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Von Neumann orbit equivalence

Ishan Ishan and Aoran Wu

Abstract. We generalize the notion of orbit equivalence to the non-commutative setting by introducing a new equivalence relation on groups, which we call von Neumann orbit equivalence (vNOE). We prove the stability of this equivalence relation under taking free products and graph products of groups. To achieve this, we introduce vNOE of tracial von Neumann algebras, show that two countable discrete groups Γ and Λ are vNOE if and only if the corresponding group von Neumann algebras $L\Gamma$ and $L\Lambda$ are vNOE, and that vNOE of tracial von Neumann algebras is stable under taking free products and graph products of tracial von Neumann algebras.

1 Introduction

Let Γ and Λ be two countable discrete groups with free probability measure-preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ on standard probability measure spaces (X, μ) and (Y, ν), respectively. An *orbit equivalence* (OE) for the actions is a measurable isomorphism $\theta : X \to Y$ such that $\theta(\Gamma x) = \Lambda\theta(x)$ for almost every $x \in X$. In this case, the two actions are called *orbit equivalent*. Two groups are said to be *orbit equivalent* if they admit orbit equivalent actions. Singer [Sin55] showed that for two free probability measure-preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$, being orbit equivalent is equivalent to the existence of an isomorphism $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ which preserves the Cartan subalgebras $L^\infty(X)$ and $L^\infty(Y)$. OE theory saw some development in the 1980s (see [CFW81, OW80, Zim84]), and has been an area of active research over the last two decades (see [Fur11, Gab10]). These advances in part have been stimulated by the success of the deformation/rigidity theory approach to the classification of II_1 factors developed by Popa and others (see [Ioa13, Pop08, Vae10]).

The study of OE can also be motivated from an entirely different point of view, being a measurable counterpart to quasi-isometry of groups. Gromov [Gro93] introduced measure equivalence (ME) for countable discrete groups as a measurable analog of quasi-isometry and since then, this notion has proven to be an important tool in geometric group theory with connections to ergodic theory and operator algebras. Two infinite countable discrete groups Γ and Λ are *measure equivalent* if there is a standard infinite measure space (Ω, m) with commuting, measure-preserving actions $\Gamma \curvearrowright (\Omega, m)$ and $\Lambda \curvearrowright (\Omega, m)$, so that both the actions admit finite-measure

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fundamental domains $Y, X \in \Omega$, that is, $m(Y), m(X) < \infty$ and

$$\Omega = \bigsqcup_{\gamma\in\Gamma} \gamma Y = \bigsqcup_{\lambda\in\Lambda} \lambda X.$$

The space (Ω, m) is called an ME-*coupling* between Γ and Λ, and the *index* of such a coupling is

$$[\Gamma : \Lambda]_\Omega := \frac{m(X)}{m(Y)}.$$

Notably, ME was used by Furman in [Fur99a, Fur99b] to prove strong rigidity results for lattices in higher rank simple Lie groups. ME relates back to OE because of the following fact, observed by Zimmer and Furman: for two discrete groups Γ and Λ, admitting free OE actions is equivalent to having an ME-coupling so that the fundamental domains coincide (see, for example, [Gab05, $\mathbf{P}_{\mathrm{ME}}\mathbf{5}$] and [Fur11, Theorem 2.5]).

If $X \subset \Omega$ is a Borel fundamental domain for the action $\Gamma \curvearrowright (\Omega, m)$, then on the level of function spaces, the characteristic function 1_X gives a projection in $L^\infty(\Omega, m)$ such that the collection $\{1_{\gamma X}\}_{\gamma\in\Gamma}$ forms a partition of unity, i.e., $\sum_{\gamma\in\Gamma} 1_{\gamma X} = 1$. This notion generalizes quite nicely to the non-commutative setting, and using this, Peterson, Ruth, and the first named author, in [IPR24], defined that a *fundamental domain* for an action on a von Neumann algebra $\Gamma \curvearrowright^\sigma \mathcal{M}$ is a projection $p \in \mathcal{M}$ such that $\sum_{\gamma\in\Gamma} \sigma_\gamma(p) = 1$, where the convergence is in the strong operator topology. Using this perspective for a fundamental domain, they generalized the notion of ME by considering actions on non-commutative spaces.

Definition 1.1 [IPR24] Two countable discrete groups Γ and Λ are *von Neumann equivalent* (vNE), written $\Gamma \sim_{\mathrm{vNE}} \Lambda$, if there exists a von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace Tr and commuting, trace-preserving actions of Γ and Λ on $\mathcal{M}$ such that the Γ- and Λ-actions individually admit a finite-trace fundamental domain. The semi-finite von Neumann algebra $\mathcal{M}$ is called a *von Neumann coupling* between Γ and Λ.

Like ME, vNE is stable under taking the direct product of groups. But neither ME nor vNE is stable under taking free products. For instance, since any two finite groups are ME (and hence vNE), and amenability is preserved under both ME and vNE, one gets that $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ (amenable) is neither ME nor vNE to $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ (non-amenable). However, as suggested in [MS06, Remark 2.28], and proved in [Gab05, $\mathbf{P}_{ME}$ **6**], stability under taking free products hold if one requires the additional assumption that groups are ME with a common fundamental domain. In other words, OE is stable under taking free products. This raises a natural question: *Is vNE, with common fundamental domain, stable under taking free products?* We obtain an affirmative answer to this question and introduce the following definition.

Definition 1.2 Two countable discrete groups Γ and Λ are said to be *von Neumann orbit equivalent* (vNOE), denoted $\Gamma \sim_{\mathrm{vNOE}} \Lambda$, if there exists a von Neumann coupling between Γ and Λ with a common fundamental domain.

Theorem 1.3 *If $\Gamma_i, \Lambda_i,\ i = 1, 2$ are countable discrete groups such that $\Gamma_i \sim_{\mathrm{vNOE}} \Lambda_i,\ i = 1, 2$, then $\Gamma_1 * \Gamma_2 \sim_{\mathrm{vNOE}} \Lambda_1 * \Lambda_2$.*

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Green [Gre90], in her Ph.D. thesis, introduced graph products of groups, another important group-theoretical construction. If $\mathcal{G} = (V, E)$ is a simple, non-oriented graph with vertex set V and edge set E, then the graph product of a family, $\{\Gamma_v\}_{v\in V}$, of groups indexed by V is obtained from the free product $*_{v\in V}\Gamma_v$ by adding commutator relations determined by the edge set E. Depending on the graph, free products and direct products are special cases of the graph product construction. Adapting the ideas of [Gab05], Horbez and Huang [HH22, Proposition 4.2] proved the stability of OE under taking graph products over finite simple graphs (see also [Dem22]). To further explore the study of graph products within the context of measured group theory, we would like to draw the reader's attention to the article [EH24]. In the current article, we also prove the stability of vNOE under taking graph products.

Theorem 1.4 *Let $\mathcal{G} = (V, E)$ be a simple graph with at most countably infinite vertices. Let Γ and Λ be two graph products over $\mathcal{G}$, with countable vertex groups $\{\Gamma_v\}_{v\in V}$ and $\{\Lambda_v\}_{v\in V}$, respectively. If $\Gamma_v \sim_{\text{vNOE}} \Lambda_v$ for every $v \in V$, then $\Gamma \sim_{\text{vNOE}} \Lambda$.*

In attempting to prove the above theorems, if one tries to adapt the techniques from one of [Gab05, HH22, Dem22], an immediate problem is presented by the lack of "point perspective" in the theory of von Neumann (orbit) equivalence. The lack of any natural non-commutative analog of the notion of OE/ME cocycles, or that of measured equivalence relation, can be considered as a few problems presented by the lack of point perspective. This often leads one to consider genuinely new techniques and different alternatives (see, e.g., [Bat23a, Bat23b, IPR24, Ish24]). To overcome this obstruction, we introduce the notion of vNOE for tracial von Neumann algebras that is "compatible" with vNOE of groups (see Definition 1.5 and Theorem 3.11), and prove the analogs of Theorems 1.3 and 1.4 at the level of tracial von Neumann algebras.

The notion of von Neumann equivalence admits a generalization in the setting of finite von Neumann algebras [IPR24, Section 8], and relates to vNE for groups as follows: $\Gamma \sim_{\text{vNE}} \Lambda$ if and only if $L\Gamma \sim_{\text{vNE}} L\Lambda$ [IPR24, Theorem 1.5]. In parallel to this, one might attempt to define two tracial von Neumann algebras to be vNOE if they are vNE and admit a "common" fundamental domain, and identify a correct meaning of "common." However, we take a slightly different approach, and motivated by the recently defined notion of ME of finite von Neumann algebras by Berendschot and Vaes in [BV25], we introduce the following definition.

Definition 1.5 Let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras. We say that (A, τ_A) and (B, τ_B) are *vNOE*, denoted $(A, \tau_A) \sim_{\text{vNOE}} (B, \tau_B)$, if there exists a tracial von Neumann algebra (Q, τ_Q), a Hilbert $A\overline{\otimes}Q - B$-bimodule $\mathcal{H}$, and a vector $\xi \in \mathcal{H}$ such that:

(1) $\langle (a \otimes x)\xi, \xi\rangle = \tau_A(a)\tau_Q(x)$, and $\langle y\xi b, \xi\rangle = \tau_Q(y)\tau_B(b)$ for every $a \in A,\ x, y \in Q$, and $b \in B$.
(2) $\overline{\text{Span}((A\overline{\otimes}Q)\xi)} = \mathcal{H} = \overline{\text{Span}(Q\xi B)}$.

We prove in Proposition 3.5 that vNOE is indeed an equivalence relation. We should remark that, in the above definition, $\mathcal{H}$ can also be considered as an $A - B\overline{\otimes}Q^{\text{op}}$-bimodule satisfying conditions analogous to the two mentioned in the definition. This essentially is the reason for the symmetry of vNOE, even though the definition seems asymmetric at first. To prove transitivity, inspired by [BV25,

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Lemma 5.11], we establish an equivalent characterization of vNOE in Theorem 3.1, and show in Theorem 3.11 that $\Gamma \sim_{\text{vNOE}} \Lambda$ if and only if $L\Gamma \sim_{\text{vNOE}} L\Lambda$. Since $L(\Gamma * \Lambda) \cong L\Gamma * L\Lambda$, Theorem 1.3 follows from the following theorem, which we prove in Section 3. We should remark that the above definition, as stated, depends on the choice of the traces τ_A and τ_B. Outside of the case of finite factors, it is not immediately clear whether the above definition is independent of the choice of the traces for general finite von Neumann algebras.

Theorem 1.6 *If $(A_i, \tau_{A_i}), (B_i, \tau_{B_i})$, $i = 1, 2$ are tracial von Neumann algebras such that $(A_i, \tau_{A_i}) \sim_{\text{vNOE}} (B_i, \tau_{B_i})$, $i = 1, 2$, then, $(A_1 * A_2, \tau_{A_1} * \tau_{A_2}) \sim_{\text{vNOE}} (B_1 * B_2, \tau_{B_1} * \tau_{B_2})$.*

Similar to free products, one also has that the group von Neumann algebra of a graph product of groups is isomorphic to the (von Neumann algebraic) graph product of the group von Neumann algebras, and hence Theorem 1.4 follows from the following theorem.

