Mathematika 59 (2013) 493–496 © University College London
doi:10.1112/S0025579313000016

ON POLYTOPAL UPPER BOUND SPHERES

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Abstract. Generalizing a result (the case $k = 1$) due to M. A. Perles, we show that any polytopal upper bound sphere of odd dimension $2k + 1$ belongs to the generalized Walkup class $\mathcal{K}_k(2k + 1)$, i.e., all its vertex links are k-stacked spheres. This is surprising since it is far from obvious that the vertex links of polytopal upper bound spheres should have any special combinatorial structure. It has been conjectured that for $d \neq 2k + 1$, all $(k + 1)$-neighborly members of the class $\mathcal{K}_k(d)$ are tight. The result of this paper shows that the hypothesis $d \neq 2k + 1$ is essential for every value of $k \geqslant 1$.

A simplicial complex is said to be *l-neighborly* if each set of l vertices of the complex spans a face. As a well-known consequence of the Dehn–Sommerville equations, any triangulated sphere of odd dimension $d = 2k + 1$ can be at most $(k + 1)$-neighborly (unless it is the boundary complex of a simplex). A $(2k + 1)$-dimensional triangulated sphere is said to be an *upper bound sphere* if it is $(k + 1)$-neighborly. This is because, by the celebrated upper bound theorem, any such sphere maximizes the face vector componentwise among all $(2k + 1)$-dimensional triangulated closed manifolds with a given number of vertices [9].

A simplicial complex is said to be a *polytopal sphere* if it is isomorphic to the boundary complex of a simplicial convex polytope. For $n \geqslant 2k + 3$, the boundary complex of an n-vertex $(2k + 2)$-dimensional cyclic polytope P (defined as the convex hull of any set of n points on the moment curve $t \mapsto (t, t^2, \ldots, t^{2k+2})$) is an example of an n-vertex polytopal upper bound sphere of dimension $2k + 1$.

A triangulated homology sphere S is said to be *k-stacked* if there is a triangulated homology ball B bounded by S all whose faces of codimension $k + 1$ are in the boundary S. The generalized lower bound conjecture (GLBC) due to McMullen and Walkup [7] states that a k-stacked d-sphere S minimizes the face-vector componentwise among all triangulated d-spheres T such that $f_i(T) = f_i(S)$ for $0 \leqslant i < k$. (Here, as usual, the *face-vector* $(f_0(T), \ldots, f_d(T))$ of a d-dimensional simplicial complex T is given by defining $f_i(T)$ to be the number of i-dimensional faces of T.) For polytopal spheres T, this conjecture was proved by Stanley [10] and McMullen [6]. Recently, Murai and Nevo [8] proved that a polytopal sphere (more generally, a triangulated homology sphere with the

Received 31 May 2012, published online 28 March 2013.
MSC (2010): 52B11, 52B05 (primary), 52B22 (secondary).
The second author was partially supported by grants from UGC Centre for Advanced Study.

https://doi.org/10.1112/S0025579313000016 Published online by Cambridge University Press

weak Lefschetz property) of dimension $d \geqslant 2k+1$ satisfies equality in GLBC only if it is k-stacked.

A triangulated homology ball B is said to be *k-stacked* if all its faces of codimension $k+1$ are in its boundary ∂B. Thus, a triangulated (homology) d-sphere S is k-stacked if and only if there is a k-stacked (homology) $(d+1)$-ball B such that $\partial B = S$. As an aside, we mention that in [8, Theorem 2.3(ii)], Murai and Nevo prove the following proposition.

PROPOSITION 1. *If S is a k-stacked triangulated homology sphere of dimension $d \geqslant 2k$ then there is a unique k-stacked homology $(d+1)$-ball B such that $\partial B = S$. It is the largest simplicial complex (in the sense of set inclusion) whose k-skeleton agrees with that of S. That is, B is given by the formula*

$$B = \left\{ \alpha \subseteq V(S) : \binom{\alpha}{\leqslant k+1} \subseteq S \right\}.$$

(Here $V(S)$ is the vertex set of S and $\binom{\alpha}{\leqslant k+1}$ denotes the set of all subsets of α of size at most $k+1$. Actually, Murai and Nevo give this formula with $d-k+1$ in place of $k+1$. But, their proof shows that it also holds with $k+1$ in place of $d-k+1$, and, of course, in view of the uniqueness statement the two formulas give the same $(d+1)$-ball.)

