J. Plasma Physics (1998), *vol.* 59, *part* 4, *pp.* 727–736. Printed in the United Kingdom
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Hamiltonian vortices and reconnection in a magnetized plasma

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(Received 29 September 1997 and in revised form 22 October 1997)

Hamiltonian vortices and reconnection in magnetized plasmas are investigated analytically and numerically using a two-fluid model. The equations are written in the Lagrangian form of three fields that are advected with different velocities. This system can be considered as a generalization and extension of the two-dimensional Euler equation for an ordinary fluid. It is pointed out that these equations allow solutions in the form of singular current-vortex filaments, drift-Alfvén vortices and magnetic islands, and admit collisionless magnetic reconnection where magnetic flux is converted into electron momentum and ion vorticity.

1. Introduction

Nonlinear phenomena in magnetized plasmas can be conveniently described and understood using approximate models for plasma motion. The model of reduced magnetohydrodynamics (RMHD) was developed by Kadomtsev and Pogutse (1974) and Strauss (1976) for the description of low-β phenomena in strongly magnetized, toroidal confinement geometries. Lately RMHD has been extended in a number of papers to include the effects of high β values (Morrison and Hazeltine 1984), of the collisionless skin depth (Kadomtsev and Pogutse 1984) and of electron inertia together with drift effects (Kuvshinov *et al.* 1994; Schep *et al.* 1994). In this paper we extend the two-dimensional analysis of Schep *et al.* (1994) to perturbations that move along the magnetic field with constant velocity. This model describes low-frequency nonlinear phenomena with spatial scales that may range from MHD lengths to the electron inertia skin depth. For simplicity, we use the cold-ion approximation, which means that the ion gyroradius is smaller than the characteristic perturbation length. We do not impose, however, any restrictions on the ion-sound gyroradius, which may be larger than the characteristic length.

The two-fluid model of plasma motion is introduced in Sec. 2. In Sec. 3 we cast the equations in the Lagrangian form of three fields that are advected with different velocities. We also derive the Hamiltonian functional of the system. Solutions to this system in the form of localized, nonlinear structures are discussed in Secs 4–6. The type of structure depends on the relationships between the fields and the velocity potentials that occur in the Lagrangian equations. To a large extent, these relationships are determined by the profiles of the plasma equilibrium. In Sec. 4

https://doi.org/10.1017/S0022377898006655 Published online by Cambridge University Press

we consider a homogeneous equilibrium and show that our model admits solutions in the form of point vortices, which represent singularities in the current density, the vorticity and in the plasma density. In Sec. 5 stationary propagating structures are analysed. In the case of linear relationships between fields and potentials, one obtains vortex solutions. For a plasma with a uniform density gradient for example dipole vortices are found. If the equilibrium depends quadratically on the spatial coordinates, the solutions consist of monopoles and quadrupoles (Jovanović *et al.* 1998). In a plasma with magnetic shear, the relationships between fields and potentials are nonlinear. In this case solutions in the form of propagating magnetic islands exist. The phenomenon of collisionless magnetic reconnection in a magnetized plasma is investigated in Sec. 6. It is pointed out that during collisionless reconnection, magnetic flux is converted into electron momentum and ion vorticity, while the magnetic energy is transformed into forms of electron energy.

2. Generalized two-fluid model

We consider a plasma with a homogeneous temperature and a homogeneous equilibrium magnetic field in the z direction. The plasma pressure is assumed to be low ($\beta \ll 1$), and the perturbation of the z component of the magnetic field is neglected. In this approximation the total electric and magnetic fields are written as

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathscr{A}}{\partial t}\mathbf{e}_z, \quad \mathbf{B} = B_0\mathbf{e}_z + \boldsymbol{\nabla}\mathscr{A} \times \mathbf{e}_z, \tag{2.1}$$

where ϕ and $\mathscr{A}$ are the electrostatic and the vector potential respectively.