Theorem 1.7 *Let $\mathcal{G} = (V, E)$ be a simple graph with at most countably infinite vertices. Let (A, τ_A) and (B, τ_B) be two graph products over $\mathcal{G}$, with tracial vertex von Neumann algebras $\{(A_v, \tau_{A_v})\}_{v \in V}$ and $\{(B_v, \tau_{B_v})\}_{v \in V}$, respectively. If $(A_v, \tau_{A_v}) \sim_{\text{vNOE}} (B_v, \tau_{B_v})$ for every $v \in V$, then $(A, \tau_A) \sim_{\text{vNOE}} (B, \tau_B)$.*

Remark 1.8 Since graph product over a totally disconnected graph, i.e., a graph with no edges, gives free product, Theorem 1.6 follows from Theorem 1.7. In particular, Theorem 1.7 shows that Theorem 1.6 holds for free products of countably many tracial von Neumann algebras similar to the result obtained by Gaboriau in [Gab05, $\mathbf{P}_{ME}$ **6***]. Furthermore, Horbez and Huang [HH22, Proposition 4.2] proved the stability of OE under taking graph products over finite simple graphs, but our result holds for an arbitrary, countably infinite graph product. We include a proof of Theorem 1.6 for two reasons. Firstly, the notation is a little less involved compared to the proof of Theorem 1.7. Secondly, for the convenience of a reader who might be interested in the result but is not familiar with graph products.

In Proposition 3.14, we show that vNOE tracial von Neumann algebras are vNE in the sense of [IPR24]. We should remark that vNE does not imply vNOE in general.

In the final section, we obtain a partial analog of Singer's theorem [Sin55] for OE in the setting of vNOE of groups. As noted in [IPR24, Example 5.2], if Γ and Λ are countable discrete groups with trace-preserving actions $\Gamma \curvearrowright (A, \tau_A)$ and $\Lambda \curvearrowright (B, \tau_B)$ on tracial von Neumann algebras (A, τ_A) and (B, τ_B), respectively, and if $\theta : B \rtimes \Lambda \to A \rtimes \Gamma$ is a trace-preserving isomorphism such that $\theta(B) = A$, then $\Gamma \sim_{\text{vNOE}} \Lambda$. As a partial converse to this, we prove the following theorem.

Theorem 1.9 *If Γ and Λ are countable discrete groups such that $\Gamma \sim_{\text{vNOE}} \Lambda$, then there exist tracial von Neumann algebras (A, τ_A), (B, τ_B), trace-preserving actions $\Gamma \curvearrowright A$, $\Lambda \curvearrowright B$, and a trace-preserving isomorphism $\theta : B \rtimes \Lambda \to A \rtimes \Gamma$.*

2 Preliminaries and notations

We set up the notations and collect some facts that will be needed in this article.

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2.1 Tracial von Neumann algebras and the standard form

A tracial von Neumann algebra A is endowed with a trace, i.e., a faithful, normal, unital linear functional $\tau : A \to \mathbb{C}$ such that $\tau(xy) = \tau(yx)$ for all $x, y \in A$. The trace τ induces an inner product on A given by $\langle x, y\rangle = \tau(y^*x), x, y \in A$, and we let $L^2(A)$ denote the Hilbert space completion of A with respect to this inner product. When we view an element $x \in A$ as a vector in $L^2(A)$, we denote it by $\hat{x}$. An element $x \in A$ defines a bounded linear operator on $L^2(A)$ given by $L_x(\hat{y}) = \widehat{xy}, y \in A$, and thus we have a representation of A in $\mathcal{B}(L^2(A))$, called the *standard representation*. There is also a canonical anti-linear conjugation operator $J : L^2(A) \to L^2(A)$, defined by $J(\hat{x}) = \widehat{x^*}, x \in A$, and we have that $A' = JAJ$, where A' is the commutant of M inside $\mathcal{B}(L^2(A))$. By the tracial property of τ, for $x \in A$, the operator R_x on $L^2(A)$ defined by $R_x(\hat{y}) = \widehat{yx}, y \in A$ is bounded. If we let $\rho(A) = \{R_x : x \in A\} \subset \mathcal{B}(L^2(A))$, then $\rho(A) = A'$.

2.2 Free product and amalgamated free product

Let (A_1, τ_1) and (A_2, τ_2) be tracial von Neumann algebras. The *free product* of A_1 and A_2 is the unique, up to isomorphism, tracial von Neumann algebra (A, τ) containing A_1 and A_2, and is such that $\tau|_{A_i} = \tau_i, i = 1, 2$, A is generated by $A_1 \cup A_2$, and A_1, A_2 are *free* inside A, i.e., $\tau(a_1a_2 \dots a_k) = 0$, whenever $a_i \in A_{n_i}$ with $n_i \neq n_{i+1}$ for all $i \in \{1, \dots, k-1\}$, $n_i \in \{1, 2\}$ and $\tau_{n_i}(a_i) = 0$ for all $1 \leq i \leq k$. We will say that an element $a_1a_2 \dots a_k$ of the algebraic free product $A_1 *_{\text{alg}} A_2$ is an *alternating centered word* with respect to τ if $a_i \in A_{n_i}$ with $n_i \neq n_{i+1}$ for all $i \in \{1, \dots, k-1\}$, $n_i \in \{1, 2\}$ and $\tau_{n_i}(a_i) = 0$ for all $1 \leq i \leq k$.

For $i = 1, 2$, let (A_i, τ_i) be tracial von Neumann algebras, $Q \subset A_i$ be a common von Neumann subalgebra, and $E_i : A_i \to Q$ be faithful, normal conditional expectations. The *amalgamated free product* $(A, E) = (A_1, E_1) *_Q (A_2, E_2)$ is a pair of a von Neumann algebra A generated by A_1 and A_2 and a faithful normal conditional expectation $E : A \to Q$ such that A_1 and A_2 are *freely independent* with respect to E: $E(a_1a_2 \dots a_k) = 0$ whenever $a_{n_i} \in A_{n_i}$ with $n_i \in \{1, 2, \}, E_{n_i}(a_i) = 0$ for all $1 \leq i \leq k$, and $n_i \neq n_{i+1}$ for all $i \in \{1, \dots, k-1\}$. An element $a_1a_2 \dots a_k \in A$ will be called an *alternating centered word* with respect to E if $a_{n_i} \in A_{n_i}$ with $n_i \in \{1, 2\}, E_{n_i}(a_i) = 0$ for all $1 \leq i \leq k$, and $n_i \neq n_{i+1}$ for all $i \in \{1, \dots, k-1\}$.

For the construction and further details on (amalgamated) free products, we refer the reader to [Pop93, Ued99, VDN92, Voi85].

2.3 Graph product

Let $\mathcal{G} = (V, E)$ be a simple graph with the vertex set V and the edge set $E \subseteq V \times V \backslash \{(v, v) : v \in V\}$. We assume that the graph $\mathcal{G}$ is non-oriented, i.e., $(v, w) \in E$ if and only if $(w, v) \in E$. A word $v_1v_2 \dots v_n$ of vertices in V is called *reduced* if it satisfies the following property: if there exist $k < l$ such that $v_k = v_l$, then there is some $k < j < l$ such that $(v_k, v_j) \notin E$. Let $\mathcal{G} = (V, E)$ be a simple graph, (A, τ) be a tracial von Neumann algebra, and $\{(A_v, \tau_v) : v \in V\}$ be a family of tracial von Neumann subalgebras of (A, τ) such that $\tau|_{A_v} = \tau_v$ for all $v \in V$. We say that the family $\{(A_v, \tau_v) : v \in V\}$ is $\mathcal{G}$-*independent* if the following property holds: if $v_1 \dots v_n$ is a reduced word and

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$a_1, \ldots, a_n \in A$ are such that $a_i \in A_{v_i}$ and $\tau(a_i) = 0$, then $\tau(a_1 \ldots a_n) = 0$. On the other hand, given a simple graph $\mathcal{G} = (V, E)$ and a family of tracial von Neumann algebras $\{(A_v, \tau_v) : v \in V\}$, there is a unique, up to isomorphism, tracial von Neumann algebra (A, τ), called the *graph product* over $\mathcal{G}$ of the family $\{(A_v, \tau_v) : v \in V\}$, and trace-preserving inclusions $\varphi_v : A_v \hookrightarrow A$ such that the family $\{\varphi_v(A_v) : v \in V\}$ is $\mathcal{G}$-independent and generates A as a von Neumann algebra (see [Mło04, CF17]). We denote the graph product (A, τ) of the family $\{(A_v, \tau_v) : v \in V\}$ by

$$(A, \tau) = \bigstar_{v \in V} (A_v, \tau_v).$$

Remark 2.1 If $\mathcal{G} = (V, E)$ is a simple graph, and $\{\Gamma_v : v \in V\}$ is a family of countable discrete groups, then $L(\bigstar_{v \in V} \Gamma_v) = \bigstar_{v \in V} L\Gamma_v$ (see [CF17, Remark 3.23]).

2.4 Modules over tracial von Neumann algebras

For details on the proofs of the facts collected in this section, we refer the reader to [AP21, Chapter 8].

Definition 2.2 Given von Neumann algebras A and B,

(1) a *left A-module* is a pair $(\mathcal{H}, \pi_A)$, where $\mathcal{H}$ is a Hilbert space and $\pi_A : A \to \mathcal{B}(\mathcal{H})$ is a normal unital $*$-homomorphism.
(2) a *right B-module* is a pair $(\mathcal{H}, \pi_B)$, where $\mathcal{H}$ is a Hilbert space and $\pi_B : B \to \mathcal{B}(\mathcal{H})$ is a normal unital $*$-anti-homomorphism, i.e., $\pi_B(xy) = \pi_B(y)\pi_B(x)$ for all $x, y \in B$. In other words, $\mathcal{H}$ is a left B^{op}-module, where B^{op} is the opposite algebra.
(3) an *$A - B$-bimodule* is a triple $(\mathcal{H}, \pi_A, \pi_B)$ such that $(\mathcal{H}, \pi_A)$ is a left A-module, $(\mathcal{H}, \pi_B)$ is a right B-module, and the representations π_A and π_B commute. For $\xi \in \mathcal{H}$, $x \in A$, and $y \in B$, we will write $x\xi y$ instead of $\pi_A(x)\pi_B(y)\xi\,(= \pi_B(y)\pi_A(x)\xi)$.

Definition 2.3 Let (A, τ_A), (B, τ_B) be tracial von Neumann algebras and let $\mathcal{H}$ be an $A - B$-bimodule. A vector $\xi \in \mathcal{H}$ is called:

(1) *tracial* if $\langle x\xi, \xi\rangle = \tau_A(x)$ for every $x \in A$, and $\langle \xi y, \xi\rangle = \tau_B(y)$ for every $y \in B$;
(2) *bi-tracial* if $\langle x\xi y, \xi\rangle = \tau_A(x)\tau_B(y)$ for all $x \in A$, $y \in B$;
(3) *cyclic* if $\overline{\mathrm{Span}\{x\xi y : x \in A, y \in B\}} = \mathcal{H}$.

Let (Q, τ_Q) be a tracial von Neumann algebra. Given two left Q-modules $\mathcal{H}$ and $\mathcal{K}$, we denote by ${}_Q\mathcal{B}(\mathcal{H}, \mathcal{K})$ the space of left Q-linear bounded maps from $\mathcal{H}$ into $\mathcal{K}$, that is,

$${}_Q\mathcal{B}(\mathcal{H}, \mathcal{K}) = \{T \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : T(x\xi) = x(T\xi) \text{ for all } x \in Q, \xi \in \mathcal{H}\}.$$

We set ${}_Q\mathcal{B}(\mathcal{H}) = {}_Q\mathcal{B}(\mathcal{H}, \mathcal{H})$. It is straightforward to check that ${}_Q\mathcal{B}(\mathcal{H}) = Q' \cap \mathcal{B}(\mathcal{H})$. Moreover, ${}_Q\mathcal{B}(\mathcal{H})$ is a semi-finite von Neumann algebra equipped with a specific semi-finite trace Tr, depending on τ_Q. Before stating the result that characterizes Tr, observe that, given $S, T \in {}_Q\mathcal{B}(L^2Q, \mathcal{H})$, we have $TS^* \in {}_Q\mathcal{B}(\mathcal{H})$, and $S^*T \in JQJ$, where $J : L^2Q \to L^2Q$ is the canonical conjugation operator. The following is a translation of [AP21, Proposition 8.4.2] for left Q-modules.

