor product with basis vectors $\{e_1, \ldots, e_k\}$ in non-decreasing order – or, for aficionados, the number of semistandard tableaux with one row of length d and content ranging from 1 to k. This gives $\dim V_{\text{sym}} = \binom{k+d-1}{d}$ and Lemma 2.1 follows. □

To illustrate some ideas that will recur in the next section, we give an alternative proof.

Alternative proof of Lemma 2.1. First, note that $\operatorname{tr} \pi$ is the number of ways to label each of π's cycles with a basis vector ranging from 1 to k, or $k^{c(\pi)}$ where $c(\pi)$ denotes the number of cycles (including fixed points, which are cycles of length 1). Thus

$$\operatorname{tr} \sum_{\pi \in S_d} \pi = \sum_{\pi \in S_d} k^{c(\pi)}. \tag{2.4}$$

To compute this generating function, we use the fact that each permutation $\pi \in S_d$ appears once in the following product, where 1 denotes the identity permutation, and τ_{ij} denotes

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the transposition of the ith and jth object:

$$\sum_{\pi \in S_d} \pi = 1(1 + \tau_{1,2})(1 + \tau_{1,3} + \tau_{2,3}) \cdots (1 + \tau_{1,d} + \tau_{2,d} + \cdots + \tau_{d-1,d}). \tag{2.5}$$

This product works by describing a permutation π_t of t objects inductively as a permutation π_{t-1} of the first $t-1$ objects, composed either with the identity or with a transposition swapping the tth object with one of the previous $t-1$. If we apply the identity, then the tth object is a fixed point, and $c(\pi_t) = c(\pi_{t-1}) + 1$, gaining a factor of k in (2.4); but if we apply a transposition, then $c(\pi_t) = c(\pi_{t-1})$. Thus counting cycles turns (2.5) into

$$\sum_{\pi \in S_d} k^{c(\pi)} = k(k+1)(k+2) \cdots (k+d-1) = \frac{(k+d-1)!}{(k-1)!}.$$

Comparing with $\operatorname{tr} X_d = 1$ again gives (2.3). □

Proof of Theorem 1.1. All that remains is to interpret the operators X_{d_v}, and the contraction of $\bigotimes_v X_{d_v}$, diagrammatically. Lemma 2.1 asserts that X_{d_v} is proportional to a sum over all $d_v!$ ways to wire v's incoming edges to its outgoing edges. But doing this at each vertex gives us a sum over all cycle partitions of G. Contracting these tensors gives the number of ways to label each cycle in each partition with a basis vector ranging from 1 to k, so each cycle contributes a factor of k. Along with the scaling factor in (2.3), this completes the proof of Theorem 1.1. □

Next we show that the cycle partition polynomial $j(G;z)$ is $\#P$-hard, and therefore that $q(G;z)$ is $\#P$-hard as well, for any $z > 1$. To our knowledge, the following theorem first appeared in [8]; we prove it here for completeness.

Theorem 2.2. *For any fixed $z > 1$, computing $j(G;z)$ from G is $\#P$-hard under Turing reductions.*

Proof. Recall that the *Tutte polynomial* of an undirected graph $G = (V, E)$ can be written as a sum over all subsets S of E,

$$T(G;x,y) = \sum_{S \subseteq E} (x-1)^{c(S)-c(G)} (y-1)^{c(S)+|S|-|V|}. \tag{2.6}$$

Here $c(G)$ denotes the number of connected components in G. Similarly, $c(S)$ denotes the number of connected components in the spanning subgraph (V, S), including isolated vertices. When $x = y$, we have

$$T(G;x,x) = \sum_{S \subseteq E} (x-1)^{c(S)+\ell(S)-c(G)}, \tag{2.7}$$

where $\ell(S) = c(S) + |S| - |V|$ is the total excess of the components of S, *i.e.*, the number of edges that would have to be removed to make each one a tree.

