J. Appl. Prob. **53,** 938–945 (2016)
*doi:*10.1017/jpr.2016.52
© *Applied Probability Trust* 2016

UNIQUENESS OF THE NULL SOLUTION TO A NONLINEAR PARTIAL DIFFERENTIAL EQUATION SATISFIED BY THE EXPLOSION PROBABILITY OF A BRANCHING DIFFUSION

K. BRUCE ERICKSON,* *University of Washington*

Abstract

The explosion probability before time t of a branching diffusion satisfies a nonlinear parabolic partial differential equation. This equation, along with the natural boundary and initial conditions, has only the trivial solution, i.e. explosion in finite time does not occur, provided the creation rate does not grow faster than the square power at ∞.

Keywords: Quasi-linear; branching diffusion; Bessel; probability of explosion

2010 Mathematics Subject Classification: Primary 60J80
Secondary 35A02; 35K59; 60K99

1. Introduction

The probability, $u = u(t, x)$, of explosion before time t of a branching diffusion process on an interval I initiated at place x satisfies a nonlinear equation of the form

$$\frac{\partial u}{\partial t} = Au + q(1 - u - F[\cdot, 1 - u]), \qquad t > 0,\ x \in I^{\mathrm{o}},\ u(0+, \cdot) \equiv 0,\ 0 \le u \le 1,$$

where A denotes the local generator of the diffusion, F is the generating function for the offspring probabilities, and $q = q(x)$ is the particle creation rate. (See Section 2 for a quick derivation of the equation.) It is therefore of some interest to determine when $u \equiv 0$ is the only solution. In this paper we establish this conclusion in a special case in which A is a Bessel operator, $I = [0, \infty)$, and for the mean offspring times the creation rate does not grow too rapidly.

We make use of the following items and assumptions.

(i) For each x, $\{f_n(x); n = 0, 2, 3, \ldots\}$ is a probability distribution on the integers $0, 2, 3, \ldots$. (Measurability assumptions will go unstated.)

(ii) $F[x, z] = \sum_{n \ne 1} f_n(x) z^n$.

(iii) $M(x) = \partial F/\partial z|_{z=1} = 2f_2(x) + 3f_3(x) + \cdots$, the mean number of offspring produced by any particle that dies at x. We assume that this is finite and continuous. Note that $M(x) \ge 1 - f_0(x)$.

(iv) $q(x)$ is a nonnegative continuous function *not necessarily bounded*.

(v) $u = u(t, x)$ is twice continuously differentiable in $x > 0$ and continuously differentiable in $t > 0$.

Received 1 October 2015; revision received 4 December 2015.
* Postal address: Department of Mathematics, University of Washington, Seattle, WA 98195, USA.
Email address: kbe@u.washington.edu

https://doi.org/10.1017/jpr.2016.52 Published online by Cambridge University Press

Theorem 1. *If $u = u(t, x)$ satisfies $t \mapsto u(t, x)$ is nondecreasing for each $x > 0$ and for some number $r \geq 1$ (not necessarily an integer),*

$$\frac{\partial u}{\partial t} = x^{-r}\frac{\partial}{\partial x}x^{r}\frac{\partial u}{\partial x} + q(x)(1 - u - F[x, 1 - u]), \qquad t > 0,\ x > 0,\ 0 \leq u \leq 1,\ u(0+, \cdot) \equiv 0, \tag{1}$$

and if

$$q(x)|M(x) - 1| = \mathrm{O}(x^2), \qquad x \to \infty, \tag{2}$$

then $u \equiv 0$.

Remark 1. (i) We call the function $q(x)(M(x) - 1)$ by the somewhat clumsy (and not entirely accurate) '*the particle creation rate at x*'. The function M is the expected number of offspring produced when a particle dies at place x and $q(x)$ represents the rate at which particles die there. Thus, $q(M-1)$ is the rate of production above population sustainability (the subtraction of 1) and this could be negative. The process we originally studied had $f_2 \equiv 1$, then $q(x)(M(x) - 1) = q(x)$ and the right-hand side of (1) can be written as $x^{-r}(\partial/\partial x)(x^r \partial u/\partial x) + q(x)(u - u^2)$.

(ii) The requirement that $t \mapsto u(t, \cdot)$ does not decrease is automatically satisfied by the probability of explosion before t and is, thus, a natural assumption. It seems almost essential to the proof.

