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Constructing projective varieties in weighted flag varieties II

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Abstract

We give the construction of weighted Lagrangian Grassmannians $w\,\mathrm{LGr}(3,6)$ and weighted partial A_3 flag variety $w\,\mathrm{FL}_{1,3}$ coming from the symplectic Lie group $\mathrm{Sp}(6,\mathbb{C})$ and the general linear group $\mathrm{GL}(4,\mathbb{C})$ respectively. We give general formulas for their Hilbert series in terms of Lie theoretic data. We use them as key varieties (Format) to construct some families of polarized 3-folds in codimension 7 and 9. Finally, we list all the distinct weighted flag varieties in codimension $4 \leqslant c \leqslant 10$.

1. *Introduction*

This paper is a continuation of joint work [**17**] with Balázs Szendrői. In that paper, we proved a general formula for the Hilbert series of weighted flag varieties and discussed a computer aided method to find their defining equations. We also applied them to construct families of projective varieties as quasilinear sections of certain weighted flag varieties in codimensions 8 and 6. In [**17**], we used a weighted G_2 variety as a key variety(ambient weighted projective variety) to construct families of polarized varieties in codimension eight and weighted Grassmannian $w\,\mathrm{Gr}(2,6)$ for the construction of varieties in codimension six.

In this paper, we explicitly discuss the construction of two new weighted flag varieties $(w\Sigma, \mathcal{O}_{w\Sigma}(1))$. The first one is the weighted Lagrangian Grassmannian $w\,\mathrm{LGr}(3,6)$; a homogeneous variety for the symplectic Lie group $\mathrm{Sp}(6,\mathbb{C})$. The second one is the weighted partial flag variety $w\,\mathrm{FL}_{1,3}$; a partial flag variety for the general linear group $\mathrm{GL}(4,\mathbb{C})$. The $w\,\mathrm{LGr}(3,6)$ is a six dimensional variety and has an embedding in weighted projective space

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$w\mathbb{P}^{13}$: a codimension seven embedding. The weighted partial flag variety $w\,\mathrm{FL}_{1,3}$ is a five dimensional variety and has an embedding in $w\mathbb{P}^{14}$: a codimension nine embedding.

We use these key varieties to exhibit some potentially new families of Calabi–Yau 3-folds in codimension seven and nine as weighted complete intersections of these varieties. The constructed Calabi–Yau 3-folds X have canonical singularities which can be resolved by crepant resolutions $Y \to X$ using standard theory [**19**]. The desingularisation Y may lead to new examples of Calabi–Yau 3-folds but since we do not compute the topological invariants such as Betti and Hodge numbers, we have no evidence.

The explicit constructions of $w\,\mathrm{LGr}(3, 6)$ and $w\,\mathrm{FL}_{1,3}$ will be worked out by computing their corresponding graded ring structures in terms of generators and relations as well as using the Lie theoretic data of the associated Lie groups. Then we use their graded rings and Hilbert series to construct some families of polarized varieties as weighted completed intersections inside them. The graded rings of the constructed polarized varieties are induced from the graded rings of the ambient key varieties $w\,\mathrm{LGr}(3, 6)$ and $w\,\mathrm{FL}_{1,3}$.

We construct projective varieties (X, D) polarized by $\mathbb{Q}$-ample Weil divisor (i.e. nD is a Cartier divisor) D with finitely generated graded ring

$$R(X, D) = \bigoplus_{n \geqslant 0} H^0(X, nD).$$

The embedding

$$i : X = \mathrm{Proj}\, R(X, D) \hookrightarrow \mathbb{P}[w_0, \ldots, w_n]$$

is provided by surjective morphism

$$\mathbb{C}[x_0, \ldots, x_n] \twoheadrightarrow R(X, D)$$

from a free graded ring $S = \mathbb{C}[x_0, ..., x_n]$. The graded ring S is generated by variables x_i of weights w_i. The divisorial sheaf $\mathcal{O}_X(D)$ of X is isomorphic to $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}}(1)$.

In the past, such examples have been computed in [**2**, **12**], where the cases of codimension at most 3 are discussed. The codimension 4 case was initially discussed in [**1**] and studied recently more rigorously in [**5**]; see Corti and Reid [**8**] for examples in codimension 5 and [**17**, **18**] for examples in codimension 6 and 8.

We construct examples in codimensions 7 and 9 by taking quasilinear sections of weighted flag varieties $(w\Sigma, \mathcal{O}_{w\Sigma}(1))$ embedded in weighted projective space $w\mathbb{P}V_\lambda$ by their natural Plücker-type embeddings. We compute the Hilbert series of a given weighted flag variety to find the canonical divisor class of $w\Sigma$. Then we take quasilinear sections (general hypersurfaces of the appropriate degree in weighted projective space) of $w\Sigma$ or of projective cone(s) over it to get a variety with the desired canonical or anticanonical class. We need the defining equations of flag varieties to understand the nature of singularities. The defining ideals of flag varieties are given in [**11**, section 1]. We work out the equations by following the algorithmic approach of DeGraaf [**9**], coded in computer algebra system GAP4 [**17**, appendix A].

We fix the notations and give the necessary definitions in Section 2. We also provide a quick review of the weighted flag varieties, the formula for their Hilbert series $P_{w\Sigma}$, the defining ideals of flag varieties, and the general method of constructing families of polarized varieties as quasi-linear sections of $w\Sigma$. In Section 3, we study weighted flag varieties associated to the symplectic Lie group $\mathrm{Sp}(6, \mathbb{C})$, leading to the codimension seven varieties. The case of weighted partial flag variety $w\,\mathrm{FL}_{1,3}$ in codimension nine is discussed in Section 4.

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In Section 5, we give the list of all possible distinct flag varieties, which have embeddings in codimension c, for $4 \leqslant c \leqslant 10$.

2. *Definitions and conventions*

We work over a field $\mathbb{C}$ of complex numbers. A pair (X, D), where X is a normal projective algebraic variety and D is an ample $\mathbb{Q}$-Cartier Weil divisor on X is called polarized variety. All our varieties appear as projective subvarieties of some weighted projective space.

The standard notation $\mathbb{P}[w_0, w_1, \ldots, w_n]$ denotes the weighted projective space; sometimes we will write $w\mathbb{P}$ if no confusion can arise. The weighted projective space $\mathbb{P}^n[w_i]$ is called well-formed, if no $n-1$ of $w_0, \ldots, w_n$ have a common factor.

A polarized variety $X \subset \mathbb{P}^n[w_i]$ of codimension c is called well-formed, if $\mathbb{P}^n[w_i]$ is well-formed and X does not contain a codimension $c+1$ singular stratum of $\mathbb{P}[w_i]$. X is called quasi-smooth if the affine cone $\widetilde{X} \subset \mathbb{A}^{n+1}$ of X is smooth outside its vertex $\underline{0}$. If X is quasi-smooth, then it will only have quotient singularities induced by the singularities of $\mathbb{P}[w_i]$. We assume that polarization is provided by the restriction of the tautological ample divisor $\mathcal{O}_{\mathbb{P}}(1)$.

The Hilbert series of a polarized projective variety (X, D) is

$$P_{(X,D)}(t) = \sum_{n \geqslant 0} \dim H^0(X, nD)t^n.$$

We will sometimes write $P_X(t)$ if no confusion can arise. Appropriate Riemann–Roch formulas, together with vanishing, can be used to compute $h^0(X, nD) = \dim H^0(X, nD)$ in favourable cases.