The electron fluid is described by the parallel component of the momentum balance and by the continuity equation

$$m_e n\frac{dv_z}{dt} = -enE_\parallel - \nabla_\parallel p_e, \quad \frac{dn}{dt} + \nabla_\parallel n v_z = 0, \tag{2.2}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_E \cdot \boldsymbol{\nabla}, \text{ with } \mathbf{v}_E = \frac{c}{B_0}\mathbf{e}_z \times \boldsymbol{\nabla}\phi,$$

and

$$\nabla_\parallel = \frac{\partial}{\partial z} + \frac{1}{B_0}\boldsymbol{\nabla}\mathscr{A} \times \mathbf{e}_z \cdot \boldsymbol{\nabla}$$

is the derivative along the magnetic field. The parallel electron velocity v_z is transported with the $\mathbf{E} \times \mathbf{B}$ velocity, which is due to cancellation of the gyroviscosity stress tensor with the contribution of the pressure gradient drift to the inertia $\mathbf{v}_\perp \cdot \boldsymbol{\nabla}$ term (Mikhailovskii 1967; Schep *et al.* 1994).

The parallel ion velocity is much smaller than the electron velocity v_z, so that

$$v_z \approx -\frac{J_z}{en_0} = \frac{c}{4\pi e n_0}\nabla_\perp^2 \mathscr{A},$$

where J_z is the parallel current, n_0 is a reference density value and $\nabla_\perp^2 = \nabla^2 - \partial^2/\partial z^2$. Assuming that the electron temperature is constant and taking (2.1) into

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account, one obtains from (2.2) the following nonlinear equations (Schep *et al.* 1994):

$$\frac{\partial}{\partial\tau}(A-\lambda_e^2\nabla_\perp^2 A)+\left[\Phi-\ln\frac{n}{n_0},\ A-\lambda_e^2\nabla_\perp^2 A\right]-\left[\ln\frac{n}{n_0},\ \lambda_e^2\nabla_\perp^2 A\right]=-\frac{\partial}{\partial z}\left(\Phi-\ln\frac{n}{n_0}\right), \tag{2.3}$$

$$\frac{\partial}{\partial\tau}\ln\frac{n}{n_0}+\left[\Phi,\ \ln\frac{n}{n_0}\right]-[A,\ \nabla_\perp^2 A]=-\frac{\partial}{\partial z}\nabla_\perp^2 A. \tag{2.4}$$

Here we have introduced the normalized variables

$$\Phi=\frac{e\phi}{T_e},\ A=\frac{ec_A}{cT_e}\mathscr{A},\ \tau=\omega_i t,$$

$$\left(\frac{x}{\rho_s},\frac{y}{\rho_s}\right)\to(x,y),\ \frac{\omega_i}{c_A}z\to z,$$

where ω_i is the ion gyro-frequency and c_A is the Alfvén velocity in the field B_0, $\rho_s=(T_e/m_i\omega_i^2)^{1/2}$ is the ion gyroradius defined at the electron temperature, $\lambda_e=d_e/\rho_s$, $d_e=c/\omega_{pe}$ being the electron inertial skin depth, and the brackets are defined by

$$[f,\ g]=\mathbf{e}_z\cdot\boldsymbol{\nabla}f\times\boldsymbol{\nabla}g.$$

The electron density is related to the ion density through the quasineutrality condition $\ln(n/n_0)=\ln(n_i/n_0)$. The ion response is derived in the cold-ion approximation (Kuvshinov *et al.* 1994):

$$\frac{\partial G_0}{\partial\tau}+[\Phi_0,\ G_0]=0, \tag{2.5}$$

where

$$G_0\equiv\nabla_\perp^2\Phi-\ln\frac{n}{n_0},\quad \Phi_0\equiv\Phi. \tag{2.6}$$

Equations (2.3)–(2.5) describe phenomena with frequencies below the ion cyclotron frequency and the magnetosonic frequency, and take into account finite electron pressure, finite parallel electron inertia and drift effects related to the density gradient. The spatial scales are larger then the ion gyroradius and may range from MHD lengths to the electron inertia skin depth. Neglecting drift effects and using the linear ion response (see e.g. Pegoraro and Schep 1986) instead of (2.5), the system (2.3)–(2.5) coincides with the model of Kadomtsev and Pogutse (1984). The reduced MHD model is recovered if one neglects the terms containing v_z and p_e in (2.2), which is equivalent to neglecting terms with λ_e and $\ln(n/n_0)$ in (2.3), and taking $G_0=0$ instead of (2.5).

