The most elementary proof that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$?

NICK LORD

Here we present a simplification of one of the standard proofs that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. We then look at extensions of the new approach, and add comments on the nature of the simplification (which relates to Step 1 below) and finally on the literature.

As in most proofs of the result, we shall actually prove that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$; because

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{k^2},$$

this is equivalent to the result sought.

Our only prerequisites are:

(i) The identity $\operatorname{cosec}\left(\frac{\pi}{2} + x\right) = \sec x$, for $0 \leqslant x < \frac{\pi}{2}$;

(ii) The identity $\operatorname{cosec}(\pi - x) = \operatorname{cosec} x$, for $0 < x \leqslant \frac{\pi}{2}$;

(iii) The identity $\operatorname{cosec}^2 x + \sec^2 x = 4 \operatorname{cosec}^2 2x$, valid when x is not an integral multiple of $\frac{\pi}{2}$;

(iv) The inequality $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$ from which it follows that $\operatorname{cosec}^2 x - 1 < \frac{1}{x^2} < \operatorname{cosec}^2 x$, for $0 < x \leqslant \frac{\pi}{2}$;

(v) The identity $\operatorname{cosec}^4 x + \sec^4 x = 16 \operatorname{cosec}^4 2x - 8 \operatorname{cosec}^2 2x$, valid when x is not an integral multiple of $\frac{\pi}{2}$.

The proof

Step 1: For $n \geqslant 1$, let $C_n = \displaystyle\sum_{k=1}^{2^{n-1}} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n}$.

Then $C_1 = 1$. Also, for $n > 1$ we have that

$$C_n = \sum_{k=1}^{2^{n-2}} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n} + \sum_{k=2^{n-2}+1}^{2^{n-1}} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n}$$

$$= \sum_{k=1}^{2^{n-2}} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n} + \sum_{k=1}^{2^{n-2}} \sec^2 \frac{(2k-1)\pi}{2^n} \text{ (using (i))}$$

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$$= \sum_{k=1}^{2^{n-2}} \left[\operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n} + \sec^2 \frac{(2k-1)\pi}{2^n} \right]$$

$$= 4 \sum_{k=1}^{2^{n-2}} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^{n-1}} \text{ (using (iii))}$$

$$= 4C_{n-1}.$$

It follows by induction that $C_n = 4^{n-1}$.

Next, for $n \geqslant 2$, let $C_n' = \sum_{k=1}^{2^{n-2}} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n}$. Then we have

$$C_n' = \frac{1}{2}\left[\sum_{k=1}^{2^{n-2}} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n} + \sum_{k=2^{n-2}+1}^{2^{n-1}} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n} \right] \text{(using (ii))}$$

$$= \frac{1}{2} C_n$$

$$= \frac{4^n}{8} \text{ (using the earlier evaluation of } C_n).$$

Step 2: It follows from (iv) that, for $n \geqslant 2$ and $1 \leqslant k \leqslant 2^{n-2}$, we have

$$\operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n} - 1 < \frac{4^n}{\pi^2} \frac{1}{(2k-1)^2} < \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n}.$$

Summing these inequalities for $1 \leqslant k \leqslant 2^{n-2}$, we obtain

$$C_n' - 2^{n-2} < \frac{4^n}{\pi^2} \sum_{k=1}^{2^{n-2}} \frac{1}{(2k-1)^2} < C_n'.$$

Substituting $\frac{4^n}{8}$ for C_n' (using the result of Step 1) and multiplying throughout by $\frac{\pi^2}{4^n}$ gives

$$\frac{\pi^2}{8} - \frac{\pi^2}{2^{n+2}} < \sum_{k=1}^{2^{n-2}} \frac{1}{(2k-1)^2} < \frac{\pi^2}{8},$$

and then letting $n \to \infty$ completes the proof.