If G is planar, then we can define a directed medial graph G_m as in Figure 2. Each vertex of G_m corresponds to an edge of G, edges of G_m correspond to shared vertices in

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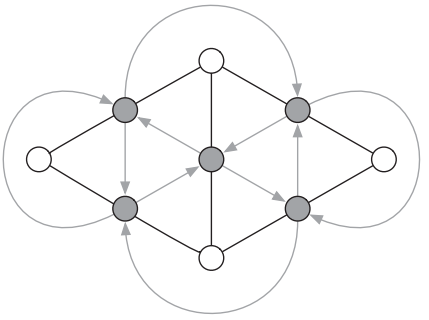


Figure 2. A planar graph G (black) and its oriented medial graph G_m (grey).

G, and we orient the edges of G_m so that they go anticlockwise around the faces of G. Each vertex of G_m has $d^{\text{in}} = d^{\text{out}} = 2$, so G_m is Eulerian.

The following identity is due to Martin [11]; see also [12], or [2], for a review.

$$j(G_m; z) = z^{c(G)}\, T(G; z+1, z+1). \tag{2.8}$$

We prove this using a one-to-one correspondence between subsets $S \subseteq E$ and circuit partitions of G_m. Let v be a vertex of G_m, corresponding to an edge e of G. Then the circuit partition connects each of v's incoming edge to the outgoing edge on the same side of e if $e \in S$, and crosses over to the other side if $e \notin S$. It is easy to prove by induction that the number of circuits is then $c(S) + \ell(S)$, in which case (2.7) yields (2.8).

The theorem then follows from the fact, proved by Vertigan [14], that the Tutte polynomial for planar graphs is $\#P$-hard under Turing reductions, except on the hyperbolas $(x-1)(y-1) \in \{1, 2\}$ or when $(x, y) \in \{(1,1), (-1,-1), (\omega, \omega^*), (\omega^*, \omega)\}$ where $\omega = \mathrm{e}^{2\pi i/3}$. Thus computing $j(G; z)$ as a function of G for any fixed $z > 1$ is $\#P$-hard, even in the special case where G is planar and where every vertex has $d^{\text{in}} = d^{\text{out}} = 2$. □

3. Real-valued vectors

We now consider the case where the x_v are real-valued and chosen uniformly from the set of vectors in $\mathbb{R}^k$ of norm 1. In this case, the inner product $\langle x_u, x_v \rangle$ is symmetric, so we define $q(G; k)$ for an undirected graph G. We might then expect $q(G; k)$ to be related to the circuit partition polynomial for undirected circuits, and indeed this is the case.

We again wish to compute the tensor $X_d = \mathbb{E}_x\, |x^{\otimes d}\rangle\langle x^{\otimes d}|$. First, let M_d denote the set of perfect matchings of $2d$ objects; note that

$$|M_d| = (2d-1)!! = (2d-1)(2d-3)\cdots 5 \cdot 3 \cdot 1 = \frac{(2d)!}{2^d d!}.$$

We can identify each matching $\mu \in M_d$ with a linear operator on $(\mathbb{R}^k)^{\otimes d}$, where the first d objects correspond to upper indices, and the last d correspond to lower indices. However, in addition to permutations that wire upper indices to lower ones with a bipartite matching, we now also have 'cups' and 'caps' that wire two upper indices, or two lower indices, to each other; these are allowed since the vectors x are now self-dual. For instance, if

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$d = 2$ then M_d includes three operators, corresponding to the three perfect matchings of 4 objects:

$$\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} = \left|\ \right|, \quad \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1} = \times, \quad \text{and} \quad \delta^{\alpha_1,\alpha_2}\delta_{\beta_1,\beta_2} = \cup\cap. \tag{3.1}$$

The first two of these operators correspond to the identity permutation and the transposition $\tau_{1,2}$ respectively, as in the previous section. The third one is a *cupcap*; it is the outer product of the vector $\sum_{i=1}^{k} e_i \otimes e_i$ with itself, where e_i denotes the ith basis vector in $\mathbb{R}^k$. We denote it by $\gamma_{1,2}$, and more generally $\gamma_{ij} = \delta^{\alpha_i,\alpha_j}\delta_{\beta_i,\beta_j}$.

Now, in the real-valued case, Lemma 2.1 becomes the following.

