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Constructions in elliptic dynamics

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Dedicated to the memory of Michel Herman

Abstract. We present an overview and some new applications of the approximation by conjugation method introduced by Anosov and Katok more than 30 years ago (*Trans. Moscow Math. Soc*. **23** (1970), 1–35). Michel Herman made important contributions to the development and applications of this method beginning from the construction of minimal and uniquely ergodic diffeomorphisms jointly with Fathi (*Asterisque* **49** (1977),37–59) and continuing with exotic invariant sets of rational maps of the Riemann sphere (*J. London Math. Soc*. (2) **34** (1986), 375–384) and the construction of invariant tori with non-standard and unexpected behavior in the context of KAM theory (Pitman Research Notes Mathematical Series 243 (1992); *Proc. Int. Congr. Mathematicians* (Berlin, 1998) Vol. 11, 797–808). Recently the method has been experiencing a revival. Some of the new results presented in the paper illustrate variety of uses for tools available for a long time, others exploit new methods, in particular the possibility of mixing in the context of Liouvillean dynamics discovered by the first author (*Ergod. Th. & Dynam. Sys*. **22** (2002) 437–468; *Proc. Amer. Math. Soc*. **130** (2002), 103–109).

1. *Elliptic dynamics: Diophantine and Liouvillean behavior*

In this paper we present and discuss a variety of examples of C^∞ volume-preserving diffeomorphisms of compact manifolds which, by any reasonable count, should be viewed as exotic if not pathological. The methods which produce these examples are discussed in the next section and specific constructions carried out in the rest of the paper. Before doing this, we would like to present a brief justification for paying attention to these admittedly very special situations by putting them into a certain general perspective along the lines of [**19**, §7], where a more detailed discussion and appropriate references can be found.

The term 'elliptic dynamics' is often, although not universally, used to denote the study of recurrent (conservative) behavior in dynamical systems at the other end of the spectrum from 'chaotic'. While, in everyday usage, the opposite of 'chaotic' would be

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'ordered', in the dynamics context the preferred counterpart is 'stable'. Stable asymptotic behavior in the context of conservative motions is best represented by quasiperiodic motions (translations and linear flows) on tori, which appear, for example, in completely integrable Hamiltonian systems. Such a system splits on most of its phase space into invariant tori which are determined by the fixed values of the action integrals in the action-angle coordinates. On each torus the motion is linear in the angle coordinates with the frequency vector determined by the action vector (see [**29**, §1.5] for an introductory discussion). It is quite remarkable that, under a suitable non-degeneracy assumption, many (in fact, most in terms of phase volume) invariant tori survive slightly distorted under any small Hamiltonian perturbation of the system. This is the principal content of KAM theory to which Michel Herman made major contributions [**21**]. This prevalence of stability which also appears in other contexts (see, e.g., [**20**]) is, of course, a major manifestation of elliptic phenomena.

Stability appears when the linear model, the translation (or rotation if multiplicative notations are used) of the torus $\mathbb{T}^k$,

$$T_\alpha x = x + \alpha, \quad \mod 1,$$

or the linear flow on $\mathbb{T}^k$,

$$T^t_\omega x = x + \omega t, \quad \mod 1, \quad t \in \mathbb{R},$$

is of a *Diophantine* type, i.e. the translation vector α or the frequency vector ω is not too well approximated by rationals. A typical Diophantine condition in the discrete-time case is

$$|\langle \alpha, m\rangle - n| > C\|m\|^{-s}$$

for any $m \in \mathbb{Z}^k$, $n \in \mathbb{Z}$ and some positive numbers C and s independent of m and n. Similarly, in the continuous-time case, one has, for some positive C and s and any $m \in \mathbb{Z}^k$,

$$|\langle \omega, m\rangle| > C\|m\|^{-s}.$$

The opposite case of exceptionally good rational approximation of α or ω is usually called *Liouvillean*. The Liouvillean behavior is associated with instability: circle diffeomorphisms with Liouvillean rotation numbers usually have singular invariant measures and, hence, are conjugate to the corresponding rotations via non-differentiable continuous maps, time changes of linear flows of tori with Liouvillean frequency vectors often and, in fact, typically exhibit unexpected ergodic properties [**10**], the invariant tori of completely integrable Hamiltonian systems with Liouvillean frequency vectors usually do not survive after perturbations (or may survive with greatly distorted dynamics, see [**23**]).

This leads to another possible opposite to chaotic behavior. Since the latter is associated with the exponential growth of orbit complexity with time (positive entropy, exponential growth of the number of periodic orbits, etc.; see, e.g., [**19**, §6]) one may consider the slow growth of various characteristics of such complexity as a hallmark of elliptic dynamics. Note that the translations and linear flows of a torus are isometries; hence, there is no divergence of orbits and, in a certain sense, no growth of orbit complexity with time at all. In the Diophantine case, this behavior is often stable under small perturbations of

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the system while in the Liouvillean case perturbations of the linear models often produce substantive qualitative effects but only after long periods of time. To understand what type of effects may result from this instability, a constructive approach is quite useful. While there is no chance of describing all possible effects which appear in nonlinear perturbations of Liouvillean behavior, the variety of possibilities is astonishing. In a number of cases, such perturbations provide the only known way to exhibit certain types of dynamical behavior, e.g. ergodic behavior accompanied by only finitely many periodic points on such simple manifolds as the two-dimensional disc or two-dimensional sphere. See §3.1 for a more detailed discussion of those cases.

