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APPROXIMATION OF THE TAIL PROBABILITIES FOR BIDIMENSIONAL RANDOMLY WEIGHTED SUMS WITH DEPENDENT COMPONENTS

XINMEI SHEN AND MINGYUE GE
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
E-mail: xshen@dlut.edu.cn; gemingyue@mail.dlut.edu.cn

KE-ANG FU
School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China
E-mail: fukeang@hotmail.com

Let $\{\boldsymbol{X}_k = (X_{1,k}, X_{2,k})^\top, k \geq 1\}$ be a sequence of independent and identically distributed random vectors whose components are allowed to be generally dependent with marginal distributions being from the class of extended regular variation, and let $\{\boldsymbol{\Theta}_k = (\Theta_{1,k}, \Theta_{2,k})^\top, k \geq 1\}$ be a sequence of nonnegative random vectors that is independent of $\{\boldsymbol{X}_k, k \geq 1\}$. Under several mild assumptions, some simple asymptotic formulae of the tail probabilities for the bidimensional randomly weighted sums $(\sum_{k=1}^n \Theta_{1,k}X_{1,k}, \sum_{k=1}^n \Theta_{2,k}X_{2,k})^\top$ and their maxima $(\max_{1\leq i\leq n}\sum_{k=1}^i \Theta_{1,k}X_{1,k}, \max_{1\leq i\leq n}\sum_{k=1}^i \Theta_{2,k}X_{2,k})^\top$ are established. Moreover, uniformity of the estimate can be achieved under some technical moment conditions on $\{\boldsymbol{\Theta}_k, k \geq 1\}$. Direct applications of the results to risk analysis are proposed, with two types of ruin probability for a discrete-time bidimensional risk model being evaluated.

Keywords: asymptotics, bidimensional randomly weighted sums, copula, extended regular variation

1. INTRODUCTION

Consider the randomly weighted sums $S_n := \sum_{k=1}^n \Theta_k X_k, n \geq 1$ and their maxima $M_n := \max_{1\leq i\leq n} S_i$, $n \geq 1$, where $\{X_k, k \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with a common distribution function $F = 1 - \overline{F}$ and $\{\Theta_k, k \geq 1\}$ is a sequence of nonnegative random variables independent of the sequence $\{X_k, k \geq 1\}$. The randomly weighted sums S_n and their maxima M_n are frequently encountered in insurance and finance theories. For instance, in insurance theory, the random variables $\{X_k, k \geq 1\}$ and $\{\Theta_k, k \geq 1\}$ are often interpreted as a liability and financial risks, respectively. More specifically, if each X_k is regarded as the net loss within the period k and each Θ_k as the discount factor from time k to time 0, then S_n can be represented as the total discounted amount of the net loss of an insurer accumulated from time 0 to n. In

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this way, the tail probability $\mathbb{P}(M_n > x)$ can be understood as the probability of ruin by time n, where x is the initial surplus of the insurer.

In recent decades, analysis of the problem of approximating the tail probabilities $\mathbb{P}(S_n > x)$ and $\mathbb{P}(M_n > x)$ has been an attractive research topic in the literature on applied probability due to their practical importance; see Tang and Tsitsiashvili [17], Chen *et al.* [2], Shen *et al.* [16], Zhang *et al.* [22], Gao and Wang [6], Hazra and Maulik [7], Olvera-Cravioto [14], and Tang and Yuan [19], among others. Notably, in all these references, the authors aim to show that the relations

$$\mathbb{P}(M_n > x) \sim \mathbb{P}(S_n > x) \sim \sum_{k=1}^{n} \mathbb{P}\left(\Theta_k X_k > x\right) \tag{1.1}$$

hold under certain technical conditions.

However, note that, in reality, an unexpected claim event might yield more than one type of claim. A typical situation is one in which certain damages or catastrophic events cause losses in different branches, for example, liability automobile insurance for material damage and bodily injury or natural catastrophe insurance for claims distributed by geographical zones. Hence, the risk theory for multidimensional risk models is of great interest, particularly the analysis of multidimensional risk models. However, most works on this topic focus on typical risk models with assumed independence among claim sizes from different businesses subject to mathematical tractability; see Li *et al.* [13], Chen *et al.* [3], Zhang and Wang [21], Chen *et al.* [4], Hu and Jiang [8], Yang and Li [20], and Li [11], among others.

However, the assumption of independence among claim sizes is clearly unrealistic in practice, as the components of claim vectors are genuinely dependent on each other. The investigation of multidimensional models considering dependent claim sizes is quite limited. Among the few contributions, Shen and Zhang [15] considered a bidimensional discrete-time risk model in which claim sizes from different businesses follow a dependence structure given in terms of copulas. Assuming that the interest rate is constant, the authors obtained some asymptotic results for both finite-time and infinite-time ruin probabilities. Huang *et al.* [9] established some asymptotic estimates for ruin probabilities in terms of the upper tail dependence function of a net loss vector with regularly varying tails. Li [10] proposed a multidimensional time-dependent risk model with the stochastic return. Assuming that the claim sizes are multivariate regularly varying, Li [10] obtained some precise asymptotic expansions for two types of ruin probabilities. Li [12] studied the asymptotic joint tail probabilities of randomly weighted sums, in which the primary random variables are strongly asymptotically independent.

Inspired by the above works, in this paper, we aim at extending (1.1) to the bidimensional case. That is, we focus on establishing some asymptotic formulae for the tail probabilities of bidimensional randomly weighted sums of random vectors and their maxima, under the assumptions that the marginal distributions of the summand random vectors have extended regularly varying tails and that the components are generally dependent.

To close this section, we recall two important classes of heavy-tailed distributions. A tail distribution function $\overline{F}$ belongs to the class of extended regular variation if there are two constants α and β, with $0 < \alpha \le \beta < \infty$, such that the following holds for all $v \ge 1$,

$$v^{-\beta} \le \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \le \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \le v^{-\alpha}.$$

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We denote the above by $\overline{F} \in ERV(-\alpha, -\beta)$. The ERV (extended regular variation) class contains the $\mathcal{R}$ class of distributions with regular variation characterized by the relation

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$$

for some $\alpha > 0$ and all $y > 0$. In this case, we write $\overline{F} \in \mathcal{R}_{-\alpha}$.

The remainder of this paper consists of two sections. In Section 2, a bidimensional risk model and some asymptotic results, along with applications of approximating the ruin probabilities, are presented. Section 3 states some lemmas and presents the proof of the main results.

2. MAIN RESULTS AND APPLICATIONS

Throughout this paper, all limit relationships hold as $x \to \infty$ unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \lesssim b(x)$ if $\limsup_{x\to\infty} a(x)/b(x) \le 1$, $a(x) \gtrsim b(x)$ if $\liminf_{x\to\infty} a(x)/b(x) \ge 1$, and $a(x) \sim b(x)$ if both hold. We use the symbol $x^+ = \max\{x, 0\}$ to denote a real number x.

2.1. Main results

Let $\{\boldsymbol{X}_k = (X_{1,k}, X_{2,k})^\top, k \ge 1\}$ be a sequence of i.i.d. copies of $\boldsymbol{X} = (X_1, X_2)^\top$, which has a survival copula $\hat{C}(\cdot,\cdot)$, marginal distributions F_1 (of X_1) and F_2 (of X_2), joint distribution $\boldsymbol{F}$, and joint survival function $\overline{\boldsymbol{F}}(x_1, x_2) = \mathbb{P}(X_1 > x_1, X_2 > x_2)$. Let $\{\boldsymbol{\Theta}_k = (\Theta_{1,k}, \Theta_{2,k})^\top, k \ge 1\}$ be a sequence of nonnegative random but not necessarily independent vectors (with independence guaranteed neither among vectors nor among their components). Assuming that $\{\boldsymbol{X}_k, k \ge 1\}$ are independent of $\{\boldsymbol{\Theta}_k, k \ge 1\}$, the present paper considers the tail probabilities of bidimensional randomly weighted sums

$$\boldsymbol{S}_n = \begin{pmatrix} S_{1,n} \\ S_{2,n} \end{pmatrix} := \sum_{k=1}^{n} \boldsymbol{\Theta}_k \boldsymbol{X}_k = \begin{pmatrix} \sum_{k=1}^{n} \Theta_k X_{1,k} \\ \sum_{k=1}^{n} \Theta_k X_{2,k} \end{pmatrix}, \quad n \ge 1, \tag{2.1}$$

and their maxima

$$\boldsymbol{M}_n = (M_{1,n}, M_{2,n})^\top := \max_{1\le i\le n} \boldsymbol{S}_i = \left(\max_{1\le i\le n} S_{1,i}, \max_{1\le i\le n} S_{2,i} \right)^\top, \quad n \ge 1.$$

In the following, all operations between vectors are interpreted componentwise.

For notational convenience, we state the following assumptions regarding the sequences $\{\boldsymbol{X}_k, k \ge 1\}$ and $\{\boldsymbol{\Theta}_k, k \ge 1\}$.

H1. $\overline{F}_i \in ERV(-\alpha, -\beta)$ for some constants α, β such that $0 < \alpha \le \beta < \infty$.

