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K-theory for ring C*-algebras attached to function fields with only one infinite place

by

XIN LI*

Abstract

We study the K-theory of ring C*-algebras associated to rings of integers in global function fields with only a single infinite place. First, we compute the torsion-free part of the K-groups of these ring C*-algebras. Secondly, we show that, under a certain primeness condition, the torsion part of K-theory determines the inertia degrees at infinity of our function fields.

Key Words: K-theory, ring C*-algebra, function field.

Mathematics Subject Classification 2000: Primary 46L05, 46L80; Secondary 14H05.

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1. Introduction

We generalize the work of J. Cuntz and the author in [Cu-Li3], where K-theory has been computed for ring C*-algebras attached to $\mathbb{F}_q[T]$, the polynomial ring over the finite field with q elements. Here q is a prime power, i.e. $q = p^\nu$ for some prime p and positive integer ν. By the (reduced) ring C*-algebra of a ring without left zero divisors, we mean the C*-algebra generated by the left regular representation of the ring. The details of constructing ring C*-algebras are recalled in the next section.

In the present paper, we consider more general global function fields, i.e. finite separable field extensions of $\mathbb{F}_p(T)$, namely those which have only one single infinite place. In other words, we focus on function fields that do not split at infinity. Here, our convention is to call the place of $\mathbb{F}_p(T)$ corresponding to the inverse of the indeterminant T the infinite place (as it is usually done). A place of a function field is then called infinite if it sits above this infinite place of $\mathbb{F}_p(T)$.

Given any function field, we can consider the ring C*-algebra attached to its ring of integers. By the ring of integers, we mean the integral closure of $\mathbb{F}_p[T]$ in this function field. Our present goal is to study the K-theory of this ring C*-algebra in the situation of a function field with only a single infinite place.

In our previous work [Cu-Li3], we considered specific examples, namely the ring of integers $\mathbb{F}_q[T]$ in the function field $\mathbb{F}_q(T)/\mathbb{F}_p(T)$. So the setting in [Cu-Li3] was very concrete. On the one hand, this had the advantage that the K-theoretic computation could be done in a very explicit way. This led to explicit generators for the K-groups of interest. On the other hand, it was not clear in [Cu-Li3] how to generalize the work, i.e. what happens in the case of more general function fields.

The present paper clarifies the situation and helps to understand what is going on in the general case. Our investigations reveal which properties of the function field are needed for the approach developed in [Cu-Li3] to work out, and where the obstacles to more general K-theoretic computations lie.

Moreover, new phenomena arise in the more general situation. Most notably, we will see that the K-theory of ring C*-algebras attached to function fields (or rather their rings of integers) can carry torsion. This is not the case for the ring C*-algebra of $\mathbb{F}_q[T]$ as shown in [Cu-Li3]. Unfortunately, the appearance of torsion leads to extension problems in our computations that cannot be solved in general. For this reason, we can in general only determine the torsion-free part of the K-groups. However, it is at the same time an interesting problem to find out what sort of information about the function fields can be extracted from the torsion part of the K-groups of the associated ring C*-algebras. It turns out that the torsion in K-theory encodes the inertia degrees at infinity of our function fields.

We state now our main results:

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Let $L/\mathbb{F}_p(T)$ be a finite separable field extension. Assume that $\mathbb{F}_q$ is the field of constants in L, i.e. $\mathbb{F}_q$ is the algebraic closure of $\mathbb{F}_p$ in L. We assume that L has only a single infinite place. Let f be the inertia degree at infinity of $L/\mathbb{F}_q(T)$. Define $Q^f := \frac{q^f-1}{q-1}$. Moreover, the multiplicative group $L^\times$ always admits a decomposition $L^\times = \mathbb{F}_q^\times \times \Gamma$ where Γ is a free abelian subgroup of $L^\times$. Note that Γ is of countably infinite rank. Now let R be the ring of integers of L. The ring C*-algebra of R, i.e. the C*-algebra generated by the left regular representation of R (see Section 2 for details), is denoted by $\mathfrak{A}[R]$. In this situation, we have

Theorem 1.1 *After inverting Q^f, we obtain for the K-theory of $\mathfrak{A}[R]$:*

$$K_*(\mathfrak{A}[R]) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}] \cong \left(\tilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)\right) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}].$$

$\tilde{K}_0$ stands for reduced K-theory, and we know that $\tilde{K}_0(C^*(\mathbb{F}_q^\times)) \cong \mathbb{Z}^{q-2}$. $\Lambda^*(\Gamma)$ is the exterior $\mathbb{Z}$-algebra over Γ. Moreover, K_* denotes the direct sum of K_0 and K_1 and is viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group. So we are identifying $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. On $\mathbb{Z}[\frac{1}{Q^f}]$ and $\tilde{K}_0(C^*(\mathbb{F}_q^\times))$, we take the trivial grading, and on $\Lambda^*(\Gamma)$ we take the canonical grading. All the tensor products are graded tensor products.

For our second main result, take two function fields L_1 and L_2. We assume that both of them have the field of constants $\mathbb{F}_q$ and only a single infinite place. Let f_i be the inertia degree at infinity of $L_i/\mathbb{F}_q(T)$ for $i = 1,2$. R_i again denotes the ring of integers in $L_i/\mathbb{F}_q(T)$ for $i = 1,2$. Under the additional assumption that $(f_i, q-1) = 1$ for $i = 1,2$, we obtain for the ring C*-algebras $\mathfrak{A}[R_1]$ and $\mathfrak{A}[R_2]$ of R_1 and R_2:

Theorem 1.2 *If $K_*(\mathfrak{A}[R_1]) \cong K_*(\mathfrak{A}[R_2])$, then the inertia degrees at infinity of the extensions $L_1/\mathbb{F}_q(T)$ and $L_2/\mathbb{F}_q(T)$ coincide, i.e. we have $f_1 = f_2$.*

This paper is organized as follows: In Section 2, we explain the construction of ring C*-algebras for rings of integers in function fields. Moreover, we review the work in [Cu-Li3] in order to recall the strategy of the K-theoretic computations in the case of polynomial rings over finite fields. As in [Cu-Li3], it is more convenient to work over the finite place of $\mathbb{F}_p(T)$ corresponding to the indeterminant T itself instead of the infinite place corresponding to T^{-1}. We explain the necessary reduction step in Section 3. Our original task can be reformulated as computing K-theory for a certain crossed product by the $ax + b$-group over a function field. We start with the action of the additive group and of the roots of unity in Section 4. It then remains to treat the full multiplicative group. This is achieved in two steps: The first step is to adjoin a prime element (Section 5). The second step is to add the remaining part of the multiplicative group. We show that it is at least possible to do this step rationally (Section 6), so that we obtain a proof of Theorem 1.1. Moreover,

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under a certain primeness condition, it is possible to say a bit more about the torsion part of K-theory. This observation, together with a little number theoretic lemma, gives a proof of Theorem 1.2 (Section 7).

I would like to thank U. Hartl for helpful comments about global function fields.

2. Preliminaries

2.1. Ring C-algebras*

The theory of ring C*-algebras was initiated in [Cun]. From that point of departure, it has been further developed in [Cu-Li1], [Li] and [Cu-Li2]. We refer the reader to the references mentioned above for the general theory of ring C*-algebras. Now, let us explain the construction in the concrete case of function fields.

By a (global) function field, we mean a finite separable field extension $L/\mathbb{F}_p(T)$. Here p is a prime number and $\mathbb{F}_p(T)$ is the quotient field of the polynomial ring over the finite field with p elements. The ring of integers R in $L/\mathbb{F}_p(T)$ is the integral closure of $\mathbb{F}_p[T]$ in L, i.e. R is the following subring of L:

$$\{x \in L\colon \exists\, a_0,\ldots,a_{m-1} \in \mathbb{F}_p[T] \text{ with } x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 = 0\}.$$

The reader may consult [Ro] for an introduction to the theory of function fields. The theory of places and adeles can be found in [Weil], Chapters I-IV.

To construct the (reduced) ring C*-algebra of R, consider the Hilbert space $\ell^2(R)$ of square summable functions on R. Denote by $\{\varepsilon_r\colon r \in R\}$ the canonical orthonormal basis of $\ell^2(R)$, i.e. $\varepsilon_r(s) = \delta_{r,s}$ for s in R ($\delta_{r,s} = 1$ if $r = s$ and $\delta_{r,s} = 0$ if $r \neq s$). The ring structure of R allows us to construct two natural families of bounded operators on $\ell^2(R)$, namely

- additive shifts determined by $U^b\varepsilon_r = \varepsilon_{b+r}$ for every $b \in R$, $r \in R$,
- multiplicative shifts given by $S_a\varepsilon_r = \varepsilon_{ar}$ for all $0 \neq a \in R$, $r \in R$.

The ring C*-algebra of R is then given by the C*-subalgebra of $\mathcal{L}(\ell^2(R))$ generated by $\{U^b\colon b \in R\}$ and $\{S_a\colon 0 \neq a \in R\}$. It is denoted by $\mathfrak{A}[R]$, i.e. we have

$$\mathfrak{A}[R] := C^*(\{U^b\colon b \in R\} \cup \{S_a\colon 0 \neq a \in R\}) \subseteq \mathcal{L}(\ell^2(R)).$$

In our situation of function fields, the C*-algebra $\mathfrak{A}[R]$ is purely infinite and simple (compare [Cu-Li1], Theorem 3.6). Moreover, it admits – up to Morita equivalence – a crossed product description

$$\mathfrak{A}[R] \sim_M C_0(\mathbb{A}_f) \rtimes L \rtimes L^\times. \tag{1}$$

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This follows more or less directly from our constructions (see [Cu-Li1], Section 5). Here $\mathbb{A}_f$ is the finite adele ring over L. Our function field L embeds diagonally into $\mathbb{A}_f$, so there is a canonical action of the $ax+b$-group $L \rtimes L^\times$ over L on $C_0(\mathbb{A}_f)$ by affine transformations. This is precisely the action which gives rise to the crossed product on the right hand side of (1).

2.2. Previous K-theoretic computations

The K-theory of $\mathfrak{A}[R]$ has been computed in the case of the function field $\mathbb{F}_q(T)/\mathbb{F}_p(T)$ in [Cu-Li3], where q is a power of the prime number p and we take the canonical embedding $\mathbb{F}_p(T) \hookrightarrow \mathbb{F}_q(T)$. For this function field $\mathbb{F}_q(T)/\mathbb{F}_p(T)$, the ring of integers R is the polynomial ring with coefficients in $\mathbb{F}_q$, i.e. $R = \mathbb{F}_q[T]$. The main result in [Cu-Li3] says that the K-theory of $\mathfrak{A}[\mathbb{F}_q[T]]$ can be described as follows:

The multiplicative group $\mathbb{F}_q(T)^\times$ admits a decomposition $\mathbb{F}_q(T)^\times = \mathbb{F}_q^\times \times \Gamma$ where Γ is a free abelian subgroup of $\mathbb{F}_q(T)^\times$ (Γ is actually a free abelian group of countably infinite rank). Theorem 7.2 in [Cu-Li3] tells us that

$$K_*(\mathfrak{A}[\mathbb{F}_q[T]]) \cong \tilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) \tag{2}$$

as $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. Here we use the same notation as in Theorem 1.1.