A polarized Calabi–Yau 3-fold (X, D) is a Gorenstein, normal, projective three dimensional algebraic variety with $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. We allow (X, D) to have at worst canonical quotient singularities, consisting of points and curves on X.

Let G be a reductive Lie group with its Lie algebra being denoted by $\mathfrak{g}$. We denote by B a Borel subgroup of G, by P a parabolic subgroup of G, and by T a maximal torus, such that $T \subset B \subseteq P \subset G$ and $\mathfrak{t} \subset \mathfrak{b} \subseteq \mathfrak{p} \subset \mathfrak{g}$ denote the corresponding Lie algebras inclusions.

Let $\Lambda_W = \mathrm{Hom}(T, \mathbb{C}^*)$ be the weight lattice and V_λ denote the G-representation with highest weight λ such that $\nabla(V)$ represents the set of weights of the representation V_λ. The quotient $\Sigma = G/P$ of the Lie group G by a parabolic subgroup P is called (generalized or partial) flag variety; if $P = B$ then we call Σ a complete flag variety. The flag varieties $\Sigma = G/P_\lambda$ are projective subvarieties of $\mathbb{P}V_\lambda$, where $P = P_\lambda$ is the parabolic corresponding to the set of simple roots ∇_0 of G which are orthogonal to the weight λ in the weight lattice Λ_W.

For example, if $G = \mathrm{SL}(n, \mathbb{C})$, then B is the subgroup of upper triangular matrices. If P is the parabolic subgroup containing B, then the quotient

$$\Sigma = \mathrm{SL}(n, \mathbb{C})/P = \{0 \subset V_i \subset V_m \subset \cdots \subset V_n\}$$

is a (partial or generalised) flag variety, with P being the stabiliser of a fixed partial flag with dimension vector $I = (i, m, \ldots, n)$. Whereas for the dimension vector $I = (1, 2, \ldots, n)$, we have $P = B$ and the quotient $\Sigma = G/B$ is a complete flag variety.

Weighted flag varieties

Let $\Lambda_W^* = \mathrm{Hom}(\mathbb{C}^*, T)$ be the lattice of one-parameter subgroups of G. Choose $\mu \in \Lambda_W^*$ and a non-negative integer $u \in \mathbb{Z}$ such that

$$\langle w\lambda, \mu \rangle + u > 0$$

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for all elements w of the Weyl group W of the Lie group G, where $\langle , \rangle$ denotes the perfect pairing between Λ_W and Λ_W^*. We recall the definition of weighted flag variety by Grojnowski and Corti–Reid.

Definition 2·1 [**8**]. Let Σ be a flag variety. Take the affine cone $\widetilde{\Sigma} \subset \widetilde{V_\lambda}$ of the embedding $\Sigma \hookrightarrow \mathbb{P}V_\lambda$, then the quotient of $V_\lambda \backslash \{0\}$ by the $\mathbb{C}^*$-action given by

$$(\varepsilon \in \mathbb{C}^*) \longmapsto (v \longmapsto \varepsilon^u(\mu(\varepsilon) \circ v))$$

is called weighted flag variety. We also use the term key variety or Format for $w\Sigma$. We denote this variety by $w\Sigma(\mu, u)$ or in short $w\Sigma$, if no confusion can arise.

The Hilbert series of a weighted flag variety

$$P_{w\Sigma}(t) = \sum_{m \geqslant 0} \dim \left(H^0(w\Sigma, \mathcal{O}(mD))\right) t^m$$

can be computed by using the following theorem.

THEOREM 2·1 [**17**, theorem 3·1]. The Hilbert series of the weighted flag variety $(w\Sigma(\mu, u), D)$ has the following closed form.

$$P_{w\Sigma}(t) = \frac{\displaystyle\sum_{w \in W} (-1)^w \frac{t^{\langle w\rho, \mu \rangle}}{(1 - t^{\langle w\lambda, \mu \rangle + u})}}{\displaystyle\sum_{w \in W} (-1)^w t^{\langle w\rho, \mu \rangle}}. \tag{2·1}$$

Here ρ is the Weyl vector, half the sum of the positive roots of G, and $(-1)^w = 1$ or -1 depending on whether w consists of an even or odd number of simple reflections in the Weyl group W, and $D = \mathcal{O}_{w\Sigma}(1)$ under the embedding $w\Sigma \subset w\mathbb{P}V_\lambda$.

Remark 2·2. By the standard Hilbert–Serre theorem [**3**, theorem 11·1], the Hilbert series of the weighted flag variety has a reduced expression

$$P_{w\Sigma}(t) = \frac{N(t)}{\displaystyle\prod_{\lambda_i \in \nabla(V_\lambda)} (1 - t^{<\lambda_i, \mu> + u})}. \tag{2·2}$$

The polynomial $N(t)$ is called the Hilbert numerator of the Hilbert series and contains some information about the free resolution of the graded ring $R(w\Sigma, \mathcal{O}(mD))$.

The flag variety $\Sigma = G/P \hookrightarrow \mathbb{P}V_\lambda$ is defined by an ideal $I = \langle Q \rangle$ of quadratic equations [**11**, 2·1]. The second symmetric power of the contragradient representation V_λ^* has a decomposition

$$Z = V_{2\nu} \oplus V_1 \oplus \cdots \oplus V_n$$

into irreducible direct summands as a G-representation, with ν being the highest weight of the representation V_λ^*. The generators of the linear subspace $Q \subset Z = S^2V_\lambda^*$ consisting of all the summands except $V_{2\nu}$, gives the defining equations of the flag variety. The equations of $w\Sigma$ can be readily computed from this information using computer algebra [**17**].

Constructing polarized varieties

We start by constructing some key variety $w\Sigma$ (weighted flag variety) embedded into some weighted projective space. Then we find the Hilbert series of the given variety, which

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gives us some information about the graded ring $R(w\Sigma, D)$. Under suitable conditions, we can compute the canonical divisor class of $w\Sigma$. Then we take quasilinear sections (general hypersurfaces of appropriate degree in weighted projective space) of $w\Sigma$ or of projective cones over it, to get a variety with the desired canonical or anticanonical class. Then we study different aspects, such as singularities, well-formedness and quasi-smoothness of the resulting variety to establish the existence of an appropriate model of the variety. At the end, using the orbifold Riemann–Roch formula of [**7**, section 3], we compute the invariants of our polarized variety (X, D) from the first few values of $h^0(nD)$, and verify that the same Hilbert series can be recovered. More details can be found in [**17**].

3. *Weighted Lagrangian Grassmannian wLGr(3,6) varieties*

3·1. *Generalities*

The symplectic Lie group $\mathrm{Sp}(6, \mathbb{C})$ is the group of automorphisms A of $\mathbb{C}^6$ preserving a nondegenerate, skew-symmetric, bilinear form called the symplectic form J, that is

$$J(Av, Aw) = J(v, w) \text{ for all } v, w \in V.$$

By using a change of basis, J has the normal form:

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The symplectic group has the embedding in $\mathrm{GL}(6, \mathbb{C})$:

$$\mathrm{Sp}(6, \mathbb{C}) = \left\{ M \in \mathrm{GL}(6, \mathbb{C}) : M^t J M = J \right\}.$$

Let G be the symplectic Lie group $\mathrm{Sp}(6, \mathbb{C})$ with maximal torus T. The weight lattice Λ_W of $\mathrm{Sp}(6, \mathbb{C})$ is a rank 3 lattice $\Lambda_W = \langle e_1, e_2, e_3 \rangle$. The simple roots in the weight lattice of the Lie algebra $\mathfrak{sp}_6$ are

$$\alpha_1 = e_1 - e_2, \ \alpha_2 = e_2 - e_3, \text{ and } \alpha_3 = 2e_3.$$

The dominant fundamental weights of $\mathrm{Sp}(6, \mathbb{C})$ are given by $\omega_i = e_1 + \cdots + e_i$, for $1 \leqslant i \leqslant 3$. The sum of the fundamental weights, also known as Weyl vector, is given by $\rho = 3e_1 + 2e_2 + e_3$.