(iii) The theorem also holds when $0 \leq r < 1$ provided u also satisfies the boundary condition

$$\lim_{x \to 0+} x^r \frac{\partial u(t, x)}{\partial x} = 0,$$

which holds automatically when $r \geq 1$. The proof of the theorem for $0 \leq r < 1$ requires a little more fiddling and is left to the reader.

(iv) The operator $A = x^{-r}\partial/\partial x(x^r \partial/\partial x)$ with $r \geq 1$ is the differential generator of a uniquely determined conservative nonsingular diffusion on $[0, \infty)$. If $0 \leq r < 1$ then A does not determine a unique process since the origin is an accessible boundary. Indeed it is both an exit and entrance in the terminology of [3]. To obtain a unique conservative process with continuous paths, the simplest idea places a *reflecting* barrier at the origin. Analytically, this means that the *scale* derivative at 0 of functions in the domain of the operator must be equal to 0 which translates to the limit of the preceding remark. See [3, Section 4.1].

(v) If $q(M - 1)$ grows faster than x^2, it is likely that explosion of the branching process in finite-time has positive probability and that (1) has a nontrivial solution. See, for example, [3, Section 5.15]. However, we do not have a proof that $q(x)[M(x) - 1]/x^\gamma \to \infty$, $x \to \infty$, for a $\gamma \geq 2$, necessarily implies explosion.

(vi) Changing the variable x yields trivial generalizations of Theorem 1. For example, if for numbers a, b, and $c > 0$ with $b \geq 1$, $a - b + 2 > 0$, and $a + b \geq 0$, the function $u = u(t, x)$, nondecreasing in t, satisfies $u_t = c^2 x^{-a}\partial_x(x^b u_x) + q(x)(1 - u - F[x, 1 - u])$ (subscripts denoting derivatives), $u(0+, \cdot) \equiv 0, 0 \leq u \leq 1$, and if $q(x)(M(x)-1) = \mathrm{O}(x^{a-b+2})$, $x \to \infty$, then $u \equiv 0$. For the proof, put $y = kx^n$ and $w(t, y) = u(t, x)$, $n = (a - b + 2)/2 > 0$, $k = (nc)^{-1}$. Then $w_t = y^{-r}\partial_y(y^r w_y) + q_*(y)(1 - w - F_*[y, 1 - w])$ with $q_*(y) = q(x)$, $F_*[y, \cdot] = F[x, \cdot]$, $x = (y/k)^{1/n}$, and $r = (b + n - 1)/n$. The bound at ∞ becomes $q_*(y)[M_*(y) - 1] = \mathrm{O}(y^2)$, and since $b \geq 1$ implies that $r \geq 1$, the conclusion follows

https://doi.org/10.1017/jpr.2016.52 Published online by Cambridge University Press

from Theorem 1. (A previous remark implies that $u \equiv 0$ when $b < 1$ if also $y^r w_y = (k^{r-1}/n)x^b u_x \to 0$ as $x \to 0+$.)

The literature on the explosion problem for various types of branching processes seems quite extensive and a survey of it in this paper will not be attempted. We emphasize that the reference to the explosion problem (and the included derivation of (1) from a branching diffusion model) is meant only as a good motive for the study of such nonlinear partial differential equations. However, such nonlinear equations have sufficient interest in their own right. See, for example, [4].

Some work most relevant to the study here in connection with the explosion problem are exemplified by [1], [2], and [5] and the references therein. In these studies various restrictions on the size of q, and/or on the domain of the process are imposed. In particular, unbounded q are not allowed. Removing the boundedness of q was (and remains) our motivation. The function k which appears in Savits [5] corresponds to the function q here. In all of Savits' theorems on explosion, k is bounded; moreover, the state-space on which the particles live is often required to be compact. On the other hand, the motion processes are much more general than the diffusion processes considered here.

2. Derivation of the equation for explosion

The following is meant to be a brief and intuitive derivation of (1). A more (nearly) rigorous derivation requires a careful description of the process among other matters. We partly follow [3, p. 210], however, and the reader may judge for him/herself the degree of rigor (here and there).

We derive the equation in the case of a one-dimensional branching process with motion, for every particle, governed by a diffusion operator of the form

$$A = D_\mu D_\sigma = x^{-r} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^r \frac{\mathrm{d}}{\mathrm{d}x}\right)$$

with state-space $[0, \infty)$. The exact form of A is not crucial, however.