Proposition 2.4 *If $\mathcal{H}$ is a left Q-module over a tracial von Neumann algebra (Q, τ_Q), then the commutant ${}_Q\mathcal{B}(\mathcal{H}) = Q' \cap \mathcal{B}(\mathcal{H})$ is a semi-finite von Neumann algebra*

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equipped with a canonical faithful normal semi-finite trace Tr *characterized by the equation*

$$\mathrm{Tr}(TT^*) = \tau_Q(JT^*TJ)$$

for every left Q-linear bounded operator $T : L^2Q \to \mathcal{H}$.

Remark 2.5 Suppose (Q, τ_Q) is a tracial von Neumann algebra and $\mathcal{H}$ is a left Q-module. If $\xi \in \mathcal{H}$ is a tracial vector, then the orthogonal projection $P : \mathcal{H} \to \overline{\mathrm{Span}(Q\xi)}$ lies in ${}_Q\mathcal{B}(\mathcal{H})$. Moreover, since ξ is tracial, the operator $U : L^2Q \to \overline{\mathrm{Span}(Q\xi)}$ given by $U\hat{x} = x\xi, x \in Q$ is a unitary. Extending U to an isometry from L^2Q into $\mathcal{H}$ in an obvious way and applying Proposition 2.4 to $T = PU : L^2Q \to \mathcal{H}$ yields that $\mathrm{Tr}(P) = \tau_Q(1) = 1$.

2.5 Actions on semi-finite von Neumann algebras

For a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace Tr, the set $\mathfrak{n}_{\mathrm{Tr}} = \{x \in \mathcal{M} \mid \mathrm{Tr}(x^*x) < \infty\}$ is an ideal. Left multiplication of $\mathcal{M}$ on $\mathfrak{n}_{\mathrm{Tr}}$ induces a normal faithful representation of $\mathcal{M}$ in $\mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$, called *the standard representation*, where $L^2(\mathcal{M}, \mathrm{Tr})$ is the Hilbert space completion of $\mathfrak{n}_{\mathrm{Tr}}$ under the inner product $\langle a, b\rangle_2 = \mathrm{Tr}(b^*a)$.

If $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ is a trace-preserving action of a countable discrete group Γ on $\mathcal{M}$, then Γ preserves the $\|\cdot\|_2$-norm on $\mathfrak{n}_{\mathrm{Tr}}$. Therefore, restricted to $\mathfrak{n}_{\mathrm{Tr}}$, the action is isometric with respect to the $\|\cdot\|_2$-norm and hence gives a unitary representation $\sigma^0 : \Gamma \to \mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$, called the *Koopman representation*. Considering $\mathcal{M} \subset \mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$ via the standard representation, we have that the action $\sigma : \Gamma \to \mathrm{Aut}(\mathcal{M}, \mathrm{Tr})$ is unitarily implemented via the Koopman representation, i.e., for $x \in \mathcal{M}$ and $\gamma \in \Gamma$, we have $\sigma_\gamma(x) = \sigma^0_\gamma x \sigma^0_{\gamma^{-1}}$ (see [Haa75, Theorem 3.2]).

3 Von Neumann orbit equivalence

In this section, we define vNOE for tracial von Neumann algebras and for countable discrete groups. We shall see that groups are vNOE if and only if the corresponding group von Neumann algebras are, and we conclude this section with the proof that vNOE tracial von Neumann algebras are vNE in the sense of [IPR24].

3.1 Von Neumann orbit equivalence for tracial von Neumann algebras

Theorem 3.1 *Let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras. Then, the following are equivalent:*

(1) *There exists a tracial von Neumann algebra (Q, τ_Q) and a pointed $A\overline{\otimes}Q - B$-bimodule $(\mathcal{H}, \xi)$ such that $\xi \in \mathcal{H}$ is a cyclic and (bi-)tracial vector for both $A\overline{\otimes}Q$-module structure, and $Q - B$-bimodule structure. That is, for all $a \in A, b \in B$, and $x \in Q$,*
   (a) $\langle (a \otimes x)\xi, \xi\rangle = \tau_A(a)\tau_Q(x)$, *and* $\langle x\xi b, \xi\rangle = \tau_Q(x)\tau_B(b)$;
   (b) $\overline{\mathrm{Span}((A\overline{\otimes}Q)\xi)} = \mathcal{H} = \overline{\mathrm{Span}(Q\xi B)}$.

(2) *There exists a tracial von Neumann algebra (Q, τ_Q), and a normal $*$-homomorphism $\phi : B \to A\overline{\otimes}Q$ such that:*

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(a) $\mathbb{E}_Q \circ \phi = \tau_B$, *where* $\mathbb{E}_Q : A \overline{\otimes} Q \to Q$ *is the normal conditional expectation;*

(b) $\overline{\mathrm{Span}\{x\phi(b) : b \in B, x \in Q\}}^{\|\cdot\|_2} = L^2(A \overline{\otimes} Q)$.

Proof Let $(\mathcal{H}, Q, \xi)$ be a triple as in (1). We thus obtain a canonical unitary $U: \mathcal{H} \to L^2(A\overline{\otimes}Q)$ such that $U(y\xi) = \hat{y}$ for all $y \in A\overline{\otimes}Q$. Hence, we can define a right action of B on $L^2(A\overline{\otimes}Q)$ by

$$\eta \cdot b = U(U^*(\eta)b), \text{ for all } \eta \in L^2(A\overline{\otimes}Q), b \in B.$$

For $b \in B$, we let $R_b \in \mathcal{B}(L^2(A\overline{\otimes}Q))$ be the operator corresponding to right multiplication by b. Since $\mathcal{H}$ is an $A\overline{\otimes}Q - B$ bimodule, and U is $A \overline{\otimes} Q$-linear, so, the right action of B commutes with the left action of $A\overline{\otimes}Q$ on $L^2(A \overline{\otimes} Q)$, and hence $R_b \in (A \overline{\otimes} Q)' \cap \mathcal{B}(L^2(A \overline{\otimes} Q))$ for every $b \in B$. Since the commutant of $A\overline{\otimes}Q$ acting on $L^2(A\overline{\otimes}Q)$ is $\rho(A\overline{\otimes}Q)$ we define $\phi: B \to A\overline{\otimes}Q$ as follows: for $b \in B$, $\phi(b)$ is the unique element in $A \overline{\otimes} Q$ such that $R_b = \rho(\phi(b))$. This is directly checked to be a normal $*$-homomorphism. Moreover, by definition of ϕ, we have that $\eta \cdot b = \eta\phi(b)$, for every $\eta \in L^2(A \overline{\otimes} Q), b \in B$. Since ξ is Q-B bi-tracial, we have for all $b \in B, x \in Q$ that

$$\tau_Q(x)\tau_B(b) = \langle x\xi b, \xi\rangle = \langle U^*(\hat{x})b, U^*(\hat{1})\rangle = \langle \widehat{x\phi(b)}, \hat{1}\rangle = \tau(x\phi(b)),$$

where τ denotes the trace on $A \overline{\otimes} Q$. Furthermore, since $\tau_Q \circ \mathbb{E}_Q = \tau$, we have

$$\tau_Q(\tau_B(b)x) = \tau(x\phi(b)) = \tau_Q(\mathbb{E}_Q(x\phi(b))) = \tau_Q(x\mathbb{E}_Q(\phi(b))),$$

for all $x \in Q, b \in B$, whence it follows that $\mathbb{E}_Q \circ \phi = \tau_B$. Finally,

$$\overline{\mathrm{Span}\{x\phi(b) : x \in Q, b \in B\}}^{\|\cdot\|_2} = U(\overline{\mathrm{Span}\{x\xi b : x \in Q, b \in B\}}) = U(\mathcal{H}) = L^2(A\overline{\otimes}Q).$$

Conversely, suppose condition (2) holds, and let $\mathcal{H} = L^2(A\overline{\otimes}Q)$, and $\xi = \hat{1}$. Define a right action of B on $\mathcal{H}$ by $\eta \cdot b = \eta\phi(b)$. Then, for $x \in Q$, $b \in B$, we have

$$\overline{\mathrm{Span}\{x\xi \cdot b : x \in Q, b \in B\}}^{\|\cdot\|_2} = \overline{\mathrm{Span}\{x\phi(b) : x \in Q, b \in B\}}^{\|\cdot\|_2} = L^2(A\overline{\otimes}Q),$$

and for $x \in Q, b \in B$, we have

$$\langle x\xi \cdot b, \xi\rangle = \langle \widehat{x\phi(b)}, \hat{1}\rangle = \tau(x\phi(b)) = \tau_Q(x\mathbb{E}_Q(\phi(b))) = \tau_Q(x)\tau_B(b).$$

■

Remark 3.2 Since the operation of taking adjoints is isometric on $L^2(A \overline{\otimes} Q)$, in condition (2) of Theorem 3.1, one might equivalently require that $\overline{\mathrm{Span}\{\phi(b)x : b \in B, x \in Q\}}^{\|\cdot\|_2} = L^2(A\overline{\otimes}Q)$.

Definition 3.3 Let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras. We say that (A, τ_A) is *vNOE* to (B, τ_B), denoted $(A, \tau_A) \sim_{\mathrm{vNOE}} (B, \tau_B)$, if either of the two equivalent conditions in Theorem 3.1 is satisfied. If (A, τ_A) is vNOE to (B, τ_B), then the triple $(\mathcal{H}, Q, \xi)$ or the pair (Q, ϕ) of Theorem 3.1 will be called a vNOE-*coupling* between (A, τ_A) and (B, τ_B).

Remark 3.4 As mentioned in the introduction, the above definition depends on the choice of the traces τ_A and τ_B. When in a situation where the traces are fixed, and

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there is no risk of confusion, we will often drop them, and simply say that A and B are vNOE and write $A \sim_{\mathrm{vNOE}} B$.

Proposition 3.5 *vNOE is an equivalence relation.*

Proof If (A, τ_A) is a tracial von Neumann algebra, then taking $Q = \mathbb{C}, \mathcal{H} = L^2(A)$, and $\xi = \hat{1} \in L^2(A)$ in condition (1) of Theorem 3.1 shows that $A \sim_{\mathrm{vNOE}} A$. To see symmetry, let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras satisfying condition (1) of Theorem 3.1, and let $\mathcal{H}, (Q, \tau_Q)$, and $\xi \in \mathcal{H}$ be as in the condition. Note that we can view $\mathcal{H}$ as an $A - B \overline{\otimes} Q^{\mathrm{op}}$-bimodule. Consider the conjugate Hilbert space $\overline{\mathcal{H}}$, and the corresponding canonical $B \overline{\otimes} Q^{\mathrm{op}} - A$ bimodule structure on $\overline{\mathcal{H}}$. Then, it is straightforward to check that the triple $(\overline{\mathcal{H}}, Q^{\mathrm{op}}, \bar{\xi})$ satisfies (1) of Theorem 3.1 and thus, $B \sim_{\mathrm{vNOE}} A$. To show transitivity, we will use condition (2) of Theorem 3.1. To this end, let $(A, \tau_A), (B, \tau_B)$, and (C, τ_C) be tracial von Neumann algebras. Let Q_1, Q_2, and $\phi_1 : B \to A \overline{\otimes} Q_1, \phi_2 : C \to B \overline{\otimes} Q_2$ be as in (2) of Theorem 3.1. Let $Q = Q_1 \overline{\otimes} Q_2$ and let $\phi : C \to A \overline{\otimes} Q$ be given by

$$\phi(c) = (\phi_1 \otimes \mathrm{id}_{Q_2})(\phi_2(c)), \quad c \in C,$$

where $\phi_1 \otimes \mathrm{id}_{Q_2} : B \overline{\otimes} Q_2 \to A \overline{\otimes} Q_1 \overline{\otimes} Q_2$ is the natural extension of $\phi_1 : B \to A \overline{\otimes} Q_1$. Let $\mathbb{E}_{Q_1} : A \overline{\otimes} Q_1 \to Q_1, \mathbb{E}_{Q_2} : B \overline{\otimes} Q_2 \to Q_2$, and $\mathbb{E}_Q : A \overline{\otimes} Q_1 \overline{\otimes} Q_2 \to Q_1 \overline{\otimes} Q_2$ be normal conditional expectations. Consider the map $\mathbb{E}_{Q_1} \otimes \mathrm{id}_{Q_2} : A \overline{\otimes} Q_1 \overline{\otimes} Q_2 \to Q_1 \overline{\otimes} Q_2$. Note that $\mathbb{E}_Q = \mathbb{E}_{Q_1} \otimes \mathrm{id}_{Q_2}$. Therefore,

$$\mathbb{E}_Q \circ \phi = (\mathbb{E}_{Q_1} \otimes \mathrm{id}_{Q_2}) \circ ((\phi_1 \otimes \mathrm{id}_{Q_2}) \circ \phi_2) = \mathbb{E}_{Q_2} \circ \phi_2 = \tau_C,$$

where the second to last equality follows from the fact that the following diagram, since $\mathbb{E}_{Q_1} \circ \phi_1 = \tau_B$, is commutative:

$$\begin{array}{ccc} B \overline{\otimes} Q_2 & \xrightarrow{\phi_1 \otimes \mathrm{id}_{Q_2}} & A \overline{\otimes} Q_1 \overline{\otimes} Q_2 \\ \downarrow{\scriptstyle \mathbb{E}_{Q_2}} & & \downarrow{\scriptstyle \mathbb{E}_{Q_1} \otimes \mathrm{id}_{Q_2}} \\ Q_2 & \xrightarrow[1 \otimes \mathrm{id}_{Q_2}]{} & Q_1 \overline{\otimes} Q_2. \end{array}$$