Consider a G-representation $V = \bigwedge^3 \mathbb{C}^6$; the third exterior power of the standard representation $\mathbb{C}^6$, which is 20 dimensional. Then we have a natural contraction map $\kappa : \bigwedge^3 \mathbb{C}^6 \to \mathbb{C}^6$, obtained by contracting with elements of $\bigwedge^2 \left(\mathbb{C}^6\right)^*$. The map κ decomposing V into its summands $V = V_\lambda \oplus V_1$, the kernel of κ is an irreducible representation V_λ of G with highest weight $\lambda = \omega_3 = e_1 + e_2 + e_3$. Then the Weyl dimension formula tells us that V_λ is 14 dimensional. All the fourteen weights of V_λ appear with multiplicity one. If $\nabla_{\mathfrak{p}} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$, then the corresponding parabolic subalgebra

$$\mathfrak{p}_\lambda = \bigoplus \left(\mathfrak{t} \bigoplus_{\alpha \in \nabla_+} \mathfrak{g}_\alpha \bigoplus_{\alpha \in \nabla_{\mathfrak{p}}} \mathfrak{g}_{-\alpha} \right)$$

is 3+9+3 = 15 dimensional. Thus the corresponding flag variety $\Sigma = G/P_\lambda$ is six dimensional:

$$\dim(\Sigma) = \dim(\mathfrak{sp}_6) - \dim(\mathfrak{p}_\lambda) = 21 - 15 = 6.$$

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This flag variety is known as the Lagrangian Grassmannian LGr(3, 6) of Lagrangian subspaces in $\mathbb{C}^6$ [**10**] and also as the *symplectic Grassmannian* [**15**]. Obviously we have a codimension seven embedding $\mathrm{LGr}(3,6) \hookrightarrow \mathbb{P}^{13}V_\lambda$.

3·2. *The weighted flag variety*

The weighted flag variety $w\,\mathrm{LGr}(3,6)$ can be constructed by letting f_1, f_2, f_3 be the basis of the dual lattice Λ_W^*, dual to e_1, e_2 and e_3. Then for $\mu = a_1f_1 + a_2f_2 + a_3f_3 \in \Lambda_W^*$ and $u \in \mathbb{Z}$, to get the weighted version of LGr(3, 6):

$$w\Sigma(\mu, u) = w\,\mathrm{LGr}(3,6) \hookrightarrow w\mathbb{P}^{13}.$$

The set of weights on weighted projective space $w\mathbb{P}^{13}$ is $\{< \lambda_i, \mu > +u\}$, where λ_i are the weights of the representation V_λ. We denote an element μ of the dual lattice Λ_W^* by a vector of integers, i.e. $\mu = (a_1, a_2, a_3)$.

3·3. *Hilbert Series of* wLGr(3,6)

THEOREM 3·1. Let S_2 be the symmetric group on 2 elements and $\sigma(a) = a$ if σ is even and $\sigma(a) = -a$ if σ is odd permutation. Then the Hilbert series of the $w\,\mathrm{LGr}(3,6)$ has the compact form

$$P_{w\,\mathrm{LGr}(3,6)}(t) = \frac{1 - P_1(t)\left(t^{2u} - t^{9u}\right) + P_2(t)\left(t^{3u} - t^{7u}\right) - P_3(t)(t^{4u} - t^{6u}) - t^{10u}}{\prod_{\lambda_i \in \nabla(V_\lambda)} \left(1 - t^{<\lambda_i, \mu> + u}\right)}, \quad (3\cdot1)$$

where

$$P_1(t) = \sum_{1 \leqslant (i,j) \leqslant 3} t^{a_i - a_j} + \sum_{\sigma \in S_2} \sum_{1 \leqslant i \leqslant j \leqslant 3} t^{\sigma(a_i + a_j)},$$

$$P_2(t) = \sum_{\sigma \in S_2} \left(\sum_{1 \leqslant i < j \leqslant 3} \left(t^{\sigma(2a_i + a_j)} + t^{\sigma(2a_i - a_j)} + t^{\sigma(a_i + 2a_j)} + t^{\sigma(a_i - 2a_j)}\right) \right.$$
$$\left. + \; 2\left(t^{\sigma(a_1 + a_2 + a_3)} + t^{\sigma(a_1 + a_2 - a_3)} + t^{\sigma(a_1 - a_2 - a_3)} + t^{\sigma(a_1 - a_2 + a_3)}\right) + 4\sum_{i=1}^{3} t^{\sigma(a_i)} \right)$$

and

$$P_3(t) = \sum_{\sigma \in S_2} \left(\sum_{i=1}^{3} t^{2\sigma(a_i)} + 3 \sum_{1 \leqslant i < j \leqslant 3}^{3} \left(t^{\sigma(a_i - a_j)} + t^{\sigma(a_i + a_j)}\right) \right)$$
$$+ \sum_{\sigma \in S_2} \left(t^{\sigma(a_1 + 2a_2)} + t^{\sigma(2a_1 + a_2)} + t^{\sigma(a_1 - 2a_2)} + t^{\sigma(2a_1 - a_2)}\right) \sum_{\sigma \in S_2} t^{\sigma(a_3)}$$
$$+ \sum_{\sigma \in S_2} \left(t^{\sigma(a_1 + a_2)} + t^{\sigma(a_1 - a_2)}\right) \sum_{\sigma \in S_2} t^{2\sigma(a_3)} + 4.$$

Moreover, if $w\,\mathrm{LGr}(3,6)$ is well-formed then the canonical line bundle $K_{w\,\mathrm{LGr}(3,6)} = \mathcal{O}_{w\,\mathrm{LGr}(3,6)}(-4u)$.

Proof. The Weyl group W of the symplectic Lie algebra $\mathfrak{sp}_6$ is the semidirect product of the symmetric group S_3 with 3 copies of $\mathbb{Z}/2\mathbb{Z}$;

$$W = S_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3.$$

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Using representation theory we work out that eight of the fourteen weights appear as the image of λ under the action of the Weyl group W. Each of the eight weights appears with multiplicity six in the orbit of λ under the W-action, accounting for the 48 elements of the Weyl group W. After performing some simplifications we get the following form of the Hilbert series of $w\,\mathrm{LGr}(3,6)$.

$$P_{w\,\mathrm{LGr}(3,6)}(t) = \frac{1 + \sum_{\sigma\in S_2}\sum_{1\leqslant i\leqslant 3} t^{\sigma(a_i)}\left(t^u - t^{3u}\right) - t^{4u}}{\prod_{\lambda_i\in W\lambda}(1 - t^{<\lambda_i,\mu>+u})}, \tag{3·2}$$

where $W\lambda$ is the orbit of λ under W-action.

