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Cusp formation for the undercooled Stefan problem in two and three dimensions

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We consider in this paper the classical one-phase Stefan problem in dimensions two and three in the undercooled situation. By means of matched asymptotic expansions, a mechanism of cusp formation is presented for interfaces that are initially smooth.

1 Introduction

This paper deals with the one-phase Stefan problem

$$u_t = \Delta u \quad \text{for} \quad x \in \Omega(t) \subset \mathbb{R}^N, \quad t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \Omega(0), \tag{1.2}$$

$$u(x,t) = 0 \quad \text{when} \quad x \in \partial\Omega(t), \quad t \geqslant 0, \tag{1.3}$$

$$v_n = -\frac{\partial u}{\partial n}(x,t) \quad \text{when} \quad x \in \partial\Omega(t), \quad t \geqslant 0. \tag{1.4}$$

Here $N = 2, 3$, the function $u_0(x)$ and the set $\Omega(0)$ are given data (we shall assume henceforth that $u_0(x) \leqslant 0$), and the domain $\Omega(t)$, where (1.1) holds is *a priori* unknown. We denote by $\partial\Omega(t)$ the boundary of $\Omega(t)$ at any time $t \geqslant 0$. On such a boundary we assume that (1.4) holds, where v_n denotes the outer normal velocity at the curve $\partial\Omega(t)$.

The purpose of this work is to exhibit solutions of (1.1)–(1.4) for which the corresponding free boundary $\partial\Omega(t)$ (which is assumed to be smooth at $t = 0$) develops a cusp in a finite time $t = t^*$. Our approach will make use of matched asymptotic expansion techniques, and is therefore of a formal nature. We shall remark, however, on possible ways of making our arguments rigorous where appropriate.

Let us replace (1.1) by

$$\Delta u = 0 \quad \text{for} \quad x \in \Omega(t), \quad t > 0. \tag{1.5}$$

Problem (1.5), (1.2)–(1.4) is usually referred to as the Hele–Shaw problem (HS). Notice that in [H], as well as in the problem considered in this paper, the ill-posed or undercooled situation is studied, where one can expect the free boundary to be very unstable and to have a strong tendency to develop singularities. The existence of cusp-developing solutions for (HS) has long been known when $N = 2$, where these were obtained by means of conformal mapping techniques (cf. [H]). A related result was recently obtained in [KLV], where the authors produced a self-similar solution of (HS) that initially has a corner which remains for a long time and then eventually smooths out. The configuration considered in [KLV]

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also provides an example of singularity formation for the undercooled problem (but in contrast to [H] the singularity is corner-like not cusp-like).

Another problem closely related to (1.1)–(1.4) is the Stefan problem regularized by means of the Gibbs–Thomson law, i.e. replacing (1.3) by $u(x,t) = H$ where H is the mean curvature of $\partial\Omega(t)$. A theory of weak solutions for such problems, even in the two-phase case, can be found in [Lu].

The existence of solutions of (1.1)–(1.4) that develop singularities (i.e. points where v_n becomes infinite) is well-known for the case of one space dimension (cf. [S], [FPHO]), and the corresponding mechanism of formation of singularities is now well understood (cf. [HV1]). In this work a corresponding mechanism for $N = 2$ and $N = 3$ (in the axisymmetrical case) is provided. It will be shown to bear some analogies with the one-dimensional case considered in [HV1]: in particular, a length scale of order

$$(t^* - t)^{1/2} (\log |\log (t^* - t))^{1/2}$$

will appear. A major difference with the case $N = 1$ arises, however, in our case. Namely, in one-dimensional problems a minimum amount of undercooling is required in order to obtain singularities. For instance, if $u_0(x) > -1$ in (1.1)–(1.4), classical solutions are globally defined in time, and no singularity arises at all (in particular, v_n remains bounded for any finite time). Solutions to be described below exhibit cusps even when they originate from initial values with arbitrarily small undercooling. The possible occurrence of such behaviour was already pointed out in [LO] for the case $N = 2$, although no asymptotic description of such singularities was given therein. Finally, [L] has recently discussed the formation of cusps for the Stefan problem as $t \to \infty$ by comparison with the Hele–Shaw problem.

There is a close relation between the one-phase Stefan problem and the equation

$$c_t = \Delta c - \chi_{\Omega(t)}(x) \quad x \in \mathbb{R}^N, \quad t > 0, \tag{1.6}$$

where $\chi_{\Omega(t)}$ is the characteristic function of the domain $\Omega(t)$ for each value of t. Given a solution of the Stefan problem (1.1)–(1.4), it is possible to obtain a solution of (1.1)–(1.4) by means of the so-called Baiocchi transformation. In the undercooled case, such a transformation reads

$$c(x,t) = -\int_t^{T(x)} u(x,\xi)\, d\xi, \tag{1.7}$$

where the free boundary has been written as $t = T(x)$ for $t > 0$, and by definition $T(x) = 0$ if $u(x,0) = u_0(x) = 0$.

It is easily seen (and will be recalled in §2) that for a given $u(x,t)$ solving (1.1)–(1.4) it is possible to obtain (1.6) by means of (1.7) if $T(x)$ is defined for any $x \in \mathbb{R}^N$. We then reduce the Stefan problem to the much more regular equation (1.6). In particular, (1.6) provides a method of solving the Stefan problem in undercooled situations.

To conclude this introduction, we point out that the plan of this paper is the following. In §2, we show how to solve Stefan problems by means of the Baiocchi transformation and the auxiliary equation (1.6). Then, in §3 the manner in which cusps develop is described. The main conclusions of the paper will then be summarized in §4. The article then concludes with an appendix describing the construction of an auxiliary function that is used

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in §3. We also recall there some facts about the so-called Ivantsov solution for the Stefan problem, which plays an important role in our approach.

2 Solving the Stefan problem

The solvability of problem (1.1)–(1.4) for short times is not completely obvious if the space dimension is larger than one. In dimension one there are different approaches that allow one to solve (1.1)–(1.4) even in undercooled situations. For instance, it is possible to use fixed point arguments combined with classical parabolic regularity theory to prove existence and uniqueness (cf. [F]). The difficulty in higher dimensions arises from the fact that there is a lack of regularity in space at the free boundary that does not allow us to use fixed point arguments. For instance, let us assume that we try to solve (1.1)–(1.4) in classical spaces. Let us assume that $\partial\Omega(t)\in C^{k,\alpha}$ for each $t\geqslant 0$, with $k\geqslant 0$ integer, $0<\alpha<1$. Then u would be at most $C^{k,\alpha}$ in space by classical regularity theory, but then

$$\frac{\partial u}{\partial n}\in C^{k-1,\alpha}$$

in space, whence solving (1.4) by characteristics one would obtain at most $\partial\Omega(t)\in C^{k-1,\alpha}$ in a hypothetical fixed point argument. The way out of this problem in classical spaces was found in [M]. The basic idea is to regularize (1.1)–(1.4) by replacing (1.3) by

$$u(x,t)=\epsilon\,\mathscr{L}(\partial\Omega),$$

where $\mathscr{L}(\partial\Omega)$ is a regular second order operator that depends on $\partial\Omega(t)$ (for instance, mean curvature). Then passing to the limit as $\epsilon\to 0$ a solution of (1.1)–(1.4) is obtained if one can prove enough *a priori* estimates. Notice that this limit is not an easy matter, in particular in the undercooled case where a fine dendritic structure could be expected to appear near the free boundary that would destroy the regularity estimates obtained at the interface.

We describe here a different method of solving the Stefan problem locally in time. It is closely related to the approach used in [DF].

Let us assume that $u_0(x)$, $\Omega(0)$ are given, and that $u_0(x)=0$ at $\partial\Omega(0)$. (Actually, this condition may be relaxed just to undercooling less than -1 at the free boundary, i.e. $u_0(x)>-1$ at $\partial\Omega(0)$). We also assume that $\Omega(0)$ is smooth and bounded. Again, this condition could be relaxed supposing suitable growth conditions for $u_0(x)$.