H2. The left tail of X_i is lighter than its right tail in the sense that, for any positive constants $a, b > 0$,

$$\lim_{x\to\infty} \frac{F_i(-x)}{\min\{\overline{F}_1(ax)\overline{F}_2(bx), \overline{\boldsymbol{F}}(ax, bx)\}} = 0, \quad i = 1, 2. \tag{2.2}$$

H3. There exists a constant $\gamma \ge 1$ such that

$$\lim_{\max\{u_1,u_2\}\to 0} \frac{\hat{C}(\nu u_1, \nu u_2)}{\hat{C}(u_1, u_2)} = \nu^\gamma, \quad \nu > 0.$$

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H4. There exists a constant $\epsilon > 0$ such that $\mathbb{E}\Theta_{i,k}^{\beta\gamma'+\epsilon} < \infty$, and

$$\mathbb{P}(\boldsymbol{\Theta}_k \in B_1, \boldsymbol{\Theta}_j \in B_2) > 0, k, \quad j \geq 1,$$

where $\gamma' = \max\{\gamma, 2\}$ and B_i, $i = 1, 2$, denotes arbitrary subsets of $[0, \infty)^2$ with a positive bidimensional Lebesgue measure.

Remark 2.1: Assumption **H3**, first proposed by Shen and Zhang [15], is mild since it can be satisfied by a fair number of important families of copulas, such as Farlie–Gumbel–Morgenstern copula, Ali–Mikhail–Haq copula and Cuadras–Augé copula.

In assumption **H3**, we claim that $\gamma \geq 1$ if γ exists. Without loss of generality, suppose that $u_1 = u_2 = u$. If γ exists and $\gamma < 1$, then it follows that

$$\lim_{u \to 0} \frac{\hat{C}(\nu u, \nu u)}{\hat{C}(u, u)} = \nu^{\gamma},$$

which means that $\hat{C}(u, u)$ is regularly varying at 0 with index γ. This contradicts the fact that $\hat{C}(u, u) \leq u$.

Let $\boldsymbol{x} = (a_1 x, a_2 x)^{\top}$, with $a_1, a_2, x > 0$ and $a_1 + a_2 = 1$. We can now state our main results.

THEOREM 2.1: *Suppose that assumptions* **H1**–**H4** *are satisfied for model* (2.1). *Then, for any fixed integer* $n \in \mathbb{N}$,

$$\mathbb{P}(\boldsymbol{M}_n > \boldsymbol{x}) \sim \mathbb{P}(\boldsymbol{S}_n > \boldsymbol{x}) \sim \sum_{k=1}^{n} \sum_{j=1}^{n} \mathbb{P}(\Theta_{1,k} X_{1,k} > a_1 x, \Theta_{2,j} X_{2,j} > a_2 x). \tag{2.3}$$

COROLLARY 2.1: *Under the conditions of Theorem* 2.1, *if* $\overline{F}_i \in \mathcal{R}_{-\alpha}$, $i = 1, 2$, *for some* $0 < \alpha < \infty$, *then* (2.3) *can be reduced to the following cases when* $\gamma \neq 2$:

(a) *For* $1 \leq \gamma < 2$,

$$\mathbb{P}(\boldsymbol{M}_n > \boldsymbol{x}) \sim \mathbb{P}(\boldsymbol{S}_n > \boldsymbol{x}) \sim \sum_{k=1}^{n} \mathbb{P}(\Theta_{1,k} X_{1,k} > a_1 x, \Theta_{2,k} X_{2,k} > a_2 x). \tag{2.4}$$

(b) *For* $\gamma > 2$,

$$\mathbb{P}(\boldsymbol{M}_n > \boldsymbol{x}) \sim \mathbb{P}(\boldsymbol{S}_n > \boldsymbol{x}) \sim (a_1 a_2)^{-\alpha} \overline{F}_1(x) \overline{F}_2(x) \sum_{1 < k \neq j \leq n} \mathbb{E}(\Theta_{1,k} \Theta_{2,j})^{\alpha}. \tag{2.5}$$

PROOF: Relation (2.4) is an immediate consequence of Theorem 2.1.

Note that $X_{1,k}$ and $X_{2,j}$ are independent for $k \neq j$. Under assumption **H3**, for some $\epsilon > 0$, it follows from Hölder's inequality that

$$\mathbb{E}(\Theta_{i,k}^{\alpha+\epsilon/2}) < \infty, \quad i = 1, 2,$$

and

$$\mathbb{E}[(\Theta_{1,k} \Theta_{2,j})^{\alpha+\epsilon/2}] < \infty.$$

Hence, applying Lemma 3(iii) of Li [12] yields (2.5). ■

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THEOREM 2.2: *Suppose that assumptions* **H1**–**H3** *are satisfied for model* (2.1). *Assume that for any* $\iota > 0$, *there is* $0 < \delta < \alpha$ *such that*

$$\sum_{k=1}^{\infty} k^{(1+\iota)(\beta\gamma+\delta)} \mathbb{E}(\Theta_{i,k}^{\alpha\gamma-\delta} \vee \Theta_{i,k}^{\beta\gamma+\delta}) < \infty, \quad i = 1, 2, \tag{2.6}$$

and

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (kj)^{(1+\iota)(\beta+\delta)} \mathbb{E}[(\Theta_{1,k}^{\alpha-\delta} \vee \Theta_{1,k}^{\beta+\delta})(\Theta_{2,j}^{\alpha-\delta} \vee \Theta_{2,j}^{\beta+\delta})] < \infty. \tag{2.7}$$

Then,

$$\mathbb{P}(\boldsymbol{M}_n > \boldsymbol{x}) \sim \mathbb{P}(\boldsymbol{S}_n^+ > \boldsymbol{x}) \sim \sum_{k=1}^{n}\sum_{j=1}^{n} \mathbb{P}(\Theta_{1,k}X_{1,k} > a_1 x, \Theta_{2,j}X_{2,j} > a_2 x) \tag{2.8}$$

holds uniformly for all $n \in \mathbb{N}$.

Remark 2.2: The proofs of Theorems 2.1 and 2.2 are insensitive to the numbers of $X_{i,j}$, $i = 1, 2$. Hence, using a similar treatment, we can determine that, under the conditions of Theorem 2.2,

$$\begin{aligned}\mathbb{P}(M_{1,m} > a_1 x, M_{2,n} > a_2 x) &\sim \mathbb{P}(S_{1,m}^+ > a_1 x, S_{2,n}^+ > a_2 x) \\ &\sim \sum_{k=1}^{m}\sum_{j=1}^{n} \mathbb{P}(\Theta_{1,k}X_{1,k} > a_1 x, \Theta_{2,j}X_{2,j} > a_2 x)\end{aligned}$$

holds uniformly for all $m, n \in \mathbb{N}$.

2.2. Applications

Following the work by Tang and Tsitsiashvili [18] (in which the authors considered a unidimensional case), we consider a stochastic economic environment. In this environment, an insurer invests his surplus into both risk-free and risky assets, which may lead to negative returns. The insurer starts with an initial capital vector $\boldsymbol{x} = (a_1 x, a_2 x)^\top$ of two classes of an insurance business:

(A1) The net losses during referenced years $\boldsymbol{X}_k = (X_{1,k}, X_{2,k})^\top$, $k \geq 1$, constitute a sequence of i.i.d. random vectors with a common survival copula $\hat{C}(\cdot, \cdot)$.

(A2) $Y_{i,k}$ denotes the discount factor from time k to time $k-1$ of each business line, where $\mathbb{P}(0 < Y_{i,k} < \infty) = 1, i = 1, 2, k \geq 1$.

(A3) The two sequences $\{\boldsymbol{X}_k, k \geq 1\}$ and $\{\boldsymbol{Y}_k = (Y_{1,k}, Y_{2,k})^\top, k \geq 1\}$ are mutually independent.

Hence, if we assume that the net losses $\boldsymbol{X}_k$, $k \geq 1$, are calculated at the end of year k and write $R_{i,k} = Y_{i,k}^{-1} - 1, i = 1, 2, k \geq 1$, then $\{\mathbf{R}_k = (R_{1,k}, R_{2,k})^\top, k \geq 1\}$ denotes the stochastic return rate within period k, and the surplus, denoted by $\mathbf{U}_n$, accumulated by the end

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of year n satisfies the recurrence equation

$$\boldsymbol{U}_0 = \boldsymbol{x}, \boldsymbol{U}_n = \begin{pmatrix} U_{1,n} \\ U_{2,n} \end{pmatrix} = \begin{pmatrix} (1+R_{1,n})U_{1,n-1} \\ (1+R_{2,n})U_{2,n-1} \end{pmatrix} - \boldsymbol{X}_n, n \geq 1. \tag{2.9}$$

Now, we write

$$\boldsymbol{\Theta}_k = \begin{pmatrix} \Theta_{1,k} \\ \Theta_{2,k} \end{pmatrix} = \begin{pmatrix} \prod_{i=1}^k (1+R_{1,i})^{-1} \\ \prod_{i=1}^k (1+R_{2,i})^{-1} \end{pmatrix} = \begin{pmatrix} \prod_{i=1}^k Y_{1,i} \\ \prod_{i=1}^k Y_{2,i} \end{pmatrix}.$$

Thus, $\Theta_{i,k}$ denotes the stochastic discount factor of business line i from year k to 0, $i = 1, 2$. Then,

$$U_{i,n} = \Theta_{i,n}^{-1} \left(a_i x - \sum_{k=1}^n \Theta_{i,k} X_{i,k} \right), \quad i = 1, 2.$$

The bidimensional risk model above with constant interest rates, which is a special case of the work in the present paper can be found in Shen and Zhang [15].