Each particle has a path-dependent alarm clock (or lifetime). When the alarm rings the particle dies but is instantly replaced by $0, 2, 3, \ldots$, *independent* particles at that spot (with probabilities f_n) each of which engenders its own branching process. If τ represents a generic particle's lifetime (or split time) as determined by the clock, then the conditional probability that τ exceeds t given the path $\{\mathrm{X}_s; s \le t\}$ has the form

$$\mathbb{P}_x[\tau > t \mid \mathcal{B}(\mathrm{X}_s,\ s < \infty)] = \mathrm{e}^{-K(t)}, \qquad K(t) = \int_0^t q(\mathrm{X}_s)\,\mathrm{d}s, \tag{3}$$

where $\mathbb{P}_x$ denotes probabilities for the process initiated by a single particle starting from $\mathrm{X}_0 = x$.

If $\#(t)$ represents the number of particles in existence at time t, the process explodes if there is a finite $T > 0$ such that $\#(t) < \infty$ for $t < T$ but $\#(t) \uparrow \infty$ as $t \uparrow T$. The process is not defined beyond T.

Let $u(t, x) = \mathbb{P}_x[T \le t]$ denote the probability of explosion on or before time t. Clearly, no explosion can occur if the initial particle's alarm clock has not yet rung, i.e. if $\tau > t$. If $\tau = s \le t$ and $\mathrm{X}_s = y$, and $n \ge 2$ offspring are produced, then each of these particles initiate branching diffusions at the place y and no explosion occurs on or before t for the overall process provided none of these n independent subprocesses explode in the time remaining of duration $t - s$. If no offspring are produced, $n = 0$, then the entire process becomes extinct and certainly no explosion occurs.

https://doi.org/10.1017/jpr.2016.52 Published online by Cambridge University Press

Summing over all n and $s \le t$, we have

$$\begin{aligned}\mathbb{P}_x[T > t] &= 1 - u(t,x) \\ &= \mathbb{E}_x \mathrm{e}^{-K(t)} + \iint_{(s,y)\in[0,t]\times[0,\infty)} (F[y, 1-u(t-s,y)])\Pi_x(\mathrm{d}s\ \mathrm{d}y), \qquad (4)\end{aligned}$$

where $F[y,z] = \sum_{n\neq 1} f_n(y) z^n$, $f_n(y)$ is equal to the probability that exactly $n = 0, 2, \ldots$ offspring are produced when a particle dies at position y, and

$$\Pi_x(\mathrm{d}s, \mathrm{d}y) = \mathbb{P}_x[\tau \in \mathrm{d}s,\ X_{\tau-} \in \mathrm{d}y].$$

(As a technical matter, at time τ the particle ceases to exist, hence, the $\tau-$.) This joint distribution has the form (not derived here; see [3, p. 184, Problem 2])

$$\Pi_x(\mathrm{d}s\ \mathrm{d}y) = \mathbb{E}_x[q(\mathrm{X}_s)\mathrm{e}^{-K(s)}\,\mathrm{d}s, \mathrm{X}_s \in \mathrm{d}y] = H_s^0(x\ \mathrm{d}y)q(y)\,\mathrm{d}s,$$

where $H_s^0(x\ \mathrm{d}y) = \mathbb{E}_x[\mathrm{e}^{-K(s)}, X_s \in \mathrm{d}y]$ is the transition probability for the diffusion killed at rate q. (Note that the measure $k(\mathrm{d}x) = q(x)\mu(\mathrm{d}x) = q(x)x^r\,\mathrm{d}x$ is the *killing* measure used in [3].) The family $(H_t^0)_{t\ge 0}$ defines a semigroup of operators with infinitesimal generator $A^0 = A - q$. For a large class of functions φ, we have

$$v(t,x) = (H_t^0\varphi)(x) \equiv \int_0^\infty v(y)H_t^0(x\ \mathrm{d}y) \quad \Longrightarrow \quad \frac{\partial v}{\partial t} = A_x^0 v \equiv A_x v(t,x) - q(x)v(t,x)$$

with $v(0+,\cdot) = \varphi$.