In the sequel, let us briefly recall the steps that lead to (2).

The first and most important ingredient is the so-called duality theorem, proven in [Cu-Li2]. It says the following:

Let L be a global field, i.e. we have a finite seperable field extension $L/\mathbb{Q}$ or $L/\mathbb{F}_p(T)$. We can associate to L its finite and its infinite adele space, denoted by $\mathbb{A}_f$ and $\mathbb{A}_\infty$, respectively. Since L embeds diagonally into $\mathbb{A}_f$ and $\mathbb{A}_\infty$, there are canonical actions of the $ax+b$-group $L \rtimes L^\times$ on $C_0(\mathbb{A}_f)$ and $C_0(\mathbb{A}_\infty)$ by affine transformations. Corollary 4.2 in [Cu-Li2] tells us that the corresponding crossed products are Morita equivalent, i.e.

$$C_0(\mathbb{A}_f) \rtimes L \rtimes L^\times \sim_M C_0(\mathbb{A}_\infty) \rtimes L \rtimes L^\times. \tag{3}$$

Of course, a function field is a special case of a global field, so we can apply (3) to function fields. Combining this with (1), we obtain for every function field $L/\mathbb{F}_p(T)$ with ring of integers R that

$$\mathfrak{A}[R] \sim_M C_0(\mathbb{A}_\infty) \rtimes L \rtimes L^\times. \tag{4}$$

This is the first step in our computation; it reduces the K-theory computation of $\mathfrak{A}[R]$ to the computation of the K-theory of $C_0(\mathbb{A}_\infty) \rtimes L \rtimes L^\times$.

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The second step, another reduction step, is to switch places from the infinite place to the finite place corresponding to the indeterminant T in $\mathbb{F}_q(T)$. For the function field $\mathbb{F}_q(T)/\mathbb{F}_p(T)$ treated in [Cu-Li3], this means the following: The infinite adele space $\mathbb{A}_\infty$ of $\mathbb{F}_q(T)/\mathbb{F}_p(T)$ is given by the ring of formal Laurent series $\mathbb{F}_q((T))$ with the embedding $\mathbb{F}_q(T) \ni f(T) \mapsto f(T^{-1}) \in \mathbb{F}_q((T))$, i.e. the indeterminant is replaced by its inverse. The local field of the place of $\mathbb{F}_q(T)$ corresponding to the indeterminant T is again given by $\mathbb{F}_q((T))$, but this time with the canonical embedding $\mathbb{F}_q(T) \ni f(T) \mapsto f(T) \in \mathbb{F}_q((T))$. These two embeddings give rise to two actions of $\mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$ on $C_0(\mathbb{A}_\infty)$ and on $C_0(\mathbb{F}_q((T)))$. Thus we obtain two crossed products. In Section 3 of [Cu-Li3], we prove that

$$C_0(\mathbb{A}_\infty) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times \cong C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times.$$

Thus to compute the K-theory of $\mathfrak{A}[\mathbb{F}_q[T]]$, we have to determine the K-theory of $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$. This computation is then done in three steps:

As a first step, we treat the additive group and the roots of unity, i.e. we compute K-theory for $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q^\times$ (see [Cu-Li3], Corollary 5.9; $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q^\times$ is abbreviated by A_{-1} in [Cu-Li3]). The idea is to find suitable filtrations and to use the continuity of K-theory.

The second step is to adjoin the canonical prime element for the place corresponding to T, namely the indeterminant T itself. This means that we have to compute the K-theory of $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes (\mathbb{F}_q^\times \times \langle T \rangle)$. However, it is crucial for what comes next that we do not only compute K-theory, but even obtain explicit generators with a certain invariance property (see [Cu-Li3], Proposition 5.10 and Lemma 6.1; $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes (\mathbb{F}_q^\times \times \langle T \rangle)$ is abbreviated by A_0 in [Cu-Li3]). This explicit K-theoretic computation is achieved through a detailed analysis of the Pimsner-Voiculescu sequence applied to our situation.

Finally, the third step is to adjoin the remaining part of the multiplicative group $\mathbb{F}_q(T)^\times$. Here, it turns out to be crucial that we can find generators for this remaining part of $\mathbb{F}_q(T)^\times$, which all lie in the coset $1 + T\mathbb{F}_q[T]$, i.e. which are congruent to 1 modulo $T\mathbb{F}_q[T]$. The idea in this step is to use the Pimsner-Voiculescu sequence and to compare our situation to the one of higher dimensional tori. This comparison works out precisely because of the invariance property of the generators for K-theory in the previous step.

This is how the K-theory of $\mathfrak{A}[\mathbb{F}_q[T]]$ can be computed. In the present paper, we follow the same route. Some steps work out just as in the case of $\mathbb{F}_q(T)/\mathbb{F}_p(T)$, but sometimes we need modifications or new ideas to make things work.

Remark 2.1 *K-theory for ring C*-algebras has also been computed in the case of number fields, see [Cu-Li2]. As for function fields, the duality theorem* (3) *plays an*

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important role in the computations for number fields, but for quite different reasons.

3. Switching places from infinite to finite

Let now $L/\mathbb{F}_p(T)$ be a function field with field of constants $\mathbb{F}_q$. This means that $\mathbb{F}_q$ is the algebraic closure of $\mathbb{F}_p$ in L. Let R be the ring of integers in $L/\mathbb{F}_p(T)$. We know by (4) that the ring C*-algebra of R is Morita equivalent to $C_0(\mathbb{A}_\infty) \rtimes L \rtimes L^\times$. As in the case of polynomial rings and their quotient fields, it is more convenient to work not with the infinite place v_∞ but with the place v_T of $\mathbb{F}_p(T)$ corresponding to the prime ideal of $\mathbb{F}_p[T]$ generated by the indeterminant T. This means that we would like to work with the places of $L/\mathbb{F}_p(T)$ sitting above v_T instead of those places which sit above v_∞. It is possible to arrange this by twisting the field extension $L/\mathbb{F}_p(T)$ by an automorphism of the ground field.

Let ι be the inclusion $\mathbb{F}_p(T) \hookrightarrow L$ corresponding to the given field extension $L/\mathbb{F}_p(T)$. Consider the automorphism σ of $\mathbb{F}_p(T)$ which fixes $\mathbb{F}_p$ and sends T to T^{-1}. Compose ι with σ to obtain a new field extension $L^{(\sigma)}/\mathbb{F}_p(T)$. In other words, we take the same field L, but view $\mathbb{F}_p(T)$ as a subfield via $\iota\circ\sigma$ and not via ι as it was the case for the extension $L/\mathbb{F}_p(T)$. Since $L^{(\sigma)} = L$ as fields, $L^{(\sigma)}$ has the same field of constants (namely $\mathbb{F}_q$) and the same places as L. Moreover, it is clear from our construction that a place w of $L^{(\sigma)}$ lies above the place v_T of $\mathbb{F}_p(T) \overset{\iota\circ\sigma}{\hookrightarrow} L^{(\sigma)}$ if and only if the same place w, now viewed as a place of L, sits above the infinite place v_∞ of $\mathbb{F}_p(T) \overset{\iota}{\hookrightarrow} L$. Therefore we have

$$\prod_{w|v_T} L_w^{(\sigma)} = \prod_{w|v_\infty} L_w. \tag{5}$$

Let $\mathbb{A}_T^{(\sigma)} = \prod_{w|v_T} L_w^{(\sigma)}$. The right hand side of (5) is $\mathbb{A}_\infty$, the infinite adele space of L, so that we obtain

$$C_0(\mathbb{A}_\infty) \rtimes L \rtimes L^\times = C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (L^{(\sigma)})^\times. \tag{6}$$

Therefore, our goal is to compute the K-theory of $C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (L^{(\sigma)})^\times$.

In analogy to the case of $L/\mathbb{F}_p(T)$, we denote by $R^{(\sigma)}$ the ring of integers of the function field $L^{(\sigma)}/\mathbb{F}_p(T)$. Let $\overline{R^{(\sigma)}}$ be the closure of $R^{(\sigma)}$ in $\mathbb{A}_T^{(\sigma)}$, i.e.

$$\overline{R^{(\sigma)}} = \prod_{w|v_T} R_w^{(\sigma)},$$

where $R_w^{(\sigma)}$ is the closure of $R^{(\sigma)}$ in $L_w^{(\sigma)}$.

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4. Additive group and roots of unity

Let us first of all state the main result of this section. For non-negative integers m, let $\mathbb{1}_m$ denote the characteristic function of $T^m\overline{R^{(\sigma)}} \subseteq \mathbb{A}_T^{(\sigma)}$. As $T^m\overline{R^{(\sigma)}}$ is closed and open in $\mathbb{A}_T^{(\sigma)}$, $\mathbb{1}_m$ lies in $C_0(\mathbb{A}_T^{(\sigma)})$. In the sequel, we will often deal with crossed products by the $ax+b$-group $L^{(\sigma)} \rtimes (L^{(\sigma)})^\times$ or by subgroups of this group. In each of these crossed products, let v^b denote the unitaries in the multiplier algebra of the crossed product corresponding to the additive part, and let t_a denote the unitaries for the multiplicative part. Moreover, whenever in such a crossed product by (a subgroup of) $L^{(\sigma)} \rtimes (L^{(\sigma)})^\times$, $\mathbb{F}_q^\times$ is contained in the multiplicative part of the group, then for a character $\chi \in \widehat{\mathbb{F}_q^\times}$ let p_χ be the spectral projection $\frac{1}{q-1}\sum_{a\in\mathbb{F}_q^\times} \chi(a)t_a$ in the multiplier algebra of the crossed product.

Here is the main result of this section:

Let n be the degree of $L^{(\sigma)}$ over $\mathbb{F}_q(T)$. Moreover, $1 \in \widehat{\mathbb{F}_q^\times}$ denotes the trivial character.

Proposition 4.1 *We can identify* $K_0(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes \mathbb{F}_q^\times)$ *with*

$$\mathbb{Z}[\tfrac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$$

in such a way that the K_0*-class* $[\mathbb{1}_m]$ *corresponds to* $\begin{pmatrix} q^{-mn} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ *and the* K_0*-class* $[\mathbb{1}_m p_\chi]$ *corresponds to* $\begin{pmatrix} -q^{-mn}Q^{mn} \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ *for every* $m \in \mathbb{Z}_{\geq 0}$ *and* $1 \neq \chi \in \widehat{\mathbb{F}_q^\times}$. *Here the entry "1" in the vector corresponding to* $[\mathbb{1}_m p_\chi]$ *is the* χ*-th entry in* $\bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}}\mathbb{Z}$ *and* $Q^{mn} = \frac{q^{mn}-1}{q-1}$.