Based on the bidimensional risk model, various types of ruin probabilities can be defined. For any integer $n \geq 1$, we may consider the following two types of ruin probabilities:

$$\psi_{\text{and}}(\boldsymbol{x}, n) = \mathbb{P}\left(\bigcap_{i=1}^2 \left(\min_{0 \leq j \leq n} U_{i,j} < 0 \right) \middle| \boldsymbol{U}_0 = \boldsymbol{x} \right),$$

$$\psi_{\text{sim}}(\boldsymbol{x}, n) = \mathbb{P}(U_{1,i} < 0, U_{2,i} < 0, \text{for some } i = 1, \ldots, n | \boldsymbol{U}_0 = \boldsymbol{x}).$$

The quantity $\psi_{\text{and}}(\boldsymbol{x}, n)$ represents the probability that ruin occurs in both insurance business classes over the time horizon $[0, n]$, though not necessarily at the same time. $\psi_{\text{sim}}(\boldsymbol{x}, n)$ is the probability that ruin occurs in both insurance business classes simultaneously over the time horizon $[0, n]$.

Let us now consider the ruin probabilities of model (2.9) by applying the main results in Section 2.1.

PROPOSITION 2.1: *Suppose that* $\{\boldsymbol{X}_k, k \geq 1\}$ *and* $\{\boldsymbol{\Theta}_k = (\prod_{i=1}^k Y_{1,i}, \prod_{i=1}^k Y_{2,i})^\top, k \geq 1\}$ *satisfy the conditions in Theorem* 2.1. *Then, for any* $n \in \mathbb{N}$,

$$\psi_{\text{and}}(\boldsymbol{x}, n) \sim \psi_{\text{sim}}(\boldsymbol{x}, n) \sim \sum_{k=1}^n \sum_{j=1}^n \mathbb{P}\left(X_{1,k} \prod_{i=1}^k Y_{1,i} > a_1 x, X_{2,j} \prod_{i=1}^j Y_{2,i} > a_2 x \right). \tag{2.10}$$

PROOF: From the ruin probability definitions given above, it follows that

$$\psi_{\text{and}}(\boldsymbol{x}, n) \geq \psi_{\text{sim}}(\boldsymbol{x}, n) \geq \mathbb{P}\left(\sum_{k=1}^n X_{i,k} \prod_{j=1}^k Y_{i,j} > a_i x, \ i = 1, 2 \right). \qquad \blacksquare$$

PROPOSITION 2.2: *Suppose that* $\{\boldsymbol{X}_k, k \geq 1\}$ *satisfies the corresponding conditions imposed in Theorem* 2.1 *and that* $\{\boldsymbol{Y}_k, k \geq 1\}$ *is a sequence of i.i.d. copies of* $\boldsymbol{Y} = (Y_1, Y_2)^\top$ *satisfying*

$$\mathbb{E}(Y_i^{\alpha\gamma-\delta} \vee Y_i^{\beta\gamma+\delta}) < 1, \mathbb{E}(Y_i^{\alpha-\delta} \vee Y_i^{\beta+\delta}) < 1, \quad i = 1, 2, \tag{2.11}$$

and

$$\mathbb{E}[(Y_1^{\alpha-\delta} \vee Y_1^{\beta+\delta})(Y_2^{\alpha-\delta} \vee Y_2^{\beta+\delta})] < 1 \tag{2.12}$$

for some $0 < \delta < \alpha$. *Then,* (2.10) *holds uniformly for all* $n \in \mathbb{N}$.

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PROOF: Clearly, assumptions (2.11) and (2.12) imply (2.6) and (2.7) in Theorem 2.2. ■

Remark 2.3: From Proposition 2.2, one can derive the asymptotic estimate for the ruin probability for a unidimensional risk model. That is, if $\{X_k, k \geq 1\}$ is a sequence of i.i.d. random variables with a common distribution $\overline{F} \in ERV(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$, and if $\{Y_k, k \geq 1\}$ is a sequence of i.i.d. and positive random variables with a generic random variable Y satisfying $\mathbb{E}(Y^{\alpha-\delta} \vee Y^{\beta+\delta}) < 1$ for some $0 < \delta < \alpha$, then

$$\psi(x,n) \sim \sum_{k=1}^{n} \mathbb{P}\left(X_k \prod_{i=1}^{k} Y_i > ax\right) \tag{2.13}$$

holds uniformly for all $n \in \mathbb{N}$, where

$$\psi(x,n) = \mathbb{P}\left(\min_{0\leq j\leq n}\left(x\prod_{i=1}^{j} Y_i^{-1} - \sum_{k=1}^{j} X_k \prod_{i=k+1}^{j} Y_i^{-1}\right) < 0\right).$$

The relation here (2.13) coincides with that of Theorem 3.1 in Tang and Tsitsiashvili [18] under the same assumptions.

3. PROOF OF THE MAIN RESULTS AND LEMMAS

In the proof of our main results, we need the following series of lemmas, which will be used frequently.

LEMMA 3.1: *Suppose that* $\boldsymbol{X}$ *is a random vector satisfying assumptions* **H1** *and* **H3**. *Then, for any* $\nu \geq 1$,

$$\nu^{-\beta\gamma} \leq \liminf_{\min\{x_1,x_2\}\to\infty} \frac{\hat{C}(\overline{F}_1(\nu x_1), \overline{F}_2(\nu x_2))}{\hat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2))} \leq \limsup_{\min\{x_1,x_2\}\to\infty} \frac{\hat{C}(\overline{F}_1(\nu x_1), \overline{F}_2(\nu x_2))}{\hat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2))} \leq \nu^{-\alpha\gamma} \tag{3.1}$$

holds for any $\nu \geq 1$.

PROOF: For any $0 < \epsilon < 1$, $\nu \geq 1$ and sufficiently large x_i, we have

$$(\nu^{-\beta} - \epsilon)\overline{F}_i(x_i) \leq \overline{F}_i(\nu x_i) \leq (\nu^{-\alpha} + \epsilon)\overline{F}_i(x_i), \quad i = 1, 2.$$

By the monotonicity of copulas, we obtain

$$\begin{aligned}\limsup_{\min\{x_1,x_2\}\to\infty} \frac{\hat{C}(\overline{F}_1(\nu x_1), \overline{F}_2(\nu x_2))}{\hat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2))} &\leq \limsup_{\min\{x_1,x_2\}\to\infty} \frac{\hat{C}((\nu^{-\alpha} + \epsilon)\overline{F}_1(x_1), (\nu^{-\alpha} + \epsilon)\overline{F}_2(x_2))}{\hat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2))} \\ &= (\nu^{-\alpha} + \epsilon)^{\gamma},\end{aligned}$$

where, in the last step, we used assumption **H3**. Letting $\epsilon \to 0$ in the last display yields

$$\limsup_{\min\{x_1,x_2\}\to\infty} \frac{\hat{C}(\overline{F}_1(\nu x_1), \overline{F}_2(\nu x_2))}{\hat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2))} \leq \nu^{-\alpha\gamma}.$$

Similarly,

$$\liminf_{\min\{x_1,x_2\}\to\infty} \frac{\hat{C}(\overline{F}_1(\nu x_1), \overline{F}_2(\nu x_2))}{\hat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2))} \geq \nu^{-\beta\gamma}.$$ ■

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LEMMA 3.2: *Suppose that* $\boldsymbol{X}$ *is a random vector satisfying assumptions* **H1** *and* **H3**, $\boldsymbol{\Theta} = (\Theta_1, \Theta_2)^\top$ *is a nonnegative random vector independent of* $\boldsymbol{X}$ *satisfying* $\mathbb{P}(\Theta_i > x) = o\left(\overline{\boldsymbol{F}}(ax, bx)\right)$ *for any positive constants* $a, b > 0$ *and* $i = 1, 2$, *and* $\mathbb{P}(\boldsymbol{\Theta} \in B) > 0$ *for any subset* $B \subset [0, \infty)^2$ *with a positive bidimensional Lebesgue measure. Then, for* $\nu \geq 1$,

$$\begin{aligned} \nu^{-\beta\gamma} &\leq \liminf_{x\to\infty} \frac{\mathbb{P}(\Theta_1 X_1 > \nu ax, \Theta_2 X_2 > \nu bx)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} \\ &\leq \limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_1 X_1 > \nu ax, \Theta_2 X_2 > \nu bx)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} \leq \nu^{-\alpha\gamma}. \end{aligned} \tag{3.2}$$

Particularly, if $\overline{F}_i \in \mathcal{R}_{-\alpha}$ *for some* $0 < \alpha < \infty$, $i = 1, 2$, *then*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\Theta_1 X_1 > \nu ax, \Theta_2 X_2 > \nu bx)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} = \nu^{-\alpha\gamma}.$$