As the weights $a_i + u$, $-a_i + u$ with $1 \leqslant i \leqslant 3$ do not appear in the orbit of λ under the W-action, we multiply and divide the expression (3·2) with

$$P_L(t) = \prod_{1\leqslant i\leqslant 3}\left(1 - t^{a_i+u}\right)\left(1 - t^{-a_i+u}\right),$$

to get the full expression for the Hilbert series of $w\,\mathrm{LGr}(3,6)$. After performing some simplifications we get the required compact form (3·1). As the adjunction number of $w\,\mathrm{LGr}(3,6)$ is $10u$ and the sum of weights on $w\mathbb{P}V_\lambda$ is $14u$; the canonical divisor is given by

$$K_{w\,\mathrm{LGr}(3,6)} = \mathcal{O}_{w\,\mathrm{LGr}(3,6)}(10u - 14u) = \mathcal{O}_{w\,\mathrm{LGr}(3,6)}(-4u).$$

3·4. *Examples*

Example 3·5. Consider the Hilbert series of the straight flag variety $\mathrm{LGr}(3,6)$, which corresponds to $u = 1$ and $\mu = \underline{0}$. Then we have

$$P_{\mathrm{LGr}(3,6)}(t) = \frac{1 - 21t^2 + 64t^3 - 70t^4 + 70t^6 - 64t^7 + 21t^8 - t^{10}}{(1-t)^{14}}.$$

Since $\mathrm{LGr}(3,6)$ is a 6-dimensional well-formed and smooth variety, we can compute the canonical bundle $K_{\mathrm{LGr}(3,6)}$ to be $\mathcal{O}(-4)$. Let H_1, H_2 and H_3 be three general hyperplanes of $\mathbb{P}^{13}$ then we get a three dimensional variety

$$V = \mathrm{LGr}(3,6) \cap H_1 \cap H_2 \cap H_3 \subset \mathbb{P}^{10}$$

with $K_V = \mathcal{O}_V(-1)$. Thus V is a Fano 3-fold of genus 9, anti-canonically polarized by K_V where $(-K_V)^3 = 16$. This variety was constructed by Mukai by using the vector bundle method in [**14**].

Remark 3·6. We searched for more families of terminal $\mathbb{Q}$-Fano threefolds but we did not manage to find a new family in codimension 7. A list of 303 such $\mathbb{Q}$-Fano 3-folds with terminal singularities can be found on the graded ring data base page [**4**].

Example 3·7. Consider the following initial data:

(i) input: $\mu = (1,0,0)$, $u = 2$;

(ii) variety and weights: $w\,\mathrm{LGr}(3,6) \subset \mathbb{P}^{13}[1^5, 2^4, 3^5]$, with weights assigned to the variables x_i in the equations given in Appendix A on page 206.

Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
Weight	3	3	3	3	2	3	2	2	1	2	1	1	1	1;

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(iii) canonical class: $K_{w\,\mathrm{LGr}(3,6)} = \mathcal{O}(-8)$;
(iv) Hilbert numerator: $1 - t^2 - 4t^3 - 7t^4 + 12t^5 + \cdots - 12t^{15} + 7t^{16} + 4t^{17} + t^{18} - t^{20}$.

We take a threefold quasilinear section

$$X = w\,\mathrm{LGr}(3,6) \bigcap (3)^2 \bigcap (2) \subset \mathbb{P}^{10}[1^5, 2^3, 3^3],$$

then

$$K_X = \mathcal{O}_X\,(-8 + (2 \times 3 + 2)) \sim 0.$$

By using the equations, we can check that X does not contain any codimension 8 singular strata. Therefore X is a well-formed polarized weighted projective threefold. We work out the singularities of X induced by the weights of $w\mathbb{P}^{10}$. To make a choice of coordinates we suppose

$$\{x_3 = f_3(x_i),\ x_4 = g_3(x_i)\ , x_5 = f_2(x_i)\}.$$

Here f_3, g_3 and f_2 are general homogeneous equations of degree 3, 3 and 2 respectively in the rest of the variables. Now we check the singularities of X on each of the singular strata and find the local transverse structure of these singularities.

1/3 *singularities*. The defining locus C of this singular stratum is defined by taking

$$C = X \bigcap \{x_7 = x_8 = x_9 = x_{10} = x_{11} = x_{12} = x_{13} = x_{14} = 0\}\,.$$

Therefore

$$C := \left\{ \frac{f_3^2}{4} - g_3 x_2 + x_1 x_6 = 0 \right\} \subset \mathbb{P}^2[x_1, x_2, x_6].$$

This defines a quadratic curve in $\mathbb{P}^2$. On each point of the curve one of the $x_i \neq 0$. If we consider $x_1 \neq 0$, then by using the implicit function theorem we can eliminate the variables $x_6, x_8, x_{10}, x_{11}, x_{12}, x_{13}$ and x_{14} from equation number (A1), (A2), (A4), (A3), (A5), (A6) and (A7) respectively appearing in appendix A on page 206. Therefore x_7 and x_9 are local variables near this point and the group μ_3 acts by

$$\varepsilon : x_7, x_9 \longmapsto \varepsilon^2 x_7, \varepsilon x_9.$$

For $x_2 \neq 0$, we can eliminate the variables $x_7, x_9, x_{10}, x_{12}, x_{13}$ and x_{14} from equation number (A2), (A3), (A8), (A9), (A10) and (A11) respectively. If we suppose that g_3 contains the monomial x_6, then we eliminate x_6 from equation (1) as well. Therefore x_8 and x_{11} are the local variables on this point of the curve and the group μ_3 acts by

$$\varepsilon : x_8, x_{11} \longmapsto \varepsilon^2 x_8, \varepsilon x_{11}.$$

Similarly, near the point $x_6 \neq 0$ we work out that x_{10} and x_{14} are local variables and the group μ_3 acts by

$$\varepsilon : x_{10}, x_{14} \longmapsto \varepsilon^2 x_{10}, \varepsilon x_{14}.$$

Therefore C is a rational curve of singularities of type 1/3 (1, 2).

1/2 *singularities*. This singular locus is defined by restricting X to the locus defined by x_7, x_8 and x_{10}. The equation (A6) contains the monomial x_7^2 and no term involving x_8 and x_{10}, the equation (A11) contains a monomial x_8^2 and no term involving x_7 and x_{10} and equation (A19) contains a monomial x_{10}^2 and no term involving x_7 and x_8. Therefore, we conclude that X has no singularities along the singular stratum defined by weight 2 variables.

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Quasismooth. As we have shown, on $1/3$ and $1/2$ strata that X is locally a threefold by using the implicit function theorem. Similarly we can show that on the rest of the strata that X is locally a threefold.

Thus (X, D) is a polarized Calabi–Yau threefold with a rational curve C of singularities of type A_2. The rest of the invariants of this family, computed by using the orbifold Riemann–Roch formula of [**7**, section 3] for Calabi–Yau threefolds are given as follows.

(i) $D^3 = \dfrac{64}{9}$, $D.c_2(X) = 48$, $\deg D|_C = \dfrac{2}{3}$, $\gamma_C = -2$

4. *Weighted A_3 type partial flag variety*

4·1. *Construction of weighted* $\mathrm{FL}_{1,3}$ *variety*

Let G be the reductive Lie group $\mathrm{GL}(4, \mathbb{C})$ with corresponding Lie algebra $\mathfrak{g} = \mathfrak{gl}_4$, known as A_3-type Lie algebra. The rank of the maximal abelian subalgebra $\mathfrak{t}$ is 4, which corresponds to the maximal torus T inside G. The weight lattice of the Lie algebra $\mathfrak{gl}_4$ is a rank 4 lattice $\Lambda_W = \langle e_1, e_2, e_3, e_4\rangle$. The set of simple roots of the root system of the corresponding simple part of $\mathfrak{g}$ is

$$\alpha_i = e_i - e_{i+1} \text{ for } 1 \leqslant i \leqslant 3.$$

The Weyl group W of G is S_4, the symmetric group on 4 letters, of order 24. The Weyl vector can be taken to be $\rho = 3e_1 + 2e_2 + e_3$.