This result of Murai and Nevo gives a common generalization of [3, Propositions 2.10, 2.11 and Corollary 3.6] as well as a complete answer to [3, Question 6.4]. Notice that if S is an upper bound sphere of dimension $2k-1$ then S is trivially k-stacked. For such a sphere S, any ball B bounded by S and having $V(B) = V(S)$ is a k-stacked ball. So, there is no uniqueness in this case, unless S is the boundary complex of a simplex. Thus, the hypothesis $d \geqslant 2k$ in Proposition 1 is best possible.

For a $(d+1)$-dimensional convex polytope $P \subseteq \mathbb{R}^{d+1}$ and a point $x \notin P$, a facet F of P is said to be *visible* from x if, for any $y \in F$, $[x, y] \cap P = \{y\}$. As a consequence of the Bruggesser–Mani construction of a shelling order on a simplicial polytope (cf. [11, Theorem 8.12]), we know that if P is a simplicial polytope then the facets of P visible from any given point outside P form a (shellable) ball. The same holds for the facets which are invisible from a point outside P.

The *generalized Walkup class* $\mathcal{K}_k(d)$ consists of the triangulated d-manifolds all whose vertex-links are k-stacked homology spheres. (We also note that the vertex-links of a polytopal sphere are polytopal spheres, hence actually triangulated spheres.) The case $k=1$ of the following result is due to M. A. Perles (cf. [1, Theorem 1]).

THEOREM 1. *Let S be a polytopal upper bound sphere of dimension $2k+1$. Then S belongs to the generalized Walkup class $\mathcal{K}_k(2k+1)$.*

Proof. Let S be the boundary complex of the simplicial polytope P of dimension $2k+2$. Then P is a $(k+1)$-neighborly $(2k+2)$-polytope. Fix a vertex v of S, and let L be the link of v in S. We need to prove that L

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is k-stacked. This is trivial if P is a simplex. Otherwise, the convex hull Q, of the vertices of P excepting v, is again a $(2k+2)$-dimensional polytope. Clearly, Q is also $(k+1)$-neighborly and hence, by Radon's theorem (cf. [5, p. 124]), Q is also simplicial. Let B be the pure $(2k+1)$-dimensional simplicial complex whose facets are the facets of Q visible from v. By Bruggesser–Mani, B is a shellable ball.

Claim. $\partial B = L = S \cap B$.

Let $\mathrm{ast}_S(v) := \{\alpha \in S : v \notin \alpha\}$ be the antistar of v in S. Then $\mathrm{ast}_S(v)$ is a triangulated $(2k+1)$-ball and $\mathrm{ast}_S(v) \cap \mathrm{star}_S(v) = \partial(\mathrm{ast}_S(v)) = \mathrm{lk}_S(v) = L$ (cf. [2, Lemma 4.1]).

Let A be the pure $(2k+1)$-dimensional simplicial complex whose facets are the facets of Q which are not in B (i.e., invisible from v). By the Bruggesser–Mani construction, A is also a shellable ball. Clearly, $\partial A = \partial B = A \cap B$ and $\mathrm{ast}_S(v) \subseteq \partial Q = A \cup B$.