PROOF: Note that $\mathbb{P}(\Theta_i > x) = o\left(\overline{\boldsymbol{F}}(ax, bx)\right)$, $i = 1, 2$. Hence, by Proposition 3.1 of Zhou *et al.* [23] and Lemma 3.1, there exists an infinitely increasing function $\varphi(x)$ (i.e., $\varphi(x)$:$[0, \infty) \to [0, \infty)$ such that $\varphi(x) \nearrow \infty, x/\varphi(x) \searrow 0$) such that $\mathbb{P}(\Theta_i > x/\varphi(x)) = o\left(\overline{\boldsymbol{F}}(ax, bx)\right)$. For any $a, b > 0$ and $\nu \geq 1$, we have

$$\begin{aligned} &\mathbb{P}(\Theta_1 X_1 > \nu ax, \Theta_2 X_2 > \nu bx) \\ &= \left(\int_0^{x/\varphi(x)} \int_0^{x/\varphi(x)} + \int_0^{x/\varphi(x)} \int_{x/\varphi(x)}^{\infty} + \int_{x/\varphi(x)}^{\infty} \int_0^{x/\varphi(x)} + \int_{x/\varphi(x)}^{\infty} \int_{x/\varphi(x)}^{\infty} \right) \\ &\quad \times \overline{\boldsymbol{F}}\left(\frac{\nu ax}{s}, \frac{\nu bx}{t} \right) \mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t) \\ &=: T_1(\nu) + T_2(\nu) + T_3(\nu) + T_4(\nu). \end{aligned} \tag{3.3}$$

Now, we begin to estimate the four terms over $\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)$ correspondingly.

$$\begin{aligned} \frac{T_1(\nu)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} &= \frac{\int_0^{x/\varphi(x)} \int_0^{x/\varphi(x)} \overline{\boldsymbol{F}}(\nu ax/s, \nu bx/t) \mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)}{\int_0^{\infty} \int_0^{\infty} \overline{\boldsymbol{F}}(ax/s, bx/t) \mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)} \\ &\leq \frac{\int_0^{x/\varphi(x)} \int_0^{x/\varphi(x)} \overline{\boldsymbol{F}}(\nu ax/s, \nu bx/t) \mathbb{P}(\Theta_1 \in ds, \Theta_2 \in dt)}{\int_0^{x/\varphi(x)} \int_0^{x/\varphi(x)} \overline{\boldsymbol{F}}(ax/s, bx/t) \mathbb{P}(\Theta_1 \in ds, \Theta_2 \in \mathrm{d}t)} \\ &\leq \sup_{z_1, z_2 \geq \varphi(x)} \frac{\overline{\boldsymbol{F}}(\nu az_1, \nu bz_2)}{\overline{\boldsymbol{F}}(az_1, bz_2)}. \end{aligned} \tag{3.4}$$

Let $x_0 > 0$ such that $\mathbb{P}(\Theta_1 > x_0, \Theta_2 > x_0) > 0$. Then,

$$\begin{aligned} \frac{T_2(\nu)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} &= \frac{\int_0^{x/\varphi(x)} \int_{x/\varphi(x)}^{\infty} \overline{\boldsymbol{F}}(\nu ax/s, \nu bx/t) \mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)}{\int_0^{\infty} \int_0^{\infty} \overline{\boldsymbol{F}}(ax/s, bx/t) \mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)} \\ &\leq \frac{\int_0^{x/\varphi(x)} \int_{x/\varphi(x)}^{\infty} \overline{\boldsymbol{F}}(\nu ax/s, \nu bx/t) \mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)}{\int_{x_0}^{\infty} \int_{x_0}^{\infty} \overline{\boldsymbol{F}}(ax/s, bx/t) \mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)} \\ &\leq \frac{\mathbb{P}(\Theta_2 > x/\varphi(x))}{\overline{\boldsymbol{F}}(ax/x_0, bx/x_0) \mathbb{P}(\Theta_1 > x_0, \Theta_2 > x_0)} \\ &\to 0 \end{aligned} \tag{3.5}$$

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due to $\mathbb{P}(\Theta_2 > x/\varphi(x)) = o(\overline{\boldsymbol{F}}(ax, bx))$. Similarly,

$$\frac{T_3(\nu)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} \le \frac{\mathbb{P}(\Theta_1 > x/\varphi(x))}{\overline{\boldsymbol{F}}(ax/x_0, bx/x_0)\mathbb{P}(\Theta_1 > x_0, \Theta_2 > x_0)} \to 0, \tag{3.6}$$

and

$$\begin{aligned}\frac{T_4(\nu)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} &= \frac{\int_{x/\varphi(x)}^{\infty}\int_{x/\varphi(x)}^{\infty} \overline{\boldsymbol{F}}(\nu ax/s, \nu bx/t)\mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)}{\int_{x_0}^{\infty}\int_{x_0}^{\infty} \overline{\boldsymbol{F}}(ax/s, bx/t)\mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)} \\ &\le \frac{\mathbb{P}(\Theta_1 > x/\varphi(x), \Theta_2 > x/\varphi(x))}{\overline{\boldsymbol{F}}(ax/x_0, bx/x_0)\mathbb{P}(\Theta_1 > x_0, \Theta_2 > x_0)} \\ &\to 0. \end{aligned} \tag{3.7}$$

Thus, combining (3.1) and (3.4)–(3.7) yields

$$\limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_1 X_1 > \nu ax, \Theta_2 X_2 > \nu bx)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} \le \limsup_{x\to\infty} \sup_{z_1, z_2 \ge \varphi(x)} \frac{\overline{\boldsymbol{F}}(\nu az_1, \nu bz_2)}{\overline{\boldsymbol{F}}(az_1, bz_2)} \le \nu^{-\alpha\gamma}.$$

Next, we estimate the asymptotic lower bound:

$$\begin{aligned}&\frac{\mathbb{P}(\Theta_1 X_1 > \nu ax, \Theta_2 X_2 > \nu bx)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} \\ &\quad\ge \frac{\int_0^{x/\varphi(x)}\int_0^{x/\varphi(x)} \overline{\boldsymbol{F}}(\nu ax/s, \nu bx/t)\mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)}{\int_0^{\infty}\int_0^{\infty} \overline{\boldsymbol{F}}(ax/s, bx/t)\mathbb{P}(\Theta_1 \in \mathrm{d}s, \Theta_2 \in \mathrm{d}t)} \\ &\quad\ge \inf_{z_1, z_2 \ge \varphi(x)} \frac{\overline{\boldsymbol{F}}(\nu az_1, \nu bz_2)}{\overline{\boldsymbol{F}}(az_1, bz_2)} \cdot \frac{T_1(1)}{T_1(1) + T_2(1) + T_3(1) + T_4(1)}, \end{aligned} \tag{3.8}$$

where $T_i(\cdot)(i = 1, 2, 3, 4)$ are defined in (3.3). Again, by $\mathbb{P}(\Theta_i > x/\varphi(x)) = o(\overline{\boldsymbol{F}}(ax, bx))$, $i = 1, 2$, and $\mathbb{P}(\Theta_1 > x_0, \Theta_2 > x_0) > 0$, via a procedure similar to the proofs of (3.5)–(3.7) but with some minor modifications, we obtain

$$\begin{aligned}&\frac{T_2(1) + T_3(1) + T_4(1)}{T_1(1)} \\ &\quad\le \frac{\mathbb{P}(\Theta_2 > x/\varphi(x)) + \mathbb{P}(\Theta_1 > x/\varphi(x)) + \mathbb{P}(\Theta_1 > x/\varphi(x), \Theta_2 > x/\varphi(x))}{\overline{\boldsymbol{F}}\left(\frac{ax}{x_0}, \frac{bx}{x_0}\right)\mathbb{P}(x_0 < \Theta_1 < x/\varphi(x), x_0 < \Theta_2 < x/\varphi(x))} \to 0. \end{aligned} \tag{3.9}$$

Now, combining (3.9) with (3.8) and (3.1) yields

$$\liminf_{x\to\infty} \frac{\mathbb{P}(\Theta_1 X_1 > \nu ax, \Theta_2 X_2 > \nu bx)}{\mathbb{P}(\Theta_1 X_1 > ax, \Theta_2 X_2 > bx)} \ge \liminf_{x\to\infty} \inf_{z_1, z_2 \ge \varphi(x)} \frac{\overline{\boldsymbol{F}}(\nu az_1, \nu bz_2)}{\overline{\boldsymbol{F}}(az_1, bz_2)} \ge \nu^{-\beta\gamma}.$$

Hence, relation (3.2) follows, as desired. ■

LEMMA 3.3: *Suppose that* $\{\boldsymbol{X}_k, k \ge 1\}$ *and* $\{\boldsymbol{\Theta}_k, k \ge 1\}$ *satisfy assumptions* **H1**, **H3** *and* **H4**. *Then, for any positive constants* $a, b > 0$ *and* $k \ne j$, $\mathbb{P}(\Theta_{1,k} X_{1,k} > ax, \Theta_{2,j} X_{2,j} > bx) \in ERV(-2\alpha, -2\beta)$, *that is, for* $\delta \ge 1$,

$$\begin{aligned}\delta^{-2\beta} &\le \liminf_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,k} X_{1,k} > \delta ax, \Theta_{2,j} X_{2,j} > \delta bx)}{\mathbb{P}(\Theta_{1,k} X_{1,k} > ax, \Theta_{2,j} X_{2,j} > bx)} \\ &\le \limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,k} X_{1,k} > \delta ax, \Theta_{2,j} X_{2,j} > \delta bx)}{\mathbb{P}(\Theta_{1,k} X_{1,k} > ax, \Theta_{2,j} X_{2,j} > bx)} \le \delta^{-2\alpha}. \end{aligned}$$

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PROOF: By assumption **H4** and Hölder's inequality, for some $\epsilon > 0$, we have

$$\mathbb{E}[(\Theta_{1,k}\Theta_{2,j})^{\beta+\epsilon/2}] < \infty.$$

Now, following from Markov's inequality and the fact $x^{-(2\beta+\epsilon)} = o(\overline{F}_1(x)\overline{F}_2(x))$,

$$\mathbb{P}(\Theta_{1,k} > x, \Theta_{2,j} > x) = o(\overline{F}_1(x)\overline{F}_2(x)) \quad \text{and} \quad \mathbb{P}(\Theta_{i,k} > x) = o(\overline{F}_1(x)\overline{F}_2(x)), i = 1, 2.$$

Then, via a procedure similar to the proof of Lemma 3.2 one gets the desired result. ■

Remark 3.1: From the proof of Lemma 3.3, one can see that for any positive constants $a, b > 0$ and $i = 1, 2$,

$$\mathbb{P}(\Theta_{i,k} > x) = o(\overline{\boldsymbol{F}}(ax, bx)), \quad k \geq 1$$

when assumption **H4** holds. Hence, conditions of Lemma 3.2 hold under the assumptions of Theorem 2.1.