Consider two 4 dimensional representations of G; $V_1 = \mathbb{C}^4$ which is the standard representation of G and the representation $V_2 = \bigwedge^3 \mathbb{C}^4$. They are irreducible representations of G with highest weights $\lambda_1 = e_1$ and $\lambda_2 = e_1 + e_2 + e_3$. Then the representation $V = V_1 \otimes V_2$ is a 16 dimensional representation of G which is not irreducible. Consider the wedge product map

$$\mathcal{S} : \mathbb{C}^4 \otimes \bigwedge^3 \mathbb{C}^4 \longrightarrow \bigwedge^4 \mathbb{C}^4.$$

Then the kernel $\kappa(S)$ of the map, is the irreducible highest weight representation of G with highest weight

$$\lambda = \lambda_1 + \lambda_2 = 2e_1 + e_2 + e_3.$$

By using the Weyl dimension formula we can show that V_λ is 15 dimensional. Twelve of the weights of V_λ appear with multiplicity one and one with multiplicity three. The dimension of the Lie algebra $\mathfrak{gl}_4$ is 16. The only simple root orthogonal to the highest weight λ under in the weight lattice is α_2. Therefore, the parabolic subalgebra

$$\mathfrak{p}_\lambda = \bigoplus \left(\mathfrak{t} \bigoplus_{\alpha \in \nabla_+} \mathfrak{g}_\alpha \bigoplus \mathfrak{g}_{-\alpha_2} \right),$$

is $4 + 6 + 1 = 11$ dimensional. Hence the corresponding flag variety $\Sigma = G/P_\lambda$ is five dimensional and we get a codimension 9 embedding $\Sigma^5 \hookrightarrow \mathbb{P}^{14}[V_\lambda]$. In the notation of Section 2, we have a dimension vector $I = (1, 3, 4)$. The reductive Lie group $\mathrm{GL}(4, \mathbb{C})$ parameterizes the flags of type

$$\{0 \subset V_1 \subset V_3 \subset V_4\}.$$

We will denote this partial flag variety by $\mathrm{FL}_{1,3}$. To obtain the weighted version of $\mathrm{FL}_{1,3}$, we

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let $\Lambda_W^* = \langle f_1, f_2, f_3, f_4\rangle$ to be the dual lattice of the weight lattice. Then for any

$$\mu = \sum_{i=1}^{4} a_i f_i \in \Lambda_W^* \text{ and } u \in \mathbb{Z}$$

we get the embedding

$$w\,\mathrm{FL}_{1,3}(\mu, u) \hookrightarrow w\mathbb{P}V_\lambda[\langle \lambda_i, \mu\rangle + u], \tag{4·1}$$

where λ_i are the weights of the representation V_λ understood with multiplicities. Following the convention of the Section 3·2, element μ of the dual lattice Λ_W^* is represented by $\mu = (a_1, a_2, a_3, a_4)$. Here we can choose all $a_i s$ to be the half or quarter integers to get the embedding (4·1) but we can get the embedding with same set of weights by changing the value of u, so we only take them to be integers.

4·2. *Hilbert series of weighted* $\mathrm{FL}_{1,3}$

THEOREM 4·1. Let s be the sum of the integers in $\mu = (a_1, a_2, a_3, a_4)$. Then the Hilbert series of the weighted flag variety $w\,\mathrm{FL}_{1,3}$ has the following compact form:

$$P_{w\,\mathrm{FL}_{1,3}}(t) = \frac{1 + \sum_{k=1}^{4}(-1)^k P_k(t) t^{ks+(k+1)u} + \sum_{k=5}^{8}(-1)^k P_k(t) t^{(k+1)s+(k+2)u} + t^{12(s+u)}}{\prod_{w_i \in \nabla(V_\lambda)} \left(1 - t^{<w_i, \mu> + u}\right)}, \tag{4·2}$$

where

$$\begin{aligned}
P_1(t) &= \sum_{1\leqslant i<j\leqslant 4} t^{2(a_i+a_j)} + 2\sum_{1\leqslant (i,j)\leqslant 4}\left(t^{s+a_i-a_j} - t^s\right),\\
P_2(t) &= 4\sum_{1\leqslant i<j\leqslant 4} t^{2(a_i+a_j)} + 8\sum_{1\leqslant (i,j)\leqslant 4} t^{(a_i-a_j)+s}\\
&\quad + \sum_{1\leqslant (i,j,i\neq j)\leqslant 4}\left(t^{2s-(3a_i+a_j)} + t^{3a_i+a_j}\right) - 16t^s,\\
P_3(t) &= 6\sum_{1\leqslant i<j\leqslant 4} t^{2(a_i+a_j)} + 14\sum_{1\leqslant (i,j)\leqslant 4} t^{(a_i-a_j)+s}\\
&\quad + \sum_{1\leqslant (i,j,i\neq j)\leqslant 4}\left(3t^{2s-(3a_i+a_j)} + 3t^{3a_i+a_j} + t^{2(a_i-a_j)}\right) - 29t^s,\\
P_4(t) &= 4\sum_{1\leqslant i<j\leqslant 4} t^{2(a_i+a_j)} + 12\sum_{1\leqslant (i,j)\leqslant 4} t^{(a_i-a_j)+s}\\
&\quad + \sum_{1\leqslant (i,j,i\neq j)\leqslant 4}\left(3t^{2s-(3a_i+a_j)} + 3t^{3a_i+a_j} + 2t^{2(a_i-a_j)+s}\right) - 24t^s.
\end{aligned}$$

If $w\,\mathrm{FL}_{1,3}$ is well-formed then

$$K_{w\,\mathrm{FL}_{1,3}} = \mathcal{O}_{w\,\mathrm{FL}_{1,3}}(-3(s+u)).$$

Proof. The Weyl group W of the symmetric group S_4. By using representation theory we work out that twelve of the fifteen weights are in the orbit of λ under the action of the Weyl group. To compute the Hilbert series of $w\,\mathrm{FL}_{1,3}$, we evaluate the expression (2·1) for W, ρ, μ and λ as described in Section 4·1. After some simplification we get the following

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form of the Hilbert series of the weighted flag variety $w\,\mathrm{FL}_{1,3}$.

$$P_{w\,\mathrm{FL}_{1,3}}(t) = \frac{1 + 3t^{s+u} - \left(\sum_{1 \leqslant i < j \leqslant 4} t^{2(a_i+a_j)} + \sum_{1 \leqslant (i,j,i \neq j) \leqslant 4} 2t^{s+a_i-a_j} \right) t^{s+2u} + \cdots + t^{9(s+u)}}{\prod_{\lambda_i \in W\lambda} 1 - t^{<\lambda_i,\mu>+u}}, \tag{4·3}$$

where $s = \sum_{i=1}^{4} a_i$, and $W\lambda$ denote the orbit of λ under W-action. The full expression for the Hilbert series of $w\,\mathrm{FL}_{1,3}$ is obtained by multiplying and dividing (4·3) by $(1 - t^{s+u})^3$, which represent those weight spaces of $\mathbb{P}^{14}[\langle\lambda_i, \mu\rangle + u]$ which do not lie in the orbit of W-action. After further simplifying we get the required compact form (4·2) of the Hilbert series of $w\,\mathrm{FL}_{1,3}$. The adjunction number is $12(s + u)$ and the sum of the weights on $w\mathbb{P}V_\lambda$ is $15(s + u)$, thus $K_{w\,\mathrm{FL}_{1,3}} = \mathcal{O}_{w\,\mathrm{FL}_{1,3}}(-3(s + u))$.