We denote by $|A|$ the geometric carrier of A, i.e., the union of the facets in A. If $x \in \mathrm{int}(|A|)$ then $x \in |S|$ and $[v, x] \cap \mathrm{int}(Q)$ is a non-trivial interval. Therefore, $[v, x] \cap \mathrm{int}(P)$ is a non-trivial interval and hence $x \in \mathrm{int}(|\mathrm{ast}_S(v)|)$. Thus $\mathrm{int}(|A|) \subseteq |\mathrm{ast}_S(v)|$ and hence $|A| \subseteq |\mathrm{ast}_S(v)|$. If $y \in \mathrm{int}(|B|)$ then the line l containing v and y intersects $\mathrm{int}(Q)$ in an interval (y, w), where $y \in (v, w)$. So, $(v, w) \subseteq \mathrm{int}(P)$ and $y \in (v, w)$. Then $y \notin |S| \supseteq |\mathrm{ast}_S(v)|$. Thus, $\mathrm{int}(|B|) \subseteq |\partial Q| \backslash |\mathrm{ast}_S(v)|$ and hence $|\mathrm{ast}_S(v)| \subseteq |\partial Q| \backslash \mathrm{int}(|B|) = |A|$. Therefore, $|\mathrm{ast}_S(v)| = |A|$. Since both $\mathrm{ast}_S(v)$ and A are subcomplexes of S, $\mathrm{ast}_S(v) = A$. Then $\partial B = \partial A = \partial(\mathrm{ast}_S(v)) = L$. This proves the first equality of the claim.

Since $\partial B = L \subseteq S$, we have $\partial B \subseteq S \cap B$. Let $y \in |S \cap B|$. If $y \in \mathrm{int}(|B|)$ then, by the same argument as before, there exists $w \in P$ such that $y \in (v, w) \subseteq \mathrm{int}(P)$. This is not possible since $y \in |S|$. Therefore, $y \in |B| \backslash \mathrm{int}(|B|) = |\partial B|$. Thus, $|S \cap B| \subseteq |\partial B|$. Since $\partial B \subseteq S$, this implies $S \cap B = \partial B$. This completes the proof of the claim.

If α is a k-face of B then $\alpha \in S$ since S is $(k+1)$-neighborly. Thus, by the claim, $\alpha \in S \cap B = \partial B$. So, B is k-stacked. Since, by the claim, $L = \partial B$, it follows that L is k-stacked. Since L is the link in S of an arbitrary vertex of S, it follows that $S \in \mathcal{K}_k(2k+1)$. □

In [3], we defined a *k-stellated sphere* to be a triangulated sphere which may be obtained from the boundary complex of a simplex by a finite sequence of bistellar moves of index at most $k-1$. We also defined $\mathcal{W}_k(d)$ as the class of all triangulated d-manifolds with k-stellated vertex-links. An easy induction on the number of bistellar moves used shows the following proposition (cf. [3, Proposition 2.9]).

PROPOSITION 2. *For $d \geqslant 2k-1$, a triangulated d-sphere S is k-stellated if and only if S is the boundary of a shellable k-stacked $(d+1)$-ball. In consequence, all k-stellated spheres of dimension at least $2k-1$ are k-stacked. Therefore, for $d \geqslant 2k$, $\mathcal{W}_k(d) \subseteq \mathcal{K}_k(d)$.*

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Thus, the proof of the above theorem shows that the polytopal upper bound spheres of dimension $2k+1$ are ($(k+1)$-neighborly) members of the smaller class $\mathcal{W}_k(2k+1)$. Recall that a connected simplicial complex X is said to be *tight* (with respect to the field $\mathbb{F}$) if, for every induced sub-complex Y of X and for all $0 \leqslant j \leqslant \dim(X)$, the morphism $H_j(Y;\mathbb{F}) \to H_j(X;\mathbb{F})$ induced by the inclusion map $Y \hookrightarrow X$ is injective. In [4], we show that $(k+1)$-neighborly members of $\mathcal{W}_k(d)$ are tight for $d \neq 2k+1$. The theorem proved here shows that the hypothesis $d \neq 2k+1$ is essential in this tightness criterion (since, except for the boundary complex of simplices, no triangulated sphere can be tight).

In [1], the case $k=1$ of Theorem 1 was used to classify the polytopal upper bound 3-spheres with nine vertices. The case $k=2$ of this theorem may be useful in similarly classifying polytopal upper bound spheres of dimension five with few vertices.

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