According to Lemmas 3.2 and 3.3, the lemma below is a direct consequence of Theorem 3.3 (iv) of Cline and Samorodnitsky [5].

LEMMA 3.4: *Let the conditions of Lemma 3.3 hold. Then, for any positive constants $a, b > 0$,*

$$\begin{aligned} 0 &< \liminf_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,k}X_{1,k} > ax, \Theta_{2,k}X_{2,k} > bx)}{\overline{\boldsymbol{F}}(ax, bx)} \\ &\leq \limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,k}X_{1,k} > ax, \Theta_{2,k}X_{2,k} > bx)}{\overline{\boldsymbol{F}}(ax, bx)} \\ &< \infty, \quad k \geq 1, \end{aligned}$$

and

$$\begin{aligned} 0 &< \liminf_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,k}X_{1,k} > ax, \Theta_{2,j}X_{2,j} > bx)}{\overline{F}_1(ax)\overline{F}_2(bx)} \\ &\leq \limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,k}X_{1,k} > ax, \Theta_{2,j}X_{2,j} > bx)}{\overline{F}_1(ax)\overline{F}_2(bx)} < \infty, k, j \geq 1, \quad k \neq j. \end{aligned}$$

LEMMA 3.5: *Let the conditions of Lemma 3.3 hold and $A_i = \{\Theta_{1,k_1}X_{1,k_1} > a_1x, \Theta_{2,k_2}X_{2,k_2} > a_2x, \Theta_{i,l_i}X_{i,l_i} > a_ix, l_i \neq k_i\}$, $i = 1, 2$. Then, for any positive constants $a_i > 0$, $i = 1, 2$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(A_i)}{\mathbb{P}(\Theta_{1,k_1}X_{1,k_1} > a_1x, \Theta_{2,k_2}X_{2,k_2} > a_2x)} = 0. \tag{3.10}$$

PROOF: Let $\mathcal{F}_{\boldsymbol{\Theta}} = \sigma\{\boldsymbol{\Theta}_k, k \geq 1\}$.

Case 1: $k_1 = k_2 = k$. Note that

$$\mathbb{P}(A_i|\mathcal{F}_{\boldsymbol{\Theta}}) \leq \mathbb{P}(\Theta_{1,k}X_{1,k} > a_1x, \Theta_{2,k}X_{2,k} > a_2x|\mathcal{F}_{\boldsymbol{\Theta}}), \quad i = 1, 2.$$

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By Lemma 3.4, the right-hand side of the above display is integrable. Thus, applying Fatou's lemma and (3.1), we have that for $i = 1, 2$,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathbb{P}(A_i)}{\mathbb{P}(\Theta_{1,k}X_{1,k} > a_1x, \Theta_{2,k}X_{2,k} > a_2x)} \\
&\quad \le \mathbb{E}\left[\limsup_{x\to\infty} \frac{\mathbb{P}((\Theta_{1,k}+\Theta_{2,k})X_{1,k} > a_1x, (\Theta_{1,k}+\Theta_{2,k})X_{2,k} > a_2x|\mathcal{F}_{\boldsymbol{\Theta}})}{\overline{\boldsymbol{F}}(a_1x, a_2x)}\right. \\
&\qquad \cdot \limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_{i,l_i}X_{i,l_i} > a_ix|\mathcal{F}_{\boldsymbol{\Theta}})}{\overline{F}_i(a_ix)} \\
&\qquad \left. \cdot \limsup_{x\to\infty} \overline{F}_i(a_ix) \cdot \limsup_{x\to\infty} \frac{\overline{\boldsymbol{F}}(a_1x, a_2x)}{\mathbb{P}(\Theta_{1,k}X_{1,k} > a_1x, \Theta_{2,k}X_{2,k} > a_2x)}\right] \\
&\quad = 0,
\end{aligned}$$

where, in the last step, we also used Lemma 3.4 and the fact that

$$\limsup_{x\to\infty} \overline{F}_i(a_ix) = 0, \quad i = 1, 2. \tag{3.11}$$

Case 2: $k_1 \neq k_2$. Without loss of generality, we suppose that $l_i \neq k_1, k_2$, $i = 1, 2$. In fact, if $l_1 = k_2$ or $l_2 = k_1$, then the proof of (3.10) is the same as that of Case 1. Keep in mind the definition of the ERV class and (3.11). Again, by Fatou's lemma and Lemma 3.4, it follows that

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathbb{P}(A_i)}{\mathbb{P}(\Theta_{1,k_1}X_{1,k_1} > a_1x, \Theta_{2,k_2}X_{2,k_2} > a_2x)} \\
&\quad \le \mathbb{E}\left[\left(\prod_{i=1}^{2} \limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_{i,k_i}X_{i,k_i} > a_ix|\mathcal{F}_{\boldsymbol{\Theta}})}{\overline{F}_i(a_ix)}\right) \cdot \limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_{i,l_i}X_{i,l_i} > a_ix|\mathcal{F}_{\boldsymbol{\Theta}})}{\overline{F}_i(a_ix)}\right. \\
&\qquad \left. \cdot \limsup_{x\to\infty} \overline{F}_i(a_ix) \cdot \limsup_{x\to\infty} \frac{\overline{F}_1(a_1x)\overline{F}_2(a_2x)}{\mathbb{P}(\Theta_{1,k_1}X_{1,k_1} > a_1x, \Theta_{2,k_2}X_{2,k_2} > a_2x)}\right] \\
&\quad = 0.
\end{aligned}$$

■

LEMMA 3.6: *Let the conditions of Theorem 2.1 hold. Then, for any positive constants $a_i > 0$, $i = 1, 2$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,j}X_{1,j} \le -x)}{\mathbb{P}(\Theta_{1,k}X_{1,k} > a_1x, \Theta_{2,j}X_{2,j} > a_2x)} = 0, k, \quad j \ge 1.$$

PROOF: As the proof of the case $k \neq j$ is quite similar to $k = j$, we prove only that

$$\lim_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,k}X_{1,k} \le -x)}{\mathbb{P}(\Theta_{1,k}X_{1,k} > a_1x, \Theta_{2,k}X_{2,k} > a_2x)} = 0, \quad k \ge 1.$$

According to Lemma 3.4, it suffices to show that for $k \ge 1$,

$$\lim_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,k}X_{1,k} \le -x)}{\overline{\boldsymbol{F}}(a_1x, a_2x)} = 0.$$

Applying Lemma 3.1, it follows from Proposition 2.2.1 of Bingham *et al.* [1] that, for any $\epsilon > 0$, there exist some positive constants K_i and D, $i = 1, 2$, such that

$$K_1t^{\alpha\gamma-\epsilon} \le \frac{\overline{\boldsymbol{F}}(a_1x/t, a_2x/t)}{\overline{\boldsymbol{F}}(a_1x, a_2x)} \le K_2t^{\beta\gamma+\epsilon}, \tag{3.12}$$

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for all $x > x/t > D$. Take an infinitely increasing function $g(x)$ satisfying $\mathbb{P}(\Theta_{i,k} > x/g(x)) = o(\overline{\boldsymbol{F}}(a_1x, a_2x)), i = 1, 2$. Based on (3.12), for large enough x such that $g(x) > D$, we obtain

$$\frac{\mathbb{P}(\Theta_{1,j}X_{1,j} \le -x)}{\overline{\boldsymbol{F}}(a_1x, a_2x)} \le \left(\int_0^1 + \int_{x/g(x)}^{\infty}\right) \frac{F_1(-x/t)}{\overline{\boldsymbol{F}}(a_1x, a_2x)} \mathbb{P}(\Theta_{1,j} \in \mathrm{d}t) + \int_1^{x/g(x)} K_2 t^{\beta\gamma+\epsilon} \cdot \frac{F_1(-x/t)}{\overline{\boldsymbol{F}}(a_1x/t, a_2x/t)} \mathbb{P}(\Theta_{1,j} \in \mathrm{d}t).$$

