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Large amplitude circularly polarized waves in quantum plasmas

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Abstract. Some previous dispersion relations for large amplitude waves in plasmas are generalized so that also quantum effects are included.

1. Introduction

Nonlinear wave phenomena are of utmost importance in plasma physics theories (e.g. Akhiezer and Polovin 1956; Sagdeev and Galeev 1969; Shukla et al. 1986; Shukla 1999, 2004; Stenflo 2004). It turns out that investigations of a circularly polarized wave propagating along a constant external magnetic field are of particular interest. The reason is that the electric field of such a wave can exactly satisfy the fluid and Maxwell equations for any wave amplitude. Here we describe a plasma by means of the model fluid equations

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \tag{1.1}$$

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{p} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{\gamma m v_t^2}{n}\nabla n + \frac{\hbar^2}{2m}\nabla\left(\frac{1}{\sqrt{n}}\nabla^2\sqrt{n}\right), \tag{1.2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{1.3}$$

$$\nabla \times \mathbf{B} = \mu_0\left[\sum_\sigma qn\mathbf{v} + \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right], \tag{1.4}$$

where $\mathbf{p} = \gamma m\mathbf{v}$, $\gamma = (1 - v^2/c^2)^{-1/2}$, $h = 2\pi\hbar$ is the Planck constant, and v_t represents the thermal velocity which could equally well include the effect of the Fermi pressure (Shukla and Eliasson 2009). As usual we have denoted the number density, fluid velocity, charge, rest mass, velocity of light and electric and magnetic field by n, $\mathbf{v}$, q, m, c, $\mathbf{E}$ and $\mathbf{B}$, respectively. The sum stands for summation over particle species σ. For notational convenience we do not write out index σ on n, $\mathbf{v}$, q, etc.

2. The large amplitude wave

It turns out that a large amplitude circularly polarized wave with electric field $\mathbf{E} = E_0(\widehat{\mathbf{x}}\cos(\omega_0 t - k_0 z) + \widehat{\mathbf{y}}\sin(\omega_0 t - k_0 z))$, where $\widehat{\mathbf{z}}$ is the direction of the external

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magnetic field $B_{0z}\widehat{\mathbf{z}}$ satisfies (1.1)–(1.4) if the frequency ω_0 and wave vector k_0 are related by the dispersion relation (e.g. Stenflo 1976)

$$\omega_0^2 - k_0^2 c^2 = \sum_\sigma \frac{\omega_p^2\,(\omega_0 - k_0 v_d)}{\omega_0 - k_0 v_d + \omega_c} \tag{2.1}$$

where $\omega_p^2 = n_0 q^2/\gamma_0 m\varepsilon_0$, n_0 is equilibrium density, $\omega_c = qB_{0z}/\gamma_0 m$, and v_d is drift motion. We note that γ_0 here is constant as $v_0^2 = v_{0\perp}^2 + v_d^2$ is constant. Numerous previous papers (e.g. Stenflo 1976; Shukla et al. 1977; Chian and Miranda 1979; Tsintsadze et al. 1980; Goldring and Friedland 1985; Paul 1990; Kotsarenko et al. 1996; 1999; Stenflo and Shukla 2001; Shukla et al. 2008; 2009; Chang et al. 2009) have studied various aspects of (2.1). Generalizations which include quantum electrodynamical effects have also been presented (e.g. Shukla and Stenflo 2006; Brodin and Stenflo 2007; Lundin et al. 2007). Let us here therefore only point out one particular investigation which shows that the ions can be ultra-relativistic, *whereas the electrons are non-relativistic*. With undrifted species, (2.1) can then have the 'ion helicon wave' solution (Stenflo and Tsintsadze 1979)

$$\omega_0 = \frac{k^2 c^2}{\omega_{pi}^2}\Omega_E, \tag{2.2}$$

where $\Omega_E = q_i E_0/cm_i$ and index i stands for the ions. We also have the 'second ion cyclotron wave' solution

$$\omega_0 = \frac{\omega_{pi}^2}{\Omega_E}\left(1 + \frac{k^2 c^2 \omega_{ci}^2}{\omega_{pi}^4}\right). \tag{2.3}$$