Let $w(s,y) = F[y, 1-u(s,y)]q(y) (\Rightarrow w(0,\cdot) = F[\cdot,1]q = q)$; then (4) continues; thus,

$$\begin{aligned}1 - u &= \mathbb{E}\mathrm{e}^{-K(t)} + \mathbb{E}\int_0^t F[\mathrm{X}_s, 1-u(t-s,\mathrm{X}_s)]\mathrm{e}^{-K(s)}\,\mathrm{d}K(s) \\ &= H_t^0 1 + \int_0^t \mathbb{E}[\mathrm{e}^{-K(s)}w(t-s,\mathrm{X}_s)]\,\mathrm{d}s \\ &= H_t^0 1 + \int_0^t H_s^0 w(t-s,\cdot)\,\mathrm{d}s \\ &= H_t^0 1 + \int_0^t H_{t-s}^0 w(s,\cdot)\,\mathrm{d}s,\end{aligned}$$

where $H_s^0 w(t-s,x) = \int w(t-s,y)H_s^0(x\ \mathrm{d}y)$. Differentiating with respect to t, we obtain

$$\begin{aligned}-\frac{\partial u}{\partial t} &= A^0 H_t^0 1 + w + \int_0^t A^0 H_{t-s}^0 w(s,\cdot)\,\mathrm{d}s \\ &= A^0 H_t^0 1 + A^0\int_0^t H_s^0 w(t-s,\cdot)\,\mathrm{d}s + w \\ &= A^0(1-u) + w \\ &= -A^0 u - q(1 - F[\cdot, 1-u])\end{aligned}$$

since $A^0 1 = -q$. On replacing A^0 by $A - q$, we conclude that

$$\frac{\partial u}{\partial t} = Au + q(1 - u - F[\cdot, 1-u]).$$

https://doi.org/10.1017/jpr.2016.52 Published online by Cambridge University Press

Remark 2. (i) If $f_0 > 0$ on a set of positive μ-measure, then u is bounded away from 1 since the branching process may go extinct with the very first particle. This has probability $\mathbb{E}_x f_0(\mathrm{X}_{\tau-}) > 0$.

(ii) The use of the additive functional K in the definition of τ in (3) is required for the Markov property of the killed motion process governed by (H_t^0).

Proof of Theorem 1. A modification of the method in [3, Section 5.15], attributed to Norman Levinson, forms the basis of the proof. See step 2 below.

Throughout, we write $A = x^{-r}(\partial/\partial)xx^r\partial/\partial x$. Also, for any function of two (or more) variables, subscripts 1 or 2 denote the derivatives with respect to the first or second variable.

Step 1. Define

$$G(x,u) = \frac{1-u-F[x,1-u]}{u}.$$

Then

$$-1 \le G(x,u) \le M(x) - 1 \quad \text{for } 0 \le u \le 1. \tag{5}$$

The proof of step 1 follows by considering that $z \mapsto F[\cdot, z]$ is a power series with nonnegative coefficients which converges at least for $|z| \le 1$. Moreover, $0 \le F \le 1 = F[\cdot, 1]$ and $z \mapsto F_2 \equiv \partial F/\partial z$ increases with $z \in [0,1]$ and is bounded by $F_2[\cdot, 1] = M$. Therefore,

$$z - F \le \max_{z \le c \le 1} \{-(1 - F_2[x,c])\} \times (1-z) = (M(x)-1)(1-z) \quad \text{for } 0 \le z < 1.$$

Putting $z = 1-u$ gives the upper bound in (5). The lower bound is plain from $1 - F \ge 0$.

It follows from step 1 and (1) that u satisfies

$$u_1 = Au + q(x)G(x,1-u)u \le Au + q(x)(M(x)-1)u. \tag{6}$$

Step 2. Put $v = \mathrm{e}^{-t^p(x^2+1)}u$, where $0 < p < 1$ (in fact, $0 < p < \min(1, 2/r)$; see step 6), then there exists $t_0 > 0$ such that

$$v_1 \le x^{-r}\frac{\partial}{\partial x}(x^r v_2 + 4t^p x^{r+1} v), \qquad 0 < t \le t_0,\ x > 0. \tag{7}$$

The proof here is somewhat more general than needed but simplifies the working. Let μ and σ denote increasing functions on an interval I; σ is continuous. As noted above for any function of (t,x) the subscripts 1 and 2 denote ordinary derivatives with respect to t and x. Subscripts μ and σ denote derivatives in the second variable, x, with respect to μ and σ. Thus, for $m = m(t,x)$,

$$m_\sigma \equiv D_\sigma(m) = \lim_{h\to 0}\frac{m(t,x+h)-m(t,x)}{\sigma(x+h)-\sigma(x)},$$

$m_{\sigma\mu} = D_\mu(D_\sigma(m))$, etc. Obviously, $m_{\sigma\mu}$ need not be the same as $m_{\mu\sigma}$.