Now, using (2.2), $\mathbb{P}(\Theta_{i,k} > x/g(x)) = o(\overline{\boldsymbol{F}}(a_1x, a_2x)), i = 1, 2$, and Fatou's lemma leads to

$$\begin{aligned}\limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,j}X_{1,j} \le -x)}{\overline{\boldsymbol{F}}(a_1x, a_2x)} &\le \int_0^1 \limsup_{x\to\infty} \frac{F_1(-x/t)}{\overline{\boldsymbol{F}}(a_1x, a_2x)} \mathbb{P}(\Theta_{1,j} \in \mathrm{d}t) \\ &\quad + \limsup_{x\to\infty} \frac{\mathbb{P}(\Theta_{1,j} > x/g(x))}{\overline{\boldsymbol{F}}(a_1x, a_2x)} \\ &\quad + \int_1^{\infty} K_2 t^{\beta\gamma+\epsilon} \limsup_{x\to\infty} \frac{F_1(-x/t)}{\overline{\boldsymbol{F}}(a_1x/t, a_2x/t)} \mathbb{P}(\Theta_{1,j} \in \mathrm{d}t) \\ &= 0,\end{aligned}$$

which completes the proof. ■

LEMMA 3.7: *Let the assumptions in Lemma 3.3 hold. In addition, suppose that* $\{\mathbf{X}_k, k \ge 1\}$ *is a sequence of nonnegative random vectors and that* $\{\boldsymbol{\Theta}_k, k \ge 1\}$ *satisfies one of the following assumptions for some* $0 < \delta < \alpha$*:*

(i) $0 < \beta < 1$ *and*

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}[(\Theta_{1,k}^{\alpha-\delta} \vee \Theta_{1,k}^{\beta+\delta})(\Theta_{2,j}^{\alpha-\delta} \vee \Theta_{2,j}^{\beta+\delta})] < \infty;$$

(ii) $\beta \ge 1$ *and*

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \{\mathbb{E}[(\Theta_{1,k}^{\alpha-\delta} \vee \Theta_{1,k}^{\beta+\delta})(\Theta_{2,j}^{\alpha-\delta} \vee \Theta_{2,j}^{\beta+\delta})]\}^{1/(\beta+\delta)} < \infty.$$

Then,

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}(\sum_{k=n+1}^{\infty} \Theta_{1,k}X_{1,k} > a_1x, \sum_{j=1}^{n} \Theta_{2,j}X_{2,j} > a_2x)}{\overline{F}_1(a_1x)\overline{F}_2(a_2x)} = 0. \quad \textbf{(3.13)}$$

Moreover, if the sequence $\{\boldsymbol{\Theta}_k, k \ge 1\}$ *satisfies* (2.6) *and* (2.7)*, then we have*

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}(\sum_{k=n}^{\infty} \boldsymbol{\Theta}_k\boldsymbol{X}_k > \boldsymbol{x})}{\overline{\boldsymbol{F}}(a_1x, a_2x) + \overline{F}_1(a_1x)\overline{F}_2(a_2x)} = 0. \quad \textbf{(3.14)}$$

Remark 3.2: Assumptions (i) and (ii) are based on the conditions of Theorem 2 (also Lemma 7) in Li [12]. Clearly, relation (2.7) implies assumptions (i) and (ii) in Lemma 3.7.

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Remark 3.3: Using Lemma 3.2 together with Lemma 1(ii) and Lemma 2(i) of Li [12], under the assumptions of Theorem 2.2, there exist some constants K_1 and K_2 such that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{P}(\Theta_{1,k}X_{1,k} > a_1x, \Theta_{2,j}X_{2,j} > a_2x)\\
&\quad\le K_1\overline{\boldsymbol{F}}(a_1x, a_2x)\sum_{i=1}^{2}\sum_{k=1}^{\infty}\mathbb{E}(\Theta_{i,k}^{\alpha\gamma-\delta}\vee\Theta_{i,k}^{\beta\gamma+\delta}) + K_2\overline{F}_1(a_1x)\overline{F}_2(a_2x)\\
&\qquad\times\sum_{1\le k\ne j\le\infty}\mathbb{E}[(\Theta_{1,k}^{\alpha-\delta}\vee\Theta_{1,k}^{\beta+\delta})(\Theta_{2,j}^{\alpha-\delta}\vee\Theta_{2,j}^{\beta+\delta})]\\
&\qquad<\infty
\end{aligned}\tag{3.15}$$

holds for all large x and $0<\delta<\alpha$. Hence, it follows that

$$\lim_{n\to\infty}\limsup_{x\to\infty}\frac{\sum_{k=n}^{\infty}\sum_{j=1}^{\infty}\mathbb{P}(\Theta_{1,k}X_{1,k} > a_1x, \Theta_{2,j}X_{2,j} > a_2x)}{\overline{\boldsymbol{F}}(a_1x, a_2x) + \overline{F}_1(a_1x)\overline{F}_2(a_2x)} = 0.\tag{3.16}$$

Actually, from the first step in (3.15), to obtain (3.16), assumptions (2.6) and (2.7) can be reduced to (i) and (ii) of Lemma 3.7, and

$$\sum_{k=1}^{\infty}\mathbb{E}(\Theta_{i,k}^{\alpha\gamma-\delta}\vee\Theta_{i,k}^{\beta\gamma+\delta})<\infty,\quad i=1,2.$$

Proof of Lemma 3.7: Note that $X_{1,k}$ and $X_{2,j}$ are independent for $k\ne j$. By Lemma 7 of Li [12], (3.13) immediately holds. Hence, we need to show only that (3.14) follows the conditions in Lemma 3.7, including (2.6) and (2.7). For all integers n such that $\sum_{k=n}^{\infty}k^{-(1+\iota)}<1$, we derive

$$\begin{aligned}
\mathbb{P}\left(\sum_{k=n}^{\infty}\boldsymbol{\Theta}_k\boldsymbol{X}_k > \boldsymbol{x}\right) &\le \mathbb{P}\left(\sum_{k=n}^{\infty}\boldsymbol{\Theta}_k\boldsymbol{X}_k > \sum_{k=n}^{\infty}k^{-(1+\iota)}\boldsymbol{x}\right)\\
&\le \mathbb{P}\left(\bigcup_{k=n}^{\infty}\left\{\Theta_{1,k}X_{1,k} > k^{-(1+\iota)}a_1x\right\}, \bigcup_{j=n}^{\infty}\{\Theta_{2,j}X_{2,j} > j^{-(1+\iota)}a_2x\}\right)\\
&\le \sum_{k=n}^{\infty}\sum_{j=n}^{\infty}\mathbb{P}(\Theta_{1,k}X_{1,k} > k^{-(1+\iota)}a_1x, \Theta_{2,j}X_{2,j} > j^{-(1+\iota)}a_2x).
\end{aligned}$$

We now shall estimate the summands on the right-hand side of the above display. Similar to Remark 3.3, there exist some constants K_1 and K_2 such that for all large x and $0<\delta<\alpha$,

$$\begin{aligned}
&\sum_{k=n}^{\infty}\sum_{j=n}^{\infty}\mathbb{P}(\Theta_{1,k}X_{1,k} > k^{-(1+\iota)}a_1x, \Theta_{2,j}X_{2,j} > j^{-(1+\iota)}a_2x)\\
&\quad\le K_1\overline{\boldsymbol{F}}(a_1x, a_2x)\sum_{i=1}^{2}\sum_{k=n}^{\infty}k^{(1+\iota)(\beta\gamma+\delta)}\mathbb{E}(\Theta_{i,k}^{\alpha\gamma-\delta}\vee\Theta_{i,k}^{\beta\gamma+\delta})\\
&\qquad+K_2\overline{F}_1(a_1x)\overline{F}_2(a_2x)\sum_{n\le k\ne j\le\infty}(kj)^{(1+\iota)(\beta+\delta)}\mathbb{E}[(\Theta_{1,k}^{\alpha-\delta}\vee\Theta_{1,k}^{\beta+\delta})(\Theta_{2,j}^{\alpha-\delta}\vee\Theta_{2,j}^{\beta+\delta})].
\end{aligned}$$

Hence, (3.14) immediately follows from (2.6) and (2.7). ■

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We now turn to the proof of Theorems 2.1 and 2.2.

Proof of Theorem 2.1: Let $n \geq 1$ be an arbitrarily fixed integer. Since $\{\boldsymbol{\Theta}_k, k \geq 1\}$ are nonnegative, we have that

$$\sum_{k=1}^{n} \Theta_{l,k} X_{l,k} \leq \max_{0 \leq m \leq n} \sum_{k=1}^{m} \Theta_{l,k} X_{l,k} \leq \sum_{k=1}^{n} \Theta_{l,k} X_{l,k}^{+}, \quad l = 1, 2.$$

Note that $\boldsymbol{S}_n^{+} = (S_{1,n}^{+}, S_{2,n}^{+})^{\top} = (\sum_{k=1}^{n} \Theta_{1,k} X_{1,k}^{+}, \sum_{k=1}^{n} \Theta_{2,k} X_{2,k}^{+})^{\top}$. Thus, to show the main result, it suffices to state that for $n \geq 1$,

$$\mathbb{P}(\boldsymbol{S}_n^{+} > \boldsymbol{x}) \lesssim \sum_{k=1}^{n} \sum_{j=1}^{n} \mathbb{P}(\Theta_{1,k} X_{1,k} > a_1 x, \Theta_{2,j} X_{2,j} > a_2 x), \tag{3.17}$$

and

$$\mathbb{P}(\boldsymbol{S}_n > \boldsymbol{x}) \gtrsim \sum_{k=1}^{n} \sum_{j=1}^{n} \mathbb{P}(\Theta_{1,k} X_{1,k} > a_1 x, \Theta_{2,j} X_{2,j} > a_2 x). \tag{3.18}$$

When $n = 1$, (3.17) and (3.18) apparently hold. Now, suppose that $n \geq 2$.