$K_1(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes \mathbb{F}_q^\times)$ *vanishes.*

The first step in the proof of this proposition is to compute the K-theory of $C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times$. Since $L/\mathbb{F}_q(T)$ is separable, $L^{(\sigma)}/\mathbb{F}_q(T)$ is separable as well. In addition, $\mathbb{F}_q[T]$ is a principal ideal domain. Therefore, there is an $\mathbb{F}_q[T]$-basis for $R^{(\sigma)}$ (see [Neu], Chapter I, Proposition (2.10)). Let $\omega_1,\ldots,\omega_n$ be such a basis. We set

$$R_m^{(\sigma)} := \left\{ \sum_{i=1}^n f_i\omega_i \colon\ f_i \in \mathbb{F}_q[T], \deg(f_i) \leq m-1 \right\}.$$

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We know that $\overline{R^{(\sigma)}}$ can be identified with $\lim_{\longleftarrow m}\{R^{(\sigma)}/T^m R^{(\sigma)}; p_{m+1,m}\}$ where $p_{m+1,m}: R^{(\sigma)}/T^{m+1}R^{(\sigma)} \to R^{(\sigma)}/T^m R^{(\sigma)}$ is the canonical projection. Moreover, we have $R^{(\sigma)} = \bigcup_{m=1}^{\infty} R_m^{(\sigma)}$. It follows that we can identify $C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times$ with

$$\lim_{\longrightarrow m}\left\{C(R^{(\sigma)}/T^m R^{(\sigma)}) \rtimes R_m^{(\sigma)} \rtimes \mathbb{F}_q^\times; i_{m,m+1}\right\}$$

where $i_{m,m+1}: C(R^{(\sigma)}/T^m R^{(\sigma)}) \rtimes R_m^{(\sigma)} \rtimes \mathbb{F}_q^\times \to C(R^{(\sigma)}/T^{m+1}R^{(\sigma)}) \rtimes R_{m+1}^{(\sigma)} \rtimes \mathbb{F}_q^\times$ is given by

$$i_{m,m+1}(g v^b t_a) = (g \circ p_{m+1,m}) v^b t_a \tag{7}$$

for all $g \in C(R^{(\sigma)}/T^m R^{(\sigma)})$, $b \in R_m^{(\sigma)}$ and $a \in \mathbb{F}_q^\times$. This can be proven analogously to Lemma 5.3 in [Cu-Li3].

As a first step, we describe $C(R^{(\sigma)}/T^m R^{(\sigma)}) \rtimes R_m^{(\sigma)} \rtimes \mathbb{F}_q^\times$. Let V_a, for $a \in \mathbb{F}_q^\times$, be the canonical unitary generators of $C^*(\mathbb{F}_q^\times)$. Moreover, let $e_m \in C(R^{(\sigma)}/T^m R^{(\sigma)})$ be the function with value 1 at $0 + T^m R^{(\sigma)}$ and 0 everywhere else. And for every d, d' in $R_m^{(\sigma)}$, let $e_{d,d'}$ be the rank one operator in $\mathcal{L}(\ell^2(R^{(\sigma)}/T^m R^{(\sigma)}))$ given by $e_{d,d'}(\xi) = \langle \xi, \varepsilon_{d'+T^m R^{(\sigma)}}\rangle \varepsilon_{d+T^m R^{(\sigma)}}$. Here $\{\varepsilon_{b+T^m R^{(\sigma)}}\}$ is the canonical orthonormal basis of $\ell^2(R^{(\sigma)}/T^m R^{(\sigma)})$.

Lemma 4.2 *We have*

$$C(R^{(\sigma)}/T^m R^{(\sigma)}) \rtimes R_m^{(\sigma)} \rtimes \mathbb{F}_q^\times \cong \mathcal{L}(\ell^2(R^{(\sigma)}/T^m R^{(\sigma)})) \otimes C^*(\mathbb{F}_q^\times)$$

via

$$v^b e_m v^{-b'} t_a \mapsto e_{b,a^{-1}b'} \otimes V_a.$$

The proof of this lemma is analogous to that of Lemma 5.4 in [Cu-Li3].

Identifying $\mathcal{L}(\ell^2(R^{(\sigma)}/T^m R^{(\sigma)}))$ with $M_{q^{mn}}(\mathbb{C})$, this lemma leads to

Corollary 4.3

$$K_0(C(R^{(\sigma)}/T^m R^{(\sigma)}) \rtimes R_m^{(\sigma)} \rtimes \mathbb{F}_q^\times) \cong \bigoplus_{\widehat{\mathbb{F}_q^\times}} \mathbb{Z}$$

and free generators for K_0 are given by the K_0-classes $\left\{[e_m p_\chi]\colon \chi \in \widehat{\mathbb{F}_q^\times}\right\}$.

Moreover, $K_1(C(R^{(\sigma)}/T^m R^{(\sigma)}) \rtimes R_m^{(\sigma)} \rtimes \mathbb{F}_q^\times)$ vanishes.

This already shows, using continuity of K_1, that $K_1(C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times)$ vanishes.

To determine $K_0(C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times)$, we have to compute what $i_{m,m+1}$ (given by (7)) does on K_0.

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Lemma 4.4 *With respect to the generators*

$$\left\{[e_m p_\chi]\colon\ \chi \in \widehat{\mathbb{F}_q^\times}\right\} \text{ and } \left\{[e_{m+1} p_\chi]\colon\ \chi \in \widehat{\mathbb{F}_q^\times}\right\}$$

of $K_0(C(R^{(\sigma)}/T^m R^{(\sigma)}) \rtimes R_m^{(\sigma)} \rtimes \mathbb{F}_q^\times)$ *and* $K_0(C(R^{(\sigma)}/T^{m+1} R^{(\sigma)}) \rtimes R_{m+1}^{(\sigma)} \rtimes \mathbb{F}_q^\times)$, *respectively,* $(i_{m,m+1})_*$ *is of the form*

$$(i_{m,m+1})_* = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} Q^n & \cdots & Q^n \\ \vdots & Q^n & \vdots \\ Q^n & \cdots & Q^n \end{pmatrix}.$$

As above, $Q^n = \frac{q^n-1}{q-1}$.

Proof: We have to determine the K_0-class of

$$i_{m,m+1}(e_m p_\chi) = e_m p_\chi = \left(\sum_{b \in R^{(\sigma)}/TR^{(\sigma)}} v^{T^m b} e_{m+1} v^{-T^m b} \right) p_\chi$$

in $K_0(C(R^{(\sigma)}/T^{m+1} R^{(\sigma)}) \rtimes R_{m+1}^{(\sigma)} \rtimes \mathbb{F}_q^\times)$. To do so, we first construct for every ψ, χ in $\widehat{\mathbb{F}_q^\times}$ a partial isometry in $C(R^{(\sigma)}/T^{m+1} R^{(\sigma)}) \rtimes R_{m+1}^{(\sigma)} \rtimes \mathbb{F}_q^\times$ with support projection $(e_m - e_{m+1}) p_{\overline{\psi}\chi}$ and range projection $(e_m - e_{m+1}) p_\chi$. Let ψ be a character in $\widehat{\mathbb{F}_q^\times}$. Moreover, set $(R^{(\sigma)}/TR^{(\sigma)})^\times := (R^{(\sigma)}/TR^{(\sigma)}) \setminus \{0 + TR^{(\sigma)}\}$. The partial isometry mentioned above is constructed with the help of a map

$$\Psi : (R^{(\sigma)}/TR^{(\sigma)})^\times \to \mathbb{T} = \{z \in \mathbb{C}\colon\ |z| = 1\}$$

satisfying the relation

$$\Psi(a \cdot b + TR^{(\sigma)}) = \psi(a)\Psi(b + TR^{(\sigma)}) \tag{8}$$

for all $a \in \mathbb{F}_q^\times$ and $b + TR^{(\sigma)} \in (R^{(\sigma)}/TR^{(\sigma)})^\times$. Let us assume that such a function exists. Take such a function Ψ and set

$$x_\Psi^{(m+1)} := \sum_{b + TR^{(\sigma)} \in (R^{(\sigma)}/TR^{(\sigma)})^\times} \Psi(b + TR^{(\sigma)}) v^{T^m b} e_{m+1} v^{-T^m b}.$$

We have

$$\begin{aligned} x_\Psi^{(m+1)} (x_\Psi^{(m+1)})^* &= (x_\Psi^{(m+1)})^* x_\Psi^{(m+1)} \\ &= \sum_{b + TR^{(\sigma)} \in (R^{(\sigma)}/TR^{(\sigma)})^\times} v^{T^m b} e_{m+1} v^{-T^m b} = e_m - e_{m+1} \end{aligned} \tag{9}$$

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and for every $\chi \in \widehat{\mathbb{F}_q^\times}$:

$$
\begin{aligned}
& p_\chi x_\Psi^{(m+1)} \qquad (10)\\
&= \Big(\tfrac{1}{q-1}\sum_{a\in\mathbb{F}_q^\times}\chi(a)t_a\Big)\Big(\sum_{b+TR^{(\sigma)}\in(R^{(\sigma)}/TR^{(\sigma)})^\times}\Psi(b+TR^{(\sigma)})v^{T^m b}e_{m+1}v^{-T^m b}\Big)\\
&= \tfrac{1}{q-1}\sum_{a,b}\chi(a)\Psi(b+TR^{(\sigma)})v^{T^m ab}e_{m+1}v^{-T^m ab}t_a\\
&= \tfrac{1}{q-1}\sum_{a,b}\psi(a)\Psi(b+TR^{(\sigma)})v^{T^m ab}e_{m+1}v^{-T^m ab}\overline{\psi}(a)\chi(a)t_a\\
&\overset{(8)}{=} \sum_{a,b}\Psi(ab+TR^{(\sigma)})v^{T^m ab}e_{m+1}v^{-T^m ab}\tfrac{1}{q-1}\overline{\psi}(a)\chi(a)t_a\\
&= \sum_{a\in\mathbb{F}_q^\times} x_\Psi^{(m+1)}\tfrac{1}{q-1}\overline{\psi}(a)\chi(a)t_a\\
&= x_\Psi^{(m+1)}p_{\overline{\psi}\chi}.
\end{aligned}
$$

The partial isometry with the desired properties is then given by $p_\chi x_\Psi^{(m+1)}$, since

$$
\begin{aligned}
(p_\chi x_\Psi^{(m+1)})^*(p_\chi x_\Psi^{(m+1)}) &\overset{(10)}{=} p_{\overline{\psi}\chi}(x_\Psi^{(m+1)})^* x_\Psi^{(m+1)}p_{\overline{\psi}\chi}\\
&\overset{(9)}{=} p_{\overline{\psi}\chi}(e_m-e_{m+1})p_{\overline{\psi}\chi} = (e_m-e_{m+1})p_{\overline{\psi}\chi}
\end{aligned}
$$

and

$$
\begin{aligned}
(p_\chi x_\Psi^{(m+1)})(p_\chi x_\Psi^{(m+1)})^* &= p_\chi x_\Psi^{(m+1)}(x_\Psi^{(m+1)})^* p_\chi\\
&\overset{(9)}{=} p_\chi(e_m-e_{m+1})p_\chi = (e_m-e_{m+1})p_\chi.
\end{aligned}
$$

This implies that the K_0-classes of $(e_m-e_{m+1})p_{\overline{\psi}\chi}$ and $(e_m-e_{m+1})p_\chi$ coincide, i.e. we have for every ψ, χ in $\widehat{\mathbb{F}_q^\times}$

$$
\Big[(e_m-e_{m+1})p_{\overline{\psi}\chi}\Big] = \Big[(e_m-e_{m+1})p_\chi\Big] \text{ in } K_0(C(R^{(\sigma)}/T^{m+1}R^{(\sigma)})\rtimes R_{m+1}^{(\sigma)}\rtimes\mathbb{F}_q^\times)). \qquad (11)
$$

With (11) in mind, we can now compute that

$$
\big[i_{m,m+1}(e_m p_\chi)\big] = \big[e_m p_\chi\big] = \big[e_{m+1}p_\chi\big] + \big[(e_m-e_{m+1})p_\chi\big] \qquad (12)
$$

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and

$$(q-1)\big[(e_m - e_{m+1})p_\chi\big] \tag{13}$$

$$\overset{(11)}{=} \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} \Big[(e_m - e_{m+1})p_{\overline{\psi}\chi}\Big] = [e_m - e_{m+1}] \tag{14}$$

$$= (q^n-1)[e_{m+1}] = (q^n-1)\sum_{\psi\in\widehat{\mathbb{F}_q^\times}} \big[e_{m+1}p_\psi\big].$$

(13) implies that for every χ in $\widehat{\mathbb{F}_q^\times}$, we have

$$\big[(e_m - e_{m+1})p_\chi\big] = Q^n \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} \big[e_{m+1}p_\psi\big]. \tag{15}$$

Inserting (15) in (12), we obtain

$$\big[i_{m,m+1}(e_m p_\chi)\big] = \big[e_{m+1}p_\chi\big] + Q^n \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} \big[e_{m+1}p_\psi\big].$$

Therefore, $(i_{m,m+1})_*$ is indeed of the desired form.