We define a class $\mathscr{C}$ of admissible initial data $(u_0(x),\Omega(0))$ as follows. Given $u_0(x)$ defined in $\Omega(0)$, $u_0(x)\leqslant 0$ in $\Omega(0)$ and $u_0(x)=0$ at $\partial\Omega(0)$, we say that $(u_0(x),\Omega(0))\in\mathscr{C}$ if there exist functions $h(x)\in C^\infty(\mathbb{R}^N)$ and

$$c_0\in C^{1,1}(\mathbb{R}^N)\cap C^\infty\quad(\mathbb{R}^N\backslash\partial\Omega(0))$$

such that:

$$-u_0(x)+h(x)=-\Delta c_0+1\quad\text{in}\quad\mathbb{R}^N\backslash\partial\Omega(0), \tag{2.1}$$

$$\operatorname{supp}(c_0)=\operatorname{supp}(u_0)=\overline{\Omega(0)}, \tag{2.2}$$

$$\operatorname{supp}(h)\subset\Omega(0), \tag{2.3}$$

$$c_0(x)\geqslant 0\quad\text{in}\quad\mathbb{R}^N. \tag{2.4}$$

Notice that (2.3) implies in particular that $\operatorname{supp}(h)\cap\partial\Omega(0)=\varnothing$.

We first remark that the class $\mathscr{C}$ contains a large family of initial data. In particular, it

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contains all the analytic data $u_0(x) \leqslant 0$ in $\Omega(0)$ with $\partial\Omega(0)$ analytic. Indeed, given such $u_0(x)$, $\Omega(0)$, we may solve the problem

$$-u_0(x) = -\Delta \tilde{v}_0(x) + 1 \quad \text{in} \quad \mathscr{U}, \tag{2.5}$$

$$\tilde{v}_0(x)|_{\partial\Omega(0)} = 0, \tag{2.6}$$

$$\left.\frac{\partial \tilde{v}_0(x)}{\partial n}\right|_{\partial\Omega(0)} = 0, \tag{2.7}$$

where $\mathscr{U}$ is an open neighbourhood of $\partial\Omega(0)$. By classical Cauchy–Kowaleskaya theory, the problem (2.5)–(2.7) is uniquely solvable if $\mathscr{U}$ is small enough. We now define the function $v_0(x)$ as

$$v_0(x) = 0 \quad \text{if} \quad x \notin \Omega(0),$$

$$v_0(x) = \tilde{v}_0(x) \quad \text{if} \quad x \in \Omega(0).$$

Again, by standard Cauchy–Kowaleskaya theory, we have that

$$v_0(x) \sim \tfrac{1}{2}(d_{(x)})^2 \quad \text{as} \quad x \to \partial\Omega(0), \quad x \in \Omega(0), \tag{2.8}$$

where $d_{(x)}$ is the distance of x to $\partial\Omega(0)$.

We then pick two non-negative cut-off functions $\xi_1(x)$, $\xi_2(x)$ such that $\xi_1(x) = 1$ in a neighbourhood of $\partial\Omega(0)$ contained in $\mathscr{U}$, $\xi_1(x) = 0$ for $x \notin \mathscr{U}$, $\xi_2(x) = 1$ in $\Omega(0)\backslash\mathscr{U}$, and $\xi_2(x) = 0$ in a set that strictly contains $\mathbb{R}^N\backslash\Omega(0)$.

We then define

$$c_0(x) = \xi_1(x)\, v_0(x) + \xi_2(x), \quad x \in \mathbb{R}^N.$$

Taking into account (2.8), we readily obtain that if $\mathscr{U}$ is close enough to $\partial\Omega(0)$, then $c_0(x) > 0$ for $x \in \Omega(0)$, whence (2.2), (2.4) are true. Moreover, if we define

$$c(x) = -\Delta\, c_0(x) + 1 + u_0(x),$$

we easily obtain from (2.5) and our choice of $\xi_1(x)$, $\xi_2(x)$ that (2.3) is satisfied. On the other hand (2.6), (2.7) imply that $u_0 \in C^{1,1}(\mathbb{R}^N)$.

Given $(c_0(x), \Omega(0)) \in \mathscr{C}$, we solve the equation

$$c_t = \Delta c - \chi_{\Omega(t)} + h(x), \quad x \in \mathbb{R}^N, \quad t > 0, \tag{2.9}$$

$$c(x, 0) = c_0(x), \quad x \in \mathbb{R}^N \tag{2.10}$$

with

$$c(\cdot, t) \in C^1(\mathbb{R}^N) \quad \text{for any} \quad t \geqslant 0.$$

Taking into account that $c_0(x) > 0$ for $x \in \Omega(0)$ we have that for short times (2.9), (2.10) amounts to the problem

$$c_t = \Delta c - H(c) + h(x), \quad x \in \mathbb{R}^N, \quad t > 0 \tag{2.11}$$

with initial value given by (2.10).

We can solve (2.9), (2.10) as long as the region $\{c < 0\} \cap \text{supp}\,(h)$ does not intersect $\partial\Omega(t)$, although the problem (2.9), (2.10) is different from (2.10), (2.11) as soon as the region $\{c < 0\}$ becomes non-empty.

Assuming that we have found a solution of (2.9), (2.10) for $0 < t < t^*$, in such a way that

$$\{c < 0\} \cap \partial\Omega(t) = \varnothing,$$

and assuming that $\partial\Omega(t)$ is smooth for $0 \leqslant t < t^*$, we obtain a solution of the Stefan problem (1.1)–(1.4) that may be calculated as

$$u(x, t) = c_t(x, t) \quad x \in \Omega(t), \quad 0 \leqslant t < t^*. \tag{2.12}$$

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It is easily seen that $u(x,t)$ satisfies (1.1), and, taking into account (2.1) and (2.9), (1.2) follows.

On the other hand, if we assume that $\partial\Omega(t)$ is smooth, the asymptotic behaviour of $c(x,t)$ near the free boundary is

$$c(x,t) \sim \tfrac{1}{2}(d_{(x)})^2,$$

where $x \in \Omega(t)$ and $d_{(x)}$ is now the distance from x to $\partial\Omega(t)$. Then

$$\Delta c - \chi_{\Omega(t)}(x) \to 0 \quad \text{as} \quad x \to \partial\Omega(t),$$

whence (1.3) holds.

Taking into account (2.9) and the fact that $c_0(x) \leqslant 0$, we easily obtain that $c_t = u < 0$ at $\Omega(t)$ for $t > 0$. Thus, the family of domains $\Omega(t)$ decreases. Moreover, they decrease strictly (assuming that $\partial\Omega(t)$ is smooth). In fact, if there exists $x_0 \in \mathbb{R}^N$ such that

$$x_0 \in \partial\Omega(t) \quad \text{for} \quad t \in [t_0, t_0+\delta], \quad \delta > 0,$$

we would have

$$c(x_0,t) = \nabla c(x_0,t) = 0, \quad t \in [t_0, t_0+\delta],$$

but then

$$c_t(x_0,t) = \nabla(c_t)(x_0,t) = 0,$$

and since $c_t < 0$ at $\Omega(t)$ we would derive a contradiction by the Hopf lemma.

We can then write the free boundary $\partial\Omega(t)$, at least locally near each point of the boundary, as

$$t = T(x)$$

where by definition.

$$c(x, T(x)) = \nabla c(x, T(x)) = 0. \tag{2.13}$$

Notice that for points that are close to the free boundary we have

$$c(x,t) = -\int_t^{T(x)} u_t(x,\lambda)\,d\lambda = -\int_t^{T(x)} u(x,\lambda)\,d\lambda. \tag{2.14}$$

On the other hand,

$$\nabla c = -\int_t^{T(x)} \nabla u(x,\lambda)\,d\lambda,$$

$$\begin{aligned}\Delta c &= -\int_t^{T(x)} \Delta u(x,\lambda)\,d\lambda - \Delta u(x,T(x))\cdot\nabla T(x)\\ &= u(x,t) - \nabla u(x,T(x))\cdot\nabla T,\end{aligned}$$

and taking into account (2.9), (2.12), we would obtain

$$-1 = c_t - \Delta c = \Delta u(x,T(x))\cdot\nabla T(x) = -\frac{\partial u}{\partial n}|\nabla T|,$$

where n is the outer normal. On the other hand,

$$V_n = -\frac{1}{|\nabla T|},$$

so that (1.4) holds, whence $c(x,t)$, $\Omega(t)$ is a solution of (1.1)–(1.4).