First, we prove (3.17). Note that there exists a constant ν, with $1/2 < \nu < 1$, such that $0 < y = (1-\nu)/(n-1) < 1/n < \nu$. Then, we have that

$$\begin{aligned}
\mathbb{P}(\boldsymbol{S}_n^{+} > \boldsymbol{x}) \leq & \sum_{k=1}^{n} \sum_{j=1}^{n} \mathbb{P}(\Theta_{1,k} X_{1,k}^{+} > \nu a_1 x, \Theta_{2,j} X_{2,j}^{+} > \nu a_2 x) \\
& + \mathbb{P}\left(\boldsymbol{S}_n^{+} > \boldsymbol{x}, \bigcap_{k=1}^{n} \{\Theta_{1,k} X_{1,k}^{+} \leq \nu a_1 x\}, \bigcap_{j=1}^{n} \{\Theta_{2,j} X_{2,j}^{+} \leq \nu a_2 x\} \right) \\
& + \mathbb{P}\left(\boldsymbol{S}_n^{+} > \boldsymbol{x}, \bigcap_{k=1}^{n} \{\Theta_{1,k} X_{1,k}^{+} \leq \nu a_1 x\}, \bigcup_{j=1}^{n} \{\Theta_{2,j} X_{2,j}^{+} > \nu a_2 x\} \right) \\
& + \mathbb{P}\left(\boldsymbol{S}_n^{+} > \boldsymbol{x}, \bigcup_{k=1}^{n} \{\Theta_{1,k} X_{1,k}^{+} > \nu a_1 x\}, \bigcap_{j=1}^{n} \{\Theta_{2,j} X_{2,j}^{+} \leq \nu a_2 x\} \right) \\
=: & K_1 + K_2 + K_3 + K_4.
\end{aligned}$$

From Lemmas 3.2 and 3.3, the following can be easily obtained for $1/2 < \nu < 1$:

$$\limsup_{x \to \infty} \frac{\mathbb{P}(\Theta_{1,k} X_{1,k} > \nu a_1 x, \Theta_{2,j} X_{2,j} > \nu a_2 x)}{\mathbb{P}(\Theta_{1,k} X_{1,k} > a_1 x, \Theta_{2,j} X_{2,j} > a_2 x)} \leq \begin{cases} \nu^{-2\beta}, & k \neq j, \\ \nu^{-\beta\gamma}, & k = j. \end{cases} \tag{3.19}$$

Because $\mathbb{P}(\Theta_{1,k} X_{1,k}^{+} > a_1 x, \Theta_{2,j} X_{2,j}^{+} > a_2 x) = \mathbb{P}(\Theta_{1,k} X_{1,k} > a_1 x, \Theta_{2,j} X_{2,j} > a_2 x)$ for $x > 0$, $a_i > 0$ and $k \geq 1$, we can obtain

$$\lim_{\nu \nearrow 1} \limsup_{x \to \infty} \frac{K_1}{\sum_{k=1}^{n} \sum_{j=1}^{n} \mathbb{P}(\Theta_{1,k} X_{1,k} > a_1 x, \Theta_{2,j} X_{2,j} > a_2 x)} = 1 \tag{3.20}$$

by (3.19) and the inequality

$$\min\left\{ \frac{c_1}{d_1}, \ldots, \frac{c_n}{d_n} \right\} \leq \frac{\sum_{i=1}^{n} c_i}{\sum_{i=1}^{n} d_i} \leq \max\left\{ \frac{c_1}{d_1}, \ldots, \frac{c_n}{d_n} \right\},$$

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with c_i, d_i, $i = 1, \ldots, n$, being positive. Next, we estimate K_2. Note that

$$\begin{aligned}
K_2 &= \mathbb{P}\Bigg(\boldsymbol{S}_n^+ > \boldsymbol{x}, \bigcap_{k=1}^n \{\Theta_{1,k} X_{1,k}^+ \le \nu a_1 x\}, \bigcap_{j=1}^n \{\Theta_{2,j} X_{2,j}^+ \le \nu a_2 x\}, \\
&\qquad \max_{1\le k\le n} \Theta_{1,k} X_{1,k}^+ > \frac{a_1 x}{n}, \max_{1\le j\le n} \Theta_{2,j} X_{2,j}^+ > \frac{a_2 x}{n}\Bigg) \\
&\le \sum_{k=1}^n \sum_{j=1}^n \mathbb{P}\Big(\boldsymbol{S}_n^+ > \boldsymbol{x}, \Theta_{1,k} X_{1,k}^+ \le \nu a_1 x, \Theta_{2,j} X_{2,j}^+ \le \nu a_2 x, \\
&\qquad \Theta_{1,k} X_{1,k}^+ > \frac{a_1 x}{n}, \Theta_{2,j} X_{2,j}^+ > \frac{a_2 x}{n}\Big) \\
&\le \sum_{k=1}^n \sum_{j=1}^n \mathbb{P}\Bigg(\sum_{s_1\ne k} \Theta_{1,s_1} X_{1,s_1}^+ > (1-\nu) a_1 x, \Theta_{1,k} X_{1,k}^+ > \frac{a_1 x}{n}, \\
&\qquad \sum_{s_2\ne j} \Theta_{2,s_2} X_{2,s_2}^+ > (1-\nu) a_2 x, \Theta_{2,j} X_{2,j}^+ > \frac{a_2 x}{n}\Bigg) \\
&\le \sum_{k=1}^n \sum_{j=1}^n \sum_{s_1\ne k} \sum_{s_2\ne j} \mathbb{P}\Big(\Theta_{1,s_1} X_{1,s_1}^+ > y a_1 x, \Theta_{2,s_2} X_{2,s_2}^+ > y a_2 x, \\
&\qquad \Theta_{1,k} X_{1,k}^+ > y a_1 x, \Theta_{2,j} X_{2,j}^+ > y a_2 x\Big).
\end{aligned}$$

Consequently, it follows from Lemma 3.5 that

$$\lim_{x\to\infty} \frac{K_2}{\sum_{k=1}^n \sum_{j=1}^n \mathbb{P}(\Theta_{1,k} X_{1,k} > a_1 x, \Theta_{2,j} X_{2,j} > a_2 x)} = 0. \tag{3.21}$$

Also observe that

$$\begin{aligned}
K_3 &\le \mathbb{P}\Bigg(S_{1,n}^+ > a_1 x, \bigcap_{k=1}^n \{\Theta_{1,k} X_{1,k}^+ \le \nu a_1 x\}, \\
&\qquad \bigcup_{j=1}^n \{\Theta_{2,j} X_{2,j}^+ > \nu a_2 x\}, \max_{1\le k\le n} \Theta_{1,k} X_{1,k}^+ > \frac{a_1 x}{n}\Bigg) \\
&\le \sum_{k=1}^n \sum_{j=1}^n \mathbb{P}\Big(S_{1,n}^+ > a_1 x, \Theta_{1,k} X_{1,k}^+ \le \nu a_1 x, \Theta_{2,j} X_{2,j}^+ > \nu a_2 x, \Theta_{1,k} X_{1,k}^+ > \frac{a_1 x}{n}\Big) \\
&\le \sum_{k=1}^n \sum_{j=1}^n \mathbb{P}\Bigg(\sum_{s_1\ne k} \Theta_{1,s_1} X_{1,s_1}^+ > (1-\nu) a_1 x, \Theta_{2,j} X_{2,j}^+ > \nu a_2 x, \Theta_{1,k} X_{1,k}^+ > \frac{a_1 x}{n}\Bigg) \\
&\le \sum_{k=1}^n \sum_{j=1}^n \sum_{s_1\ne k} \mathbb{P}(\Theta_{1,s_1} X_{1,s_1}^+ > y a_1 x, \Theta_{2,j} X_{2,j}^+ > y a_2 x, \Theta_{1,k} X_{1,k}^+ > y a_1 x).
\end{aligned}$$

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Similar to the computation of (3.21), by applying Lemma 3.5 again, we can obtain

$$\lim_{x\to\infty}\frac{K_3}{\sum_{k=1}^{n}\sum_{j=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k}>a_1x,\Theta_{2,j}X_{2,j}>a_2x)}=0. \tag{3.22}$$

Likewise, the following is obtained:

$$\lim_{x\to\infty}\frac{K_4}{\sum_{k=1}^{n}\sum_{j=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k}>a_1x,\Theta_{2,j}X_{2,j}>a_2x)}=0. \tag{3.23}$$

Hence, a combination of (3.20)–(3.23) implies (3.17).