Now, consider $V = \overline{\mathrm{Span}\{\phi(c)x : x \in Q, c \in C\}}^{\|\cdot\|_2}$, and note that V is invariant under multiplication on the right by elements of $Q = Q_1 \overline{\otimes} Q_2$. In the light of Remark 3.2, it suffices to show that $V = L^2(A \overline{\otimes} Q)$, and for this, since V is invariant under right multiplication by Q, it suffices to show that $A \otimes 1 \otimes 1 \subseteq V$. Recall that

$$\overline{\mathrm{Span}\{\phi_2(c)x_2 : c \in C, x_2 \in Q_2\}}^{\|\cdot\|_2} = L^2(B \overline{\otimes} Q_2) \supseteq B \overline{\otimes} Q_2.$$

Hence, we have

$$\overline{\mathrm{Span}\{(\phi_1 \otimes \mathrm{id}_{Q_2})(\phi_2(c)(1 \otimes x_2)) : c \in C, x_2 \in Q_2\}}^{\|\cdot\|_2} \supseteq (\phi_1 \otimes \mathrm{id}_{Q_2})(B \overline{\otimes} Q_2).$$

Moreover, since ϕ_1 is unital and $\phi_1 \otimes \mathrm{id}_{Q_2}$ is a homomorphism, we also have that

$$(\phi_1 \otimes \mathrm{id}_{Q_2})(\phi_2(c)(1 \otimes x_2)) = (\phi_1 \otimes \mathrm{id}_{Q_2})(\phi_2(c))(1 \otimes x_2) \quad \text{for all } c \in C, x_2 \in Q_2.$$

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Finally, since $\overline{\mathrm{Span}\{\phi_1(b)x_1 : b \in B, x_1 \in Q_1\}}^{\|\cdot\|_2} = L^2(A \overline{\otimes} Q_1) \supseteq A \overline{\otimes} Q_1$, the following computation completes the proof:

$$\begin{aligned} V &\supseteq \overline{\mathrm{Span}\{(\phi_1 \otimes \mathrm{id}_{Q_2})(\phi_2(c))(x_1 \otimes x_2) : c \in C, x_1 \in Q_1, x_2 \in Q_2\}}^{\|\cdot\|_2} \\ &\supseteq \overline{\mathrm{Span}\{(\phi_1 \otimes \mathrm{id}_{Q_2})((\phi_2(c)(1 \otimes x_2))(x_1 \otimes 1) : c \in C, x_1 \in Q_1, x_2 \in Q_2\}}^{\|\cdot\|_2} \\ &\supseteq \overline{\mathrm{Span}\{(\phi_1 \otimes \mathrm{id}_{Q_2})(b \otimes 1)(x_1 \otimes 1) : b \in B, x_1 \in Q_1\}}^{\|\cdot\|_2} \\ &\supseteq A \otimes 1 \otimes 1. \end{aligned}$$

■

Checking $\overline{\mathrm{Span}\{\phi(b)x : b \in B, x \in Q\}}^{\|\cdot\|_2} = L^2(A \overline{\otimes} Q)$ might not be easy in general. However, the following lemma simplifies verifying it in certain examples.

Lemma 3.6 *Let $(A, \tau_A), (B, \tau_B)$, and (Q, τ_Q) be tracial von Neumman algebras. Let $\phi: B \to A \overline{\otimes} Q$ be a $*$-homomorphism satisfying $\mathbb{E}_Q \circ \phi = \tau_B$, where $\mathbb{E}_Q : A \overline{\otimes} Q \to Q$ is the normal conditional expectation. Let $V = \overline{\mathrm{Span}\{\phi(b)x : b \in B, x \in Q\}}^{\|\cdot\|_2} \subset L^2(A \overline{\otimes} Q)$. Then, $N = \{a \in A : a \otimes 1 \in V\}$ is an SOT-closed subalgebra of A.*

Proof The fact that N is SOT-closed follows from the fact that SOT-convergence in A implies $\|\cdot\|_2$-convergence. First note that V is invariant under left multiplication by elements of $\phi(B)$ and right multiplication by elements of Q. We prove the following claim, whence the lemma follows immediately. ■

Claim *For $\eta \in V$ and $a \in N$, we have that $\eta(a \otimes 1) \in V$.*

Proof of Claim Given $\eta \in V$ and $a \in N$, let $\{x_n\}_{n \in \mathbb{N}} \subset \mathrm{Span}\{\phi(b)x : b \in B, x \in Q\}$ be such that $\|x_n - \eta\|_2 \to 0$. Since $a \otimes 1$ is bounded, it follows that

$$\|x_n(a \otimes 1) - \eta(a \otimes 1)\|_2 \leq \|x_n - \eta\|_2 \|a\| \to 0,$$

as $n \to \infty$. Since V is $\|\cdot\|_2$-closed, it suffices to show that $x_n(a \otimes 1) \in V$ for all $n \in \mathbb{N}$. To this end, fix $n \in \mathbb{N}$, and write $x_n = \sum_{j=1}^{k} \phi(b_j) y_j$ with $b_j \in B$, $y_j \in Q$. Then,

$$x_n(a \otimes 1) = \sum_{j=1}^{k} \phi(b_j) y_j (a \otimes 1) = \sum_{j=1}^{k} \phi(b_j)(a \otimes 1) y_j,$$

where, in the last equality, we use that A and Q commute in $A \overline{\otimes} Q$. Since we already noted that V is invariant under left multiplication by $\phi(B)$ and right multiplication by Q, and $a \otimes 1 \in V$, it follows that $x_n(a \otimes 1) \in V$. ■

Remark 3.7 We do *not* know if N is a $*$-subalgebra.

Theorem 3.8 *If $(A_i, \tau_{A_i}), (B_i, \tau_{B_i})$, $i = 1, 2$ are tracial von Neumann algebras such that $(A_i, \tau_{A_i}) \sim_{\mathrm{vNOE}} (B_i, \tau_{B_i})$, $i = 1, 2$, then, $(A_1 * A_2, \tau_{A_1} * \tau_{A_2}) \sim_{\mathrm{vNOE}} (B_1 * B_2, \tau_{B_1} * \tau_{B_2})$.*

Proof Since vNOE is an equivalence relation, it suffices to show that if $A \sim_{\mathrm{vNOE}} B$ and if (C, τ_C) is another tracial von Neumann algebra, then $A * C \sim_{\mathrm{vNOE}} B * C$. Let (Q, τ_Q) be a tracial von Neumann algebra and $\phi: B \to A \overline{\otimes} Q$ be a $*$-homomorphism as in condition (2) of Theorem 3.1. For tracial von Neumann algebra (C, τ_C), we view $(A * C) \overline{\otimes} Q$ as $(A \overline{\otimes} Q) *_Q (C \overline{\otimes} Q)$. Define $\widetilde{\phi}_0: B *_{\mathrm{alg}} C \to (A \overline{\otimes} Q) *_Q (C \overline{\otimes} Q)$ by

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declaring that $\widetilde{\phi}_0(b) = \phi(b) *_Q 1$ for $b \in B$ and $\widetilde{\phi}_0(c) = 1 *_Q c$ for $c \in C$. Let $\mathbb{E}_Q : (A * C) \overline{\otimes} Q \to Q$ be the normal conditional expectation. Note that $\mathbb{E}_Q \circ \phi = \tau_B$ and $\mathbb{E}_Q|_{C\overline{\otimes}Q} = \tau_C \otimes \mathrm{id}_Q$. Therefore, if $x \in B *_{\mathrm{alg}} C$ is an alternating centered word with respect to τ_{B*C}, then $\widetilde{\phi}_0(x)$ is an alternating centered word with respect to $\mathbb{E}_Q$. Hence, $\mathbb{E}_Q \circ \widetilde{\phi}_0 = \tau_{B*C}$. Since $\mathbb{E}_Q$ is trace-preserving, it follows that $\widetilde{\phi}_0$ is trace-preserving, and thus extends to a unique trace-preserving $*$-homomorphism $\widetilde{\phi}: B * C \to (A\overline{\otimes}Q) *_Q (C\overline{\otimes}Q)$. Moreover, by continuity, we still have $\mathbb{E}_Q \circ \widetilde{\phi} = \tau_{B*C}$. In light of Remark 3.2, it thus remains to check that $\overline{\mathrm{Span}\{\widetilde{\phi}(x)y : x \in B * C, y \in Q\}}^{\|\cdot\|_2} = L^2((A * C)\overline{\otimes} Q)$. To this end, set $V = \overline{\mathrm{Span}\{\widetilde{\phi}(x)y : x \in B * C, y \in Q\}}^{\|\cdot\|_2}$. Since V is invariant under right multiplication by Q, to show that $V = L^2((A * C)\overline{\otimes} Q)$, it suffices to show that $(A * C) \otimes 1 \subseteq V$. For this, by Lemma 3.6, it suffices to show that V contains $A \otimes 1$ and $C \otimes 1$. That $C \otimes 1 \subseteq V$ follows from the fact that $\widetilde{\phi}$ takes the copy of C in $B * C$ to the copy of C in $(A * C)\overline{\otimes}Q$, and since $\overline{\mathrm{Span}\{\phi(b)y : b \in B, y \in Q\}}^{\|\cdot\|_2} = L^2(A\overline{\otimes}Q)$, we also have that $A \otimes 1 \subseteq V$. ∎

We conclude this section by proving Theorem 1.7, which we recall below.