However, to make all this work, we still need to construct a function Ψ with property (8).

Lemma 4.5 *For every character ψ in $\widehat{\mathbb{F}_q^\times}$ there exists a function*

$$\Psi : (R^{(\sigma)}/TR^{(\sigma)})^\times \to \mathbb{T} = \{z\in\mathbb{C}\colon |z| = 1\}$$

satisfying the relation (8)*, i.e.*

$$\Psi(a\cdot b + TR^{(\sigma)}) = \psi(a)\Psi(b+TR^{(\sigma)}) \text{ for all } a\in\mathbb{F}_q^\times, b+TR^{(\sigma)}\in(R^{(\sigma)}/TR^{(\sigma)})^\times.$$

Proof: Let $\omega_1,\dots,\omega_n$ be a $\mathbb{F}_q[T]$-basis of $R^{(\sigma)}$ (see the beginning of this section, right after Proposition 4.1). Then a complete system of representatives in $R^{(\sigma)}/TR^{(\sigma)}$ is given by $R_1^{(\sigma)} = \big\{\sum_{i=1}^n b_i\omega_i\colon b_i\in\mathbb{F}_q\big\}$. Of course, we have $\sum_{i=1}^n b_i\omega_i + TR^{(\sigma)} = 0 + TR^{(\sigma)}$ if and only if $b_1 = \cdots = b_n = 0$. Thus every $b + TR^{(\sigma)}$ in $(R^{(\sigma)}/TR^{(\sigma)})^\times$ is of the form $\sum_{i=i_0}^n b_i\omega_i$ with an index $i_0\in\{1,\dots,n\}$ such that $b_{i_0}\neq 0$. Now a function Ψ with the desired property is for example given by

$$\Psi\Big(\sum_{i=i_0}^n b_i\omega_i + TR^{(\sigma)}\Big) := \psi(b_{i_0}).$$

□

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This completes the proof of Lemma 4.4. □

Using

$$C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times \cong \varinjlim_m \left\{ C(R^{(\sigma)}/T^m R^{(\sigma)}) \rtimes R_m^{(\sigma)} \rtimes \mathbb{F}_q^\times; i_{m,m+1} \right\}$$

and continuity of K_0, the previous lemma yields

Lemma 4.6 *We can identify* $K_0(C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times)$ *with* $\mathbb{Z}[\frac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times} \setminus \{1\}} \mathbb{Z}$ *such that for every* $m \in \mathbb{Z}_{\geq 0}$ *and* $\chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$, $[\mathbb{1}_m]$ *corresponds to* $\begin{pmatrix} q^{-mn} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ *and* $[\mathbb{1}_m p_\chi]$ *corresponds to* $\begin{pmatrix} q^{-mn}Q^{mn} \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. *Here, as in Proposition 4.1, the entry "1" in the vector corresponding to* $[\mathbb{1}_m p_\chi]$ *is the* χ*-th entry in* $\bigoplus_{\widehat{\mathbb{F}_q^\times} \setminus \{1\}} \mathbb{Z}$, *and* $Q^{mn} = \frac{q^{mn}-1}{q-1}$.

Now, it is only a small step from this lemma to our proposition. We just have to invert $\mathbb{F}_q[T]^\times = \mathbb{F}_q[T] \setminus \{0\}$ in $R^{(\sigma)}$. Namely, we know that

$$\mathbb{A}_T^{(\sigma)} = (\mathbb{F}_q[T]^\times)^{-1} \overline{R^{(\sigma)}} \text{ and } L^{(\sigma)} = (\mathbb{F}_q[T]^\times)^{-1} R^{(\sigma)}.$$

This implies

$$C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes \mathbb{F}_q^\times \cong \varinjlim_{\mathbb{F}_q[T]^\times} \left\{ C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times; \mu \right\}$$

where μ is the multiplicative action of the semigroup $\mathbb{F}_q[T]^\times$ on $C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times$ via endomorphisms. This observation already implies, using continuity of K_1, that $K_1(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes \mathbb{F}_q^\times)$ vanishes, since we know that $K_1(C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times)$ vanishes.

Moreover, it turns out that for every f in $\mathbb{F}_q[T]^\times$, μ_f is an isomorphism on $K_0(C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times)$. Namely, with the help of Lemma 4.6, it is easy to compute that

$$(\mu_f)_* = \text{id on } K_0(C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times) \text{ for every } f \in \mathbb{F}_q[T]^\times \text{ with } v_T(f) = 0$$

and

$$(\mu_T)_* = \begin{pmatrix} q^{-n} & -q^{-n}Q^n & \dots & -q^{-n}Q^n \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$$

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with respect to the identification $K_0(C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times) \cong \mathbb{Z}[\frac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$ from Lemma 4.6. Again, $Q^n = \frac{q^n-1}{q-1}$.

But this implies, again by continuity of K_0, that the canonical homomorphism

$$C(\overline{R^{(\sigma)}}) \rtimes R^{(\sigma)} \rtimes \mathbb{F}_q^\times \to C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes \mathbb{F}_q^\times$$

induces an isomorphism on K_0.

Proposition 4.1 follows immediately from this observation and Lemma 4.6.

5. Adjoining a prime element

To pass from $C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes \mathbb{F}_q^\times$ to $C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (L^{(\sigma)})^\times$, we have to treat the multiplicative action, say μ. We proceed in two steps which occupy the next two sections.

From now on, we assume that L has only a single infinite place. This means that $L^{(\sigma)}$ has only a single place w sitting above v_T.

Our first goal is to treat the multiplicative action of a prime element for w. Take such a prime element in $R^{(\sigma)}$, say τ. We observe that we can always find a free abelian subgroup (of countably infinite rank) Γ' of $(L^{(\sigma)})^\times$ such that $(L^{(\sigma)})^\times = \mathbb{F}_q^\times \times \langle\tau\rangle \times \Gamma'$. Later on, we will choose a particular Γ' with special properties.

Let us now determine $(\mu_\tau)_*$ on $K_0(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes \mathbb{F}_q^\times)$. We first fix a few notations. Let f be the inertia degree at w, i.e. $(R^{(\sigma)} : \tau R^{(\sigma)}) = q^f$. Moreover, for a subset Y of $\mathbb{A}_T^{(\sigma)}$, we denote by $\mathbb{1}_Y$ the characteristic function of $Y \subseteq \mathbb{A}_T^{(\sigma)}$. Recall that we have already introduced the notation $\mathbb{1}_m := \mathbb{1}_{T^m\overline{R^{(\sigma)}}}$. In addition, let $\mathbb{1} := \mathbb{1}_0 = \mathbb{1}_{\overline{R^{(\sigma)}}}$.

Here is the result:

Proposition 5.1 *With respect to the identification*

$$K_0(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes \mathbb{F}_q^\times) \cong \mathbb{Z}[\tfrac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$$

in Proposition 4.1, $(\mu_\tau)_*$ *is of the form*

$$(\mu_\tau)_* = \begin{pmatrix} q^{-f} & -q^{-f}Q^f & \dots & -q^{-f}Q^f \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

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Proof: Proposition 4.1 tells us that the copy of $\mathbb{Z}[\frac{1}{q}]$ is generated by the K_0-classes $\{[\mathbb{1}_m]\colon m \in \mathbb{Z}_{\geq 0}\}$ and that the $q-2$ copies of $\mathbb{Z}$ (i.e. $\bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}}\mathbb{Z}$) are generated by the K_0-classes $\left\{[\mathbb{1}p_\chi]\colon 1 \neq \chi \in \widehat{\mathbb{F}_q^\times}\right\}$ with respect to the identification

$$K_0(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes \mathbb{F}_q^\times) \cong \mathbb{Z}[\tfrac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$$

in Proposition 4.1.

We have

$$\mu_\tau(\mathbb{1}_m) = \mathbb{1}_{\tau T^m \overline{R^{(\sigma)}}} = \sum_{b \in \tau R^{(\sigma)}/TR^{(\sigma)}} v^{T^m b}\mathbb{1}_{m+1}v^{-T^m b}.$$

As each of the pairwise orthogonal projections $v^{T^m b}\mathbb{1}_{m+1}v^{-T^m b}$ gives the same class as $\mathbb{1}_{m+1}$ in K_0, we obtain

$$\begin{aligned}(\mu_\tau)_*([\mathbb{1}_m]) &= (\tau R^{(\sigma)} : TR^{(\sigma)})[\mathbb{1}_{m+1}] \\ &= (R^{(\sigma)} : TR^{(\sigma)})(R^{(\sigma)} : \tau R^{(\sigma)})^{-1}[\mathbb{1}_{m+1}] \\ &= q^n q^{-f}[\mathbb{1}_{m+1}] = q^{n-f}[\mathbb{1}_{m+1}].\end{aligned}$$

As $[\mathbb{1}_m]$ corresponds to q^{-mn} and $[\mathbb{1}_{m+1}]$ corresponds to $q^{-(m+1)n}$ with respect to the identification in Proposition 4.1 (only looking at the copy of $\mathbb{Z}[\frac{1}{q}]$), we obtain that $(\mu_\tau)_*$ acts by multiplication with q^{-f} on the copy of $\mathbb{Z}[\frac{1}{q}]$.