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Notice that the main point that we have left unproved in the previous argument is the regularity of the free boundary $\partial\Omega(t)$ for the solutions of (2.9) with the additional assumption $u \in C^1(\mathbb{R}^N)$. For the well-posed Stefan problem such a regularity result has been obtained in [C] under the assumption of some flatness of the free boundary (in a very weak measure-theoretical form). However, it is very natural to assume this type of results (flatness implies regularity) to be satisfied for the equation (2.9) with the free boundary condition $u \in C^1(\mathbb{R}^N)$, even in the supercooled case. In the description of the cusp development mechanism it will be assumed in the rest of the paper that this type of result holds true.

We intend to investigate this assumption elsewhere.

3 Description of the cusp formation mechanism

3.1 Two-dimensional case

We introduce self-similar variables as follows:

$$c(x,t) = (t^* - t)\,\Phi(y,\tau) \tag{3.1}$$

where
$$y = \frac{x}{\sqrt{t^* - t}}, \quad \tau = -\log(t^* - t), \tag{3.2}$$

and where $t^* > 0$ is the time of formation of the cusp. It is then readily seen that Φ satisfies the equation

$$\Phi_\tau = \Delta\Phi - \frac{y\nabla\Phi}{2} + \Phi - H(\Phi), \quad y \in \mathbb{R}^2, \quad \tau > -\log(T). \tag{3.3}$$

A linear operator that plays an important role is

$$B\phi = \Delta\phi - \frac{y\cdot\nabla\phi}{2} + \phi, \tag{3.4}$$

defined in
$$L^2_\omega(\mathbb{R}^2) = \left\{ f: \int_{\mathbb{R}^2} |f|^2\, d\mu(y) \equiv \|f\| < +\infty \right\},$$

where
$$d\mu(y) = \mathrm{e}^{-\frac{|y|^2}{4}} d^2 y,$$

and with domain

$$D(B) = H^2_\omega(\mathbb{R}^N) = \{ f: \nabla f, \nabla^2 f \in L^2_\omega(\mathbb{R}^2) \}.$$

The spectrum of B is given by

$$\sigma(B) = \sigma_p(B) = \left\{ 1 - \frac{n}{2}: n = 0, 1, \ldots \right\},$$

with eigenfunctions

$$\phi_\alpha(y) = H_{\alpha_1}(y_1)\, H_{\alpha_2}(y_2), \quad y = (y_1, y_2),$$
$$\alpha = (\alpha_1, \alpha_2) \quad \alpha_i = 0, 1, 2, \ldots, \tag{3.5}$$

where
$$H_n(\xi) = (\eta^{n/8}(4\pi)^{1/4}(n!)^{1/2})^{-1}\, \tilde{H}_n\left(\frac{\xi}{2}\right);$$

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here $\tilde{H}_n$ is the standard nth Hermite polynomial and

$$B\psi_\alpha = \left(1 - \frac{|\alpha|}{2}\right)\psi_\alpha,$$

where $$|\alpha| = \alpha_1 + \alpha_2.$$

Notice that ψ_α is normalized so as to have

$$\|\psi_\alpha\| = 1.$$

The cusp mechanism that is described here has as asymptotic limit

$$\Phi(y,\tau) \to \tfrac{1}{2}y_2^2 - A\,\psi_{(0,2)}(y) \equiv \bar{\Phi}(y).$$

Notice that

$$\psi_{(0,2)}(y) = \Gamma(y_2^2 - 2), \quad \Gamma = \frac{1}{4\sqrt{2\pi}}.$$

We impose $2\Gamma A < 1$. Then $\bar{\Phi}(y) > 0$, and

$$\bar{\Phi}(y) \sim (\tfrac{1}{2} - A\Gamma)\,y_2^2$$

as $y_2 \to +\infty$. Notice that $2\Gamma A = 1$ would give $\bar{\Phi}(y) = 1$. This is the asymptotic limit that was obtained in the one-dimensional case (cf. [HV1]). The one-dimensional mechanism of formation of singularities is also possible in higher dimensional situations just by the addition of new spatial variables. However, we will restrict our attention in this paper to the case $2\Gamma A < 1$, since the one-dimensional mechanism has been studied in [HV1]. The main difference between the mechanisms of formation of singularities with $2\Gamma A < 1$ and with $2\Gamma A = 1$ is, as will be described below, that solutions with $2\Gamma A = 1$ give, locally near the interface at the singularity, an undercooling that takes exactly the value one, whereas the solutions with $2\Gamma A < 1$ allow for a continuous gradation of undercooling from zero to one. In a remark at the end of §3.1 we shall describe in detail the variation of the shape of the cusp as $2\Gamma A$ approaches 1 from below.

We obtain our mechanism of cusp formation as a perturbation of the one-dimensional solution $\bar{\Phi}(y)$. More precisely, we write

$$\Phi(y,\tau) = \bar{\Phi}(y) + \psi(y,\tau) \tag{3.6}$$

Let us denote by $D(\tau)$ the region where $\Phi = 0$. As $\bar{\Phi}(y) > 0$, we expect the region $D(\tau)$ to escape to $|y| \to \infty$ as $\tau \to \infty$. More precisely, we expect $D(\tau)$ to be concentrated near the region $|y_2| \to 0$, $y_1 \in (a(\tau), +\infty)$, where $a(\tau) \to \infty$ as $\tau \to \infty$. Notice that the function $\psi(y,\tau)$ satisfies

$$\begin{aligned}\psi_\tau &= \Delta\psi - \frac{y\nabla\psi}{2} + \psi + (1 - H(\Phi)) \\ &= \Delta\psi - \frac{y\cdot\nabla\Psi}{2} + \psi + \chi_{D(\tau)}.\end{aligned} \tag{3.7}$$

The dynamics of ψ should be controlled by the linear operator A previously defined. On the other hand, the modes associated to the eigenfunctions $\psi_{(0,0)}(y)$, $\psi_{(1,0)}(y)$, $\psi_{(0,1)}(y)$

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yield unstable behaviours, and as in other singularity formation problems (see, for instance, [HV1], [HV2]) they should be subdominant.

In fact, these unstable eigenvalues are related to the fact that a small change of initial data would change the time and the point of formation of the cusp. The instability associated with the eigenfunctions could be eliminated by changing either T or the point of formation of the cusp. The generic mechanism of formation for the singularities should be driven by zero eigenvalues, and this is precisely the case that we will consider here.

More precisely, assuming for simplicity that we consider symmetric solutions with respect to the y_1-axis we expect the following behaviour:

$$\psi(y,\tau) \sim \alpha(\tau)\,\psi_{(2,0)}(y) + \beta(\tau)\,\psi_{(0,2)}(y) \quad \text{as} \quad \tau \to \infty, \tag{3.8}$$

where $|\alpha(\tau)| + |\beta(\tau)| \to 0$ as $\tau \to \infty$.

Putting together (3.8) and the definition of $\Phi(y)$, we would obtain

$$\Phi(y,\tau) \sim (\tfrac{1}{2}y_2^2 - (A + \beta(\tau))\,\psi_{(0,2)}(y)) + \alpha(\tau)\,\psi_{(2,0)}(y)$$

as $\tau \to \infty$.

This expression shows that the Fourier coefficient of $\psi_{(0,2)}(y)$ that measures the undercooling locally near the point $x = 0$ changes in time. It will be checked later on that it is possible to obtain consistent expansions where $\beta(\tau) \to 0$ as $\tau \to \infty$, and where $\beta(\tau)$ is subdominant compared with $|\alpha(\tau)|$. This means that the asymptotic behaviour of $\Phi(y,\tau)$ is basically

$$\Phi(y,\tau) \sim (\tfrac{1}{2}y_2^2 - A\psi_{(0,2)}(y)) + \alpha(\tau)\,\psi_{(2,0)}(y).$$

Let us write

$$\psi(y,\tau) = \alpha(\tau)\,\psi_{(2,0)}(y) + \beta(\tau)\,\psi_{(0,2)}(y) + R(y,\tau), \tag{3.9}$$

where $\langle R, \psi_\alpha \rangle = 0$ for $|\alpha| = 2$.