Theorem 3.9 *Let $\mathcal{G} = (V, E)$ be a simple graph, with at most countably infinite vertices. Let (A, τ_A) and (B, τ_B) be two graph products over $\mathcal{G}$, with tracial vertex von Neumann algebras $\{(A_v, \tau_{A_v})\}_{v\in V}$ and $\{(B_v, \tau_{B_v})\}_{v\in V}$, respectively. If $(A_v, \tau_{A_v}) \sim_{\mathrm{vNOE}} (B_v, \tau_{B_v})$ for every $v \in V$, then $(A, \tau_A) \sim_{\mathrm{vNOE}} (B, \tau_B)$.*

Proof Let (Q_v, ϕ_v) be a vNOE-coupling between (A_v, τ_{A_v}) and (B_v, τ_{B_v}), where $\phi_v : A_v \to B_v \overline{\otimes} Q_v$ is a normal unital $*$-homomorphism satisfying

$$\mathbb{E}_v \circ \phi_v = \tau_{A_v} \qquad \text{and} \qquad \overline{\mathrm{Span}\{\phi_v(a)x : a \in A_v, x \in Q_v\}}^{\|\cdot\|_2} = L^2(B_v \overline{\otimes} Q_v),$$

here, $\mathbb{E}_v : B_v \overline{\otimes} Q_v \to Q_v$ is the normal conditional expectation. Define (Q, τ_Q) to be the tensor product $\overline{\otimes}_{v\in V}(Q_v, \tau_{Q_v})$. We now define a $*$-homomorphism $\phi : A \to B\overline{\otimes} Q$ and show that it has the desired properties. To this end, first note that if v and w are two distinct vertices connected by an edge, then B_v commutes with B_w and Q_v commutes with Q_w as $v \neq w$. Thus, $\phi_v(A_v)$ and $\phi_w(A_w)$ commute inside $B\overline{\otimes} Q$, and therefore, we can define ϕ on the algebraic graph product (i.e., the universal unital $*$-algebra generated by the $\{A_v\}_{v\in V}$ subject to the relation that A_v commutes with A_w whenever v and w are connected by an edge). In particular, note that $\phi(a) = \phi_v(a)$, whenever $a \in A_v$. Next, we show that ϕ extends to a normal unital $*$-homomorphism on A and satisfies $\mathbb{E}_Q(\phi(a)) = \tau_A(a)$ for all $a \in A$, where $\mathbb{E}_Q : B\overline{\otimes} Q \to Q$ is the normal conditional expectation. To see this, we note that for any reduced word $v = v_1 \dots v_n$ of vertices, and elements $a_i \in A_{v_i}$ of trace zero, $1 \leq i \leq n$, we have that

$$\mathbb{E}_Q(\phi_{v_1}(a_1) \dots \phi_{v_n}(a_n)) = 0.$$

Indeed, for every $i \in \{1, \dots, n\}$, we have $\mathbb{E}_{v_i}(\phi_{v_i}(a_i)) = \tau_{A_{v_i}}(a_i) = 0$, so, $\phi_{v_i}(a_i)$ can be approximated by linear combinations of elements of the form $b_{v_i} \otimes q_{v_i}$, where $b_{v_i} \in B_{v_i}, q_{v_i} \in Q_{v_i}$ with $\tau_{B_{v_i}}(b_{v_i}) = 0$. Thus, $\mathbb{E}_Q(\phi_{v_1}(a_1) \dots \phi_{v_n}(a_n)) = 0$ and hence it follows that the map ϕ, defined on the algebraic graph product, is trace-preserving. Therefore, by [CF17, Proposition 3.22], ϕ extends to a well-defined normal unital $*$-homomorphism on A. Furthermore, if either $a = 1$ or $a \in A$ is any reduced

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operator (see [CF17, Definition 3.10]), then we have seen that $\mathbb{E}_Q(\phi(a)) = 0 = \tau_A(a)$. Since the linear span of 1 and reduced operators is a dense $*$-subalgebra of A, we conclude that $\mathbb{E}_Q(\phi(a)) = \tau_A(a)$ for all $a \in A$. Finally, to show that $\overline{\mathrm{Span}\{\phi(a)x : a \in A, x \in Q\}}^{\|\cdot\|_2} = L^2(B \overline{\otimes} Q)$, it suffices to show that

$$(*) \qquad \overline{\mathrm{Span}(\phi_{v_1}(A_{v_1}) \dots \phi_n(A_{v_n})(1 \otimes Q))}^{\|\cdot\|_2} = \overline{\mathrm{Span}(B_{v_1} \dots B_{v_n}(1 \otimes Q))}^{\|\cdot\|_2},$$

for any reduced word $v = v_1 \dots v_n$ of vertices and for all n. We proceed by induction. It is straightforward to see that for any vertex v, $\overline{\mathrm{Span}(\phi_v(A_v)(1 \otimes Q))}^{\|\cdot\|_2} = L^2(B_v \overline{\otimes} Q) = \overline{\mathrm{Span}(B_v(1 \otimes Q))}^{\|\cdot\|_2}$. Suppose that $(*)$ holds for any reduced word $v_1 \dots v_n$ of length n and let $v_1 \dots v_n v_{n+1}$ be any reduced word of length $n+1$. Then,

$$\begin{aligned}
&\overline{\mathrm{Span}(\phi_{v_1}(A_{v_1}) \dots \phi_{v_{n+1}}(A_{n+1})(1 \otimes Q))}^{\|\cdot\|_2} \\
&= \overline{\mathrm{Span}(\phi_{v_1}(A_{v_1}) B_{v_2} \dots B_{v_{n+1}}(1 \otimes Q))}^{\|\cdot\|_2} \\
&= \overline{\mathrm{Span}(\phi_{v_1}(A_{v_1})(1 \otimes Q)(B_{v_2} \dots B_{v_{n+1}} \otimes 1))}^{\|\cdot\|_2} \\
&= \overline{\mathrm{Span}(B_{v_1}(1 \otimes Q)(B_{v_2} \dots B_{v_{n+1}} \otimes 1))}^{\|\cdot\|_2} \\
&= \overline{\mathrm{Span}(B_{v_1} B_{v_2} \dots B_{v_{n+1}}(1 \otimes Q))}^{\|\cdot\|_2}.
\end{aligned}$$

Thus, $(A, \tau_A) \sim_{\mathrm{vNOE}} (B, \tau_B)$. ∎

3.2 Von Neumann orbit equivalence for groups

Let $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ be an action of a countable discrete group Γ on a von Neumann algebra $\mathcal{M}$. A *fundamental domain* for the action is a projection $p \in \mathcal{M}$ such that the projections $\{\sigma_\gamma(p)\}_{\gamma \in \Gamma}$ are pairwise orthogonal and $\sum_{\gamma \in \Gamma} \sigma_\gamma(p) = 1$, where the sum converges in the strong operator topology. Two countable discrete groups Γ and Λ are said to be *vNE*, denoted $\Gamma \sim_{\mathrm{vNE}} \Lambda$, if there exists a semi-finite von Neumann algebra $(\mathcal{M}, \mathrm{Tr})$ with a faithful normal semi-finite trace Tr, and commuting trace-preserving actions $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ and $\Lambda \curvearrowright^{\alpha} \mathcal{M}$ such that each action admits a finite-trace fundamental domain. Such an $\mathcal{M}$ is called a *von Neumann coupling* between Γ and Λ.

Definition 3.10 Two countable groups Γ and Λ are said to be *vNOE*, denoted $\Gamma \sim_{\mathrm{vNOE}} \Lambda$, if there exists a von Neumann coupling between Γ and Λ with a common fundamental domain.

Theorem 3.11 *For countable discrete groups* Γ *and* Λ, $\Gamma \sim_{\mathrm{vNOE}} \Lambda$ *if and only if* $L\Gamma \sim_{\mathrm{vNOE}} L\Lambda$.

Proof First, suppose that $L\Gamma \sim_{\mathrm{vNOE}} L\Lambda$, and let $(\mathcal{H}, Q, \xi)$ be a triple as in condition (1) of Theorem 3.1. Set $A = L\Gamma$ and $B = L\Lambda$, and consider $\mathcal{M} = Q' \cap \mathcal{B}(\mathcal{H}) = {}_Q\mathcal{B}(\mathcal{H})$. For $\gamma \in \Gamma$, let $u_\gamma \in L\Gamma$ be the corresponding unitary and for $T \in \mathcal{B}(\mathcal{H})$, define $\sigma_\gamma(T) = u_\gamma T u_\gamma^*$. Since $L\Gamma$- and Q-actions on $\mathcal{H}$ commute, it follows that $\mathcal{M}$ is invariant under σ_γ for each $\gamma \in \Gamma$, and thus we have an action $\Gamma \curvearrowright^{\sigma} \mathcal{M}$. Similarly, since $\mathcal{H}$ is a $Q - B$-bimodule, we have an action $\Lambda \curvearrowright^{\alpha} \mathcal{M}$ given by $\alpha_s(T) = v_s^* T v_s, s \in \Lambda, T \in \mathcal{M}$, where

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$v_s \in L\Lambda$ is the unitary corresponding to $s \in \Lambda$. It is clear that the actions $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ and $\Lambda \curvearrowright^{\alpha} \mathcal{M}$ commute. Let Tr be the canonical trace on $\mathcal{M}$ given by Proposition 2.4. Since $L\Gamma$- and Q-actions on $\mathcal{H}$ commute, we have that for every $\gamma \in \Gamma$, u_γ is a Q-linear operator on $\mathcal{H}$ and hence belongs to $\mathcal{M}$. Therefore, it now follows from the tracial property that $\mathrm{Tr}(\sigma_\gamma(T)) = \mathrm{Tr}(u_\gamma T u_\gamma^*) = \mathrm{Tr}(T)$ for all $\gamma \in \Gamma$, $T \in \mathcal{M}$, and hence $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ is trace-preserving. Similarly, $\Lambda \curvearrowright^{\alpha} \mathcal{M}$ too is trace-preserving.

Let $P \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection from $\mathcal{H}$ onto $\overline{\mathrm{Span}(Q\xi)}$. It is straightforward to see that P is Q-linear and thus, $P \in \mathcal{M}$. Moreover, it follows from Remark 2.5 that $\mathrm{Tr}(P) = 1$. Therefore, it only remains to show that P is a fundamental domain for both Γ- and Λ-actions. To see that P is a Γ-fundamental domain, we first note that, since ξ is tracial for the $A \overline{\otimes} Q$-module structure, we have, for $a \in A, x \in Q$, that

$$P((a \otimes x)\xi) = \tau_A(a)x\xi.$$

Furthermore, if $a = \sum_{\gamma\in\Gamma} a_\gamma u_\gamma$ is the Fourier series expansion of $a \in A$, then we recall that $\tau_A(a) = a_e$, and thus $P((a \otimes x)\xi) = a_e x\xi$. Therefore,

$$\begin{aligned}\sigma_\gamma(P)((a \otimes x)\xi) &= u_\gamma P u_\gamma^*((a \otimes x)\xi)\\ &= u_\gamma P\left(\left(\sum_{g\in\Gamma} a_g u_{\gamma^{-1}g} \otimes x\right)\xi\right)\\ &= a_\gamma(u_\gamma \otimes x)\xi.\end{aligned}$$

If we let P_γ be the orthogonal projection from $\mathcal{H}$ onto $u_\gamma(\overline{\mathrm{Span}(Q\xi)})$, then it follows from the above calculation that $\sigma_\gamma(P) = P_\gamma$, and it is straightforward to check that the projections $\{P_\gamma\}_{\gamma\in\Gamma}$ are pairwise orthogonal. Moreover, since $\mathcal{H} = \overline{\mathrm{Span}(A \overline{\otimes} Q)\xi}$, we also get that $\sum_{\gamma\in\Gamma} \sigma_\gamma(P) = 1$ and hence P is a Γ-fundamental domain. Since we also have that ξ is bi-tracial for the $Q - B$-bimodule structure, we observe that, for $b \in B, x \in Q$,

$$P(x\xi b) = \tau_B(b)x\xi.$$

If $b = \sum_{t\in\Lambda} b_t v_t$ is the Fourier series expansion of $b \in B$, then $\tau_B(b) = b_e$, and hence $P(x\xi b) = b_e x\xi$. For $s \in \Lambda$, let P_s be the orthogonal projection from $\mathcal{H}$ onto $(\overline{\mathrm{Span}(Q\xi)})v_s$. Now, the following calculation shows that $\alpha_s(P) = P_s$, so, $\{\alpha_s(P)\}_{s\in\Lambda}$ are pairwise orthogonal, and thus P is a Λ-fundamental domain since $\mathcal{H} = \overline{\mathrm{Span}(Q\xi B)}$.