Remark 4·3. Due to the Gorenstein symmetry of the resolution of the graded ring corresponding to $w\,\mathrm{FL}_{1,3}$, we have

$$P_i(t) = P_{c-i}(t), \quad \text{where} \quad c = \mathrm{codim}(w\,\mathrm{FL}_{1,3}), \quad i = 1, 2, 3, 4.$$

The 36 defining quadrics are visible in the Hilbert numerator of the Hilbert series of $w\,\mathrm{FL}_{1,3}$ from $P_1(t)$.

4·4. *Examples*

Example 4·5. First we consider the case of the straight partial flag variety $\mathrm{FL}_{1,3}$. We evaluate the expression (4·2) for $\mu = \underline{0}$ and $u = 1$ to get the Hilbert series of the straight A_3 flag variety $\mathrm{FL}_{1,3}$;

$$P_{\mathrm{FL}_{1,3}}(t) = \frac{1 - 36t^2 + 160t^3 - 315t^4 + 288t^5 - 288t^7 + 315t^8 - 160t^9 + 36t^{10} - t^{12}}{(1 - t)^{15}}.$$

The canonical divisor class of $\mathrm{FL}_{1,3}$ can be read off from the Hilbert series, which is $K_{\mathrm{FL}_{1,3}} = \mathcal{O}_{\mathrm{FL}_{1,3}}(-3)$. Then the intersection of $\mathrm{FL}_{1,3}$ with two general hyperplanes

$$V = \mathrm{FL}_{1,3} \bigcap H_1 \bigcap H_2 \subset \mathbb{P}^{12}$$

is a non-prime Fano 3-fold, polarized by its anti-canonical class $-K_V$. The degree of the embedding is given by $(-K_V)^3 = 20$ and by using $(-K_V)^3 = 2g - 2$ it is evident that the genus of V is 11. This variety is listed in [**13**] as non-prime Fano threefold. The non-primeness also follows from the fact that $\mathrm{Pic}(\Sigma) = \mathbb{Z}^2$, by using standard theory [**6**, section 6·3]. Our description of this variety as a linear section of a partial flag variety does not seem to appear in any of Mukai's articles.

Remark 4·6. We searched for more families of terminal $\mathbb{Q}$-Fano threefolds but like previous attempts we did not manage to find a new family in codimension 9. A list of all possible 93 $\mathbb{Q}$-Fano 3-folds with terminal singularities can be found on Gavin Brown's graded ring data base page [**4**].

Example 4·7. Consider the following initial data:

(i) input: $\mu = (0, 0, 1, 1)$, $u = 0$;
(ii) variety and weights: $w\,\mathrm{FL}_{1,3} \subset \mathbb{P}^{14}[1^4, 2^7, 3^4]$, with weights assigned to the variables

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x_i in the equations given in Appendix B:

$$\begin{array}{l} \text{Variable } x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12}\ x_{13}\ x_{14}\ x_{15} \\ \text{Weight } \ 1\ \ 1\ \ 1\ \ 2\ \ 2\ \ 1\ \ 2\ \ 2\ \ 2\ \ 3\ \ \ 2\ \ \ 2\ \ \ 3\ \ \ 3\ \ \ 3 \end{array};$$

(iii) canonical class: $K_{w\,\mathrm{FL}_{1,3}} = \mathcal{O}(-6)$;
(iv) Hilbert numerator: $1 - t^2 - 8t^3 - 10t^4 + 32t^5 + \cdots - 32t^{19} + 10t^{20} + 8t^{21} + t^{22} - t^{24}$.

We take a projective cone over $w\,\mathrm{FL}_{1,3}$, so that we get the embedding

$$\mathcal{C}w\,\mathrm{FL}_{1,3} \subset \mathbb{P}^{15}[1^5, 2^7, 3^4], \text{ and } K_{\mathcal{C}w\,\mathrm{FL}_{1,3}} = \mathcal{O}(-7).$$

We take a threefold complete intersection

$$X = \mathcal{C}w\,\mathrm{FL}_{1,3} \bigcap (2)^2 \bigcap (3) \subset \mathbb{P}^{12}[1^5, 2^5, 3^3],$$

then $K_X \sim 0$. Then by using the equations we can show that (X, D) is a well-formed and quasismooth Calabi–Yau threefold with a rational curve of singularities of type 1/3 (1, 2). The rest of the invariants of (X, D) are listed below.

$$D^3 = \frac{76}{9},\ D.c_2(X) = 48,\ \deg D|_C = \frac{2}{3},\ \gamma_C = 10 \qquad (*)$$

Example 4·8. Consider the following initial data:

(i) input: $\mu = (0, 1, 1, 1)$, $u = 2$;
(ii) variety and weights: $w\,\mathrm{FL}_{1,3} \subset \mathbb{P}^{14}[1^3, 2^9, 3^3]$, with weights assigned to the variables x_i in the equations given in Appendix B:

$$\begin{array}{l} \text{Variable } x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12}\ x_{13}\ x_{14}\ x_{15} \\ \text{Weight } \ 1\ \ 1\ \ 2\ \ 1\ \ 2\ \ 2\ \ 2\ \ 2\ \ 2\ \ 3\ \ \ 2\ \ \ 3\ \ \ 2\ \ \ 3\ \ \ 3 \end{array};$$

(iii) canonical class: $K_{w\,\mathrm{FL}_{1,3}} = \mathcal{O}(-6)$;
(iv) Hilbert numerator: $1 - 9t^3 - 15t^4 + 33t^5 + 58t^6 - \cdots - 58t^{18} - 33t^{19} + 15t^{20} + 9t^{21} - t^{24}$.

We take a projective cone over $w\,\mathrm{FL}_{1,3}$, so that we get the embedding

$$\mathcal{C}w\,\mathrm{FL}_{1,3} \subset \mathbb{P}^{15}[1^4, 2^9, 3^3], \text{ and } K_{\mathcal{C}w\,\mathrm{FL}_{1,3}} = \mathcal{O}(-7).$$

Consider a threefold quasilinear section

$$X = \mathcal{C}w\,\mathrm{FL}_{1,3} \bigcap (2)^2 \bigcap (3) \subset \mathbb{P}^{12}[1^4, 2^7, 3^2],$$

then the canonical bundle $K_X \sim 0$. Then (X, D) is a well-formed and quasismooth polarized Calabi–Yau threefold with two rational curves of singularities, C of type A_2 and E of type A_1. The rest of the invariants of (X, D) are listed below.

$$D^3 = \frac{127}{18},\ D.c_2(X) = 46,\ \deg D|_C = \frac{1}{3},\ \gamma_C = 8,\ \deg D|_E = 3,\ \gamma_E = 1. \qquad (*)$$

5. *List of flag varieties in codimension* $4 \leqslant c \leqslant 10$

Given a reductive Lie group G there is one-to-one correspondence between flag varieties $\Sigma = G/P_\lambda$ and irreducible highest weight representations V_λ. We list all highest weights λ which leads to the embedding of Σ in codimension $4 \leqslant c \leqslant 10$. The list does not qualify to be the complete classification of such flag varieties but one can deduce the classification