3. Hamiltonian formulation of nonlinear plasma dynamics

In what follows we shall restrict the analysis to solutions that propagate with a constant velocity u_z in the z direction, so that all functions depend on the normalized variables $\tau-\lambda_z z$ and (x,y), where $\lambda_z\equiv c_A/u_z$. Equations (2.3) and (2.4) can be written in the Lagrangian conservation form

$$\frac{\partial G_\pm}{\partial\tau}+[\Phi_\pm,\ G_\pm]=0, \tag{3.1}$$

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where

$$\pm 2\lambda_e G_\pm \equiv -A + \lambda_z \Phi + (\pm\lambda_e - \lambda_z)\left(\ln\frac{n}{n_0} \pm \lambda_e \nabla_\perp^2 A\right), \tag{3.2a}$$

$$\Phi_\pm \equiv \frac{\lambda_e \Phi \mp A}{\lambda_e \mp \lambda_z}. \tag{3.2b}$$

The strictly two-dimensional (2D) case corresponds to the limit $\lambda_z \to 0$, i.e. to $u_z \to \infty$. In this limit the equations reduce to those given by Schep *et al.* 1994. In the limit $\lambda_z \to \infty$, (3.1) yields the Boltzmann distribution for the electron density, $\ln(n/n_0) = \Phi$. On substituting this distribution into the ion equation (2.6), one obtains the Hasegawa–Mima (1977) equation. We consider either cold electrons $\lambda_e/\lambda_z = v_z/v_T > 1$ (where v_T is the electron thermal velocity) or isothermal electrons $\lambda_e/\lambda_z < 1$. For $\lambda_z \approx \lambda_e$, the fluid model is invalid because of electron Landau damping.

Equations (2.5) and (3.1) are in Lagrangian form, and can be viewed as a generalized 2D Euler system for three fields G_α, where $\alpha = (0, +, -)$, that are advected with 'velocity' fields described by potentials Φ_α. The scalars G_α are pointwise-conserved and frozen into the velocity fields $\mathbf{e}_z \times \boldsymbol{\nabla}\Phi_\alpha$. This structure of the equations implies that any functional of G_α is conserved, i.e. the system has three distinct sets of conserved quantities (Casimirs)

$$\int d^2x\, F_\alpha(G_\alpha) = \text{const}, \tag{3.3}$$

where the integration is over any domain that moves with the appropriate velocity field. This elegant structure of the equations is lost in the limit of reduced MHD. The energy integral of the system (2.3)–(2.5) is obtained after multiplying (2.3) by $\nabla_\perp^2 A$, (2.4) by $\Phi - \ln(n/n_0)$, (2.5) by Φ, and adding the results:

$$W = \frac{1}{2}\int_D d^2x \left\{ |\boldsymbol{\nabla}_\perp A|^2 + \lambda_e^2 |\nabla_\perp^2 A|^2 + \left(\ln\frac{n}{n_0}\right)^2 + |\boldsymbol{\nabla}_\perp \Phi|^2 - 2\lambda_z \left(\ln\frac{n}{n_0} - \Phi\right)\nabla_\perp^2 A \right\}. \tag{3.4}$$

The contributions represent the magnetic energy, the kinetic energy of the parallel motion of the electrons, the internal energy of the electrons, the kinetic energy of the $\mathbf{E}\times\mathbf{B}$ motion of the ions, and the time-integrated divergence of the z components of the Poynting vector and of the electron thermal energy flux, respectively. The integrand in (3.4) can be written as a combination of $\Phi_\alpha G_\alpha$ and $G_\pm^2$. Integrals of the quadratic terms $G_\pm^2$ are Casimirs (cross-helicity integrals), and hence can be subtracted. As a result, one obtains the conserved quantity

$$H = -\frac{1}{2}\sum_\alpha \int_D d^2x\, G_\alpha \Phi_\alpha. \tag{3.5}$$

Functional (Fréchet) derivatives of H are $\delta H/\delta G_\alpha = -\Phi_\alpha$. If one defines the Poisson bracket as