$$\begin{aligned}\alpha_s(P)(x\xi b) &= v_s^* P v_s\left(x\xi \sum_{t\in\Lambda} b_t v_t\right)\\ &= v_s^*(b_{s^{-1}} x\xi)\\ &= b_{s^{-1}} x\xi v_{s^{-1}}.\end{aligned}$$

Conversely, suppose $\Gamma \sim_{\mathrm{vNOE}} \Lambda$, and let $(\mathcal{M}, \mathrm{Tr})$ be a von Neumann coupling between Γ and Λ with common fundamental domain $p \in \mathcal{M}$ for both $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ and $\Lambda \curvearrowright^{\alpha} \mathcal{M}$. Let $A = L\Gamma, B = L\Lambda, \mathcal{H} = L^2(\mathcal{M}, \mathrm{Tr}) \overline{\otimes} \ell^2(\Lambda), Q = \mathcal{M}^\Gamma \rtimes \Lambda$, and $\xi = p \otimes \delta_e$. Let τ be the trace on $\mathcal{M}^\Gamma$, which we recall is given by $\tau(x) = \mathrm{Tr}(pxp)$ (see [IPR24,

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Proposition 4.2]). Consider the action of $L\Gamma$ on $\mathcal{H}$ given by

$$u_\gamma \eta = (\sigma^0_\gamma \otimes \mathrm{id})\eta, \qquad \gamma \in \Gamma, \eta \in \mathcal{H},$$

where σ^0_γ is the Koopman representation of Γ into $\mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$. The action of $\mathcal{M}^\Gamma$ on $\mathcal{H}$ is given by

$$x\eta = (x \otimes \mathrm{id})\eta, \qquad x \in \mathcal{M}^\Gamma, \eta \in \mathcal{H},$$

and Λ acts on $\mathcal{H}$ on the left by

$$v_s \eta = (\alpha^0_s \otimes \lambda_\Lambda(s))\eta, \qquad s \in \Lambda, \eta \in \mathcal{H},$$

where $\lambda_\Lambda : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the left regular representation, and $\alpha^0_s : \Lambda \to \mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$ is the Koopman representation implementing the Λ-action. Since the Γ- and Λ-actions on $\mathcal{M}$ commute, the actions defined above make $\mathcal{H}$ into a left $L\Gamma \overline{\otimes} (\mathcal{M}^\Gamma \rtimes \Lambda)$-module. Furthermore, for $g \in \Gamma, s \in \Lambda$, and $x \in \mathcal{M}^\Gamma$, we have

$$\begin{aligned}
\langle (u_g \otimes x v_s)\xi, \xi \rangle &= \langle (u_g \otimes x v_s)(p \otimes \delta_e), p \otimes \delta_e \rangle \\
&= \langle \sigma_g(x\alpha_s(p)) \otimes \delta_s, p \otimes \delta_e \rangle \\
&= \delta_{s,e} \mathrm{Tr}(x\sigma_g(p)p) \\
&= \delta_{s,e}\delta_{g,e} \mathrm{Tr}(pxp) \\
&= \delta_{s,e}\delta_{g,e}\tau(x) \\
&= \tau_A(u_g)\tau_Q(xv_s).
\end{aligned}$$

Thus, it follows that ξ is tracial for the left $L\Gamma \overline{\otimes} (\mathcal{M}^\Gamma \rtimes \Lambda)$-module structure. For a fixed $s \in \Lambda$, we have

$$\begin{aligned}
&\overline{\mathrm{Span}\{(u_g \otimes x v_s)\xi : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} \\
&= \overline{\mathrm{Span}\{\sigma_g(x\alpha_s(p)) \otimes \delta_s : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} \\
&= \overline{\mathrm{Span}\{x\alpha_s(\sigma_g(p)) \otimes \delta_s : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} \\
&= \overline{\mathrm{Span}\{\alpha_s(\alpha_{s^{-1}}(x)\sigma_g(p)) \otimes \delta_s : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} \\
&= \overline{\mathrm{Span}\{\alpha_s(y\sigma_g(p)) \otimes \delta_s : g \in \Gamma, y \in \mathcal{M}^\Gamma\}} \\
&= (\alpha^0_s \otimes \lambda_\Lambda(s))(\overline{\mathrm{Span}\{y\sigma_g(p) \otimes \delta_e : g \in \Gamma, y \in \mathcal{M}^\Gamma\}}) \\
&= L^2(\mathcal{M}, \mathrm{Tr}) \overline{\otimes} \mathbb{C}\delta_s,
\end{aligned}$$

where the last equality follows from the fact that $\overline{\mathrm{Span}\{x\sigma_g(p) : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} = L^2(\mathcal{M}, \mathrm{Tr})$ [IPR24, Proposition 4.2]. Therefore, we have

$$\begin{aligned}
\overline{\mathrm{Span}((A \overline{\otimes} Q)\xi)} &= \overline{\mathrm{Span}\{(u_g \otimes x v_s)\xi : g \in \Gamma, x \in \mathcal{M}^\Gamma, s \in \Lambda\}} \\
&= \overline{\mathrm{Span}(L^2(\mathcal{M}, \mathrm{Tr}) \overline{\otimes} \mathbb{C}\delta_s : s \in \Lambda)} = \mathcal{H}.
\end{aligned}$$

Finally, the right action of $L\Lambda$ on $\mathcal{H}$ given by

$$\eta v_s = (\mathrm{id} \otimes \rho_\Lambda(s^{-1}))\eta, \qquad s \in \Lambda, \eta \in \mathcal{H},$$

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where $\rho_\Lambda : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the right regular representation, makes $\mathcal{H}$ into a $Q - L\Lambda$-bimodule. For $x \in \mathcal{M}^\Gamma$ and $s, t \in \Lambda$, we have

$$\langle xv_s(p \otimes \delta_e)v_t, p \otimes \delta_e\rangle = \langle x\alpha_s(p) \otimes \delta_{st}, p \otimes \delta_e\rangle = \delta_{s,e}\delta_{t,e}\mathrm{Tr}(pxp) = \tau_Q(xv_s)\tau_B(v_t),$$

and hence, ξ is a tracial vector for the $Q - B$-bimodule structure. We recall from the proof of [IPR24, Proposition 4.2], that, since p is Λ-fundamental domain, we have a direct sum decomposition $L^2(\mathcal{M}, \mathrm{Tr}) = \sum_{s\in\Lambda} L^2(\mathcal{M}, \mathrm{Tr})\alpha_s(p)$. Thus, to show that $\overline{\mathrm{Span}(Q\xi B)} = \mathcal{H}$, it suffices to show that, for $s \in \Lambda$, $\overline{\mathrm{Span}\{xv_s\xi v_t : x \in \mathcal{M}^\Gamma, t \in \Lambda\}} = L^2(\mathcal{M}, \mathrm{Tr})\alpha_s(p)\overline{\otimes}\ell^2\Lambda$. To this end, fix $s \in \Lambda$ and note that

$$\begin{aligned}\overline{\mathrm{Span}\{xv_s(p \otimes \delta_e)v_t : x \in \mathcal{M}^\Gamma, t \in \Lambda\}} &= \overline{\mathrm{Span}\{\alpha_s(\alpha_{s^{-1}}(x)p) \otimes \delta_{st} : x \in \mathcal{M}^\Gamma, t \in \Lambda\}}\\ &= (\alpha_s^0 \otimes \lambda_\Lambda(s))(\overline{\mathrm{Span}\{yp \otimes \delta_t : y \in \mathcal{M}^\Gamma, t \in \Lambda\}})\\ &= (\alpha_s^0 \otimes \lambda_\Lambda(s))(L^2(\mathcal{M}, \mathrm{Tr})p\overline{\otimes}\ell^2\Lambda)\\ &= L^2(\mathcal{M}, \mathrm{Tr})\alpha_s(p)\overline{\otimes}\ell^2\Lambda.\end{aligned}$$

■

3.3 Relationship to von Neumann equivalence

Definition 3.12 [IPR24] Let A and B be tracial von Neumann algebras and let $\mathcal{M}$ be a semi-finite von Neumann algebra such that $A \subset \mathcal{M}$ and $B^{\mathrm{op}} \subset \mathcal{M}$.

(1) A *fundamental domain for A inside of* $\mathcal{M}$ consists of a realization of the standard representation $A \subset \mathcal{B}(L^2(A))$ as an intermediate von Neumann subalgebra $A \subset \mathcal{B}(L^2(A)) \subset \mathcal{M}$. The fundamental domain is *finite* if finite-rank projections in $\mathcal{B}(L^2(A))$ are mapped to finite projections in $\mathcal{M}$.
(2) $\mathcal{M}$ is a *von Neumann coupling between A and B* if $B^{\mathrm{op}} \subset A' \cap \mathcal{M}$ and each inclusion $A \subset \mathcal{M}$ and $B^{\mathrm{op}} \subset \mathcal{M}$ has a finite fundamental domain.

Definition 3.13 [IPR24] Two tracial von Neumann algebras A and B are *vNE*, denoted $A \sim_{\mathrm{vNE}} B$, if there exists a von Neumann coupling between them.

Proposition 3.14 *Let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras. If $A \sim_{\mathrm{vNOE}} B$, then $A \sim_{\mathrm{vNE}} B$.*

Proof Suppose $(\mathcal{H}, Q, \xi)$ is a triple as in condition (1) of Theorem 3.1. As in the proof of (1) implies (2) in Theorem 3.1, let $U : \mathcal{H} \to L^2(A\overline{\otimes}Q)$ be the unitary such that $U(x\xi) = \hat{x}$ for all $x \in A\overline{\otimes}Q$, and let $\phi : B \to A\overline{\otimes}Q$ be the $*$-homomorphism obtained therein. Let $\mathcal{M} = Q' \cap \mathcal{B}(\mathcal{H}) = \mathcal{B}(L^2(A))\overline{\otimes}Q^{\mathrm{op}}$. We will show that $\mathcal{M}$ is a von Neumann coupling between A and B. It is clear that the inclusion $A \subset \mathcal{M}$ has a finite fundamental domain. We recall that the argument used in defining ϕ shows that we have an inclusion $B^{\mathrm{op}} \subset \mathcal{M}$ and moreover, since the left A- and right B-actions on $\mathcal{H}$ commute, we have that $B^{\mathrm{op}} \subset A' \cap \mathcal{M}$. Thus, it only remains to show that the inclusion $B^{\mathrm{op}} \subset \mathcal{M}$ has a finite fundamental domain. To this end, note that we can also view $\mathcal{H}$ as an $A - B\overline{\otimes}Q^{\mathrm{op}}$-bimodule, and ξ is tracial and cyclic for the right $B\overline{\otimes}Q^{\mathrm{op}}$-module structure. Thus, by the same construction as above, we get an inclusion $B^{\mathrm{op}} \subset Q^{\mathrm{op}\prime} \cap \mathcal{B}(\mathcal{H}) = \mathcal{B}(L^2(B))\overline{\otimes}Q$. Since $\mathcal{M} = Q' \cap \mathcal{B}(\mathcal{H}) = Q^{\mathrm{op}\prime} \cap \mathcal{B}(\mathcal{H})$, we get that the inclusion $B^{\mathrm{op}} \subset \mathcal{M}$ admits a finite fundamental domain and hence $A \sim_{\mathrm{vNE}} B$. ■

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4 Toward an analog of Singer's theorem

The main goal of this section is to prove the following theorem.