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Table 1. *List of flag varieties in codimension* $4 \leqslant c \leqslant 10$

c	Type of G	λ	Embedding	Number of Eqs
4	GL(3)	$\omega_1 + \omega_2$	$\mathrm{FL}_{1,2} \subset \mathbb{P}^7$	9
5	SO(10)	ω_5	$\mathrm{OGr}(5, 10) \subset \mathbb{P}^{15}$	10
6	GL(6)	ω_2	$\mathrm{Gr}(2, 6) \subset \mathbb{P}^{14}$	15
7	Sp(6)	ω_3	$\mathrm{LGr}(3, 6) \subset \mathbb{P}^{13}$	21
8	G_2	ω_2	$\Sigma \subset \mathbb{P}^{13}$	28
9	GL(4)	$\omega_1 + \omega_3$	$\mathrm{FL}_{1,3} \subset \mathbb{P}^{14}$	36
10	E_6	ω_1	$\Sigma \subset \mathbb{P}^{26}$	27
10	GL(7)	ω_2	$\mathrm{Gr}(2, 7) \subset \mathbb{P}^{20}$	35
10	GL(6)	ω_3	$\mathrm{Gr}(3, 6) \subset \mathbb{P}^{19}$	35

with a bit more effort. We only list those λ which lead to a distinct embedding of some flag variety Σ in $\mathbb{P}V_\lambda$.

We call a flag variety Σ to be *distinct* if:

(1) it cannot be recovered as a complete linear section of some other flag variety $\Sigma_1 = G_1/P_{\lambda_1}$;
(2) it is not isomorphic to some other flag variety $\Sigma_1 = G_1/P_{\lambda_1}$.

For example the Lagrangian Grassmannian $\mathrm{LGr}(2, 6)$ is a hyperplane section of the the Grassmannian $\mathrm{Gr}(2, 6)$ in codimension 6. Thus we will only list $\mathrm{Gr}(2, 6)$ as a variety in codimension 6. We briefly discuss the possibilities in each codimension c, for $4 \leqslant c \leqslant 10$:

(i) $c = 4$. We have a flag variety of the Lie group $\mathrm{GL}(3, \mathbb{C})$ with highest weight $\lambda = 2e_1 + e_2$. The corresponding fixed flag parameterised by G is $\{0 \subset V_1 \subset V_2 \subset V_3\}$. Thus we have a complete flag variety $\mathrm{FL}_{1,2}$;

(ii) $c = 5$. For $G = \mathrm{SO}(10, \mathbb{C})$ we have a codimension five embedding of

$$\mathrm{OGr}(5, 10) \subset \mathbb{P}^{15}(S^+)$$

for highest weight ω_5, discussed in [**8**]. For $G = \mathrm{SO}(9, \mathbb{C})$ we have a codimension five homogeneous variety $\mathrm{OGr}(4, 9)$ corresponding to highest weight ω_4 which is isomorphic to $\mathrm{OGr}(5, 10)$;

(iii) $c = 6$. For $G = \mathrm{GL}(6, \mathbb{C})$ and highest weight ω_2 we get a codimension 6 embedding of the Grassmannian $\mathrm{Gr}(2, 6) \hookrightarrow \mathbb{P}^{14}$, discussed in [**17**]. For the symplectic Lie group $\mathrm{Sp}(6, \mathbb{C})$ we have a codimension six embedding of the isotropic Lagrangian Grassmannian $\mathrm{LGr}(2, 6) \hookrightarrow \mathbb{P}^{13}$ for the highest weight ω_2, which is just a general linear section of $\mathrm{Gr}(2, 6)$;

(iv) $c = 7$. There is only one case which we discussed above in Section 3;

(v) $c = 8$. For $G = G_2$ we have a codimension 8 embedding $\Sigma^5 \hookrightarrow \mathbb{P}^{13}$ corresponding to the highest weight ω_2, which appeared in [**17**];

(vi) $c = 9$. The only case is discussed in the Section 4 above;

(vii) $c = 10$. In this case we get three distinct homogeneous varieties:

(1) let G be the simple Lie group of type E_6 and V_λ be the highest weight representation of G with $\lambda = \omega_1$ or ω_6. Then the corresponding homogeneous variety Σ has a codimension 10 embedding $\Sigma \hookrightarrow \mathbb{P}^{26}V_\lambda$. The F_4 homogeneous variety corresponding to the highest weight representation V_λ with $\lambda = e_1$, discussed in [**16**, Chapter 7], is just a general linear section of this E_6 variety;
(2) for $G = \mathrm{GL}(7, \mathbb{C})$ we have a representation $\bigwedge^2 \mathbb{C}^7$, which leads to the Plücker

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embedding of Grassmannian $\mathrm{Gr}(2,7) \hookrightarrow \mathbb{P}^{20}$; which is a codimension 10 embedding;

(3) for $G = \mathrm{GL}(6,\mathbb{C})$ we have a representation $\bigwedge^3 \mathbb{C}^6$, which leads to the Plücker embedding of the Grassmannian $\mathrm{Gr}(3,6) \hookrightarrow \mathbb{P}^{19}$, which is also a codimension 10 embedding.

Remark 5·1. Computing the Hilbert series and expected syzygies does suggest that Gr(3, 6) might be a linear section of Gr(2, 7) as they have exactly the same Hilbert numerator, propagating the same number of generators and syzygies in each degree. In fact it is not the case, the $H^6(\mathrm{Gr}(2,7))$ is a 2-dimensional and $H^6(\mathrm{Gr}(3,6))$ is a 3-dimensional, so by Lefschetz hyperplane section principle Gr(3, 6) is not a hyperplane section of Gr(2, 7).

In Table 1 we list all the distinct flag varieties in codimension $4 \leqslant c \leqslant 10$. The last column represents the number of defining equations of flag variety Σ.

A. Equations of the Lagrangian Grassmannian LGr(3,6)

We compute the defining equations of LGr(3, 6) by the GAP4 code given in [**17**, appendix]. The code finds the decomposition of the second symmetric power of the dual highest weight representation V_λ^* into its direct summands as a module over the symplectic Lie algebra $\mathfrak{sp}_6$. The second symmetric power of the dual representation V_λ^* has a decomposition $S^2V_\lambda^* = \bigoplus(\oplus V_1 \oplus V_e)$ into 84 and 21 dimensional representations. The 21 quadrics generate the 21 dimensional subspace V_e of $S^2V_\lambda^*$ and they are listed as follows:

$$x_1x_6 - x_2x_4 + \tfrac{1}{4}x_3^2 \tag{A·1}$$
$$x_1x_8 - x_2x_7 + \tfrac{1}{2}x_3x_5 \tag{A·2}$$
$$x_1x_{11} - x_2x_9 + \tfrac{1}{4}x_5^2 \tag{A·3}$$
$$x_1x_{10} - \tfrac{1}{2}x_3x_7 + x_4x_5 \tag{A·4}$$
$$x_1x_{12} - x_3x_9 + \tfrac{1}{2}x_5x_7 \tag{A·5}$$
$$x_1x_{13} - x_4x_9 + \tfrac{1}{4}x_7^2 \tag{A·6}$$
$$x_1x_{14} - \tfrac{1}{4}x_5x_{10} + \tfrac{1}{4}x_7x_8 - x_6x_9 \tag{A·7}$$
$$x_2x_{10} - \tfrac{1}{2}x_3x_8 + x_5x_6 \tag{A·8}$$
$$x_2x_{12} - x_3x_{11} + \tfrac{1}{2}x_5x_8 \tag{A·9}$$
$$x_2x_{13} - \tfrac{1}{4}x_3x_{12} + \tfrac{1}{4}x_7x_8 - x_6x_9 \tag{A·10}$$
$$x_2x_{14} - x_6x_{11} + \tfrac{1}{4}x_8^2 \tag{A·11}$$
$$x_3x_{10} - 2x_4x_8 + 2x_6x_7 \tag{A·12}$$
$$x_3x_{12} + x_5x_{10} - 4x_4x_{11} + 4x_6x_9 \tag{A·13}$$
$$x_3x_{13} - x_4x_{12} + \tfrac{1}{2}x_7x_{10} \tag{A·14}$$
$$x_3x_{14} - x_6x_{12} + \tfrac{1}{2}x_8x_{10} \tag{A·15}$$
$$x_5x_{12} - 2x_7x_{11} + 2x_8x_9 \tag{A·16}$$
$$x_5x_{13} - \tfrac{1}{2}x_7x_{12} + x_9x_{10} \tag{A·17}$$
$$x_5x_{14} - \tfrac{1}{2}x_8x_{12} + x_{10}x_{11} \tag{A·18}$$
$$x_4x_{14} - x_6x_{13} + \tfrac{1}{4}x_{10}^2 \tag{A·19}$$
$$x_7x_{14} - x_8x_{13} + \tfrac{1}{2}x_{10}x_{12} \tag{A·20}$$
$$x_9x_{14} - x_{11}x_{13} + \tfrac{1}{4}x_{12}^2. \tag{A·21}$$

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B. Equations of partial A_3 flag variety $\mathrm{FL}_{1,3}$

Following the description given in [**11**], we have the following decomposition of the second symmetric power of the contragradient representation V_λ^*:

$$S^2V_\lambda^* = V_1 \oplus V_{e_1} \oplus V_{e_2} \oplus V_{e_3}.$$

The vector spaces V_1, V_{e1}, V_{e_2} and V_{e_3} are 84, 20, 15 and 1 dimensional respectively. The defining equations of $\mathrm{FL}_{1,3}$, are the basis of the linear subspaces of dimension 20, 15 and 1 of $S^2V_\lambda^*$.

$$I = \langle Q \rangle = \langle V_{e_1} \rangle \cup \langle V_{e_2} \rangle \cup \langle V_{e_3} \rangle \subset S^2V_\lambda^*.$$

We compute these quadratic equations by using the GAP4 code in [**17**, appendix] and they are listed below:

$$x_1x_6 - x_2x_3 \tag{B·1}$$
$$x_1x_9 - x_1x_7 - x_1x_8 - x_3x_4 + x_2x_5 \tag{B·2}$$
$$x_1x_{12} - x_3x_7 - x_3x_8 + x_5x_6 \tag{B·3}$$
$$x_1x_{11} + x_2x_9 - x_2x_8 - x_4x_6 \tag{B·4}$$
$$x_1x_{10} - x_5x_4 \tag{B·5}$$
$$x_1x_{14} + x_3x_{10} - x_5x_9 \tag{B·6}$$
$$x_1x_{13} - x_2x_{10} + x_4x_7 \tag{B·7}$$
$$x_1x_{15} + x_5x_{11} - x_4x_{12} - x_6x_{10} + x_9x_7 + x_9x_8 - x_7x_8 - x_8^2 \tag{B·8}$$
$$x_3x_{11} + x_2x_{12} - x_6x_8 \tag{B·9}$$
$$x_3x_{14} - x_5x_{12} \tag{B·10}$$
$$x_3x_{13} - x_2x_{14} - x_5x_{11} + x_4x_{12} - x_9x_8 + x_7x_8 + x_8^2 \tag{B·11}$$
$$x_3x_{15} - x_6x_{14} + x_7x_{12} \tag{B·12}$$
$$x_2x_{13} - x_4x_{11} \tag{B·13}$$
$$x_2x_{15} + x_6x_{13} - x_9x_{11} \tag{B·14}$$
$$x_5x_{13} + x_4x_{14} - x_8x_{10} \tag{B·15}$$
$$x_5x_{15} + x_9x_{14} - x_8x_{14} - x_{12}x_{10} \tag{B·16}$$
$$x_4x_{15} - x_7x_{13} - x_8x_{13} + x_{11}x_{10} \tag{B·17}$$
$$x_6x_{15} - x_{12}x_{11} \tag{B·18}$$
$$x_9x_{15} - x_7x_{15} - x_8x_{15} - x_{12}x_{13} + x_{11}x_{14} \tag{B·19}$$
$$x_{10}x_{15} - x_{14}x_{13} \tag{B·20}$$
$$x_1x_9 + x_1x_7 - 2x_3x_4 - 2x_2x_5 \tag{B·21}$$
$$x_1x_{12} - \tfrac{1}{2}x_3x_9 + \tfrac{1}{2}x_3x_7 - x_5x_6 \tag{B·22}$$
$$x_1x_{11} - \tfrac{1}{2}x_2x_9 + \tfrac{1}{2}x_2x_7 + x_4x_6 \tag{B·23}$$
$$x_1x_{14} - x_3x_{10} + \tfrac{1}{2}x_5x_9 - \tfrac{1}{2}x_5x_7 - x_5x_8 \tag{B·24}$$
$$x_1x_{13} + x_2x_{10} + \tfrac{1}{2}x_4x_9 - \tfrac{1}{2}x_4x_7 - x_4x_8 \tag{B·25}$$
$$x_1x_{15} - x_3x_{13} - x_2x_{14} + x_6x_{10} - \tfrac{1}{2}x_9x_7 + \tfrac{1}{2}x_7^2 + x_7x_8 \tag{B·26}$$
$$x_3x_{11} - x_2x_{12} + \tfrac{1}{2}x_6x_9 + \tfrac{1}{2}x_6x_7 \tag{B·27}$$
$$x_3x_{13} + x_2x_{14} - x_5x_{11} - x_4x_{12} + \tfrac{1}{2}x_9^2 - \tfrac{1}{2}x_9x_8 - \tfrac{1}{2}x_7^2 - \tfrac{1}{2}x_7x_8 \tag{B·28}$$

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$$x_3x_{15} + x_6x_{14} + \tfrac{1}{2}x_9x_{12} - \tfrac{1}{2}x_7x_{12} - x_8x_{12} \tag{B·29}$$

$$x_2x_{15} - x_6x_{13} + \tfrac{1}{2}x_9x_{11} - \tfrac{1}{2}x_7x_{11} - x_8x_{11} \tag{B·30}$$

$$x_5x_{11} + x_4x_{12} - \tfrac{1}{4}x_9^2 + \tfrac{1}{4}x_7^2 \tag{B·31}$$

$$x_5x_{13} - x_4x_{14} + \tfrac{1}{2}x_9x_{10} + \tfrac{1}{2}x_7x_{10} \tag{B·32}$$

$$x_5x_{15} - \tfrac{1}{2}x_9x_{14} + \tfrac{1}{2}x_7x_{14} + x_{12}x_{10} \tag{B·33}$$

$$x_4x_{15} - \tfrac{1}{2}x_9x_{13} + \tfrac{1}{2}x_7x_{13} - x_{11}x_{10} \tag{B·34}$$

$$x_9x_{15} + x_7x_{15} - 2x_{12}x_{13} - 2x_{11}x_{14} \tag{B·35}$$

$$x_1x_{15} - x_3x_{13} + x_2x_{14} - x_5x_{11} + x_4x_{12} - x_6x_{10} - \tfrac{3}{8}(x_9^2 + x_7^2) + \frac{1}{4}x_9x_7 + \frac{1}{2}(x_9x_8 - x_7x_8 - x_8^2) \tag{B·36}$$

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