$$\{f, g\} = \sum_\alpha \int d^2x\, G_\alpha \left[\frac{\delta f}{\delta G_\alpha}, \frac{\delta g}{\delta G_\alpha}\right],$$

the integral H will be the Hamiltonian for the system consisting of (2.5) and (3.1), which can now be written in the form

$$\frac{\partial G_\alpha}{\partial t} = \{G_\alpha,\ H\}. \tag{3.6}$$

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4. Singular current-vortex filaments

In a homogeneous plasma, the equilibrium values of the potentials Φ and A and of the density $\ln(n/n_0)$ are constant, and solutions of (2.5) and (3.1) exist in which the convected scalar quantities G_α are distributed in singular current-vortex filaments of the form

$$G_\alpha = \sum_i \kappa_{\alpha,i}\delta(\mathbf{r} - \mathbf{r}_{\alpha,i}), \tag{4.1}$$

where $\mathbf{r} = (x, y)$ and $\kappa_{\alpha,i}$ are the filament intensities. Equation (4.1) together with the definitions (2.6) and (3.2a) of the G_α lead to a set of differential equations for $\Phi - \lambda_z A$ and $-A + \lambda_z \Phi$ that can easily be solved. The three velocity potentials are found to be

$$\begin{aligned}\Phi_\beta(\mathbf{r}) &= (1-\lambda_z^2)^{-1}\left[\Phi - \lambda_z A + c_\beta(-A + \lambda_z\Phi)\right] \\ &= \frac{1}{2\pi(1-\lambda_z^2)}\sum_{\alpha,i}\kappa_{\alpha,i}[\ln|\mathbf{r} - \mathbf{r}_{\alpha,i}| + c_\alpha c_\beta K_0(\gamma|\mathbf{r} - \mathbf{r}_{\alpha,i}|)],\end{aligned} \tag{4.2}$$

where

$$c_\pm = \frac{1 \mp \lambda_z\lambda_e}{\pm\lambda_e - \lambda_z},\ c_0 = -\lambda_z,\ \gamma^2 = \frac{1-\lambda_z^2}{\lambda_e^2 - \lambda_z^2}.$$

Localized solutions require $\gamma^2 > 0$. Equations (4.2) generalize the well-known hydrodynamical point-vortex model (see Lamb 1932) to the case of a magnetized plasma. In contrast to hydrodynamics, we have three types of filament, each moving in a distinct velocity field. We shall refer to the different types as κ_0, κ_+ and, κ_- filaments. As can be seen from the definitions (2.6) and (3.2a), the κ_0 filaments describe singularities in the vorticity and the density, while the $\kappa_\pm$ filaments represent singularities in the current and the density. For generality, all types will be referred to as current-vortex filaments. Using (4.2), the Hamiltonian functional (3.5) becomes

$$\hat{H} = -\frac{1}{4\pi(1-\lambda_z^2)}\sum_{\alpha,i}\sum_{\beta,j}\kappa_{\alpha,i}\kappa_{\beta,j}[\ln|\mathbf{r}_{\beta,j} - \mathbf{r}_{\alpha,i}| + c_\alpha c_\beta K_0(\gamma|\mathbf{r}_{\beta,j} - \mathbf{r}_{\alpha,i}|)], \tag{4.3}$$

with $(\alpha, i) \neq (\beta, j)$. Equations (2.5) and (3.1) reduce to the Hamiltonian equations of motion of current-vortex filaments with the usual definition of the Poisson bracket:

$$\frac{d\mathbf{r}_{\alpha,i}}{d\tau} = \{\mathbf{r}_{\alpha,i}, \hat{H}\}, \quad \{f, g\} = \sum_{\alpha,i}\frac{1}{\kappa_{\alpha,i}}\left(\frac{\partial f}{\partial x_{\alpha,i}}\frac{\partial g}{\partial y_{\alpha,i}} - \frac{\partial f}{\partial y_{\alpha,i}}\frac{\partial g}{\partial x_{\alpha,i}}\right). \tag{4.4}$$