Lemma 3.1. *If x is uniform in the set of vectors in $\mathbb{R}^k$ of norm 1, then*

$$X_d = \frac{1}{k(k+2)(k+4)\cdots(k+2d-2)} \sum_{\mu \in M_d} \mu = \frac{(k-2)!!}{(k+2d-2)!!} \sum_{\mu \in M_d} \mu. \tag{3.2}$$

Here $n!! = n(n-2)(n-4)\cdots 6 \cdot 4 \cdot 2$ if n is even and $n(n-2)(n-4)\cdots 5 \cdot 3 \cdot 1$ if n is odd, and by convention $0!! = -1!! = 1$.

Proof. Analogous to the complex case, X_d is a member of the commutant of the group $O(k)$ of $k \times k$ orthogonal matrices, since these preserve the uniform measure. That is, X_d commutes with $O^{\otimes d}$ for any $O \in O(k)$. The commutant of $O(k)$ is the *Brauer algebra*; namely, the algebra consisting of linear combinations of the operators $\mu \in M_d$. Thus X_d is of the form $\sum_{\mu \in M_d} a_\mu \mu$.

In addition to being fixed under permutations as in the complex case, X_d is also fixed under partial transposes, which switch some upper indices with some lower ones. Since any two matchings can be transformed into each other with a sequence of permutations and partial transposes, the coefficients a_μ must be equal, and X_d is proportional to the uniform superposition $\sum_{\mu \in M_d} \mu$. Proceeding as in the alternative proof of Lemma 2.1, we expand $\sum \mu$ as an inductive product in the Brauer algebra and compare traces to determine the constant of proportionality.

As in the case of permutations, the trace of an operator $\mu \in M_d$ is $k^{c(\mu)}$, where $c(\mu)$ is the number of loops in the diagram resulting from joining the upper indices to the lower ones. For instance, for the operators in (3.1), we have $\operatorname{tr} 1 = k^2$, $\operatorname{tr} \tau_{1,2} = k$, and $\operatorname{tr} \gamma_{1,2} = k$. Thus we wish to calculate

$$\operatorname{tr} \sum_{\mu \in M_d} \mu = \sum_{\mu \in M_d} k^{c(\mu)}. \tag{3.3}$$

We can write $\sum_{\mu \in M_d}$ as a product, analogous to (2.5):

$$\sum_{\pi \in S_d} \pi = 1(1 + \tau_{1,2} + \gamma_{1,2})(1 + \tau_{1,3} + \gamma_{1,3} + \tau_{2,3} + \gamma_{2,3}) \cdots (1 + \tau_{1,d} + \gamma_{1,d} + \cdots + \tau_{d-1,d} + \gamma_{d-1,d}).$$

This product describes a matching μ_t of $2t$ objects inductively as a matching μ_{t-1} of the first $2(t-1)$ objects, composed either with the identity, or with a transposition or cupcap connecting the tth upper object with the ith lower one and the tth lower object with the

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ith upper one, or *vice versa*. If we apply the identity, then the tth upper object is matched to the tth lower one, and $c(\mu_t) = c(\mu_{t-1}) + 1$, gaining a factor of k in (3.3); but if we apply a transposition or cupcap, then $c(\mu_t) = c(\mu_{t-1})$. Thus counting cycles as before turns (3.3) into

$$\sum_{\pi \in S_d} k^{c(\pi)} = k(k+2)(k+4)\cdots(k+2d-2) = \frac{(k+2d-2)!!}{(k-2)!!}.$$

We again have $\operatorname{tr} X_d = \mathbb{E}_x |x|^{2d} = 1$, and comparing traces gives (3.2). □

Now that there is no distinction between incoming and outgoing edges, we use d_v to denote the degree of v. In that case, $q(G;k)$ is a contraction of $\bigotimes_v X_{d_v/2}$. Applying Lemma 3.1 to each v sums over all the ways to match v's edges with each other, and hence sums over all possible partitions of G's edges into undirected cycles. The trace of the resulting diagram is again the number of ways to label each cycle with a basis vector. So, if we define the undirected cycle polynomial $j_{\text{undirected}}(G;z) = \sum_{t=1}^{\infty} r_t z^t$, where r_t is the number of partitions with t cycles, we have the following analogue of Theorem 1.1.