3. The nonlinear dispersion relation

We now consider the propagation of a small amplitude wave in the presence of our large amplitude wave (2.1). Assuming that all waves propagate in the z-direction, we can then derive the *exact* dispersion relation, valid for any magnitude of the pump wave (Stenflo 1976). Here we shall for simplicity just present the dispersion relation for an undrifted non-relativistic plasma (Stenflo and Shukla 2000)

$$\frac{\epsilon}{1 + c_0\epsilon/c_+ c_-} \equiv -\left(\frac{c_+^2}{N_+} + \frac{c_-^2}{N_-}\right) E_0^2, \tag{3.1}$$

where

$$\epsilon = 1 + \sum \chi,$$

$$\chi = -\frac{\omega_p^2}{\omega^2 - k^2 v_T^2},$$

$$v_T^2 = v_t^2 + \frac{\hbar^2 k^2}{4m^2} + \frac{2q^2 k_0^2 \omega^2 \omega_c E_0^2}{m^2 k^2 \omega_0^2 (\omega_0 + \omega_c)[\omega^2 - (\omega + \omega_c)^2]},$$

$$c_0 = \sum \frac{q^2}{m^2}\frac{\chi}{(\omega_0 + \omega_c)^2}\left[k - \frac{k_0\omega_c\omega}{\omega_0(\omega_0 - \omega + \omega_c)}\right]\left[k - \frac{k_0\omega_c\omega}{\omega_0(\omega_0 + \omega + \omega_c)}\right],$$

$$c_\pm = \mp \sum \frac{q}{m}\frac{\chi}{(\omega_0 + \omega_c)}\left[k - \frac{k_0\omega_c\omega}{\omega_0(\omega_0 \mp \omega + \omega_c)}\right],$$

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and

$$N_{\pm}=(k\mp k_0)^2\,c^2-(\omega\mp\omega_0)^2+\sum_{\sigma}\frac{\omega_p^2\,(\omega_0\mp\omega)}{\omega_0\mp\omega+\omega_c}$$

$$-\left\{\sum\frac{q^2}{m^2}\frac{\chi}{(\omega_0+\omega_c)^2}\left[k-\frac{k_0\omega_c\omega}{\omega_0(\omega_0\mp\omega+\omega_c)}\right]^2+\frac{c_{\pm}c_0}{c_{\mp}}\right\}E_0^2.$$

In the particular case where ϵ as well as N_+ and N_- are close to zero we can reduce (3.1) to the result for the comparatively simple three wave interaction process in a cold magnetized plasma (Sjölund and Stenflo 1967). Another case of interest concerns the parametric excitation in a single cold beam. If $v_d \ll c$ and ω/k is very close to v_d, we can then derive the instability criterium (Tsintsadze and Stenflo 1974)

$$\left(1+\frac{\omega_p^2}{k^2c^2}\right)^{-1}<\frac{\nu_0^2}{1+\nu_0^2}<\frac{k^2c^2}{\omega_p^2}, \tag{3.2}$$

where $\nu_0 = qE_0/mc\omega_0$. The general nonlinear dispersion relation generalizing (3.1) to both the relativistic as well as to the fully kinetic case has been presented elsewhere (Stenflo 1981). In the particular case where the excited longitudinal wave as well as the sideband waves are natural modes we can reduce that dispersion relation so that it describes the three wave interaction processes in a Vlasov plasma (Stenflo 1970). Some authors who were not familiar with the previous literature, have later (sometimes incorrectly) re-derived the results mentioned above. It was thus necessary to further elucidate these matters (Shukla and Stenflo 1999; Stenflo and Shukla 1999, 2004, 2005).

4. Finite amplitude Alfvén waves

We now consider the limiting case where $v_d = 0$, the plasma is unrelativistic, and the frequency satisfies $\omega_0 \ll \omega_{ce}$ (where index e represents electrons). This leads to Alfvén-ion-cyclotron waves. Equation (2.1) then reduces to

$$k_0^2c_A^2=\frac{\omega_0^2\omega_{ci}}{(\omega_0+\omega_{ci})}, \tag{4.1}$$

where $c_A = [B_{0z}^2/\mu_0 m_i n_0]^{1/2}$ is the Alfvén velocity. In the same limit the dispersion relation (3.1) reduces to (c.f. Brodin and Stenflo 1988)