Besides the Hamiltonian (4.3), (4.4) have three other constants of motion related to the invariance of $\hat{H}$ under continuous translations and rotations of the coordinates. These constants are respectively

$$P = \sum_{\alpha,i}\kappa_{\alpha,i}x_{\alpha,i}, \quad Q = \sum_{\alpha,i}\kappa_{\alpha,i}y_{\alpha,i}, \quad I = \frac{1}{2}\sum_{\alpha,i}\kappa_{\alpha,i}(x_{\alpha,i}^2 + y_{\alpha,i}^2). \tag{4.5}$$

In the general case only the three invariants H, I and $P^2 + Q^2$ are in involution, as can be seen from their Poisson brackets. Hence the motion of three vortices is always integrable. In the special case of vanishing total vorticity ($\sum_{\alpha,i}\kappa_{\alpha,i} = 0$) and of vanishing momenta ($P = 0, Q = 0$) all the conserved quantities H, P, Q and I are in involution, and the four-filament problem becomes integrable. A

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detailed analysis of integrable motion of four filaments, of filament chains and of the collisional dynamics of vortex pairs will be presented elsewhere.

It is known that three or more point vortices in ideal fluids can merge producing a single vortex (Novikov and Sedov 1979). This phenomenon, which is called vortex collapse, indicates that under some initial conditions the uniqueness of Euler's equation is lost. One could expect similar phenomena to occur in plasmas. Indeed, at scales below the ion-sound gyroradius or for $\gamma \to 0$, i.e. $\lambda_z \to \pm 1$, which corresponds to propagation with the Alfvén velocity, the Bessel function in (4.3) can be replaced by a logarithm: $K_0(\gamma|\mathbf{r}_{\beta,j} - \mathbf{r}_{\alpha,i}|) \to -\ln|\mathbf{r}_{\beta,j} - \mathbf{r}_{\alpha,i}|$. Then, on the basis of their scale invariance under $\tau \to a^2\tau$ and $\mathbf{r} \to a\mathbf{r}$, (4.4) admit the self-similar solutions

$$r_{\alpha,j}(t) = \left(1 - \frac{t}{t_*}\right)^{1/2} r_{\alpha,j}(0), \quad \phi_{\alpha,j}(t) = \phi_{\alpha,j}(0) - \Omega t_* \ln\left(1 - \frac{t}{t_*}\right), \tag{4.6}$$

where, $r_{\alpha,j}$ and $\phi_{\alpha,j}$ are the polar coordinates of the filaments, $t = 0$ corresponds to the initial time, and Ω and t_* are constants. All filaments merge at $r = 0$, the collapse time being t_*. The necessary conditions for such a collapse follow from the conservation laws. They are similar to those obtained in Novikov and Sedov (1979) for the case of ideal MHD, and read as $P = 0$, $Q = 0$ and $I = 0$, while the conservation of the Hamiltonian requires

$$(1 - \lambda_z^2)^{-1} \sum_{\alpha,i} \sum_{\beta,j} \kappa_{\alpha,i}\kappa_{\beta,j}(1 - c_\alpha c_\beta) = 0, \quad (\alpha, i) \neq (\beta, j). \tag{4.7}$$

If the sum of intensities for each type of filament $\kappa_\alpha = \sum_i \kappa_{\alpha,i}$ vanishes then the collapse results in the total annihilation of vortices. The annihilation condition cannot be satisfied by a system of filaments of a single type. For this reason annihilation is impossible in hydrodynamics. In contrast to fluids, the conservation laws in plasmas do not forbid the annihilation of vortex filaments.