We then easily obtain

$$\dot{\alpha}(\tau) = \langle \psi_{(2,0)}(\cdot), \quad \chi_{D(\tau)}(\cdot) \rangle,$$
$$\dot{\beta}(\tau) = \langle \psi_{(0,2)}(\cdot), \quad \chi_{D(\tau)}(\cdot) \rangle$$

and
$$R_\tau = AR + \chi_{D(\tau)} - \psi_{(2,0)} \langle \psi_{(2,0)}, \chi_{D(\tau)} \rangle - \psi_{(0,2)} \langle \psi_{(0,2)}, \chi_{D(\tau)} \rangle.$$

By assumption $|\alpha(\tau)| + |\beta(\tau)| \to 0$ as $\tau \to \infty$. Then

$$\alpha(\tau) = -\int_\tau^{+\infty} \langle \psi_{(2,0)}(\cdot), \quad \chi_{D(s)}(\cdot) \rangle\, ds, \tag{3.10a}$$

$$\beta(\tau) = -\int_\tau^{+\infty} \langle \psi_{(0,2)}(\cdot), \quad \chi_{D(s)}(\cdot) \rangle\, ds. \tag{3.10b}$$

An important role in the mechanism of singularity formation is played by some travelling waves that yield the matching mechanism. Let us consider the equation for such waves for (1.1) with speed (1/2):

$$\Delta \bar{G} - \frac{1}{2}\frac{\partial \bar{G}}{\partial \xi_1} = H(\bar{G}). \tag{3.11}$$

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Equation (3.11) may be solved integrating as in (1.7) the classical Ivantsov solution for the Stefan problem (cf. [I], [P]). The corresponding computation, which is carried out in detail in the appendix, yields

$$\bar{G}(\xi) = \left(\frac{1}{2} - \frac{K}{4}\right)\xi_2^2 - K\xi_1 + \gamma + o(1) \tag{3.12}$$

as $r \to \infty$, where $\xi = r(\cos(\theta), \sin(\theta))$, $\theta \in (0, 2\pi)$; the convergence is uniform in compact sets of $(0, 2\pi)$, $K \in (0, 2)$, and γ is a number depending on K. Moreover, the solutions $G(\xi)$ described above have a vanishing set Σ contained in a parabola. More precisely, Σ is the set of points where $\bar{G}(\xi) = 0$ and

$$\Sigma = \left\{\xi \in \mathbb{R}^2 \colon K\xi_1 \leqslant \left(\frac{1}{2} - \frac{K}{4}\right)\xi_2^2 + \gamma\right\}.$$

Let us denote by $a(\tau)$ the smallest value of y_1 such that there are points $(y_1, y_2) \in D(\tau)$. There is an important relationship between $a(\tau)$, and $\alpha(\tau)$, namely

$$A\psi_{(0,2)}(a(\tau), 0) \sim \alpha(\tau)\,\psi_{(2,0)}(a(\tau), 0). \tag{3.13}$$

The reason for (3.13) is the following. If (3.13) is not true then in a neighbourhood of the tip $(a(\tau), 0)$, Φ would be strictly positive or negative. In either case, this would imply a contradiction with the definition of $(a(\tau), 0)$ and there would be a very fast adjustment of the free boundary in order to make (3.13) hold.

We need to compute the scalar products in (3.10). Let us assume that $D(\tau)$ is very close to the line given by $(a(\tau), +\infty)$. Assuming that $a(\tau) \to \infty$ as $\tau \to \infty$ we can expect $\Delta\Phi$ and $y\nabla\Phi/2$ to be the largest terms near the tip of $D(\tau)$. A standard dominated balance argument suggests the introduction of the following new scales:

$$\Phi(y, \tau) = \left(\frac{1}{a(\tau)}\right)^2 G(\xi, \tau) \tag{3.14}$$

where
$$\xi = a(\tau)(y - a(\tau)\,e), \quad e = (1, 0). \tag{3.15}$$

With this change of variables (3.3) becomes

$$\frac{1}{a^2}G_\tau - \frac{2\dot{a}}{a^3}G + \frac{\dot{a}}{a^3}\xi\nabla_\xi G - \frac{\dot{a}}{a}G_\xi = \Delta_\xi G - \frac{1}{2}\left(ae + \frac{\xi}{a}\right)\cdot\frac{1}{a}\nabla_\xi G + \frac{1}{a^2}G - H(G). \tag{3.16}$$

Notice that $a(\tau) \to \infty$ as $\tau \to \infty$. Taking into account that the dynamic is driven by a zero eigenvalue, we can naturally expect that $|\dot{a}|/|a| \to 0$ as $\tau \to \infty$.

In any case, to lower order, (3.16) becomes, as $\tau \to \infty$

$$\Delta_\xi G = \frac{1}{2}\frac{\partial G}{\partial \xi_1} + H(G). \tag{3.17}$$

We now match the asymptotic behaviour of Φ given by (3.6), (3.8) with the region near $y \sim a(\tau)\,e$ described by (3.16) and (3.17) to lowest order.

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Taking into account (3.6), (3.8) we easily obtain that

$$\Phi(y,\tau) \sim \tfrac{1}{2}y_2^2 - A\Gamma(y_2^2-2) + \alpha(\tau)\,\Gamma(y_1^2-2)$$

as $\tau \to \infty$, and, if we introduce ξ given by (3.15), we then obtain

$$\Phi(y,\tau) \sim \frac{1}{a^2}(\tfrac{1}{2}\xi_2^2 - A\Gamma\xi_2^2) + 2A\Gamma + \alpha(\tau)\,\Gamma(a^2-2) + 2\alpha(\tau)\,\Gamma\xi_1 + \Gamma\alpha(\tau)\frac{\xi_1^2}{a^2} - 2\Gamma\alpha(\tau) \quad (3.18)$$

as $\tau \to \infty$. In view of (3.13) we can cancel the second and the third term in (3.18). On the other hand (3.13) means asymptotically that

$$2A \sim -\alpha(\tau)\,a^2 \quad \text{as} \quad t \to \infty. \quad (3.19)$$

We then obtain from (3.18) that

$$\Phi(y,\tau) \sim \frac{1}{a^2}[(\tfrac{1}{2} - A\Gamma)\,\xi_2^2 - 4A\Gamma\xi_1 + 4A\Gamma]$$

as $\tau \to \infty$ or, in another words, the function $G(\xi,\tau)$ defined in (3.14) satisfies a matching condition

$$G(\xi,\tau) \sim (\tfrac{1}{2} - A\Gamma)\,\xi_2^2 - 4A\Gamma(\xi_1 - 1) \quad (3.20)$$

as $|\xi| \to \infty$ (except in the region $|\xi_2| \ll |\xi_1|$ where a different analysis is required). We can then match the asymptotic expansion (3.20) for $G(\xi,\tau)$ with the asymptotic behaviour for $\bar{G}(\xi)$ that follows from (3.12) if we take $k = 4A\Gamma$.

On the other hand, the largest contribution to the right-hand side of (3.10) for large values of s is due to the part of $D(s)$ close to $y = a(\tau)\,e$. More precisely, let us write

$$\langle \psi_\alpha(\cdot), \chi_{D(s)}(\cdot)\rangle = \int_{D(s)} \psi_\alpha(y)\,\mathrm{e}^{-\frac{|y|^2}{4}}\,d^2y \equiv J_\alpha,$$

where α may take the values $(2,0)$ and $(0,2)$.

We change variables in the integral J_α as follows:

$$y = ae + \frac{1}{a}\xi,$$

$$J_\alpha = \frac{1}{a^2}\int_{S_\sigma} \psi_\alpha\left(ae + \frac{1}{a}\xi\right)\exp\left(-\frac{a^2}{4} - \frac{\xi_1}{2} - \frac{1}{4a^2}|\xi|^2\right)d^2\xi,$$

where $S_\sigma = a(D(s) - ae) \cap \{y_1 > 0\}$. Taking into account that $a(D(s) - a(s)\,e) \sim \Sigma$ as $s \to \infty$, we readily obtain that

$$J_\alpha \sim \frac{\mathrm{e}^{-\frac{a^2}{4}}}{a^2}\psi_\alpha(ae)\int_\Sigma \mathrm{e}^{-\frac{\xi_1}{2}}d^2\xi \quad \text{as} \quad s \to \infty.$$

Making use of (3.5), we obtain

$$J_{(2,0)} \sim \gamma_k \frac{\mathrm{e}^{-a^2/4}\,H_2(a)\,H_0(y_2)}{a^2} \quad \text{as} \quad s \to \infty,$$

$$J_{(0,2)} \sim \gamma_k \frac{\mathrm{e}^{-a^2/4}\,H_0(y_1)\,H_2(0)}{a^2} \quad \text{as} \quad s \to \infty;$$

here $\gamma_k \equiv \int_\Sigma e^{-\frac{\xi_1}{2}}d^2\xi$ depends on the value of k in (3.12) that will be fixed by the value of A that characterizes the limit function $\bar{\Phi}(y)$.