Temporarily, $u = u(t,x)$ denotes a smooth function that satisfies

$$u_1 \le (D_\mu D_\sigma)_x u + Qu, \qquad (t,x) \in (0,\infty) \times I^{\mathrm{o}}, \tag{8}$$

for some locally bounded $Q = Q(x)$. The function u will become the function of Theorem 1 shortly.

https://doi.org/10.1017/jpr.2016.52 Published online by Cambridge University Press

For a smooth $m = m(t, x)$, put $v = \mathrm{e}^{-m}u$ and $u = \mathrm{e}^{m}v$. Then $u_1 = \mathrm{e}^{m}(v_1 + m_1 v)$, and

$$\begin{aligned} v_1 + m_1 v - Qv &\le \mathrm{e}^{-m} D_\mu D_\sigma u \\ &= \mathrm{e}^{-m} D_\mu (\mathrm{e}^{m} D_\sigma v + \mathrm{e}^{m} m_\sigma v) \\ &= D_\mu D_\sigma v + m_\mu D_\sigma v + m_\mu m_\sigma v + m_{\sigma\mu} v + m_\sigma D_\mu v. \end{aligned}$$

Next

$$\begin{aligned} v_1 &\le D_\mu(v_\sigma + 2m_\sigma v) - 2m_\sigma \mu v - 2m_\sigma v_\mu + m_\mu v_\sigma \\ &\quad + m_\mu m_\sigma v + m_{\sigma\mu} v + m_\sigma v_\mu + Qv - m_1 v \\ &= D_\mu(v_\sigma + 2m_\sigma v) + m_\mu v_\sigma - m_\sigma v_\mu + Sv, \end{aligned}$$

and

$$S = m_\mu m_\sigma + Q - m_{\sigma\mu} - m_1.$$

Note that $m_\mu v_\sigma - m_\sigma v_\mu = (m'/\mu')(v'/\sigma') - (m'/\sigma')(v'/\mu') \equiv 0$, (where '=' denotes $\mathrm{d}/\mathrm{d}x$), in the case that all four functions are differentiable.

Now suppose that u satisfies the conditions of Theorem 1, then, as noted in step 1, u satisfies (8) with $Q = q(M - 1)$ and

$$\mu'(x) = x^r, \qquad \sigma'(x) = x^{-r}.$$

Hence, $m_\mu v_\sigma - m_\sigma v_\mu = 0$, $v_1 \le x^{-r}(\mathrm{d}/\mathrm{d}x)(x^r v_2 + 2x^r m_2 v) + Sv$, and

$$S = \frac{m_2^2}{(\mu'\sigma')} + Q - \frac{(m_2/\sigma')'}{\mu'} - m_1 = m_2^2 + q(M-1) - m_{22} - \left(\frac{r}{x}\right) m_2 - m_1,$$

If $m = t^p(x^2 + 1)$ then $v_1 \le x^{-r}(\mathrm{d}/\mathrm{d}x)(x^r v_2 + 4t^p x^{r+1} v) + Sv$ with

$$S = 4t^{2p}x^2 + q(M-1) - 2(r+1)t^p - pt^{p-1}(x^2+1),$$

and (7) will follow from $S \le 0$.

By local boundedness and (2), there is a $C > 0$ such that

$$q(x)(M(x) - 1) \le C \max(x^2, 1)(x \ge 0).$$

Consequently, for $0 < t \le 1$ and all $x > 0$ (and $0 < p < 1$), we have

$$\begin{aligned} S &\le t^{p-1} \begin{cases} (4t^{p+1} - p)x^2 + [Ct^{1-p} - 2(r+1)t - p] & \text{for } x \le 1, \\ (4t^{p+1} + Ct^{1-p} - p)x^2 - 2(r+1)t - p] & \text{for } x \ge 1, \end{cases} \\ &\le C + 4 - pt^{p-1} < 0 \quad \text{for } 0 < t \le t_0 = [p/(4+c)]^{1/(1-p)} (< 1). \end{aligned}$$