$$\omega^2-k^2c_s^2-\frac{\hbar^2k^4}{4m_em_i}-\frac{q_i^2B_{\perp0}^2\omega_{ci}\omega^2}{m_i^2(\omega_0+\omega_{ci})[\omega^2-(\omega_0+\omega_{ci})^2]}$$

$$=\frac{q_i^2B_{\perp0}^2\omega_0^2}{2m_i^2(\omega_0+\omega_{ci})^2k_0^2}\left\{\frac{\left[k\omega_0-k_0\omega\left(1+\frac{(\omega_0-\omega)\omega_{ci}}{\omega_0(\omega_0-\omega+\omega_{ci})}\right)\right]^2}{\varepsilon_+}\right.$$

$$\left.+\frac{\left[k\omega_0-k_0\omega\left(1+\frac{(\omega_0+\omega)\omega_{ci}}{\omega_0(\omega_0+\omega+\omega_{ci})}\right)\right]^2}{\varepsilon_-}\right\}, \tag{4.2}$$

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where $B_{\perp 0}^2 = k_0^2 E_0^2/\omega_0^2$, c_s is the ion-sound velocity (including the Fermi pressure as well) and where $\varepsilon_\pm = [\omega_{ci}(\omega_0 \mp \omega)^2/(\omega_0 \mp \omega + \omega_{ci})] - (k \mp k_0)^2 c_A^2$. Equation (4.2) generalizes other results (e.g. Araneda 1998; Nariyuki and Hada 2007; Stenflo and Shukla 2007; Ruderman and Caillol 2008).

In the low-frequency case ($\omega \ll \omega_0 \ll \omega_{ci}$), (4.2) reduces to

$$\omega^2 - k^2 c_s^2 - \frac{\hbar^2 k^4}{4 m_e m_i} = -\frac{B_{\perp 0}^2 \omega_0^2 k^2}{B_{0z}^2 (k^2 - 4k_0^2)}. \tag{4.3}$$

We note from (4.3) that the wave is unstable if k is sufficiently close to $2k_0$ or $-2k_0$.

5. Alfvén waves in anisotropic plasmas

Here we consider the low-frequency case ($\omega, \omega_0 \ll \omega_{ci}$ and $kc_A, k_0 c_A \ll \omega_{ci}$), and improve the pressure term in order to include pressure anisotropies in terms of the Chew–Goldberger–Low (CGL) model. The Alfvén wave dispersion relation is then (Brodin and Lundberg 1990)

$$\omega_0^2 = k_0^2 \left(c_A^2 - \frac{B_{0z}^2}{B_0^2} \left(c_{sz}^2 - c_{s\perp}^2 \right) \right), \tag{5.1}$$

where $B_0^2 = B_{0\perp}^2 + B_{0z}^2$, and c_{sz}^2 and $c_{s\perp}^2$ are derived from the parallel and perpendicular part of the background pressure respectively. We here stress that the firehose-instability can be suppressed if $B_{0\perp}^2$ is comparable to B_{0z}^2. Finally we note that the nonlinear dispersion relation in the low-frequency low-amplitude limit reduces to

$$\omega^2 - 3k^2 c_{sz}^2 - \frac{\hbar^2 k^4}{4 m_e m_i} = -\frac{B_{0\perp}^2}{B_0^2} \frac{k^2 \left[k_0 \left(c_A^2 + 2c_{sz}^2 \right) + k \left(3c_{sz}^2 - c_{s\perp}^2 \right) \right]^2}{(k^2 - 4k_0^2)\left(c_A^2 - c_{sz}^2 + c_{s\perp}^2 \right)}, \tag{5.2}$$

where a quantum term has been added to the left hand side of (5.2). We point out that the numerical factors in front of c_{sz}^2 and $c_{s\perp}^2$ are dependent on the assumption of adiabatic particle orbits made in the CGL model. Obviously there is an instability if $k^2 \approx 4k_0^2$.

6. Summary

By means of some typical examples, we have shown that previous dispersion relations for wave propagation in the presence of a large amplitude circularly polarized electromagnetic wave in a plasma can be easily generalized, so that also quantum plasma effects can be included. Our expression for the susceptibility χ in Section 3 shows that quantum effects are important if the frequency ω of the excited longitudinal quasimode is comparable to $\hbar k^2/m$.

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