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Taking into account (3.10), we then obtain

$$\alpha(\tau) \sim -2\gamma_k H_0 \Gamma \int_\tau^{+\infty} \mathrm{e}^{-\frac{(a(s))^2}{4}} ds, \quad \text{as} \quad \tau \to \infty,$$

$$\beta(\tau) \sim 4\gamma_k H_0 \Gamma \int_\tau^{+\infty} \frac{\mathrm{e}^{-\frac{(a(s))^2}{4}}}{(a(s))^2} ds, \quad \text{as} \quad \tau \to \infty$$

and by (3.19), we readily derive the integral equation

$$\frac{2A}{a^2} \sim \gamma_k H_0 \Gamma \int_\tau^{+\infty} \mathrm{e}^{-\frac{(a(s))^2}{4}} ds. \tag{3.21}$$

This yields at once the asymptotic behaviour

$$a(\tau) \sim 2\sqrt{\log(\tau)} \quad \text{as} \quad \tau \to \infty, \tag{3.22}$$

as well as

$$\alpha(\tau) \sim -\frac{A}{2\log(\tau)} \quad \text{as} \quad \tau \to \infty, \tag{3.23}$$

the term $\beta(\tau)$ being of lower order. Notice that the term $R(\cdot, \tau)$ is subdominant; in fact, we can decompose it as follows

$$R(y, \tau) = \sum_{k=0}^{1} D_k(\tau) \psi_{(0,k)}(y) + W(y, \tau),$$

where $\langle W(\cdot, \tau), \psi_{(0,k)}(\cdot) \rangle = 0$, $k = 0, 1$.

We then have that

$$D_k(\tau) = -\int_\tau^{+\infty} ds\, \mathrm{e}^{(1-\frac{k}{2})(\tau - s)} \langle \psi_{(0,k)}(\cdot), \chi_{D(s)}(\cdot) \rangle, \quad k = 0, 1$$

and taking into account (3.22) we easily obtain

$$|D_k(\tau)| \leqslant \frac{C}{\tau^{1-\delta}} \quad \text{as} \quad \tau \to \infty \quad \text{for some small} \quad \delta > 0.$$

It is readily seen that $\|R(\,, \tau)\|$ has a similar power-like rate of decay:

$$\|R(\,, \tau)\| \leqslant \frac{C}{\tau^{(1-\delta)}} \quad \text{as} \quad \tau \to \infty \quad \text{for some small} \quad \delta > 0.$$

We now need to consider the profile of the domain $D(\tau)$ for $y_1 > a(\tau)$. To this end, we introduce a new set of variables

$$\eta_1 = \frac{y_1}{a(\tau)}, \tag{3.24a}$$

$$\eta_2 = y_2, \tag{3.24b}$$

$$\Phi(y, \tau) = Q(\eta, \tau). \tag{3.25}$$

Equation (2.3) becomes

$$Q_\tau - \frac{\dot{a}}{a} \eta_1 \frac{\partial Q}{\partial \eta_1} = \frac{1}{(a(\tau))^2} \frac{\partial^2 Q}{\partial \eta_1^2} + \frac{\partial^2 Q}{\partial \eta_1^2} - \frac{\eta_1}{2} \cdot \frac{\partial Q}{\partial \eta_1} - \frac{\eta_2}{2} \frac{\partial Q}{\partial \eta_2} + Q - H(Q).$$

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To lowest order, we then expect for $\eta_1 > 0$ and $\tau \to \infty$ that

$$\frac{\eta_1}{2}\cdot\frac{\partial Q}{\partial \eta_1} = \frac{\partial^2 Q}{\partial \eta_2^2} - \frac{\eta_2}{2}\cdot\frac{\partial Q}{\partial \eta_2} + Q - H(Q), \quad \eta_1 > 1 \tag{3.26}$$

with the matching condition

$$Q(1, \eta_2) = (\tfrac{1}{2} - A\Gamma)\,\eta_2^2, \tag{3.27}$$

where to obtain (3.27) we have used the lowest order terms in the expansion (3.18). We will prove in an Appendix that the solution of (3.26) is unique and is given by

$$Q(\eta_1, \eta_2) = (\eta_1^2 - 1)\, M\left(\frac{\eta_2}{\sqrt{\eta_1^2 - 1}}\right), \quad \lambda = \frac{\eta_2}{\sqrt{\eta_1^2 - 1}}, \tag{3.28}$$

where M is the unique solution of the problem

$$M_{\lambda\lambda} + \tfrac{1}{2}\lambda M_\lambda - M - H(M) = 0, \tag{3.29}$$

such that $M \in C^1(\mathbb{R})$, and M satisfies

$$M(\lambda) \sim (\tfrac{1}{2} - A\Gamma)\,\lambda^2 \quad \text{as} \quad |\lambda| \to \infty. \tag{3.30}$$

It is also proved in the appendix that M vanishes for $|\lambda| \leqslant \lambda_0(A)$, where $\lambda_0(A) > 0$ and

$$\lambda_0(A) \to 0 \quad \text{if} \quad A \to 0, \quad A > 0 \tag{3.31a}$$

$$\lambda_0(A) \to \infty \quad \text{if} \quad A \to \left(\frac{1}{2\Gamma}\right), \quad A < \frac{1}{2\Gamma}. \tag{3.31b}$$

For $\tau \to \infty$ the asymptotic behaviour of $D(\tau)$ would be

$$|\eta_2| \leqslant \lambda_0(A)\sqrt{\eta_1^2 - 1}$$

or taking into account (3.24) we deduce that

$$D(\eta) \sim \left\{ y : |y_2| \leqslant \lambda_0(A)\frac{\sqrt{y_1^2 - (a(\tau))^2}}{(a(\tau))} \right\} \quad \text{as} \quad \tau \to \infty. \tag{3.32}$$

Notice that (3.31) is consistent with our previous assumption that $D(\tau)$ approaches the line $y_2 = 0$, $y_1 \in (a(\tau), +\infty)$ as $\tau \to \infty$.

We now describe the shape of the cusp once it has developed. We fix $\bar{\tau}$ sufficiently large and $|y| > B\,a(\bar{\tau})$ for $B \gg 1$ large enough. Then, the shape of the region where $u = 0$ is given by

$$|y_2| \leqslant \lambda_0(A)\frac{|y_1|}{a(\bar{\tau})}.$$

In this set, the evolution of the free boundary is rather slow and we then assume that the region where $u = 0$ is basically frozen until $t = t^*$. Then the free boundary remains in the region

$$\left\{ |x_2| \leqslant \lambda_0(A)\frac{|x_1|}{a(\bar{\tau})} \right\},$$

where

$$|a(\bar{\tau})| \sim \frac{|y_1|}{B} = \frac{|a(\bar{\tau})|\, e^{+\bar{\tau}/2}}{B}|x_1|.$$

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One then approximately has that

$$\bar{\tau} \sim -2\log(|x_1|) \quad \text{if} \quad |x_1| \to 0.$$

(Notice that the particular choice of B is irrelevant if it is large enough.) Then the profile of the cusp as $(x_1) \to 0$ is given by

$$|x_2| \sim \frac{\lambda_0(A)}{2} \cdot \frac{(x_1)_+}{\sqrt{\log(|\log|x_1||)}} \quad \text{as} \quad |x_1| \to 0. \tag{3.33}$$

Finally, we rewrite all the previous results in terms of the solutions of the original Stefan problem. We find $c(x,t)$ by means of (2.5). Taking into account (3.1) we then obtain

$$u(x,t) = -\Phi + \tfrac{1}{2} y \nabla \Phi + \Phi_\tau. \tag{3.34}$$

To lowest order

$$\Phi(y,\tau) \sim (\tfrac{1}{2} - A\,\Phi)\, y_2^2 + 2\Gamma A$$

as $\tau \to \infty$ at least in the region

$$\Omega_{\epsilon_0}(\{|y_2| \geqslant \epsilon_0 y_1\} \cup \{y_1 \leqslant (1-\epsilon_0)\, a(\tau)\}) \cap \{x : |x| \text{ small}\},$$

where $\epsilon_0 > 0$ is an arbitrarily small number. Then, by (2.5) we obtain that, in the region Ω_{ϵ_0},

$$u(x,t) \sim -2\Gamma A \quad \text{as} \quad |x| \to 0, \quad t \uparrow t^*, \quad \text{where} \quad 2\Gamma A \in (0,1). \tag{3.35}$$

Notice that the undercooling that generates the cusp may be arbitrarily small, a possibility indicated in [LO]. The amount of undercooling with this cusp-generation mechanism is smaller than 1. On the other hand, in the one-dimensional case the amount of undercooling at the singularly is exactly 1 as indicated above (cf. [HV1]).