Extensions of the new approach

A similar, but slightly more fiddly, argument may be used to show directly that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Starting with $S_n = \sum_{k=1}^{2^n-1} \operatorname{cosec}^2 \frac{k\pi}{2^n}$, the method of Step 1 yields the recurrence relation $S_n = 1 + 4S_{n-1}$ with $S_1 = 1$,

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which has the solution $S_n = \dfrac{4^n - 1}{3}$. Then the method of Step 2, with $S_n' = \displaystyle\sum_{k=1}^{2^{n-1}} \operatorname{cosec}^2 \frac{k\pi}{2^n} = \frac{1}{2}(S_n + 1)$, gives

$$\lim_{n\to\infty} \sum_{k=1}^{2^{n-1}} \frac{1}{k^2} = \lim_{n\to\infty} \frac{\pi^2}{4^n} \frac{S_n}{2} = \frac{\pi^2}{6}.$$

The added complication here arises from the fact that the sum S_n contains an odd number of terms, so they pair up apart from the term corresponding to $k = 2^{n-2}$ which contributes 1 to the sum.

We can also use an argument similar to our original one to show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$, where instead of (iii) we use the trigonometrical identity (v).

For $n \geqslant 2$, let $D_n = \displaystyle\sum_{k=1}^{2^{n-1}} \operatorname{cosec}^4 \frac{(2k-1)\pi}{2^n}$, with $C_n = \displaystyle\sum_{k=1}^{2^{n-1}} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2^n}$, as before. Then it follows from (i) and (v) that

$$D_n = 16D_{n-1} - 8C_{n-1},$$

so, since $C_n = 4C_{n-1}$, we have

$$D_n - \tfrac{2}{3}C_n = 16\left(D_{n-1} - \tfrac{2}{3}C_{n-1}\right).$$

From this we deduce that $D_n - \frac{2}{3}C_n = \frac{1}{3}16^{n-1}$ and (using the result that $C_n = 4^{n-1}$) we then get

$$D_n = \tfrac{1}{3}\,16^{n-1} + \tfrac{2}{3}\,4^{n-1} \text{ (which, in fact, is always an integer).}$$

Next, for $n \geqslant 2$, let $D_n' = \displaystyle\sum_{k=1}^{2^{n-2}} \operatorname{cosec}^4 \frac{(2k-1)\pi}{2^n}$. By an argument similar to that in the original proof, we get

$$D_n' = \frac{D_n}{2} = \frac{16^n}{96} + \frac{4^n}{12}.$$

Now if we square throughout the inequalities in (iv), we get

$$\operatorname{cosec}^4 x - 2\operatorname{cosec}^2 x + 1 < \frac{1}{x^4} < \operatorname{cosec}^4 x.$$

Substituting $x = \dfrac{(2k-1)\pi}{2^n}$ and summing these inequalities for $1 \leqslant k \leqslant 2^{n-2}$ then gives

$$D_n' - 2C_n' + 2^{n-2} < \frac{16^n}{\pi^4} \sum_{k=1}^{2^{n-2}} \frac{1}{(2k-1)^4} < D_n'.$$

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It follows that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \lim_{n\to\infty} \frac{\pi^4}{16^n} D_n' = \frac{\pi^4}{96}$$

and so

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{16}{15} \times \frac{\pi^4}{96} = \frac{\pi^4}{90}.$$

Comments

The idea of using $\sum \operatorname{cosec}^2 \frac{k\pi}{n}$ as a catalyst for a proof that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ is not a new one. In the existing proofs, Step 2 is essentially as above, but Step 1 is replaced with the results of more general summations such as $\sum_{k=1}^{m} \operatorname{cosec}^2 \frac{k\pi}{2m+1}$ in [1; pp. 209-210] and $\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1}$ in [2, 3, 4, 5, 6]. These are evaluated via the respective trigonometrical identities

$$\sin(2m+1)x = (2m+1)\sin^{2m+1} x P_m(\operatorname{cosec}^2 x)$$

$$\text{and } \sin(2m+1)x = \sin^{2m+1} x Q_m(\cot^2 x)$$

where P_m and Q_m are polynomials of degree m whose coefficients are explicitly known. Such evaluations involve de Moivre's theorem and the relationship between the roots and coefficients of polynomials. It is worth noting that, although using the odd number $2m+1$ makes these proofs run smoothly, the summations involved are particular cases of the overarching results $\sum_{k=1}^{n-1} \operatorname{cosec}^2 \frac{k\pi}{n} = \frac{n^2-1}{3}$ and $\sum_{k=1}^{n} \operatorname{cosec}^2 \frac{(2k-1)\pi}{2n} = n^2$ (and their counterparts in terms of $\cot^2\theta$) which are valid for all n. These may be obtained either by collating some exercises in [1; p.228, p.230] or by making the respective choices $\phi \to 0$ and $\phi = \frac{\pi}{2n}$ in Formula (B) on page 216 of [7]. Such sums and their generalisations have attracted considerable attention during recent years in papers such as [8] where several further historical references are given.