It remains to determine $(\mu_\tau)_*([\mathbb{1}p_\chi])$ for $1 \neq \chi \in \widehat{\mathbb{F}_q^\times}$. To do so, we construct for every ψ, χ in $\widehat{\mathbb{F}_q^\times}$ a partial isometry with support projection $(\mathbb{1}_{\tau\overline{R^{(\sigma)}}} - \mathbb{1}_{T\overline{R^{(\sigma)}}})p_{\overline{\psi}\chi}$ and range projection $(\mathbb{1}_{\tau\overline{R^{(\sigma)}}} - \mathbb{1}_{T\overline{R^{(\sigma)}}})p_\chi$. The procedure is very similar to the one in the proof of Lemma 4.4. What we need is an element $x_\Psi \in C_0(\mathbb{A}_T^{(\sigma)})$ corresponding to $\psi \in \widehat{\mathbb{F}_q^\times}$ with the properties

$$x_\Psi x_\Psi^* = x_\Psi^* x_\Psi = \mathbb{1}_{\tau\overline{R^{(\sigma)}}} - \mathbb{1}_{T\overline{R^{(\sigma)}}}$$

and

$$p_\chi x_\Psi = x_\Psi p_{\overline{\psi}\chi} \text{ for all } \chi \in \widehat{\mathbb{F}_q^\times}.$$

Let $(\tau R^{(\sigma)}/TR^{(\sigma)})^\times$ be $(\tau R^{(\sigma)}/TR^{(\sigma)})\setminus\{0 + TR^{(\sigma)}\}$. A computation similar to that in Lemma 4.4 shows that an element x_Ψ with the desired properties is given by

$$x_\Psi = \sum_{b+TR^{(\sigma)} \in (\tau R^{(\sigma)}/TR^{(\sigma)})^\times} \Psi(b + TR^{(\sigma)})v^b \mathbb{1}_{T\overline{R^{(\sigma)}}}v^{-b}$$

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provided that Ψ is a function $(\tau R^{(\sigma)}/TR^{(\sigma)})^\times \to \mathbb{T}$ satisfying

$$\Psi(a\cdot b + TR^{(\sigma)}) = \psi(a)\Psi(b+TR^{(\sigma)}) \text{ for all } a\in\mathbb{F}_q^\times.$$

Such a function Ψ exists by Lemma 4.5: We just have to restrict the function constructed in that lemma to

$$(\tau R^{(\sigma)}/TR^{(\sigma)})^\times \subseteq (R^{(\sigma)}/TR^{(\sigma)})^\times.$$

Thus, there is a partial isometry with the desired properties. This implies

$$\left[(\mathbb{1}_{\overline{\tau R^{(\sigma)}}} - \mathbb{1}_{\overline{TR^{(\sigma)}}})p_{\overline{\psi}\chi}\right] = \left[(\mathbb{1}_{\overline{\tau R^{(\sigma)}}} - \mathbb{1}_{\overline{TR^{(\sigma)}}})p_\chi\right] \text{ for all } \psi,\chi\in\widehat{\mathbb{F}_q^\times}. \tag{16}$$

We can then compute as in the proof of Lemma 4.4:

$$\left[\mu_\tau(\mathbb{1}p_\chi)\right] = \left[\mathbb{1}_{\overline{\tau R^{(\sigma)}}}p_\chi\right] = \left[(\mathbb{1}_{\overline{\tau R^{(\sigma)}}} - \mathbb{1}_{\overline{TR^{(\sigma)}}})p_\chi\right] + \left[\mathbb{1}_{\overline{TR^{(\sigma)}}}p_\chi\right] \tag{17}$$

and

$$\begin{aligned}&(q-1)\left[(\mathbb{1}_{\overline{\tau R^{(\sigma)}}} - \mathbb{1}_{\overline{TR^{(\sigma)}}})p_\chi\right] \\ &\overset{(16)}{=} \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} \left[(\mathbb{1}_{\overline{\tau R^{(\sigma)}}} - \mathbb{1}_{\overline{TR^{(\sigma)}}})p_{\overline{\psi}\chi}\right] \\ &= \left[\mathbb{1}_{\overline{\tau R^{(\sigma)}}} - \mathbb{1}_{\overline{TR^{(\sigma)}}}\right] = (q^{n-f}-1)\left[\mathbb{1}_{\overline{TR^{(\sigma)}}}\right].\end{aligned} \tag{18}$$

(18) implies that

$$\left[(\mathbb{1}_{\overline{\tau R^{(\sigma)}}} - \mathbb{1}_{\overline{TR^{(\sigma)}}})p_\chi\right] = Q^{n-f}\left[\mathbb{1}_{\overline{TR^{(\sigma)}}}\right] \text{ for all } \chi\in\widehat{\mathbb{F}_q^\times}. \tag{19}$$

Inserting (19) in (17) yields

$$\left[\mu_\tau(\mathbb{1}p_\chi)\right] = Q^{n-f}\left[\mathbb{1}_{\overline{TR^{(\sigma)}}}\right] + \left[\mathbb{1}_{\overline{TR^{(\sigma)}}}p_\chi\right].$$

In the identification

$$K_0(C_0(\mathbb{A}_T^{(\sigma)})\rtimes L^{(\sigma)}\rtimes\mathbb{F}_q^\times) \cong \mathbb{Z}[\tfrac{1}{q}]\oplus\bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}}\mathbb{Z}$$

in Proposition 4.1, $\left[\mathbb{1}_{\overline{TR^{(\sigma)}}}\right] = [\mathbb{1}_1]$ corresponds to $\begin{pmatrix} q^{-n}\\ 0\\ \vdots\\ 0\end{pmatrix}$ and $\left[\mathbb{1}_{\overline{TR^{(\sigma)}}}p_\chi\right] = \left[\mathbb{1}_1 p_\chi\right]$ corresponds to $\begin{pmatrix} q^{-n}Q^n\\ 0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}$. We therefore obtain that $\left[\mu_\tau(\mathbb{1}p_\chi)\right]$ corresponds

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to

$$\begin{pmatrix} q^{-n}Q^{n-f}-q^{-n}Q^n \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{q^n}\frac{q^{n-f}-1-(q^n-1)}{q-1} \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} -q^{-f}Q^f \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with respect to the identification in Proposition 4.1. □

Going through the Pimsner-Voiculescu sequence, we can now derive the following

Proposition 5.2 *We have*

$$K_0(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle\tau\rangle)) \cong (\mathbb{Z}/Q^f\mathbb{Z}) \oplus \bigoplus_{\chi \in \widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$$

and

$$K_1(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle\tau\rangle)) \cong \bigoplus_{\chi \in \widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}.$$

Moreover, the K_0-class $[\mathbb{1}]$ is a generator for the copy of $\mathbb{Z}/Q^f\mathbb{Z}$, and generators in K_0 of $\bigoplus_{\chi \in \widehat{\mathbb{F}_q^\times}\setminus\{1\}}\mathbb{Z}$ are given by the classes $\left\{[\mathbb{1}p_\chi]\colon \chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}\right\}$. Generators for K_1 are given by the classes $\left\{[w_\chi]\colon \chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}\right\}$ where w_χ are unitaries in the unitalization of $C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle\tau\rangle)$ of the form

$$w_\chi = t_\tau(\mathbb{1}p_1) + x_X p_\chi + (\mathbb{1}p_\chi)t_\tau^* + (1 - \mathbb{1}p_1 - \mathbb{1}p_\chi). \tag{20}$$

Here, we can take for x_X any element in $C_0(\mathbb{A}_T^{(\sigma)})$ satisfying

$$x_X x_X^* = x_X^* x_X = \mathbb{1} - \mathbb{1}_{\overline{\tau R^{(\sigma)}}}$$

and

$$x_X p_\chi = p_1 x_X.$$

The proof of this proposition is analogous to that of Proposition 5.10 in [Cu-Li3]. However, in our present situation, we obtain torsion in K_0 (if $Q^f > 1$). This did not happen in [Cu-Li3].

Remark 5.3 *The element x_X can be chosen as*

$$x_X = \sum_{b+\tau R^{(\sigma)} \in (R^{(\sigma)}/\tau R^{(\sigma)})^\times} X(b + \tau R^{(\sigma)}) v^b \mathbb{1}_{\overline{\tau R^{(\sigma)}}} v^{-b}$$

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where X is a function $X : (R^{(\sigma)}/\tau R^{(\sigma)})^\times \to \mathbb{T}$ with the property that

$$X(a \cdot b + \tau R^{(\sigma)}) = \overline{\chi}(a)X(b + \tau R) \text{ for all } a \in \mathbb{F}_q^\times.$$

Such a function exists since $R^{(\sigma)}/\tau R^{(\sigma)} \cong \mathbb{F}_{q^f}$ and we can identify $\mathbb{F}_{q^f}$ with $(\mathbb{F}_q)^f$ as $\mathbb{F}_q$-vector spaces. Then we just have to apply the analogous construction as in the proof of Lemma 4.5.

Later on, we will choose particular elements x_X satisfying an invariance property.

In the case of polynomial rings and their quotient fields, it was possible to go on from this point and to compute the K-theory of the associated ring C*-algebras. The idea was to compare the situation with higher dimensional tori and to use the Pimsner-Voiculescu sequence. Since there was no torsion, everything could be computed.

Now, as we have seen, torsion can appear. This leads do extension problems which we cannot solve up to now. Another complication that arises in our more general situation is that the field of constants $\mathbb{F}_q$ in L (or $L^{(\sigma)}$) does not need to coincide with the field of constants of the local field at infinity (or at the place of $L^{(\sigma)}$ over v_T). Therefore, it is not clear any more whether we can choose generators for the remaining part of the multiplicative group which fix the unitaries w_χ (at the level of C*-algebras, not just in K-theory). This step was however crucial in our computations for polynomial rings. Note that both complications arise if and only if the inertia degree at infinity of our function field $L/\mathbb{F}_q(T)$ (or of $L^{(\sigma)}/\mathbb{F}_q(T)$ over v_T) is strictly bigger than 1.

In the sequel, we proceed in two directions: First, it turns out that once we rationalize, these two problems disappear. This is of course not surprising for the extension problems caused by torsion, but the second problem can be solved as well. And secondly, we develop a solution of the second problem under a certain primeness condition, namely that the inertia degree f at infinity of $L/\mathbb{F}_q(T)$ (or of $L^{(\sigma)}/\mathbb{F}_q(T)$ over v_T) satisfies $(f, q-1) = 1$. In this case, we can say more about torsion, and this leads to our observation that the K-theory of ring C*-algebras detects the inertia degrees at infinity.

6. Rational K-theory computations

Our goal is to prove Theorem 1.1. Here is the statement:

Take a function field $L/\mathbb{F}_p(T)$ with field of constants $\mathbb{F}_q$. Assume that the field extension $L/\mathbb{F}_q(T)$ has only a single infinite place with inertia degree f. Define $Q^f = \frac{q^f-1}{q-1}$. Let $\Gamma \subseteq L^\times$ be a free abelian subgroup (of countably infinite rank) such that $L^\times = \mathbb{F}_q^\times \times \Gamma$.