The expansion (3.35) ceases to be valid near the free boundary. For $y \sim a(\tau)\, e$, we can use (3.14) and (3.34) to obtain

$$u(x,t) \sim \frac{1}{2} \cdot \frac{\partial \bar{G}}{\partial \xi_1} \quad \text{as} \quad \tau \to \infty \quad \text{for} \quad |\xi| \sim 1, \tag{3.36}$$

and taking into account (3.12) with $k = 4A\Gamma$ we obtain that (3.35) matches with (3.36) for $|\xi| \to \infty$. Finally, for $\eta_1 > 1$, $|\eta_2| = |y_2| \sim 1$ and the behaviour of Φ to the lowest order is given by (3.25), (3.28). Using also (3.26) and (3.34), we obtain, in the region under consideration,

$$u(x,t) \sim \frac{\partial^2 Q}{\partial \eta_2^2} - H(Q) \sim \left(\frac{\partial^2 M}{\partial \lambda^2}\right)\left(\frac{\eta_2}{\sqrt{\eta_1^2 - 1}}\right) - H(M). \tag{3.37}$$

Then $c(x,t)$ disappears at the free boundary, which behaves as

$$|\eta_2| \sim \lambda_0(A)\, \sqrt{\eta_1^2 - 1}\,;$$

on the other hand for $|\eta_2| \to \infty$ and η_1 fixed, we recover (3.35) and again we obtain a correct matching.

Remark The cusp-generating solutions for the Stefan problem described above have an arbitrary undercooling in the interval $(0,1)$. Notice that (3.31) implies that the opening of the cusp becomes much smaller if the undercooling approaches zero. If, on the contrary, the undercooling approaches 1, the opening of the cusp becomes larger, a situation that could be expected if we take into account that for such undercooling we should approximate the one-dimensional situation studied in [HV1], where cusps become planes.

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3.2 The three-dimensional case

The argument is very similar to the two-dimensional case. We shall just sketch it, and remark only on the main differences whenever they arise.

We introduce self-similar variables exactly as in (3.1), (3.2). The operator B might be defined as in (3.4) with straightforward modifications. The eigenfunctions are now

$$\left.\begin{array}{c} \psi_\alpha(y) = H_{\alpha_1}(y_1)\, H_{\alpha_2}(y_2)\, H_{\alpha_3}(y_3) \\ y = (y_1, y_2, y_3), \quad \alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \alpha_i = 0, 1, 2, \ldots, \quad i = 1, \ldots, 3. \end{array}\right\} \tag{3.38}$$

Then

$$B\psi_\alpha = \left(1 - \frac{|\alpha|}{2}\right)\psi_\alpha, |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

We now assume that

$$\Phi(y,\tau) \to \tfrac{1}{2}(y_2^2 + y_3^2) - A_2\,\psi_{(0,2,0)}(y) - A_3\,\psi_{(0,0,2)}(y) \equiv \Phi(y),$$

where $2\Gamma(A_2 + A_3) < 1$, and $\Gamma = 1/8\pi\sqrt{2}$. Then $\Phi(y) > 0$ and

$$\bar{\Phi}(y) \sim (\tfrac{1}{4} - \Gamma A_2)\, y_2^2 + (\tfrac{1}{4} - \Gamma A_3)\, y_3^2 \quad \text{as} \quad |y_2| + |y_3| \to \infty.$$

We define $\psi(y,\tau)$ as in (3.6). We now expect $D(\tau)$ to be concentrated near the segment $y_2 = y_3 = 0$, $y_1 \in (a(\tau), +\infty)$. Arguing exactly as in the two-dimensional case, we would obtain

$$\psi(y,\tau) = \alpha(\tau)\,\psi_{(2,0,0)}(y) + \beta_2(\tau)\,\psi_{(0,2,0)}(y) + \beta_3(\tau)\,\psi_{(0,0,2)}(y) + R(y,\tau)$$

where

$$\alpha(\tau) = -\int_\tau^{+\infty} < \psi_{(2,0,0)}(\cdot), \chi_{D(s)}(\cdot) > ds, \tag{3.39a}$$

$$\beta_2(\tau) = -\int_\tau^{+\infty} < \psi_{(0,2,0)}(\cdot), \chi_{D(s)}(\cdot) > ds, \tag{3.39b}$$

$$\beta_3(\tau) = -\int_\tau^{+\infty} < \psi_{(0,0,2)}(\cdot), \chi_{D(s)}(\cdot) > ds. \tag{3.39c}$$

The structure of the tip is given in this case by the Ivantsov-type translating solutions obtained in [HC], where the translating front is an ellipsoidal paraboloid. These solutions are recalled in the appendix.

Instead of (3.13) we now need to impose the condition

$$A_2\,\psi_{(0,2,0)}(a(\tau),0,0) + A_3\,\psi_{(0,0,2)}(a(\tau),0,0) \sim \alpha(\tau)\,\psi_{(2,0,0)}(a(\tau),0,0). \tag{3.40}$$

Asymptotically, this reads as

$$2(A_2 + A_3) \sim -\alpha(\tau)\,(a(\tau))^2 \quad \text{as} \quad \tau \to \infty. \tag{3.41}$$

We then select G as in (3.14), and define ξ in a similar way. One then obtains the matching condition

$$G(\xi,\tau) \sim (\tfrac{1}{4} - \Gamma A_2)\,\xi_2^2 + (\tfrac{1}{4} - \Gamma A_3)\,\xi_3^2 - 4(A_2 + A_3)\,\Gamma(\xi_1 - 1) \tag{3.42}$$

as $|\xi| \to \infty$: thus we can match with the solution $\bar{G}(\xi)$ obtained from the Ivantsov–Horvay–Cahn translating paraboloids (cf. (A 29) in the appendix).

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Arguing exactly as in the two-dimensional case, we thus obtain the integral equation

$$\frac{2(A_2+A_3)}{a^2} \sim \gamma_k H_0 \Gamma \int_\tau^{+\infty} \mathrm{e}^{-\frac{(\alpha(\xi))}{4}} ds, \tag{3.43}$$

where $\gamma_k = \int_\Sigma \mathrm{e}^{-\xi_1/2}\, d^3\xi$, and where Σ is now the Ivantsov–Horvay–Cahn paraboloid. It is then readily shown that (3.22) holds and that

$$\alpha(\tau) \sim -\frac{(A_2+A_3)}{2\log(\tau)} \quad \text{as} \quad \tau \to \infty. \tag{3.44}$$

The proof of the fact that R is subdominant is completely analogous to the two-dimensional one. Finally, the structure of the region where $u = 0$ for $y_1 > a(\tau)$ is determined from the analysis of the equation

$$\frac{\tau_1}{2}\frac{\partial Q}{\partial \eta_1} = \frac{\partial^2 Q}{\partial \eta_2^2} + \frac{\partial^2 Q}{\partial \eta_3^2} - \frac{\eta_2}{2}\frac{\partial Q}{\partial \eta_2} - \frac{\eta_3}{2}\frac{\partial Q}{\partial \eta_3} + Q - H(Q), \quad \eta_1 > 1 \tag{3.45}$$

with the matching condition

$$Q(1, \eta_2, \eta_3) = (\tfrac{1}{4} - A_2\Gamma)\,\eta_2^2 + (\tfrac{1}{4} - A_3\Gamma)\,\eta_3^2. \tag{3.46}$$