Theorem 4.1 *If Γ and Λ are countable discrete groups such that $\Gamma \sim_{\mathrm{vNOE}} \Lambda$, then there exist tracial von Neumann algebras $(A, \tau_A), (B, \tau_B)$, trace-preserving actions $\Gamma \curvearrowright A, \Lambda \curvearrowright B$, and a trace-preserving isomorphism $\theta : B \rtimes \Lambda \to A \rtimes \Gamma$.*

Before proving the theorem, we first set up some notations and recall a few facts. Let Γ be a countable discrete group and (M, τ) be a tracial von Neumann algebra. A *1-cocycle* for a trace-preserving action $\Gamma \overset{\alpha}{\curvearrowright} (M, \tau)$ is a map $w : \Gamma \to \mathcal{U}(M)$ that satisfies the following cocycle identity:

$$w_s \alpha_s(w_t) = w_{st}, \quad s, t \in \Gamma.$$

If $\Gamma \overset{\beta}{\curvearrowright} (M, \tau)$ is another trace-preserving action, then we say that α and β are *cocycle conjugate* if there exists an automorphism $\theta \in \mathrm{Aut}(M, \tau)$ and a 1-cocycle $w : \Gamma \to \mathcal{U}(M)$ for α such that

$$\theta \circ \beta_s \circ \theta^{-1} = \mathrm{Ad}(w_s) \circ \alpha_s, \quad s \in \Gamma. \tag{4.1}$$

We recall that if $\Gamma \overset{\alpha}{\curvearrowright} (M, \tau)$ and $\Gamma \overset{\beta}{\curvearrowright} (M, \tau)$ are cocycle conjugate, then $M \rtimes_\alpha \Gamma \cong M \rtimes_\beta \Gamma$. Indeed, let $\theta \in \mathrm{Aut}(M, \tau)$ and $w : \Gamma \to \mathcal{U}(M)$ be as in (4.1), and consider the map $\Theta : M \rtimes_\alpha \Gamma \to M \rtimes_\beta \Gamma$ given by

$$\Theta(x u_s) = \mathrm{Ad}(w_s)(\theta(x)) v_s, \quad x \in M, s \in \Gamma,$$

where, for $s \in \Gamma$, u_s, v_s represent the canonical group unitaries in $M \rtimes_\alpha \Gamma, M \rtimes_\beta \Gamma$, respectively. It is then straightforward to verify that Θ is an isomorphism.

Proof of Theorem 4.1 Let $\Gamma \overset{\sigma}{\curvearrowright} \mathcal{M}$ and $\Lambda \overset{\alpha}{\curvearrowright} \mathcal{M}$ be commuting, trace-preserving actions of countable discrete groups Γ and Λ on a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace Tr. Let $p \in \mathcal{M}$ be a finite-trace projection which is a common fundamental domain for both Γ- and Λ-actions, that is, $\{\sigma_\gamma(p)\}_{\gamma \in \Gamma}$ are mutually orthogonal and $\sum_{\gamma \in \Gamma} \sigma_\gamma(p) = 1$ (SOT), and similarly, $\{\alpha_\lambda(p)\}_{\lambda \in \Lambda}$ are mutually orthogonal and $\sum_{\lambda \in \Lambda} \alpha_\lambda(p) = 1$ (SOT). From [IPR24, Proposition 4.2], there exists a unitary $\mathcal{F}_p : \ell^2\Gamma \mathbin{\overline{\otimes}} L^2(\mathcal{M}^\Gamma, \tau) \to L^2(\mathcal{M}, \mathrm{Tr})$ such that

$$\mathcal{F}_p(\delta_\gamma \otimes x) = \sigma_{\gamma^{-1}}(p)x \qquad \text{for all } \gamma \in \Gamma, x \in \mathcal{M}^\Gamma,$$

where the trace τ on $\mathcal{M}^\Gamma$ is given by $\tau(x) = \mathrm{Tr}(pxp)$, $x \in \mathcal{M}^\Gamma$.[1] Furthermore, from [IPR24, Proposition 4.3], there is a trace-preserving isomorphism $\Delta_p^\Gamma : \mathcal{M} \rtimes \Gamma \to \mathcal{B}(\ell^2\Gamma) \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma$ such that for $\gamma \in \Gamma$ and $x \in \mathcal{M}$,

$$\Delta_p^\Gamma(u_\gamma) = \rho_\gamma \otimes 1, \qquad \Delta_p^\Gamma(x) = \mathcal{F}_p^* x \mathcal{F}_p.$$

If we view $\mathcal{B}(\ell^2\Gamma) \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma$ as $\mathcal{M}^\Gamma$-valued $\Gamma \times \Gamma$ matrices, then we have that for all $x \in \mathcal{M}$, $\Delta_p^\Gamma(x) = [x_{s,t}]_{s,t}$, where

$$x_{s,t} = \sum_{\gamma \in \Gamma} \sigma_\gamma(\sigma_{t^{-1}}(p) x \sigma_{s^{-1}}(p)) \in \mathcal{M}^\Gamma. \qquad \blacksquare$$

[1] Note that the formula for $\mathcal{F}_p$ in [IPR24, Proposition 4.2] requires $x \in \mathfrak{n}_\tau \subset \mathcal{M}^\Gamma$, where τ is the semi-finite trace on $\mathcal{M}^\Gamma$ given by $\tau(x) = \mathrm{Tr}(pxp)$. However, when $\mathrm{Tr}(p) < \infty$, τ is a finite trace and thus we can have $x \in \mathcal{M}^\Gamma$ in the formula for $\mathcal{F}_p$.

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Since the actions of Γ and Λ on $\mathcal{M}$ commute, we get a well-defined action of Λ on $\mathcal{M} \rtimes \Gamma$, which we denote by $\alpha \rtimes \mathrm{id}_\Gamma$, and it is given by

$$(\alpha_\lambda \rtimes \mathrm{id}_\Gamma)(xu_\gamma) = \alpha_\lambda(x)u_\gamma, \quad \lambda \in \Lambda, \gamma \in \Gamma, x \in \mathcal{M}.$$

Further, let $\mathrm{id} \otimes \alpha$ be the action of Λ on $\mathcal{B}(\ell^2\Gamma) \overline{\otimes} \mathcal{M}^\Gamma$ given by

$$(\mathrm{id} \otimes \alpha_\lambda)(T \otimes x) = T \otimes \alpha_\lambda(x), \quad \lambda \in \Lambda, T \in \mathcal{B}(\ell^2\Gamma), x \in \mathcal{M}^\Gamma.$$

Define an action $\tilde{\alpha}$ of Λ on $\mathcal{B}(\ell^2\Gamma) \overline{\otimes} \mathcal{M}^\Gamma$ by

$$\tilde{\alpha}_\lambda = \Delta_p^\Gamma \circ (\alpha_\lambda \rtimes \mathrm{id}_\Gamma) \circ (\Delta_p^\Gamma)^{-1}, \ \lambda \in \Lambda.$$

By definition, $\tilde{\alpha}$ is conjugate to $\alpha \rtimes \mathrm{id}_\Gamma$, and hence we get an isomorphism of the crossed products

$$\mathcal{M} \rtimes (\Gamma \times \Lambda) = (\mathcal{M} \rtimes \Gamma) \rtimes_{\alpha \rtimes \mathrm{id}_\Gamma} \Lambda \overset{\Delta_p^\Gamma \rtimes \mathrm{id}_\Gamma}{\underset{\cong}{\longrightarrow}} (\mathcal{B}(\ell^2\Gamma) \overline{\otimes} \mathcal{M}^\Gamma) \rtimes_{\tilde{\alpha}} \Lambda.$$

Now, we note that for any $\lambda \in \Lambda$, $\alpha_{\lambda^{-1}}(p)$ is again a Γ-fundamental domain and hence we can consider the map $\Delta^\Gamma_{\alpha_{\lambda^{-1}}(p)} : \mathcal{M} \rtimes \Gamma \to \mathcal{B}(\ell^2) \otimes \mathcal{M}^\Gamma$ defined analogously to Δ_p^Γ. For any $x \in \mathcal{M}$ and $s, t \in \Gamma$, by a slight abuse of notation, let $[\Delta_p^\Gamma(x)]_{s,t}$ be the (s,t)-entry in the matrix representation of $\Delta_p^\Gamma(x)$. Similarly, consider $[\Delta^\Gamma_{\alpha_{\lambda^{-1}}(p)}(x)]_{s,t}$. We observe that

$$\begin{aligned}
\alpha_\lambda([\Delta^\Gamma_{\alpha_{\lambda^{-1}}(p)}(x)]_{s,t}) &= \alpha_\lambda\left(\sum_{\gamma\in\Gamma} \sigma_\gamma(\sigma_{t^{-1}}(\alpha_{\lambda^{-1}}(p))x\sigma_{s^{-1}}(\alpha_{\lambda^{-1}}(p)))\right) \\
&= \sum_{\gamma\in\Gamma} \sigma_\gamma(\sigma_{t^{-1}}(p)\alpha_\lambda(x)\sigma_{s^{-1}}(p)) \\
&= [\Delta_p^\Gamma(\alpha_\lambda(x))]_{s,t},
\end{aligned}$$

whence it follows that $\Delta_p^\Gamma(\alpha_\lambda(x)) = (\mathrm{id} \otimes \alpha_\lambda)(\Delta^\Gamma_{\alpha_{\lambda^{-1}}(p)}(x))$ for all $x \in \mathcal{M}$ and $\lambda \in \Lambda$. Therefore, for any $x \in \mathcal{M}$, $\gamma \in \Gamma$, and $\lambda \in \Lambda$, we have

$$\begin{aligned}
(\Delta_p^\Gamma \circ (\alpha_\lambda \rtimes \mathrm{id}_\Gamma))(xu_\gamma) &= \Delta_p^\Gamma(\alpha_\lambda(x)u_\gamma) \\
&= \Delta_p^\Gamma(\alpha_\lambda(x))\Delta_p^\Gamma(u_\gamma) \\
&= (\mathrm{id} \otimes \alpha_\lambda)(\Delta^\Gamma_{\alpha_{\lambda^{-1}}(p)}(x))(\rho_\gamma \otimes 1) \\
&= (\mathrm{id} \otimes \alpha_\lambda)(\mathcal{F}^*_{\alpha_{\lambda^{-1}}(p)}\mathcal{F}_p\mathcal{F}_p^* x \mathcal{F}_p\mathcal{F}_p^*\mathcal{F}_{\alpha_{\lambda^{-1}}(p)})(\rho_\gamma \otimes 1) \\
&= (\mathrm{id} \otimes \alpha_\lambda)(v_\lambda \Delta_p^\Gamma(x) v_\lambda^*)(\rho_\gamma \otimes 1) \\
&= (\mathrm{id} \otimes \alpha_\lambda)(v_\lambda \Delta_p^\Gamma(xu_\gamma) v_\lambda^*),
\end{aligned}$$

where $v_\lambda = \mathcal{F}^*_{\alpha_{\lambda^{-1}}(p)}\mathcal{F}_p$. The last equality follows from the fact that $v_\lambda \in \mathcal{U}(L\Gamma \overline{\otimes} \mathcal{M}^\Gamma)$ (see [IPR24, Proposition 4.4]), and hence commutes with $\rho_\gamma \otimes 1$. Therefore, if we let $w_\lambda = (\mathrm{id} \otimes \alpha_\lambda)(v_\lambda) \in \mathcal{U}(L\Gamma \overline{\otimes} \mathcal{M}^\Gamma)$, then we have that

$$\tilde{\alpha}_\lambda = \Delta_p^\Gamma \circ (\alpha_\lambda \rtimes \mathrm{id}_\Gamma) \circ (\Delta_p^\Gamma)^{-1} = \mathrm{Ad}(w_\lambda) \circ (\mathrm{id} \otimes \alpha_\lambda).$$

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Claim *The map* $w : \Lambda \to \mathcal{U}(L\Gamma \overline{\otimes} \mathcal{M}^\Gamma)$ *defined by* $w_\lambda = (\mathrm{id} \otimes \alpha_\lambda)(v_\lambda)$ *is a 1-cocycle for* $\Lambda \curvearrowright^{\mathrm{id}\otimes\alpha} \mathcal{B}(\ell^2\Gamma) \overline{\otimes} \mathcal{M}^\Gamma$.