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Theorem 6.1 *After inverting Q^f, the K-theory of $\mathfrak{A}[R]$, the ring C*-algebra associated to the ring of integers R in $L/\mathbb{F}_q(T)$, is given by*

$$K_*(\mathfrak{A}[R]) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}] \cong \left(\tilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)\right) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}].$$

Let us come to the proof of this theorem. Since we have already computed K-theory for $C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle\tau\rangle)$, it remains to treat the multiplicative action of a subgroup Γ' of $(L^{(\sigma)})^\times$ with the property that $(L^{(\sigma)})^\times = \mathbb{F}_q^\times \times \langle\tau\rangle \times \Gamma'$. Now, let us fix such a subgroup Γ' which is necessarily free abelian of countably infinite rank. Let μ denote the multiplicative action of Γ' on $C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle\tau\rangle)$. We start with the following

Lemma 6.2 *Let $\gamma_1,\dots,\gamma_i$ and γ be linearly independent elements in Γ'. Assume that there are unitaries $\left\{w_\chi\colon\ \chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}\right\}$ as in Proposition 5.2 which are fixed by $\mu_{\gamma_1},\dots,\mu_{\gamma_i}$ and μ_γ. Then we have $(\mu_\gamma)_* = \mathrm{id}$ on $K_*(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times (\langle\tau\rangle \times \langle\gamma_1,\dots,\gamma_i\rangle)))$.*

Proof: Consider the Pimsner-Voiculescu sequence for

$$\begin{aligned} &C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times (\langle\tau\rangle \times \langle\gamma_1,\dots,\gamma_i\rangle \times \langle\gamma\rangle)) \\ \cong\ &C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times (\langle\tau\rangle \times \langle\gamma_1,\dots,\gamma_i\rangle)) \rtimes_{\mu_\gamma} \mathbb{Z}. \end{aligned}$$

Exactness of the Pimsner-Voiculescu sequence implies that our assertion is equivalent to saying that the boundary maps are surjective. To show this, we construct K-theoretic elements

$$\left[\mathbb{1}, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}\right], \left[\mathbb{1}\cdot p_\chi, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}\right], \left[w_\chi, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}\right]$$

and

$$\left[\mathbb{1}, t_{\gamma_{j_1}}, \dots, t_{\gamma_{j_l}}, t_\gamma\right], \left[\mathbb{1}\cdot p_\chi, t_{\gamma_{j_1}}, \dots, t_{\gamma_{j_l}}, t_\gamma\right], \left[w_\chi, t_{\gamma_{j_1}}, \dots, t_{\gamma_{j_l}}, t_\gamma\right]$$

for every $1 \le j_i < \dots < j_l \le i$, $0 \le j$ and $\chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$ by comparing our situation with higher dimensional tori as in [Cu-Li3], Section 6. This is possible because of our assumption that the elements w_χ are invariant under $\mu_{\gamma_1},\dots,\mu_{\gamma_i}$ and μ_γ. Now the claim is that

$$\begin{aligned} &\left\{\left[\mathbb{1}, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}\right]\colon 1 \le j_1 < \dots < j_l \le i, 0 \le l\right\} \\ \cup\ &\left\{\left[\mathbb{1}\cdot p_\chi, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}\right]\colon 1 \le j_1 < \dots < j_l \le i, 0 \le l; \chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}\right\} \\ \cup\ &\left\{\left[w_\chi, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}\right]\colon 1 \le j_1 < \dots < j_l \le i, 0 \le l; \chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}\right\} \end{aligned}$$

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are generators for $K_*(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times (\langle\tau\rangle \times \langle\gamma_1,\dots,\gamma_i\rangle)))$ and that the boundary maps in the Pimsner-Voiculescu sequence for

$$\begin{aligned} &C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times (\langle\tau\rangle \times \langle\gamma_1,\dots,\gamma_i\rangle \times \langle\gamma\rangle)) \\ \cong\, &C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times (\langle\tau\rangle \times \langle\gamma_1,\dots,\gamma_i\rangle)) \rtimes_{\mu_\gamma} \mathbb{Z} \end{aligned}$$

send

$$\begin{aligned} \left[\mathbb{1}, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}, t_\gamma\right] &\text{ to } \left[\mathbb{1}, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}\right], \\ \left[\mathbb{1}\cdot p_\chi, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}, t_\gamma\right] &\text{ to } \left[\mathbb{1}\cdot p_\chi, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}\right] \\ \text{and } \left[w_\chi, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}, t_\gamma\right] &\text{ to } \left[w_\chi, t_{\gamma_{j_1}}, t_{\gamma_{j_2}}, \dots, t_{\gamma_{j_l}}\right] \end{aligned}$$

for every $1 \le j_1 < \cdots < j_l \le i, 0 \le l; \chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$.

It actually suffices to prove the second assertion, because the first one can be deduced inductively as follows: The starting point is Proposition 5.2, and the induction step from j to $j+1$ is given by the second observation applied to $\gamma_1,\dots,\gamma_j$ and $\gamma = \gamma_{j+1}$.

The proof of the second assertion is the same as the one for Proposition 7.1 in [Cu-Li3]. □

Now, we can choose $\Gamma' \subseteq (L^{(\sigma)})^\times$ so that $(L^{(\sigma)})^{\{w\}} = \mathbb{F}_q \times \Gamma'$ where $L^{(\sigma)\{w\}} = \{a \in L^{(\sigma)} : w(a) = 0\}$. By construction, the inclusion $L^{(\sigma)} \hookrightarrow L_w^{(\sigma)} = \mathbb{A}_T^{(\sigma)}$ induces a map $\Gamma' \to R_w^{(\sigma)} = \overline{R^{(\sigma)}} \to \overline{R^{(\sigma)}}/\tau\overline{R^{(\sigma)}}$ with image in $(\overline{R^{(\sigma)}}/\tau\overline{R^{(\sigma)}})^\times$, i.e. a group homomorphism $\Gamma' \to (\overline{R^{(\sigma)}}/\tau\overline{R^{(\sigma)}})^\times; \gamma \mapsto \overline{\gamma}$. It is clear that if we choose x_X in the construction of w_χ to be of the form as in Remark 5.3, the unitaries w_χ for all $\chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$ will be fixed by μ_γ if $\overline{\gamma} = 1 + \tau\overline{R^{(\sigma)}}$. So, let us choose particular generators $\gamma_1, \gamma_2, \dots$ for a subgroup $\Gamma' \subseteq (L^{(\sigma)})^\times$ with $(L^{(\sigma)})^{\{w\}} = \mathbb{F}_q \times \Gamma'$. We can arrange that

$$\overline{\gamma_1} \text{ is a multiplicative generator of } (\overline{R^{(\sigma)}}/\tau\overline{R^{(\sigma)}})^\times$$

and that

$$\overline{\gamma_i} = 1 + \tau\overline{R^{(\sigma)}} \text{ for all } i \ge 2.$$

Set $\Gamma_i := \langle\tau, \gamma_1, \dots, \gamma_{i-1}\rangle$.

Proposition 6.3 *We have*

$$K_*(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \Gamma_i)) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^T}] \cong \left(\tilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma_i)\right) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^T}]$$

for all $i \ge 2$ and under these identifications, the canonical map

$$C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \Gamma_i) \to C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \Gamma_{i+1})$$

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induces the canonical inclusion

$$\left(\tilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma_i)\right) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}] \to \left(\tilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma_{i+1})\right) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}]$$

for every $i \geq 2$.

Proof: By the previous lemma and the Pimsner-Voiculescu sequence, we obtain

$$\begin{aligned} &K_*(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle \tau, \gamma_2, \ldots, \gamma_{i-1}\rangle)) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}] \\ &\cong \left(\tilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\langle \tau, \gamma_2, \ldots, \gamma_{i-1}\rangle)\right) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}] \text{ for all } i \geq 2 \end{aligned}$$

because $\overline{\gamma_i} = 1 + \tau\overline{R^{(\sigma)}}$ for all $i \geq 2$ implies $\mu_{\gamma_i}(w_\chi) = w_\chi$ for all $\chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$ and $i \geq 2$ as observed above.

It remains to prove that $(\mu_{\gamma_1})_* \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]} = \mathrm{id} \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]}$ on $K_*(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle \tau, \gamma_2, \ldots, \gamma_{i-1}\rangle)) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{Q^f}]$. We know that $\overline{\gamma_1} \in (\overline{R^{(\sigma)}}/\tau\overline{R^{(\sigma)}})^\times$ has order $q^f - 1$ because $(\overline{R^{(\sigma)}}/\tau\overline{R^{(\sigma)}})^\times \cong \mathbb{F}_{q^f}^\times$ is a cyclic group of order $q^f - 1$. Thus $\overline{(\gamma_1)^{q^f-1}} = 1 + \tau\overline{R^{(\sigma)}}$, and this implies $\mu_{(\gamma_1)^{q^f-1}}(w_\chi) = w_\chi$ for all $\chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$ as explained above. Hence, by our previous lemma, we have

$$(\mu_{(\gamma_1)^{q^f-1}})_* \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]} = \mathrm{id} \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]}$$

on $K_*(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle \tau, \gamma_2, \ldots, \gamma_{i-1}\rangle)) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{Q^f}]$. Thus

$$\left((\mu_{\gamma_1})_* \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]}\right)^{q^f-1} = \left((\mu_{\gamma_1})_*\right)^{q^f-1} \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]} = \mathrm{id} \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]}.$$

This tells us that $(\mu_{\gamma_1})_* \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]}$ is of order $q^f - 1$, and with the help of the Pimsner-Voiculescu sequence (proceeding inductively on i), this implies $(\mu_{\gamma_1})_* \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]} = \mathrm{id} \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{Q^f}]}$ on $K_*(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle \tau, \gamma_2, \ldots, \gamma_{i-1}\rangle)) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{Q^f}]$ for all $i \geq 2$. □

Now, let us set $\Gamma := \langle \tau \rangle \times \Gamma'$ so that $L^\times = (L^{(\sigma)})^\times = \mathbb{F}_q^\times \times \Gamma$. By continuity of K-theory, we deduce from the previous proposition that

$$K_*(C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (L^{(\sigma)})^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}] \cong \left(\tilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)\right) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{Q^f}].$$

This result, together with (4) and (6), completes the proof of Theorem 1.1.

Note that Theorem 1.1 covers the results in [Cu-Li3], since for the function fields $\mathbb{F}_q(T)/\mathbb{F}_p(T)$, the inertia degree at infinity f is 1, i.e. $Q^f = 1$.

An immediate consequence of Theorem 1.1 is

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Corollary 6.4 *In the same situation as in Theorem 1.1, let $x \in K_*(\mathfrak{A}[R])$ be a torsion element of order N. Then N divides a suitable power of Q^f, i.e. $N \mid (Q^f)^j$ for j big enough. In particular, every prime divisor of N is a prime divisor of Q^f.*

Remark 6.5 *It becomes clear why we restrict our investigations to the case of a single infinite place. The reason is that if we allow more infinite places, we will have to deal with many prime elements, one for each infinite place. But then, it is not possible to carry out the explicit computation in Section 5 for the crossed product involving the action of all these prime elements. However, the explicit description of K-theory at this stage is the basis for what we do in this section, so it is not clear how to proceed in the case of more than one infinite place.*

Of course, it would be desirable to say more about the torsion part of $K_*(\mathfrak{A}[R])$. We do so in the next section, but even there we are far away from completely determining the torsion part. So this remains an open problem.

7. Torsion in K-theory determines inertia degree

In this section, we consider the same situation as in the previous section. In addition, we assume that the inertia degree f at infinity of our function field $L/\mathbb{F}_q(T)$ is prime to the cardinality of the multiplicative group of the field of constants $\mathbb{F}_q$, i.e. $(f,q-1) = 1$. In this case, we will be able to choose a particular element x_X or rather a particular function X such that the corresponding unitaries w_χ are fixed by every element in Γ' for a suitable choice of Γ'. The unitaries w_χ were defined in Proposition 5.2, see (20), and the elements x_X were constructed in Remark 5.3.