As in the two-dimensional case, the function Q may be obtained as

$$Q(\eta_1, \eta_2, \eta_3) = (\eta_1^2 - 1)\, M\left(\frac{\eta_2}{\sqrt{\eta_1^2-1}}, \frac{\eta_3}{\sqrt{\eta_1^2-1}}\right),$$

$$\lambda_2 = \frac{\eta_2}{\sqrt{\eta_1^2-1}}, \quad \lambda_3 = \frac{\eta_3}{\sqrt{\eta_1^2-1}},$$

where M solves the problem

$$\frac{\partial^2 M}{\partial \lambda_2^2} + \frac{\partial^2 M}{\partial \lambda_3^2} + \frac{1}{2}\lambda_2\frac{\partial M}{\partial \lambda_2} + \frac{1}{2}\lambda_3\frac{\partial M}{\partial \lambda_3} - M - H(M) = 0 \tag{3.47}$$

and M has the asymptotic behaviour

$$M(\lambda) \sim (\tfrac{1}{4} - A_2\Gamma)\,\lambda_2^2 + (\tfrac{1}{4} - A_3\Gamma)\,\lambda_2^3 \quad \text{as} \quad |\lambda| \to \infty. \tag{3.48}$$

The proof of the existence and uniqueness of solutions of (3.47), (3.48) is as in the two-dimensional case. We denote the region where M vanishes as O_{A_2, A_3}. We can then analyse the shape of the cusp exactly as in the two-dimensional case to obtain the profile

$$(x_2, x_3) \sim \frac{1}{2}\frac{(x_1)_+}{\sqrt{\log(|\log|x_1||)}} \cdot \partial O_{A_2, A_3} \quad \text{as} \quad |x_1| \to 0. \tag{3.49}$$

Finally, in the original variables

$$u(x, t^*) \to -2\Gamma(A_1 + A_2) \quad \text{as} \quad |x| \to 0, \tag{3.50}$$

if $\sqrt{x_2^2 + x_3^2} \geqslant \epsilon_0 x_1$, where $2\Gamma(A_1 + A_2) \in (0, 1)$, and $\epsilon_0 > 0$ is arbitrarily small.

Notice that $c(x, t^*) = 0$ at the free boundary, the transition from this behaviour to (3.50) taking place in the region

$$\sqrt{x_2^2 + x_3^2} \sim \frac{(x_1)_+}{\sqrt{\log(|\log|x_1||)}} \quad \text{as} \quad |x_1| \to 0.$$

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4 Concluding remarks

In this paper we have formally described a mechanism of cusp development, starting from smooth solutions, for the classical Stefan problem with undercooling in the two- and three-dimensional cases. A remarkable feature is that the phenomenon of cusp formation – in non-dimensional units – can take place with an arbitrary local undercooling between 0 and 1. In particular, it is possible to have singularities for the Stefan problem with arbitrarily small undercoolings, a possibility that was indicated in [LO]. There is also a clear contrast between the mechanisms described here and the one-dimensional situation considered in [HV1], where the undercooling at the point of formation of the singularity is exactly 1.

It would be interesting to obtain rigorous proofs of the results in §3. By analogy with other singularity formation problems, such as an analysis should be feasible by using topological arguments as in [B], [HV2]. There is, however, a technical difficulty, namely the lack of a clear regularity theory for the ill-posed Stefan problem. Such a theory has been developed by Caffarelli (cf. [C], [F]), for the well-posed Stefan problem. In a very schematic way the results in [C] may be stated as saying that flatness of the free boundary implies regularity, where flatness is understood in a weak measure-theoretical way. The remarks in §2 clearly suggest that an analogous 'flatness implies regularity' statement should be true also for the ill-posed case, but no proof of such a fact is provided here.

Finally, there are remarkable differences between the solutions obtained in this paper as compared with the solutions of the Hele–Shaw problem that are known to develop cusps. In particular, in two space dimensions, there exist solutions of Hele–Shaw flow developing cusps with the form $(x_2) \sim c(x_1)_+^{3/2}$ at the interface. In this sense it would be interesting to understand the asymptotic limit of zero specific heat for the Stefan problem (i.e. replacing c_t by Dc_t in (1.1) and making $D \to 0$). The solutions of (1.1), (1.4) should converge to the solutions of Hele–Shaw and for solutions generating cusps with the 3/2-power there would be most likely a transition region connecting the 3/2-power behaviour with the behaviour

$$|x_2| \sim \frac{(x_1)_+}{\sqrt{\log(|\log|x_1||)}}$$

described here.

Appendix

A1 Analysis of the problem (3.26), (3.27)

Let us assume that $A\Gamma \in (0, 1/2)$. Existence and uniqueness of solutions for (3.26), (3.27) follows from standard parabolic theory. We can then obtain Q as indicated in (3.28) with M as in (3.29), (3.30). Notice that

$$w(x,t) = t^{1/2} M\left(\frac{x}{\sqrt{t}}\right) \tag{A 1}$$

solves the initial value problem

$$w_t = w_{xx} - H(w) \tag{A 2}$$

with initial condition

$$w(x,0) = (\tfrac{1}{2} - A\Gamma)\, x^2 \equiv \gamma x^2, \quad \gamma \in (0, \tfrac{1}{2}). \tag{A 3}$$

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By comparison with the supersolution

$$\tilde{w}(x,t) = \tfrac{1}{2}(1-\delta)(x-x_0)^2 + \delta(M-t) \tag{A 4}$$

with $\frac{1}{2}(1-\delta) > \gamma$, $0 < \delta < 1$, $|x_0| < r$, $r > 0$, and M large enough such that

$$\tilde{w}(x,0) \geqslant w(x,0) \quad \text{for any} \quad x \in \mathbb{R}^2 \tag{A 5}$$

we then readily obtain that $w(x,t) = 0$ for $t = 1$. Then, the parameter $\lambda_0(A)$ defined in §3 satisfies $\lambda_0(A) \geqslant r > 0$.

Moreover, if $\gamma \to 0$, condition (A 5) transforms into

$$\tfrac{1}{2}x^2 - x\,x_0 + \tfrac{1}{2}x_0^2 \geqslant 0 \tag{A 6}$$

and then we can obtain r arbitrarily large, whence $\lambda_0(A) \to 0$ if $A \to 1/(2\Gamma)$, $A < 1/(2\Gamma)$ as stated in §3. If $A \to 0$, $w(x,t) \to \frac{1}{2}x^2$, and then $\lambda_0(A) \to 0$ by standard continuous dependence results for parabolic equations.

A2 Ivantsov-type solutions

For convenience, we list below some properties of the solutions of (1.1)–(1.4) for which the free boundary is a translating paraboloid. Such solutions were obtained by Ivantsov in a classical paper (cf. [I] and also [P]), in the case of rotationally symmetric paraboloids, and by Horvay & Cahn [HC] in the case of ellipsoidal paraboloids. For convenience, we distinguish between two-dimensional and three-dimensional cases.

(i) Two-dimensional case

We look for solutions of (1.1), (1.3), (1.4) with the particular form

$$u(x,t) = \Phi\left(x + \frac{e}{2}t\right) = \Phi(\xi), \tag{A 7}$$

where $x = (x_1, x_2)$, $e = (1,0)$. Moreover, let us assume that in the coordinate system that moves with the free boundary this one is given as $\xi_1 = l(\xi_2)$. Then (Φ, l) solve the problem

$$\frac{1}{2}\frac{\partial \Phi}{\partial \xi_1} = \Delta \Phi \quad \mathbb{R}^2 \backslash D, \tag{A 8}$$

$$\Phi = 0 \quad \text{in} \quad \partial D, \tag{A 9}$$

$$\frac{1}{2} = \frac{\partial \Phi}{\partial \xi_1} - l'(\xi_2)\frac{\partial \Phi}{\partial \xi_2} \quad \text{in} \quad \partial D. \tag{A 10}$$

Arguing as in [P], we try to obtain a solution of (A 8)–(A 10) in the form

$$\Phi = W(r - \xi_1), \quad \text{where} \quad r = \sqrt{\xi_1^2 + \xi_2^2},$$

and where $\eta = r - \xi - 1$. After some straightforward computations, we obtain from (A 8) that W solves

$$-\tfrac{1}{2}\eta\, W'(\eta) = 2\eta\, W''(\eta) + W'(\eta). \tag{A 11}$$

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Suppose that the free boundary is at the region $n = A$. Then

$$l(\xi_2) = \frac{\xi^2}{2A} - \frac{A}{2},$$

and (A 10) becomes

$$W'(A) = -\tfrac{1}{4}. \tag{A 12}$$

Finally, (A 9) implies

$$W(A) = 0. \tag{A 13}$$

Solving (A 11)–(A 13), we readily obtain

$$W(\eta) = -\frac{A^{1/2}}{4} \int_A^\eta \frac{e^{(1/4)(A-u)}}{u^{1/2}} \cdot du. \tag{A 14}$$