Step 3. We have

$$\lim_{x \to 0+} x^r u_2(t, x) = 0 \quad \text{for all } t > 0. \tag{9}$$

First, the general solution to $Af = g$ on an interval $(0, b)$ for g integrable is given by $f = h + \int_0^x z^{-r}\,\mathrm{d}z(\int_0^z y^r g(y)\,\mathrm{d}y)$, where $h = c_1 + c_2 x^{1-r}$ if $r > 1$, $h = c_1 + c_2 \log x$, if $r = 1$. For $0 < x \le b$, we have $|\int_0^x z^{-r}\,\mathrm{d}z \int_0^z y^r g(y)\,\mathrm{d}y| \le x \int_0^b |g|\,\mathrm{d}y$, so if f is bounded near 0, then $c_2 = 0$ and, thus,

$$x^r f'(x) = \int_0^x y^r g(y)\,\mathrm{d}y = \mathrm{o}(x^r) \to 0, \qquad x \to 0+.$$

https://doi.org/10.1017/jpr.2016.52 Published online by Cambridge University Press

Next, since $u_1 \geq 0$, we have, for any $\lambda \geq 0$, $t \geq 0$, $x \geq 0$,

$$\int_0^t \mathrm{d}s \int_0^x y^\lambda u_1(s, y)\,\mathrm{d}y = \int_0^x y^\lambda u(t, y)\,\mathrm{d}y \leq \frac{x^{\lambda+1}}{\lambda+1} \leq x^{\lambda+1} \tag{10}$$

by Fubini, $0 \leq u \leq 1$, and $u(0+, \cdot) = 0$. In particular, $x \mapsto u_1(t, x)$ is integrable on $[0, b]$ for any $b > 0$ and fixed $t > 0$. (Note that, at this stage one cannot rule out the possibility that $u_1(t, x)$ becomes unbounded as x goes to 0 (ditto u_2) regardless of joint continuity on $t > 0, x > 0$.)

However, for fixed $t > 0$, $Au = u_1 - qGu$ (see step 1) and now the right-hand side is integrable in x on any $[0, b]$ and continuous at any $x > 0$, and (9) follows from the above limit for $x^r f'$ since u is bounded.

Step 4. The functions $(t, x) \mapsto \partial(x^r u_2(t, x))/\partial x$ and $(t, x) \mapsto x^r u_2(t, x)$ are absolutely integrable on any bounded rectangle $[0, a] \times [0, b]$.

Since

$$\frac{\partial}{\partial y}(y^r u_2(t, y)) = y^r u_1 - y^r q(y) G(y, u) u,$$

from (10), (6), $u_1 \geq 0$, and $0 \leq u \leq 1$, it follows that

$$\int_0^x \left| \frac{\partial}{\partial y}[y^r u_2(t, y)] \right| \mathrm{d}y \leq \int_0^x [y^r u_1(t, y) + y^r R(x) u(t, y)]\,\mathrm{d}y \leq x^{r+1} R(x), \tag{11}$$

$$R(x) = \sup_{0 \leq y \leq x} q(y)|M(y) - 1| < \infty \quad \text{for all } x > 0,$$

and this implies the joint integrability of $\partial(x^r u_2)/\partial x$.

The bound (10) implies the desired joint integrability of $y^r u_1$. Also, from (9) and the bound (11), we have

$$|x^r u_2(t, x)| = \left| \int_0^x \frac{\partial}{\partial y}[y^r u_2(t, y)]\,\mathrm{d}y \right| \leq \int_0^x y^r u_1(t, y)\,\mathrm{d}y + x^{r+1} R(x),$$

implying the stated joint integrability $x^r u_2$.