First of all, note that

$$Q^f = \tfrac{q^f-1}{q-1} = 1 + q + \cdots + q^{f-1} \equiv f \mod q-1.$$

Hence it follows from $(f,q-1) = 1$ that we also have $(Q^f,q-1) = 1$. Thus the homomorphism $\mathbb{F}_q^\times \to \mathbb{F}_q^\times; a \mapsto a^{Q^f}$ is an isomorphism.

Now let χ be a character in $\widehat{\mathbb{F}_q^\times}$. Composing χ by the inverse of the map $\mathbb{F}_q^\times \to \mathbb{F}_q^\times; a \mapsto a^{Q^f}$, we obtain a character $\tilde{\chi}$ with the property $\tilde{\chi}(a^{Q^f}) = \chi(a)$ for every a in $\mathbb{F}_q^\times$. Since $Q^f \cdot (q-1) = q^f - 1$ is the order of the cyclic group $(R^{(\sigma)}/\tau R^{(\sigma)})^\times$, we know that $(b + \tau R^{(\sigma)})^{Q^f}$ lies in $\mathbb{F}_q^\times \subseteq (R^{(\sigma)}/\tau R^{(\sigma)})^\times$ for every element $b + \tau R^{(\sigma)} \in (R^{(\sigma)}/\tau R^{(\sigma)})^\times$. Thus we can define $X : (R^{(\sigma)}/\tau R^{(\sigma)})^\times \to \mathbb{T}$ as the composition $(R^{(\sigma)}/\tau R^{(\sigma)})^\times \overset{(\cdot)^{Q^f}}{\to} \mathbb{F}_q^\times \overset{\tilde{\chi}}{\to} \mathbb{T}$, i.e.

$$X(b + \tau R^{(\sigma)}) = \tilde{\chi}((b + \tau R^{(\sigma)})^{Q^f}).$$

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By construction, we have

$$X(a\cdot b+\tau R^{(\sigma)})=\tilde{\chi}(a^{Q^f})\tilde{\chi}((b+\tau R^{(\sigma)})^{Q^f})=\chi(a)X(b+\tau R^{(\sigma)}).$$

The point is that we can now choose Γ' with particular free generators so that the unitaries w_χ are fixed by the multiplicative action of Γ' for every $\chi\in\widehat{\mathbb{F}_q^\times}\setminus\{1\}$. Recall that Γ' is defined as a subgroup of $(L^{(\sigma)})^\times$ with $(L^{(\sigma)})^\times=\mathbb{F}_q^\times\times\langle\tau\rangle\times\Gamma'$ or, equivalently, $(L^{(\sigma)})^{\{w\}}=\mathbb{F}_q\times\Gamma'$. Let us start with an arbitrary choice of Γ' together with free generators $\{\gamma_i\}$ for Γ'. We know that $(\gamma_i+\tau R^{(\sigma)})^{Q^f}$ lies in $\mathbb{F}_q^\times\subseteq(R^{(\sigma)}/\tau R^{(\sigma)})^\times$, so there exists an element a_i in $\mathbb{F}_q^\times$ such that

$$(a_i\gamma_i+\tau R^{(\sigma)})^{Q^f}=1+\tau R^{(\sigma)}\text{ in }(R^{(\sigma)}/\tau R^{(\sigma)})^\times. \tag{21}$$

(Recall that $a\mapsto a^{Q^f}$ is an isomorphism of $\mathbb{F}_q^\times$.) Thus we obtain

$$\begin{aligned}\mu_{a_i\gamma_i}(x_X)&=\sum_{b+\tau R^{(\sigma)}\in(R^{(\sigma)}/\tau R^{(\sigma)})^\times}X(b+\tau R^{(\sigma)})v^{a_i\gamma_i b}\mathbb{1}_{\overline{\tau R^{(\sigma)}}}v^{-a_i\gamma_i b}\\&=\sum_{b+\tau R^{(\sigma)}}\tilde{\chi}((b+\tau R^{(\sigma)})^{Q^f})v^{a_i\gamma_i b}\mathbb{1}_{\overline{\tau R^{(\sigma)}}}v^{-a_i\gamma_i b}\\&=\sum_{b+\tau R^{(\sigma)}}\tilde{\chi}((a_i\gamma_i+\tau R^{(\sigma)})^{Q^f})^{-1}X(a_i\gamma_i b+\tau R^{(\sigma)})v^{a_i\gamma_i b}\mathbb{1}_{\overline{\tau R^{(\sigma)}}}v^{-a_i\gamma_i b}\\&=\tilde{\chi}((a_i\gamma_i+\tau R^{(\sigma)})^{Q^f})^{-1}x_X\overset{(21)}{=}x_X.\end{aligned}$$

As $\mu_{a_i\gamma_i}$ also fixes all the remaining summands of w_χ (see (20)), we conclude that $\mu_{a_i\gamma_i}(w_\chi)=w_\chi$. So for every generator γ_i, there exists an element a_i in $\mathbb{F}_q^\times$ such that $\mu_{a_i\gamma_i}$ fixes w_χ for all χ in $\widehat{\mathbb{F}_q^\times}\setminus\{1\}$. But the elements $\{a_i\gamma_i\}$ are again free generators of a subgroup Γ' of $(L^{(\sigma)})^\times$ with $(L^{(\sigma)})^{\{w\}}=\mathbb{F}_q\times\Gamma'$.

The following lemma is proven with the help of the Pimsner-Voiculescu sequence, similarly to the proof of Lemma 6.2:

Lemma 7.1 *Assume that we can find unitaries w_χ, $\chi\in\widehat{\mathbb{F}_q^\times}\setminus\{1\}$, of the form* (20) *and free generators γ_i for a choice of Γ' (a subgroup of $(L^{(\sigma)})^\times$ with $(L^{(\sigma)})^\times=\mathbb{F}_q^\times\times\langle\tau\rangle\times\Gamma'$) such that every μ_{γ_i} fixes w_χ for every $\chi\in\widehat{\mathbb{F}_q^\times}\setminus\{1\}$. Set $\Gamma_i:=\{\tau,\gamma_1,\dots,\gamma_{i-1}\}$. Then the canonical map*

$$C_0(\mathbb{A}_T^{(\sigma)})\rtimes L^{(\sigma)}\rtimes(\mathbb{F}_q^\times\times\Gamma_i)\to C_0(\mathbb{A}_T^{(\sigma)})\rtimes L^{(\sigma)}\rtimes(\mathbb{F}_q^\times\times\Gamma_{i+1})$$

induces an injective map on K-theory for every $i\in\mathbb{Z}_{\geq 0}$.

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Now we know that

$$C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (L^{(\sigma)})^\times \cong \varinjlim_i \left\{ C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \Gamma_i) \right\}.$$

Thus the previous lemma, together with continuity of K-theory, implies that the canonical homomorphism

$$C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (\mathbb{F}_q^\times \times \langle \tau \rangle) \to C_0(\mathbb{A}_T^{(\sigma)}) \rtimes L^{(\sigma)} \rtimes (L^{(\sigma)})^\times$$

induces an injective map on K-theory. The reason, as explained above, is that we can choose Γ' with free generators γ_i for Γ' and unitaries w_χ for $\chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$ such that $\mu_{\gamma_i}(w_\chi) = w_\chi$ for all χ in $\widehat{\mathbb{F}_q^\times} \setminus \{1\}$ as required by Lemma 7.1.

This observation in combination with (6), (4) and Proposition 5.2 leads to the following

Proposition 7.2 *Assume that the inertia degree f at infinity of $L/\mathbb{F}_q(T)$ satisfies $(f, q-1) = 1$. Let R be the ring of integers in $L/\mathbb{F}_p(T)$. Set $Q^f = \frac{q^f - 1}{q-1}$. Then $K_*(\mathfrak{A}[R])$ contains nontrivial Q^f-torsion.*

Now let us come to the proof of Theorem 1.2. Let $L_1/\mathbb{F}_p(T)$, $L_2/\mathbb{F}_p(T)$ be function fields, both having the constant field $\mathbb{F}_q$. Assume that both function fields have only a single infinite place and let f_i be the inertia degree at infinity of $L_i/\mathbb{F}_q(T)$. Suppose that $(f_i, q-1) = 1$ for $i = 1,2$. Let R_1 and R_2 denote the rings of integers in $L_1/\mathbb{F}_p(T)$ and $L_2/\mathbb{F}_p(T)$. In this situation, we can prove that

Theorem 7.3 *If $K_*(\mathfrak{A}[R_1]) \cong K_*(\mathfrak{A}[R_2])$, then the inertia degrees at infinity of $L_1/\mathbb{F}_q(T)$ and $L_2/\mathbb{F}_q(T)$ must coincide, i.e. $f_1 = f_2$.*

Proof: Let f_i denote the inertia degrees, for $i = 1,2$. Since $K_*(\mathfrak{A}[R_1]) \cong K_*(\mathfrak{A}[R_2])$, we conclude by Proposition 7.2 that $K_*(\mathfrak{A}[R_2])$ contains a torsion element of order Q^{f_1}. Hence, Corollary 6.4 tells us that every prime divisor of Q^{f_1} divides Q^{f_2}. As the situation is symmetric in $L_1/\mathbb{F}_q(T)$ and $L_2/\mathbb{F}_q(T)$, we can equally well deduce that every prime divisor of Q^{f_2} divides Q^{f_1}. Thus we conclude that Q^{f_1} and Q^{f_2} have the same set of prime divisors. The assertion of the theorem now follows from the following

Lemma 7.4 *Let q be an integer with $q > 1$. Let f_1, f_2 be positive integers. Set $Q^{f_i} = \frac{q^{f_i} - 1}{q-1}$ for $i = 1,2$. If Q^{f_1} and Q^{f_2} have the same set of prime divisors, then $f_1 = f_2$.*

Proof: If Q^{f_1} and Q^{f_2} have the same prime divisors, then the set of their prime divisors is the set of prime divisors of their least common multiple (Q^{f_1}, Q^{f_2}). We first prove that

$$(Q^{f_1}, Q^{f_2}) = Q^{(f_1, f_2)}. \tag{22}$$

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Write $f = (f_1, f_2)$ and $f_1 = fg_1$, $f_2 = fg_2$ with $(g_1, g_2) = 1$. Then

$$Q^{f_1} = Q^f \cdot (1 + q^f + \cdots + (q^f)^{g_1 - 1})$$

and

$$Q^{f_2} = Q^f \cdot (1 + q^f + \cdots + (q^f)^{g_2 - 1}).$$

We have to show $(1 + q^f + \cdots + (q^f)^{g_1-1}, 1 + q^f + \cdots + (q^f)^{g_2-1}) = 1$. As $(g_1, g_2) = 1$, there are $a, b \in \mathbb{Z}_{\geq 0}$ such that $ag_1 = 1 + bg_2$. Thus

$$\begin{aligned}
&(1 + q^f + \cdots + (q^f)^{g_1-1}) \cdot (1 + (q^f)^{g_1} + \cdots + ((q^f)^{g_1})^{a-1}) \\
&= 1 + q^f + \cdots + (q^f)^{ag_1 - 1} = 1 + q^f + \cdots + (q^f)^{bg_2 - 1} + (q^f)^{bg_2} \\
&= (1 + q^f + \cdots + (q^f)^{g_2-1}) \cdot (1 + (q^f)^{g_2} + \cdots + ((q^f)^{g_2})^{b-1}) + (q^f)^{bg_2}.
\end{aligned}$$

Thus $(1+q^f+\cdots+(q^f)^{g_1-1}, 1+q^f+\cdots+(q^f)^{g_2-1})$ divides $(q^f)^{bg_2}$. If $b = 0$, then g_1 has to be 1 and we deduce $(1+q^f+\cdots+(q^f)^{g_1-1}, 1+q^f+\cdots+(q^f)^{g_2-1}) = 1$. If $b > 0$, then $(1 + q^f + \cdots + (q^f)^{g_1-1}, (q^f)^{bg_2}) = 1$, since every prime number π that divides $(q^f)^{bg_2}$ also divides q^f, thus π cannot divide $1 + q^f + \cdots + (q^f)^{g_1-1}$. This proves (22).