A point $x = (x_1, x_2)$ is reached by the boundary at the value of t defined by

$$\sqrt{(x_1 + \tfrac{1}{2}t)^2 + x_2^2} - (x_1 + \tfrac{1}{2}t) = A.$$

Thus, the free boundary is given by

$$t = T(x) = 2(x_2^2 - A^2 - 2Ax_1). \tag{A 15}$$

We define $u(x, t)$ as in (2.7). Then c satisfies (2.2) and

$$\begin{aligned} c(x,t) &= -\int_t^{T(x)} u(x,\lambda)\, d\lambda \\ &= -\int_t^{T(x)} W(\sqrt{(x_1 + \tfrac{1}{2}\lambda)^2 + x_2^2} - x_1 - \tfrac{1}{2}\lambda)\, d\lambda \end{aligned} \tag{A 16}$$

and a simple change of variables then yields $c(x, t) = \bar{G}\left(x + \frac{e}{2}t\right)$, where

$$\bar{G}(\xi_1, \xi_2) = -\int_A^{\sqrt{\xi_1^2 + \xi_2^2} - \xi_1} \left(\frac{\xi_2^2}{u^2} + 1\right) W(u)\, du \tag{A 17}$$

satisfies (3.11) in the two-dimensional case. Using (A 14) in (A 17), we obtain after integration by parts

$$\bar{G}(\xi) = J_2(\Omega; A)\, \xi_2^2 - 2J_1(\Omega; A)\, \xi_1 - J_0(\Omega; A),$$

where

$$J_n(\Omega; A) = \frac{A^{1/2}}{4} \int_A^\Omega \eta^{1/2 - n}\, e^{1/4(A-\eta)}\, d\eta, \quad n = 0, 1, 2 \ldots$$

and $\Omega = \sqrt{\xi_1^2 - \xi_2^2} - \xi_1$.

We then obtain the following asymptotic behaviour for $\bar{G}(\xi)$ as $|\xi| \to \infty$, $\Omega \to +\infty$:

$$\bar{G}(\xi) \sim J_2(\infty; A)\, \xi_2^2 - 2J_1(\infty; A)\, \xi_1 - J_0(\infty, A) \tag{A 18}$$

Let us define

$$I(A) = J_2(\infty; A) = \frac{A^{1/2}}{4} \int_A^{+\infty} \frac{e^{(1/4)(A-\eta)}}{\eta^{3/2}}\, d\eta.$$

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Integration by parts then yields

$$J_1(\infty, A) = (1-2I(A)),$$
$$J_0(\infty, A) = A-2(1-I(A)).$$

The function $I(A)$ decreases monotonically from $I(0) = 1/2$ to $I(\infty) = 0$. This yields the asymptotics (3.12).

(ii) Three-dimensional case

Following [HC], we look for solutions of (1.1)–(1.4) with the particular form

$$u(x, t) = \Phi(\omega), \tag{A 19}$$

where ω is defined as

$$\frac{x_1^2}{\omega}+\frac{x_3^2}{B+\omega}-2\,(x_1+\tfrac{1}{2}t) = \omega, \tag{A 20}$$

and $B \geqslant 0$. It is easily seen that for each (x_1, x_2, x_3, t) there is a unique positive ω satisfying (A 20). Following [HC], we would obtain after some lengthy computations

$$2\Phi''(\omega)+\Phi'(\omega)\left\{\frac{1}{2}+\frac{1}{\omega}+\frac{1}{B+\omega}\right\} = 0. \tag{A 21}$$

Suppose that the free boundary is at $\omega = A$. Then (1.3) implies that

$$\Phi(A) = 0 \tag{A 22}$$

and (1.4), (A 20) yield

$$\Phi'(A) = -\tfrac{1}{4}. \tag{A 23}$$

The solution of (A 21)–(A 23) is

$$\Phi(\omega) = -\frac{1}{4}\int_A^\omega \sqrt{\frac{A(A+B)}{\eta(\eta+B)}}\,\mathrm{e}^{(1/4)(A-\eta)}\,d\eta, \quad \omega \geqslant A. \tag{A 24}$$

The free boundary may be written as

$$t = T(x) = \frac{x_2^2}{A}+\frac{x_3^2}{B+A}-2x_1-A \tag{A 25}$$

and using (2.7), (A 19) we compute $c(x, t)$ as follows:

$$c(x, t) = -\int_t^{T(x)} c(x, \lambda)\,d\lambda = -\int_t^{T(x)} \Phi(\omega(x, \lambda))\,d\lambda,$$

where $\omega(x, \lambda)$ is defined by (A 20) with t replaced by λ there. We now use ω in (A 20) as variable of integration, whence

$$c(x, t) = \int_{\omega(x,t)}^A \Phi(\omega)\left(\frac{x_1^2}{\omega^2}+\frac{x_3^2}{(B+\omega)^2}+1\right)d\omega.$$

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Taking into account that $\omega(x,t) = \zeta(x_1, x_2, x_3 + \frac{1}{2}t)$, where $\zeta = \zeta(\xi_1, \xi_2, \xi_3)$ is defined as

$$\frac{\xi_1^2}{\zeta} + \frac{\xi_3^2}{B+\zeta} - 2\xi_1 = \zeta,$$

we readily obtain $c(x,t) = \bar{G}(x + \frac{1}{2}et)$, $e = (1,0,0)$,

$$\bar{G}(\xi) = \int_{\zeta(\xi)}^{A} \Phi(\omega) \left(\frac{\xi_1^2}{\omega^2} + \frac{\xi_2^2}{(B+\omega)^2} + 1 \right) d\omega. \tag{A 26}$$

Notice the strong analogy between (A 17) and (A 26).

Substituting (A 24) into (A 26) we obtain after some integrations by parts that

$$\bar{G}(\xi) = J_{2,1}(\zeta(\xi), A)\, \xi_1^2 + J_{1,2}(\zeta(\xi), A)\, \xi_2^2 - 2J_{1,1}(\zeta(\xi), A)\, \xi_3 - J_{0,1}(\zeta(\xi), A), \tag{A 27}$$

where

$$J_{n,m}(\zeta, A) = \frac{\sqrt{A(A+B)}}{4} \int_A^{\zeta} \eta^{1/2-n} (\eta + B)^{1/2-m}\, e^{1/4(A-\eta)}\, d\eta. \tag{A 28}$$

The asymptotic behaviour of $\bar{G}(\xi)$ as $|\xi| \to \infty$, $\zeta(\xi) \to \infty$ is given by

$$\bar{G}(\xi) = J_{2,1}(\infty, A)\, \xi_1^2 + J_{1,2}(\infty, A)\, \xi_3^2 - 2J_{1,1}(\infty, A)\, \xi_1 - J_{0,1}(\infty, A) + o(1) \quad \text{as} \quad |\xi| \to \infty, \quad \zeta(\xi) \to \infty. \tag{A 29}$$

It is easily seen that $J_{2,1}(\infty, A) > 0$, $J_{1,2}(\infty, A) > 0$, and

$$2J_{2,1}(\infty, A) + 2J_{1,2}(\infty, A) < 1.$$

Notice that, in order to make the matching with (3.42), we need to choose A, B in such a way that

$$J_{2,1} = (\tfrac{1}{4} - \Gamma A_2), \quad \zeta = +\infty,$$
$$J_{1,2} = (\tfrac{1}{4} - \Gamma A_3), \quad \zeta = +\infty,$$
$$J_{1,1} = 2\Gamma(A_2 + A_3), \quad \zeta = +\infty$$

or equivalently

$$J_{1,1} + 2(J_{1,2} + J_{2,1}) = 1, \quad \zeta = +\infty. \tag{A 30}$$

Formula (A 30) is equivalent to proving

$$\int_A^{+\infty} \left\{ \frac{1}{\eta^{1/2}(\eta+B)^{1/2}} + \frac{2}{\eta^{3/2}(\eta+B)^{1/2}} + \frac{2}{\eta^{1/2}(\eta+B)^{3/2}} \right\} \mathrm{e}^{-1/4\eta}\, d\eta = \frac{4\,\mathrm{e}^{-1/4A}}{\sqrt{A(A+B)}}. \tag{A 31}$$

To show (A 31) we just differentiate both sides with respect to A, to arrive at an identity. Then the left- and the right-hand sides of (A 31) differ by a constant that turns out to be zero by taking the limit $A \to \infty$. This concludes the proof of (A 30).

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