Step 5. Let $v(t, x) = \mathrm{e}^{-t^p(x^2+1)} u(t, x)$, as in step 2, and put

$$L(t, x) = x^r v_2(t, x) + 4t^p x^{r+1} v(t, x) = \mathrm{e}^{-m}(x^r u_2 + 2t^p x^{r+1} u), \qquad m = t^p(x^2 + 1).$$

According to (7), $v_1 \leq x^{-r} \partial L/\partial x$ and according to (9),

$$\lim_{x \to 0+} L(t, x) = 0 \quad \text{for each } t > 0. \tag{12}$$

Note that since $u_1 \geq 0$ and $0 < p < 1$, we have

$$|v_1| \leq [p(x^2 + 1)t^{-(1-p)} u + u_1]\mathrm{e}^{-m},$$

and v_1 is integrable in (t, x) over bounded rectangles $[0, a] \times [0, b]$, $a, b > 0$, by steps 3 and 4. Similar calculations show that L and $\partial L/\partial x$ are absolutely integrable over these rectangles.

Step 6. For $n \geq 1$ and $0 < t \leq t_0$, t_0 as in step 2, we have

$$(2n)^{-r} \int_{0+}^n v(t, x) x^r\,\mathrm{d}x$$

$$\leq \int_n^{n+1} y^{-r}\,\mathrm{d}y \int_{0+}^y x^r\,\mathrm{d}x \int_0^t v_1(s, x)\,\mathrm{d}s \quad (\text{as } v(0+, x) = 0)$$

https://doi.org/10.1017/jpr.2016.52 Published online by Cambridge University Press

$$
\begin{aligned}
&= \int_0^t \mathrm{d}s \int_n^{n+1} y^{-r}\,\mathrm{d}y \int_{0+}^{y} x^r v_1(s,x)\,\mathrm{d}x \\
&\le \int_0^t \mathrm{d}s \int_n^{n+1} y^{-r}\,\mathrm{d}y \int_{0+}^{y} \frac{\partial}{\partial x} L(s,x)\,\mathrm{d}x \quad \text{(by (7))} \\
&= \int_0^t \mathrm{d}s \int_n^{n+1} y^{-r}\,\mathrm{d}y L(s,y) \quad \text{(by (12))} \\
&= \int_0^t \mathrm{d}s \int_n^{n+1} [v_2(s,y) + 4s^p y v(s,y)]\,\mathrm{d}y \\
&= \int_0^t \mathrm{d}s \left\{ \mathrm{e}^{-s^p(y^2+1)} u(s,y) \Big|_{y=n}^{n+1} + \int_n^{n+1} 4s^p y \mathrm{e}^{-s^p(y^2+1)} u(s,y)\,\mathrm{d}y \right\} \\
&\le \int_0^t \mathrm{e}^{-s^p(n^2+1)} [1 + 2(n+1)s^p]\,\mathrm{d}s \\
&= \mathrm{O}(n^{-2/p}) \quad \text{for } 0 < t \le t_0.
\end{aligned}
$$

If $0 < p < \min(2/r, 1)$ then

$$
0 \le \int_{0+}^{\infty} v(t,x) x^r\,\mathrm{d}x \le \lim_{n\to\infty} \mathrm{O}(n^{r-2/p}) = 0, \qquad 0 \le t \le t_0.
$$

Since $v \ge 0$ and is jointly continuous on $t > 0$, $x > 0$, we may conclude that

$$
u(t,x) = \mathrm{e}^{t^p(x^2+1)} v(t,x) = 0
$$

for $0 \le t \le t_0$ and all $x > 0$. The function $\widehat{u}(t,x) = u(t+t_0, x)$ satisfies the same hypotheses as u, and this leads to $u(t,\cdot) \equiv 0$ for $0 \le t \le 2t_0$ etc. i.e. $u \equiv 0$.

This completes the proof of Theorem 1. □

References

[1] Fujita, H. and Watanabe, S. (1968). On the uniqueness and non-uniqueness of solutions of initial value problems for some quasi-linear parabolic equations. *Commun. Pure Appl. Math.* **21,** 631–652.

[2] Ikeda, N., Nagasawa, M. and Watanabe, S. (1969). Branching Markov processes. III. *J. Math. Kyoto Univ.* **9,** 95–160.

[3] Itô, K. and Mckean, H. P., Jr. (1974). *Diffusion Processes and Their Sample Paths*, 2nd edn. Springer, Berlin.

[4] Polyanin, A. D. and Zaitsev, V. F. (2012). *Handbook of Nonlinear Partial Differential Equations*, 2nd edn. CRC, Boca Raton, FL.

[5] Savits, T. H. (1969). The explosion problem for branching Markov process. *Osaka J. Math.* **6,** 375–395.

https://doi.org/10.1017/jpr.2016.52 Published online by Cambridge University Press