5. Propagating nonlinear structures

In a non-homogeneous plasma, (2.5) and (3.1) contain solutions in the form of localized structures that propagate in the y direction with constant velocity u_y. Such structures are described by the relations $[G_\alpha,\ \Phi_\alpha - x/\lambda_y] = 0$, which have the general solutions

$$G_\alpha = F_\alpha(\Phi_\alpha - x/\lambda_y). \tag{5.1}$$

where $\lambda_y = c_s/u_y$, $c_s = (T_e/m_i)^{1/2}$ is the ion-sound velocity, and the F_α are arbitrary functions of their arguments. The functions F_α need not be the same over all space, but may change across separatrices, which are curves $\Phi_\alpha - x/\lambda_y = \text{const}$ that connect to the singular points where $\boldsymbol{\nabla}_\perp(\Phi_\alpha - x/\lambda_y) = 0$. Equations (5.1) describe generic stationary propagating nonlinear structures. Depending on the behaviour of the fields G_α and the potentials $\Phi_\alpha - x/\lambda_y$ at infinity, these structures can be distributed vortices or magnetic islands, as we discuss below.

5.1. Dipole vortices

First of all, we consider an equilibrium with a linear density profile and vanishing A and Φ at $|x| \to \infty$, so that $(A, \Phi) \to 0$, $\ln(n/n_0) \to -x/\lambda_n$, where $\lambda_n = l_n/\rho_s$ is the normalized density scale length. The boundary conditions can be satisfied

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at $|x| \to \infty$ if the functions F_α are linear in their arguments. We shall take these functions to be linear everywhere, so that (5.1) reads $G_\alpha = F'_\alpha(\Phi_\alpha - x/\lambda_y)$. The proportionality constants F'_α cannot be the same over all space, and must be allowed to change at the separatrices. Equations (5.1) can be expressed as

$$\nabla^2 \begin{pmatrix} \Phi - \lambda_z A \\ -A + \lambda_z \Phi \end{pmatrix} = \begin{pmatrix} \dfrac{\sum\limits_\alpha F'_\alpha}{1-\lambda_z^2} & \dfrac{\sum\limits_\alpha F'_\alpha c_\alpha}{1-\lambda_z^2} \\ -\dfrac{\sum\limits_\alpha F'_\alpha c_\alpha}{1-\lambda_z^2} & \gamma^2 - \dfrac{\sum\limits_\alpha F'_\alpha c_\alpha^2}{1-\lambda_z^2} \end{pmatrix} \begin{pmatrix} \Phi - \lambda_z A - (1-\lambda_z^2)\dfrac{x}{\lambda_y} \\ -A + \lambda_z \Phi \end{pmatrix}. \tag{5.2}$$

It follows from the boundary conditions that in the outer region

$$F'_0 = -\frac{\lambda_y}{\lambda_n},\ F'_\pm = \frac{\lambda_y}{\lambda_n}\frac{\lambda_e \mp \lambda_z}{2\lambda_e}.$$

As a consequence, all elements of the matrix vanish except for the 22 element, which is equal to the eigenvalue in the external region:

$$k_e^2 = \left(1 - \frac{\lambda_y}{\lambda_n}\right)\frac{1-\lambda_z^2}{\lambda_e^2 - \lambda_z^2}. \tag{5.3}$$

Localized solutions can exist if $k_e^2 > 0$. Note that $\lambda_y/\lambda_n = u_*/u_y$, with $u_* = cT_e/eB_0 l_n$ being the diamagnetic velocity. The boundary condition at large $|x|$ implies that the solutions have the form of dipole vortices (Larichev and Reznik 1976), $\Phi_\alpha = \Phi_\alpha(r)\cos\theta$, with $(r,\ \theta)$ being polar coordinates. We suppose that we have two X points and a single separatrix circle at $r = r_0$. On this circle one of the quantities $\Phi_\alpha - x/\lambda_y$ vanishes. The eigenvalues $k_{1,2}$ of the matrix in (5.2) in the inner region (inside the separatrix) are determined by matching $\Phi - \lambda_z A$ and $-A + \lambda_z \Phi$ and their first derivatives at the separatrix circle. These matching conditions give the dispersion relation (Schep *et al.* 1995)

$$(k_1^2 - k_2^2)\hat{K}(k_e) + (k_e^2 - k_2^2)\hat{\mathscr{B}}(k_1) - (k_e^2 - k_1^2)\hat{\mathscr{B}}(k_2) = 0, \tag{5.4}$$

where

$$\hat{K}(k) = \frac{kK_2(kr_0)}{K_1(kr_0)}, \hat{\mathscr{B}}(k) = \frac{\mathrm{sgn}(k^2)|k|\mathscr{B}_2(kr_0)}{\mathscr{B}_1(kr_0)},$$