Proof of Claim First note that, for any $y \in \mathfrak{n}_{\mathrm{Tr}} \subset \mathcal{M}$, it is straightforward to verify that

$$\mathcal{F}_p^*(y) = \sum_{\gamma\in\Gamma} \delta_\gamma \otimes y_\gamma,$$

where

$$y_\gamma = \sum_{b\in\Gamma} \sigma_{b\gamma^{-1}}(p)\sigma_b(y).$$

Since $\mathrm{Tr}(p) < \infty$, so, for any $a \in \Gamma$ and $x \in \mathcal{M}^\Gamma$, we have that $\mathrm{Tr}(\sigma_{a^{-1}}(p)x) < \infty$ and hence $\sigma_{a^{-1}}(p)x \in \mathfrak{n}_{\mathrm{Tr}}$. Therefore, for any $a \in \Gamma$ and $x \in \mathcal{M}^\Gamma$, we have

$$\begin{aligned}
\mathcal{F}^*_{\alpha_{\lambda^{-1}}(p)}\mathcal{F}_p(\delta_a \otimes x) &= \mathcal{F}^*_{\alpha_{\lambda^{-1}}(p)}(\sigma_{a^{-1}}(p)x) \\
&= \sum_{\gamma\in\Gamma} \delta_\gamma \otimes \left(\sum_{b\in\Gamma} \sigma_{b\gamma^{-1}}(\alpha_{\lambda^{-1}}(p))\sigma_b(\sigma_{a^{-1}}(p)x) \right) \\
&= \sum_{\gamma\in\Gamma} \delta_\gamma \otimes \left(\sum_{b\in\Gamma} \sigma_{b\gamma^{-1}}(\alpha_{\lambda^{-1}}(p))\sigma_{ba^{-1}}(p)x \right).
\end{aligned}$$

Thus, as an $\mathcal{M}^\Gamma$-valued $\Gamma \times \Gamma$ matrix, we can write $v_\lambda = [[v_\lambda]_{s,t}]_{s,t}$, where

$$[v_\lambda]_{s,t} = \sum_{\gamma\in\Gamma} \sigma_{\gamma s^{-1}}(\alpha_{\lambda^{-1}}(p))\sigma_{\gamma t^{-1}}(p),$$

and therefore, w_λ can be written as an $\mathcal{M}^\Gamma$-valued $\Gamma \times \Gamma$ matrix $w_\lambda = [[w_\lambda]_{s,t}]_{s,t}$, where

$$[w_\lambda]_{s,t} = \alpha_\lambda([v_\lambda]_{s,t}) = \sum_{\gamma\in\Gamma} \sigma_{\gamma s^{-1}}(p)\sigma_{\gamma t^{-1}}(\alpha_\lambda(p)).$$

Finally, the following calculation verifies the cocycle identity for w. For $\lambda_1, \lambda_2 \in \Lambda$, and $s, t \in \Gamma$, we have

$$\begin{aligned}
&[w_{\lambda_1}(\mathrm{id}\otimes\alpha_{\lambda_1})(w_{\lambda_2})]_{s,t} \\
&\quad= \sum_{a\in\Gamma} [w_{\lambda_1}]_{s,a}[(\mathrm{id}\otimes\alpha_{\lambda_1})(w_{\lambda_2})]_{a,t} \\
&\quad= \sum_{a\in\Gamma} \left[\left(\sum_{\gamma\in\Gamma} \sigma_{\gamma s^{-1}}(p)\sigma_{\gamma a^{-1}}(\alpha_{\lambda_1}(p)) \right) \left(\sum_{\gamma'\in\Gamma} \sigma_{\gamma' a^{-1}}(\alpha_{\lambda_1}(p))\sigma_{\gamma' t^{-1}}(\alpha_{\lambda_1\lambda_2}(p)) \right) \right] \\
&\quad= \sum_{a\in\Gamma}\sum_{\gamma\in\Gamma} \sigma_{\gamma s^{-1}}(p)\sigma_{\gamma a^{-1}}(\alpha_{\lambda_1}(p))\sigma_{\gamma t^{-1}}(\alpha_{\lambda_1\lambda_2}(p)) \\
&\quad= \sum_{\gamma\in\Gamma}\sum_{a\in\Gamma} \sigma_{\gamma s^{-1}}(p)\sigma_{\gamma a^{-1}}(\alpha_{\lambda_1}(p))\sigma_{\gamma t^{-1}}(\alpha_{\lambda_1\lambda_2}(p)) \\
&\quad= \sum_{\gamma\in\Gamma} \sigma_{\gamma s^{-1}}(p)\sigma_{\gamma t^{-1}}(\alpha_{\lambda_1\lambda_2}(p)) \\
&\quad= [w_{\lambda_1\lambda_2}]_{s,t}.
\end{aligned}$$

■

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It now follows from the above claim that the actions $\tilde{\alpha}$ and $\mathrm{id}\otimes\alpha$ of Λ on $\mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,\mathcal{M}^\Gamma$ are cocycle conjugate, and hence we get the following isomorphisms of the crossed products:

$$(\mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,\mathcal{M}^\Gamma)\rtimes_{\tilde{\alpha}}\Lambda \xrightarrow[\cong]{\Psi^\Gamma} (\mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,\mathcal{M}^\Gamma)\rtimes_{\mathrm{id}\otimes\alpha}\Lambda \xrightarrow[\cong]{\Psi^\alpha} \mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,(\mathcal{M}^\Gamma\rtimes_\alpha\Lambda),$$

where the isomorphism Ψ^α is the canonical isomorphism and satisfies

$$\Psi^\alpha((T\otimes x)u_\lambda) = T\otimes xu_\lambda, \qquad T\in\mathcal{B}(\ell^2\Gamma), x\in\mathcal{M}^\Gamma, \lambda\in\Lambda.$$

Similarly, starting with the isomorphism $\Delta_p^\Lambda : \mathcal{M}\rtimes\Lambda\to\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda$, and performing the above analysis, yields the following isomorphisms of the crossed products:

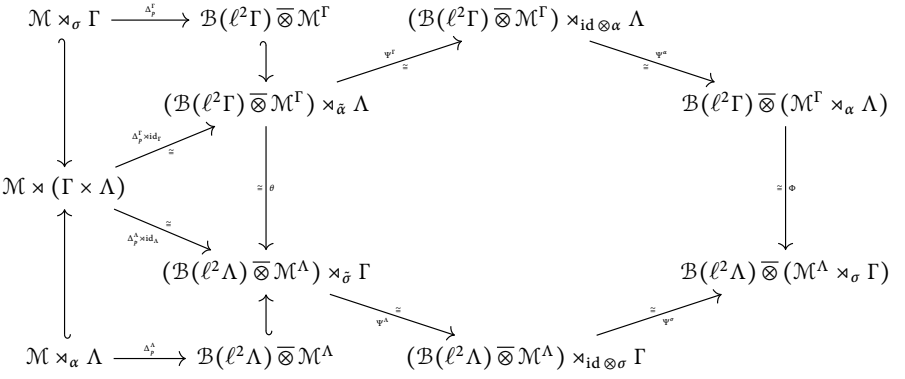
$$\mathcal{M}\rtimes(\Gamma\times\Lambda) = (\mathcal{M}\rtimes\Lambda)\rtimes_{\sigma\rtimes\mathrm{id}_\Lambda}\Gamma \xrightarrow[\cong]{\Delta_p^\Lambda\rtimes\mathrm{id}_\Lambda} (\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda)\rtimes_{\tilde{\sigma}}\Gamma,$$

$$(\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda)\rtimes_{\tilde{\sigma}}\Gamma \xrightarrow[\cong]{\Psi^\Lambda} (\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda)\rtimes_{\mathrm{id}\otimes\sigma}\Gamma \xrightarrow[\cong]{\Psi^\sigma} \mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,(\mathcal{M}^\Lambda\rtimes_\sigma\Gamma).$$

Thus, if $\Phi : \mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,(\mathcal{M}^\Gamma\rtimes_\alpha\Lambda)\to\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,(\mathcal{M}^\Lambda\rtimes_\sigma\Gamma)$ is given by

$$\Phi := \Psi^\sigma\circ\Psi^\Lambda\circ\theta\circ(\Psi^\Gamma)^{-1}\circ(\Psi^\alpha)^{-1},$$

where $\theta := \Delta_p^\Lambda\rtimes\mathrm{id}_\Lambda\circ(\Delta_p^\Gamma\rtimes\mathrm{id}_\Gamma)^{-1}$, then Φ is an isomorphism and moreover, the following diagram commutes:



Let $\omega_{e_\Gamma,e_\Gamma}\in\mathcal{B}(\ell^2\Gamma)$ (resp., $\omega_{e_\Lambda,e_\Lambda}\in\mathcal{B}(\ell^2\Lambda)$) denote the orthogonal projection onto $\mathbb{C}\delta_{e_\Gamma}$ (resp., $\mathbb{C}\delta_{e_\Lambda}$). We make the following observations:

- $\Psi^\alpha(\omega_{e_\Gamma,e_\Gamma}\otimes 1) = \omega_{e_\Gamma,e_\Gamma}\otimes 1$ and $\Psi^\sigma(\omega_{e_\Lambda,e_\Lambda}\otimes 1) = \omega_{e_\Lambda,e_\Lambda}\otimes 1$.
- $\Psi^\Gamma(x) = x$ for all $x\in\mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,\mathcal{M}^\Gamma$. This is true because of how $\tilde{\alpha}$ and $\mathrm{id}\otimes\alpha$ are cocycle conjugate. Similarly, $\Psi^\Lambda(y) = y$ for all $y\in\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda$.

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- $\Delta_p^\Gamma(p) = \omega_{e_\Gamma,e_\Gamma} \otimes 1$, $\Delta_p^\Lambda(p) = \omega_{e_\Lambda,e_\Lambda} \otimes 1$, $(\Delta_p^\Gamma \rtimes \mathrm{id}_\Gamma)(p) = \omega_{e_\Gamma,e_\Gamma} \otimes 1$, and $(\Delta_p^\Lambda \rtimes \mathrm{id}_\Lambda)(p) = \omega_{e_\Lambda,e_\Lambda} \otimes 1$. These in particular imply that

$$\theta(\omega_{e_\Gamma,e_\Gamma} \otimes 1) = \Delta_p^\Lambda \rtimes \mathrm{id}_\Lambda((\Delta_p^\Gamma \rtimes \mathrm{id}_\Gamma)^{-1}(\omega_{e_\Gamma,e_\Gamma} \otimes 1)) = \Delta_p^\Lambda \rtimes \mathrm{id}_\Lambda(p) = \omega_{e_\Lambda,e_\Lambda} \otimes 1.$$

From the observations above, we obtain that

$$\begin{aligned}\Phi(\omega_{e_\Gamma,e_\Gamma} \otimes 1) &= \Psi^\sigma(\Psi^\Lambda(\theta((\Psi^\Gamma)^{-1}((\Psi^\alpha)^{-1}(\omega_{e_\Gamma,e_\Gamma} \otimes 1))))) \\ &= \Psi^\sigma(\Psi^\Lambda(\theta(\omega_{e_\Gamma,e_\Gamma} \otimes 1))) \\ &= \Psi^\sigma(\Psi^\Lambda(\omega_{e_\Lambda,e_\Lambda} \otimes 1)) \\ &= \omega_{e_\Lambda,e_\Lambda} \otimes 1.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Phi(\mathcal{M}^\Gamma \rtimes_\alpha \Lambda) &= \Phi((\omega_{e_\Gamma,e_\Gamma} \otimes 1)(\mathcal{B}(\ell^2\Gamma) \mathbin{\overline{\otimes}} (\mathcal{M}^\Gamma \rtimes_\alpha \Lambda))(\omega_{e_\Gamma,e_\Gamma} \otimes 1)) \\ &= (\omega_{e_\Lambda,e_\Lambda} \otimes 1)(\mathcal{B}(\ell^2\Lambda) \mathbin{\overline{\otimes}} (\mathcal{M}^\Lambda \rtimes_\sigma \Gamma))(\omega_{e_\Lambda,e_\Lambda} \otimes 1) \\ &= \mathcal{M}^\Lambda \rtimes_\sigma \Gamma.\end{aligned}$$

■

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