In the following, we begin to prove (3.18). Let $\xi>1$ be a constant, and set $w=(\xi-1)/(n-1)$. Clearly, $w>0$, hence, we have

$$\begin{aligned}
\mathbb{P}(\boldsymbol{S}_n>\boldsymbol{x}) &\ge \mathbb{P}\left(\boldsymbol{S}_n>\boldsymbol{x},\max_{1\le k\le n}\Theta_{1,k}X_{1,k}>\xi a_1x,\max_{1\le j\le n}\Theta_{2,j}X_{2,j}>\xi a_2x\right)\\
&\ge \sum_{k=1}^{n}\sum_{j=1}^{n}\mathbb{P}(\boldsymbol{S}_n>\boldsymbol{x},\Theta_{1,k}X_{1,k}>\xi a_1x,\Theta_{2,j}X_{2,j}>\xi a_2x)\\
&\quad-\sum_{1\le k_1\ne k_2\le n}\sum_{j=1}^{n}\mathbb{P}(\boldsymbol{S}_n>\boldsymbol{x},\Theta_{1,k_1}X_{1,k_1}>\xi a_1x,\Theta_{1,k_2}X_{1,k_2}\\
&\qquad\qquad>\xi a_1x,\Theta_{2,j}X_{2,j}>\xi a_2x)\\
&\quad-\sum_{k=1}^{n}\sum_{1\le j_1\ne j_2\le n}\mathbb{P}(\boldsymbol{S}_n>\boldsymbol{x},\Theta_{1,k}X_{1,k}\\
&\qquad\qquad>\xi a_1x,\Theta_{2,j_1}X_{2,j_1}>\xi a_2x,\Theta_{2,j_2}X_{2,j_2}>\xi a_2x)\\
&=:\Gamma_1-\Gamma_2-\Gamma_3.
\end{aligned}$$

Obviously, by Lemma 3.5, we can obtain

$$\lim_{x\to\infty}\frac{\Gamma_i}{\sum_{k=1}^{n}\sum_{j=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k}>a_1x,\Theta_{2,j}X_{2,j}>a_2x)}=0,\quad i=2,3. \tag{3.24}$$

Regarding Γ_1, we have

$$\begin{aligned}
\Gamma_1 &\ge \sum_{k=1}^{n}\sum_{j=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1x,\Theta_{1,s_1}X_{1,s_1}>-wa_1x,1\le s_1\ne k\le n,\\
&\qquad\Theta_{2,j}X_{2,j}>\xi a_2x,\Theta_{2,s_2}X_{2,s_2}>-wa_2x,1\le s_2\ne j\le n)\\
&=\sum_{1\le k\ne j\le n}\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1x,\Theta_{2,j}X_{2,j}>\xi a_2x,\Theta_{1,j}X_{1,j}>-wa_1x,\\
&\qquad\Theta_{2,k}X_{2,k}>-wa_2x,\boldsymbol{\Theta}_s\boldsymbol{X}_s>-w\boldsymbol{x},1\le s\ne k,j\le n)\\
&\quad+\sum_{k=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1x,\Theta_{2,k}X_{2,k}>\xi a_2x,\boldsymbol{\Theta}_s\boldsymbol{X}_s>-w\boldsymbol{x},1\le s\ne k\le n).\\
&=:\Gamma_{11}+\Gamma_{12}
\end{aligned}$$

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Regarding Γ_{11}, it follows from Lemma 3.6 that

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{\Gamma_{11}}{\sum_{1\le k\neq j\le n}\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1 x,\Theta_{2,j}X_{2,j}>\xi a_2 x)}\\
&\quad\ge \liminf_{x\to\infty}\sum_{1\le k\neq j\le n}\left[\frac{\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1 x,\Theta_{2,j}X_{2,j}>\xi a_2 x)}{\sum_{1\le k\neq j\le n}\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1 x,\Theta_{2,j}X_{2,j}>\xi a_2 x)}\right.\\
&\qquad\left.-\frac{\mathbb{P}(\Theta_{1,j}X_{1,j}\le -\omega a_1 x)+\mathbb{P}(\Theta_{2,k}X_{2,k}\le -\omega a_2 x)+\sum_{i=1,2}\sum_{1\le s_i\neq k,j\le n}\mathbb{P}(\Theta_{i,s_i}X_{i,s_i}\le -\omega a_i x)}{\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1 x,\Theta_{2,j}X_{2,j}>\xi a_2 x)}\right]\\
&\quad= 1. \qquad\textbf{(3.25)}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{\Gamma_{12}}{\sum_{k=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1 x,\Theta_{2,k}X_{2,k}>\xi a_2 x)}\\
&\quad\ge \liminf_{x\to\infty}\sum_{k=1}^{n}\left[\frac{\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1 x,\Theta_{2,k}X_{2,k}>\xi a_2 x)}{\sum_{k=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1 x,\Theta_{2,k}X_{2,k}>\xi a_2 x)}\right.\\
&\qquad\left.-\frac{\sum_{i=1,2}\sum_{1\le s_i\neq k\le n}\mathbb{P}(\Theta_{i,s_i}X_{i,s_i}\le -\omega a_i x)}{\mathbb{P}(\Theta_{1,k}X_{1,k}>\xi a_1 x,\Theta_{2,k}X_{2,k}>\xi a_2 x)}\right]\\
&\quad= 1. \qquad\textbf{(3.26)}
\end{aligned}$$

Thus, applying Lemma 3.3 together with (3.25) and (3.26) ensures that

$$\lim_{\xi\searrow 1}\liminf_{x\to\infty}\frac{\Gamma_1}{\sum_{k=1}^{n}\sum_{j=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k}>a_1 x,\Theta_{2,j}X_{2,j}>a_2 x)}\ge 1. \qquad\textbf{(3.27)}$$

Therefore, the combination of (3.24) and (3.27) leads to (3.18), which completes the proof. ■

Proof of Theorem 2.2: Using Lemma 3.7, for every $\epsilon>0$, we can also choose some large integer n_0 such that

$$\begin{aligned}
&\mathbb{P}\left(\sum_{k=n_0+1}^{\infty}\Theta_{1,k}X_{1,k}^{+}>a_1 x,\sum_{j=1}^{n_0}\Theta_{2,j}X_{2,j}^{+}>a_2 x\right)\\
&\quad+\mathbb{P}\left(\sum_{k=1}^{n_0}\Theta_{1,k}X_{1,k}^{+}>a_1 x,\sum_{j=n_0+1}^{\infty}\Theta_{2,j}X_{2,j}^{+}>a_2 x\right)\\
&\quad+\mathbb{P}\left(\sum_{k=n_0}^{\infty}\boldsymbol{\Theta}_k\boldsymbol{X}_k^{+}>\boldsymbol{x}\right)+\sum_{k=n_0}^{\infty}\sum_{j=1}^{\infty}\mathbb{P}(\Theta_{1,k}X_{1,k}>a_1 x,\Theta_{2,j}X_{2,j}>a_2 x)\\
&\quad+\sum_{k=1}^{\infty}\sum_{j=n_0}^{\infty}\mathbb{P}(\Theta_{1,k}X_{1,k}>a_1 x,\Theta_{2,j}X_{2,j}>a_2 x)\\
&\lesssim \epsilon(\overline{\boldsymbol{F}}(a_1 x,a_2 x)+\overline{F}_1(a_1 x)\overline{F}_2(a_2 x)). \qquad\textbf{(3.28)}
\end{aligned}$$

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Applying Theorem 2.1 shows us that relation (2.3) holds uniformly for $1 \le n \le n_0$. We now consider the case of $n > n_0$. By (2.3) with $n = n_0$, the following holds uniformly for $n > n_0$:

$$\begin{aligned}\mathbb{P}(\boldsymbol{M}_n > \boldsymbol{x}) &\gtrsim \left(\sum_{k=1}^{n}\sum_{j=1}^{n} - \sum_{k=n_0+1}^{n}\sum_{j=1}^{n} - \sum_{k=1}^{n_0}\sum_{j=n_0+1}^{n}\right)\mathbb{P}(\Theta_{1,k}X_{1,k} > a_1x, \Theta_{2,j}X_{2,j} > a_2x)\\ &\gtrsim \sum_{k=1}^{n}\sum_{j=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k} > a_1x, \Theta_{2,j}X_{2,j} > a_2x)\\ &\quad - (\epsilon\overline{\boldsymbol{F}}(a_1x, a_2x) + 2\epsilon\overline{F}_1(a_1x)\overline{F}_2(a_2x)), \end{aligned} \tag{3.29}$$

where, in the last step, we used (3.28). Moreover, by (2.3) and (3.28), for every $0 < \rho < 1$, via a procedure similar to the proof of (37) in Li [12] but with some modifications, we can obtain

$$\begin{aligned}\mathbb{P}(\boldsymbol{S}_n^+ > \boldsymbol{x}) &\lesssim 2\epsilon[(1-\rho)\rho]^{-\beta}\overline{F}_1(a_1x)\overline{F}_2(a_2x) + \epsilon[\rho^{-\beta\gamma}\overline{\boldsymbol{F}}(a_1x, a_2x) + \rho^{-2\beta}\overline{F}_1(a_1x)\overline{F}_2(a_2x)]\\ &\quad + \sum_{k=1}^{n}\sum_{j=1}^{n}\mathbb{P}(\Theta_{1,k}X_{1,k} > (1-\rho)a_1x, \Theta_{2,j}X_{2,j} > (1-\rho)a_2x) \end{aligned} \tag{3.30}$$

uniformly in $n > n_0$. Combining (3.29) and (3.30) and then letting $\epsilon \to 0$ and $\rho \to 0$ in turn yields the desired result, that is, (2.8). ■

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