K and $\mathscr{B}$ are Bessel functions, with $\mathscr{B}_i(k) = I_i(|k|)$ if $k^2 > 0$, $\mathscr{B}_i(k) = J_i(|k|)$ if $k^2 < 0$. Equation (5.4) is a general dispersion relation for systems of the type (5.2). In Fig. 1 the roots of the dispersion relation (5.4) found numerically are shown. A second relationship between $k_{1,2}$ and k_e follows from the matrix in (5.2):

$$\frac{k_1^2}{k_e^2} + \frac{k_1^2}{k_e^2} - 1 = (1 - c_\alpha^2)\frac{k_1^2 k_2^2}{k_e^4}, \tag{5.5}$$

depending upon which of the $\Phi_\alpha - x/\lambda_y$ vanishes at $r = r_0$. The eigenvalues of the system are given by the intersection points of the curves (5.5) with the curves in Fig. 1.

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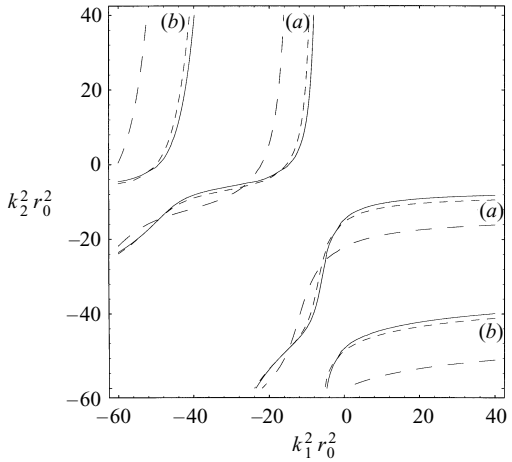


Figure 1. Solutions of the dipole vortex dispersion relation for $k_e^2 r_0^2 = 0.04$ (solid curves), $k_e^2 r_0^2 = 1$ (short-dashed curves) and $k_e^2 r_0^2 = 100$ (long-dashed curves). Labels (a) and (b) denote zeroth order and first-order roots respectively.

5.2. Magnetic islands

In a sheared magnetic field the unperturbed potential A has the form

$$A_{\rm eq} = \frac{c_A}{c_s}\frac{x^2}{2\lambda_s}, \text{ with } \lambda_s = \frac{l_s}{\rho_s}$$

where l_s is the shear length of the equilibrium field. Then, in the case of a constant density gradient, from the matching at large values of x, one obtains nonlinear (algebraic) relationships between the fields $G_\pm$ and the potentials $\Phi_\pm$ in (5.1), while the relationship involving G_0 and Φ_0 is linear. Relationships of this type, written, however, in terms of different combinations of variables, are presented by Schep *et al.* (1994) for $\lambda_z = 0$. These relationships describe the nonlinear stage of a chain of magnetic islands propagating along y, and lead to current density distributions with a characteristic dependence on $(A - A_s)^{-1/2}$. Here A_s is the value of the magnetic flux function at the outermost separatrices, and is related to the reconnected flux inside the magnetic island separatrices. This dependence indicates that the current and vorticity densities tend to concentrate at the separatrices. The square-root dependence is similar to that obtained in Kadomtsev (1991).

The nonlinear stage of a collisionless reconnection process is described by a configuration where the fields $G_\pm$ have no X points and magnetic reconnection is due to the effect of electron inertia only. On the other hand, X points of $G_\pm$ can be present if the configuration described is the nonlinear stage of a resistive reconnection processes. In the first case the relationship between the fields and the potentials found at large values of x can be extended all over the (x, y) plane. In the latter case a different relationship can be prescribed inside the separatrices of $G_\pm$. This may be of interest for example in order to describe islands that are sustained by an external source, such as the radiofrequency heating at the electron cyclotron frequency in the RTP experiment (Lopes Cardozo *et al.* 1994).