Theorem 3.2. *For any undirected graph $G = (V, E)$ where every vertex has even degree, if we define $q(G;k)$ by selecting the x_v independently and uniformly from the set of vectors in $\mathbb{R}^k$ with norm 1, then*

$$q(G;k) = \left(\prod_{v \in V} \frac{(k-2)!!}{(k+d_v-2)!!} \right) j_{\text{undirected}}(G;k). \tag{3.4}$$

To our knowledge the computational complexity of $j_{\text{undirected}}(G;z)$ for fixed z is open, although it seems likely that it is $\#P$-hard as in the directed case. However, Brightwell and Winkler [6] proved that counting Eulerian circuits of an undirected graph is #P-complete. This is the coefficient of z in $j_{\text{undirected}}(G;z)$, or equivalently $\frac{\mathrm{d}}{\mathrm{d}z} j_{\text{undirected}}(G;z)$ at $z = 0$. Since $j_{\text{undirected}}(G;z)$ is a polynomial of degree at most $|E|$, we can obtain this coefficient by interpolation from $j_{\text{undirected}}(G;z)$ at $|E|+1$ values of z. Thus, if z is allowed to vary, computing $j_{\text{undirected}}(G;z)$ or $q(G;z)$ is $\#P$-hard under Turing reductions.

4. The Gaussian distribution

Our results above assume that each x_v is chosen uniformly from the unit sphere in $\mathbb{C}^k$ or $\mathbb{R}^k$, *i.e.*, the set of vectors of norm 1. Another natural isotropic measure is induced by choosing each component of x_v independently from the Gaussian distribution with variance scaled so that $\mathbb{E}\left[|x_v|^2\right] = 1$.

In the complex-valued case, we choose the real and imaginary parts of each coordinate independently with variance $1/2k$. The probability density of the norm $|x|$ is then

$$p(|x|) = \frac{2k^{k+1}}{k!} |x|^{2k-1} \mathrm{e}^{-k|x|^2}. \tag{4.1}$$

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Compared to the case where $|x_v| = 1$, each x_v contributes a scaling factor of $|x_v|^{2d}$ to the product (1.1). The even moments of (4.1) are

$$\mathbb{E}\left[|x|^{2d}\right] = \frac{(d+k-1)!}{k^d(k-1)!},$$

so in the Gaussian distribution (1.3) becomes

$$q(G;k) = \left(\prod_{v \in V} \frac{1}{k^{d_v}}\right) j(G;k) = \frac{1}{k^m}\, j(G;k), \tag{4.2}$$

where m denotes the number of edges.

We could also have derived this directly from the Gaussian analogue of Lemma 2.1. If x is chosen according to the Gaussian distribution on $\mathbb{C}^k$, and we again let X_d denote $\mathbb{E}_x\, |x^{\otimes d}\rangle\langle x^{\otimes d}|$, then

$$X_d = \frac{1}{k^d} \sum_{\pi \in S_d} \pi. \tag{4.3}$$

Using $\sum_v d_v = m$ then gives (4.2).

Similarly, in the real-valued (undirected) case, if we choose each component of $x \in \mathbb{R}^k$ from the Gaussian distribution on $\mathbb{R}$ with variance $1/k$, then (3.4) becomes

$$q(G;k) = \frac{1}{k^m}\, j_{\text{undirected}}(G;k), \tag{4.4}$$

since

$$X_d = \frac{1}{k^d} \sum_{\mu \in M_d} \mu. \tag{4.5}$$

Both (4.3) and (4.5) can also be derived from Wick's theorem [9, 16], which states that if complex-valued random variables $y_1, \ldots, y_{2d}$ obey a multivariate Gaussian distribution with mean zero, then

$$\mathbb{E}\left[\prod_{i=1}^{2d} y_i\right] = \sum_{\mu \in M_d} \prod_{(i,j) \in \mu} \mathbb{E}[y_i y_j].$$

Each entry of X_d is a $(2d)$th moment of this form, where each y_i is one of the coordinates of $|x\rangle$ or $\langle x|$, perhaps with multiplicity. In the complex-valued case (4.3), this expansion is simplified by the fact that $\mathbb{E}[y_i y_j] = 0$ unless one of y_i, y_j belongs to $|x\rangle$ and the other belongs to $\langle x|$. Thus the relevant matchings are those that pair variables of $|x\rangle$ with the dual variables in $\langle x|$, *i.e.*, that belong to S_d.

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