The literature

The history of this method of proving $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ is a tangled one. Robin Chapman in his collection of proofs [6] cites an exercise in Apostol's book, [5], which is based on [4]. Stark in [9] traces the proof back to Yaglom and Yaglom's paper [2], which is reprised in their well-known problem book [3]. However, the proof is in Durell and Robson's school textbook [1] which takes the story back to 1930. A version of the proof is also in Bromwich's text [7; pp.218-219], the first edition of which dates

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from 1908. This contains identities equivalent to $\sum_{k=1}^{n-1}\operatorname{cosec}^2\frac{k\pi}{n}=\frac{n^2-1}{3}$ for n odd and $\sum_{k=1}^{n}\operatorname{cosec}^2\frac{(2k-1)\pi}{2n}=n^2$ for n even but, in place of Step 2 above, Tannery's theorem (a double-limit dominated convergence theorem) is used to deduce the limiting results $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ and $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$. But the originator of this proof seems to have been Cauchy who, in Note 8 near the end of [10], not only derives all odd and even cases of the identities for $\sum_{k=1}^{n-1}\operatorname{cosec}^2\frac{k\pi}{n}$ and $\sum_{k=1}^{n}\operatorname{cosec}^2\frac{(2k-1)\pi}{2n}$ mentioned above, but also points out their utility for deriving $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ and $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$. He does this by bounding the terms of the sums directly, rather than using the identity $\operatorname{cosec}^2 x=1+\cot^2 x$ which is clearly a later refinement to the proof. Thus, from $\sum_{k=1}^{\frac{1}{2}(n-1)}\operatorname{cosec}^2\frac{k\pi}{n}=\frac{n^2-1}{6}$ for n odd we have, for any $1\leqslant r<\frac{1}{2}(n-1)$,

$$\sum_{k=1}^{r}\frac{1}{k^2}\left(\frac{\frac{k\pi}{n}}{\sin\frac{k\pi}{n}}\right)^2+\sum_{k=r+1}^{\frac{1}{2}(n-1)}\frac{1}{k^2}\left(\frac{\frac{k\pi}{n}}{\sin\frac{k\pi}{n}}\right)^2=\frac{\pi^2}{n^2}\frac{n^2-1}{6}.$$

The RHS tends to $\frac{\pi^2}{6}$ as $n\to\infty$, while the LHS is sandwiched between the lower bound $\sum_{k=1}^{r}\frac{1}{k^2}$ (since $\sin x<x$ for $0<x<\frac{\pi}{2}$) and the upper bound $\frac{1}{\cos^2\frac{r\pi}{n}}\sum_{k=1}^{r}\frac{1}{k^2}+\frac{4}{r^2}\left(\frac{n-1}{2}-r\right)$ (since, for $0<x<\frac{\pi}{2}$, $\frac{x}{\sin x}$ is increasing and pointwise less than $\frac{1}{\cos x}$ and bounded above by 2). The summation $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ then ensues by choosing $n=O\left(r^{\alpha}\right)$ for some fixed $1<\alpha<2$.

Finally, we remark that Euler's general formula

$$\sum_{k=1}^{\infty}\frac{1}{k^{2n}}=(-1)^{n-1}\frac{B_{2n}2^{2n-1}\pi^{2n}}{(2n)!}$$

may be proved in this way; here the numbers B_n are the so-called *Bernoulli numbers*. This is hinted at in [2, 3] and the details are given in full by Apostol in [11].

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NICK LORD

Tonbridge School, Kent TN9 1JP

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