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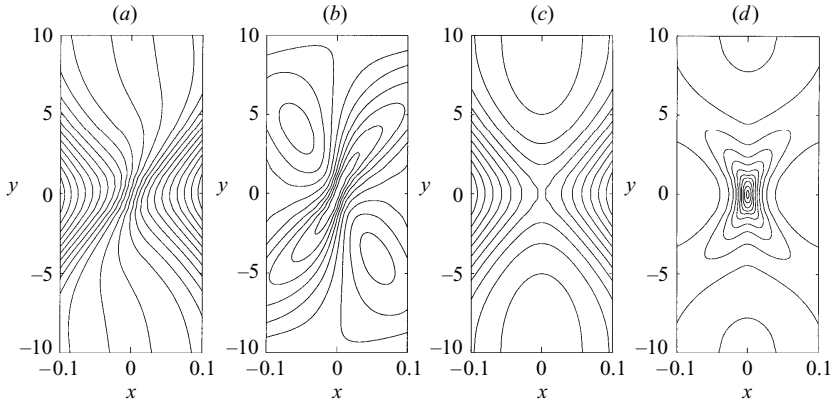


Figure 2. Forced collisionless reconnection: (*a*) the conserved field G_+; (*b*), the associated flow potential Φ_+; (*c*) the vector-potential component A; (*d*) the parallel current j.

6. Collisionless reconnection

We have numerically simulated forced collisionless reconnection on the basis of (2.5) and (3.1) with $\lambda_z = 0$. A rectangular domain $-L_x < x < L_x$, $-L_y < y < L_y$, periodic in y, was used, with initial conditions $\Phi = 0$ and $A = 0.5x^2$. Note that here the lengths are normalized on the shear length and time on the Alfvén time τ_A. A wave was excited by specifying $\Phi(y,t)$ at $x = \pm L_x$, approximately modelling radially moving, ideally conducting boundaries with positions $\pm x = L_x + \delta\cos(\pi y/L_y)\tanh(\omega t)$ (Rem and Schep 1998). The solutions A and Φ are respectively even and odd functions of both x and y. The simulations were performed with the numerical code Vlugr2 (Blom *et al.* 1996). The results are shown in Fig. 2. The parameters used are $L_x = 1$, $L_y = 10$, $\delta = 0.04$, $\rho_s^2 = 10^{-3}$, $d_e^2 = 10^{-3}$ and $\omega = 0.1\tau_A^{-1}$. The functions G_+, Φ_+, A and $j \equiv -\Delta_\perp A$ are shown on a narrow strip around $x = 0$ for $t = 93\tau_A$. Shortly after $t = 93\tau_A$, the current distribution became too narrow to be resolved by the finite-difference grid (500 points for $0 < x < 1$; 100 points for $0 < y < 10$). The functions G_- and Φ_- are obtained from G_+ and $-\Phi_+$ by reflection in the x or y axis. One sees that X-point formation has taken place in A but not in $G_\pm$. The current density j is peaked at the X point, but in addition tends to be transported along the separatrix. Also, vorticity layers are formed that tend to extend along the magnetic separatrices. These current and vorticity configurations are analogous to those found in Cafaro *et al.* (1997) in the case of the nonlinear evolution of a collisionless reconnection instability in a slab geometry.

7. Discussion

The two-fluid model of a magnetized plasma has been cast into three Lagrangian conservation laws that form a generalization of the Euler equation for an incompressible fluid to a fluid described by three fields. This model describes a wide variety of nonlinear phenomena. The existence of solutions in the form of current-vortex filaments has been pointed out, and collisionless reconnection has been demonstrated numerically. The roles of the Hamiltonian, of the conservation laws and of the topological constraints have been emphasized. A stability analysis of these configurations following Arnold's approach is a natural extension of this formulation.

https://doi.org/10.1017/S0022377898006655 Published online by Cambridge University Press

Acknowledgements

The authors thank dr H.J. de Blank for providing Fig. 2. This work was performed under the Euratom–FOM Association agreement with financial support from NWO and Euratom.

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https://doi.org/10.1017/S0022377898006655 Published online by Cambridge University Press