Therefore, we can without loss of generality assume that f_2 divides f_1, i.e. we have $f_1 = f_2 f$. (In general, we can then apply our result to f_1 and $f = (f_1, f_2)$ and to f_2 and (f_1, f_2).) Now consider the prime decompositions

$$Q^{f_2} = \pi_1^{n_1} \cdots \pi_r^{n_r} \text{ and } Q^{f_1} = \pi_1^{N_1} \cdots \pi_r^{N_r}$$

where n_i and N_i are positive integers. Here, we have used our assumption that the set of prime divisors of Q^{f_1} and Q^{f_2} coincide. Moreover, since f_2 divides f_1, we must have $N_i \geq n_i$ for all $1 \leq i \leq r$. It follows that

$$q^{f_2} - 1 = (q-1)\pi_1^{n_1} \cdots \pi_r^{n_r} \text{ and thus } q^{f_2} = 1 + (q-1)\pi_1^{n_1} \cdots \pi_r^{n_r}.$$

Analogously, we obtain $q^{f_1} = 1 + (q-1)\pi_1^{N_1} \cdots \pi_r^{N_r}$. But we also have

$$\begin{aligned}
q^{f_1} = (q^{f_2})^f &= (1 + (q-1)\pi_1^{n_1} \cdots \pi_r^{n_r})^f \\
&= \sum_{e=0}^{f} \binom{f}{e} (q-1)^e \pi_1^{n_1 e} \cdots \pi_r^{n_r e} \\
&= 1 + (q-1) \sum_{e=1}^{f} \binom{f}{e} (q-1)^{e-1} \pi_1^{n_1 e} \cdots \pi_r^{n_r e}.
\end{aligned}$$

It follows that

$$\pi_1^{N_1} \cdots \pi_r^{N_r} = \sum_{e=1}^{f} \binom{f}{e} (q-1)^{e-1} \pi_1^{n_1 e} \cdots \pi_r^{n_r e}. \tag{23}$$

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We now claim that for all $1 \leq i \leq r$ and all $2 \leq e \leq f$

$$v_{\pi_i}(f\pi_1^{n_1}\cdots\pi_r^{n_r}) < v_{\pi_i}(\binom{f}{e}(q-1)^{e-1}\pi_1^{n_1 e}\cdots\pi_r^{n_r e}). \tag{24}$$

Here, if x is an integer then $v_{\pi_i}(x)$ is defined as $\max(\{v \in \mathbb{Z}_{\geq 0}\colon \pi_i^v \mid x\})$.

To prove this, we compute $v_{\pi_i}(f\pi_1^{n_1}\cdots\pi_r^{n_r}) = v_{\pi_i}(f) + n_i$ and

$$\begin{aligned} &v_{\pi_i}(\binom{f}{e}(q-1)^{e-1}\pi_1^{n_1 e}\cdots\pi_r^{n_r e}) \\ &= v_{\pi_i}(f\cdots(f-e+1)) - v_{\pi_i}(e!) + (e-1)v_{\pi_i}(q-1) + n_i e \\ &\geq v_{\pi_i}(f) - v_{\pi_i}(e!) + (e-1)v_{\pi_i}(q-1) + n_i e. \end{aligned} \tag{25}$$

Now expand e with respect to π_i, i.e. write

$$e = a_0 + a_1\pi_i + \cdots + a_m\pi_i^m \text{ with } a_0,\ldots,a_m \in \{0,\ldots,\pi_i - 1\}.$$

By [Neu], Chapter II, Lemma (5.6), we have

$$v_{\pi_i}(e!) = \tfrac{1}{\pi_i - 1}\big((\pi_i^m - 1)a_m + (\pi_i^{m-1} - 1)a_{m-1} + \cdots + a_1\big). \tag{26}$$

If $\pi_i = 2$, then $2 \mid Q^{f_1}$ implies that q is odd. Hence for every $2 \leq e$, we must have $(e-1)v_{\pi_i}(q-1) > 0$. Inserting this into (25), we obtain

$$v_{\pi_i}(\binom{f}{e}(q-1)^{e-1}\pi_1^{n_1 e}\cdots\pi_r^{n_r e}) > v_{\pi_i}(f) - v_{\pi_i}(e!) + n_i e. \tag{27}$$

Moreover, we always have by (26) that

$$\begin{aligned} v_{\pi_i}(e!) &\leq (\pi_i^m - 1)a_m + (\pi_i^{m-1} - 1)a_{m-1} + \cdots + a_1 \\ &= e - (a_m + a_{m-1} + \cdots + a_1 + a_0) \leq e - 1. \end{aligned} \tag{28}$$

Combining (27) and (28), we obtain in the case $\pi_i = 2$:

$$v_{\pi_i}(\binom{f}{e}(q-1)^{e-1}\pi_1^{n_1 e}\cdots\pi_r^{n_r e}) > v_{\pi_i}(f) - (e-1) + n_i e.$$

But this inequality also holds for $\pi_i > 2$, since we always have

$$v_{\pi_i}(\binom{f}{e}(q-1)^{e-1}\pi_1^{n_1 e}\cdots\pi_r^{n_r e}) \geq v_{\pi_i}(f) - v_{\pi_i}(e!) + n_i e \tag{29}$$

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by (25). Moreover, if $\pi_i > 2$, then we deduce from (26) that

$$\begin{aligned} v_{\pi_i}(e!) &< (\pi_i^m - 1)a_m + (\pi_i^{m-1} - 1)a_{m-1} + \cdots + a_1 \\ &= e - (a_m + a_{m-1} + \cdots + a_1 + a_0) \leq e - 1. \end{aligned}$$

Plugging this inequality into (29), we obtain

$$v_{\pi_i}(\binom{f}{e}(q-1)^{e-1}\pi_1^{n_1 e}\cdots\pi_r^{n_r e}) > v_{\pi_i}(f) - (e-1) + n_i e.$$

Thus we have for every π_i:

$$\begin{aligned} & v_{\pi_i}(\binom{f}{e}(q-1)^{e-1}\pi_1^{n_1 e}\cdots\pi_r^{n_r e}) > v_{\pi_i}(f) - (e-1) + n_i e \\ &= v_{\pi_i}(f) + n_i + n_i(e-1) - (e-1) = v_{\pi_i}(f) + n_i + (n_i - 1)(e-1) \\ &\geq v_{\pi_i}(f) + n_i = v_{\pi_i}(f\pi_1^{n_1}\cdots\pi_r^{n_r}). \end{aligned}$$

This proves (24). As a consequence, we deduce for all $1 \leq i \leq r$ that

$$v_{\pi_i}(\sum_{e=1}^{f}\binom{f}{e}(q-1)^{e-1}\pi_1^{n_1 e}\cdots\pi_r^{n_r e}) = v_{\pi_i}(f\pi_1^{n_1}\cdots\pi_r^{n_r}) = v_{\pi_i}(f) + n_i.$$

Comparing this with (23), we conclude that

$$N_i = v_{\pi_i}(\pi_1^{N_1}\cdots\pi_r^{N_r}) = v_{\pi_i}(f) + n_i.$$

Thus $v_{\pi_i}(f) = N_i - n_i$ for all $1 \leq i \leq r$. This means that $\pi_1^{N_1-n_1}\cdots\pi_r^{N_r-n_r}$ divides f. But $\pi_1^{N_1-n_1}\cdots\pi_r^{N_r-n_r} = Q^{f_1}/Q^{f_2} = 1 + q^{f_2} + (q^{f_2})^2 + \cdots + (q^{f_2})^{f-1} > f$ if $f > 1$, because $q > 1$. Thus we obtain a contradiction unless $f = 1$, i.e. $f_1 = f_2$. □

This completes the proof of Theorem 1.2. □

Let us comment on the conditions in our main results. First of all, since our function field $L/\mathbb{F}_p(T)$ has only one single infinite place, the inertia degree f at infinity of $L/\mathbb{F}_q(T)$ always divides the degree $n = [L : \mathbb{F}_q(T)]$. Thus the primeness assumption $(f, q-1) = 1$ in Theorem 1.2 can be replaced by the stronger statement $(n, q-1) = 1$. As the latter condition just requires looking at the global fields instead of the local fields at infinity, it might be simpler to verify.

Moreover, we can explicitly construct function fields $L/\mathbb{F}_p(T)$ satisfying all the conditions needed in Theorem 1.1.

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Remark 7.5 *For every prime power q and every positive integer f, there exists a finite separable field extension $L/\mathbb{F}_p(T)$ such that*

- *the field of constants of $L/\mathbb{F}_p(T)$ is $\mathbb{F}_q$*
- *$L/\mathbb{F}_p(T)$ has only one single infinite place*
- *the inertia degree at infinity of $L/\mathbb{F}_q(T)$ is f.*

One possible construction goes as follows: Let ζ be a primitive $(q^f - 1)$-th root of unity in $(\mathbb{F}_q)^{\mathrm{alg}}$, so that $\mathbb{F}_{q^f} = \mathbb{F}_q(\zeta)$. Let $g(X) \in \mathbb{F}_q[X]$ be the minimal polynomial of ζ over $\mathbb{F}_q$. Set $L := \mathbb{F}_q(X)$ and embed $\mathbb{F}_p(T)$ into L via $T \mapsto g(X)^{-1}$. The corresponding extension $L/\mathbb{F}_p(T)$ is separable as $\mathbb{F}_q$ is perfect, so that the derivative of $g(X)$ does not vanish. Moreover, $L/\mathbb{F}_p(T)$ satisfies by construction all three conditions listed above.

Remark 7.5, in combination with Theorem 1.2, shows that unlike in the case of number fields, there are infinitely many possibilities for the K-groups of ring C*-algebras attached to rings of integers in function fields. The reason why this is not the case for number fields is that there are not too many possibilities for the inertia degree at infinity in the number field case: A real place is said to have inertia degree 1, and a complex place is said to have inertia degree 2 (compare [Neu], Chapter III, § 1).

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XIN LI xinli.math@uni-muenster.de

Mathematisches Institut
Einsteinstrasse 62
48149 Münster
Germany

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