2. *An overview of the approximation by conjugation method*

2.1. *General scheme.* Let M be a differentiable manifold with a non-trivial smooth circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$ preserving a smooth volume. Every smooth S^1 action preserves a smooth volume ν which can be obtained by taking any volume μ and averaging it with respect to the action: $\nu = \int_0^1 (S_t)_*\mu\, dt$. Similarly $\mathcal{S}$ preserves a smooth Riemannian metric on M obtained by averaging of a smooth Riemannian metric.

When this does not cause an ambiguity, we will often call the elements S_α of the circle action $\mathcal{S}$ 'rotations', correspondingly 'rational' and 'irrational', for rational or irrational values of α.

Denote by C_q the subgroup of S^1 with q elements, i.e. the qth roots of unity. Assuming, without loss of generality, that the action is effective (no element acts as identity), we see that for an open dense subset of M the stationary subgroup is trivial. Using the standard properties of isometries, one sees that the set F of fixed points of the action is a smooth submanifold of M. The fixed point set F_q of the map $S_{1/q}$ (or, equivalently, of the action of C_q) is also a smooth submanifold.

Volume-preserving maps with various interesting, often surprising, topological and ergodic properties are obtained as limits of volume-preserving periodic transformations:

$$f = \lim_{n\to\infty} f_n, \quad \text{where } f_n = H_n S_{\alpha_{n+1}} H_n^{-1} \tag{2.1}$$

with $\alpha_n = p_n/q_n \in \mathbb{Q}$ and

$$H_n = h_1 \circ \cdots \circ h_n, \tag{2.2}$$

where every h_n is a volume-preserving diffeomorphism of M that satisfies

$$h_n \circ S_{\alpha_n} = S_{\alpha_n} \circ h_n. \tag{2.3}$$

In certain versions of the method, the diffeomorphisms h_n are chosen not preserving the volume but distorting it in a controllable way: this, for example, is the only interesting situation when M is the circle (see, e.g., [**29**, §12.6]).

Usually at step n, the diffeomorphism h_n is constructed first and α_{n+1} is chosen afterwards close enough to α_n to guarantee the convergence of the construction. For example, it is easy to see that for the limit in (2.1) to exist in the C^∞ topology, it is largely sufficient to ask that

$$|\alpha_{n+1} - \alpha_n| \le \frac{1}{2^n q_n \|H_n\|_{C^n}}. \tag{2.4}$$

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The power and fruitfulness of the method depend on the fact that the sequence of diffeomorphisms f_n is made to converge while the conjugacies H_n diverge often 'wildly' albeit in a controlled (or prescribed) way. The dynamics of the circle actions and of their individual elements is simple and well understood. In particular, no element of such an action is ergodic or topologically transitive, unless the circle action itself is transitive, i.e. $M = S^1$. To provide interesting asymptotic properties of the limit, typically the successive conjugacies spread the orbits of the circle action $\mathcal{S}$ (and, hence, also those of its restriction to the subgroup C_q for any sufficiently large q) across the phase space M making them almost dense or almost uniformly distributed or approximate another type of interesting asymptotic behavior. Due to the high speed of the convergence, this remains true for sufficiently long orbit segments of the limit diffeomorphism. To guarantee an appropriate speed of approximation, extra conditions on the convergence of approximations in addition to (2.4) may be required.

There are many variations of the construction within this general scheme. In the subsequent sections, we describe several representative versions, each addressing a problem which has not been treated before. Presenting each version to the last detail and in the greatest generality would be quite tedious. Accordingly, we often choose to describe routine steps only briefly and sometimes treat representative special cases indicating generalizations afterwards.

2.2. *Almost all versus all orbits.* In different versions of the approximation by conjugation method, one may control the asymptotic behavior of almost all orbits with respect to the invariant volume or that of all orbits. Somewhat imprecisely, we will call these versions ergodic and topological.

Ergodic constructions deal with measure-theoretic (ergodic) properties with respect to a given invariant volume, such as the number of ergodic components (in particular ergodicity), rigidity, weak mixing, mixing and other spectral properties. Topological constructions deal with minimality, the number of ergodic invariant measures (e.g. unique ergodicity) and their supports, the presence of particular invariant sets, and so on. The original paper [**1**] dealt only with ergodic constructions. A topological version was announced in [**25**] (see also [**4**]) and independently developed in [**7**].

Control over the behavior of the orbits of approximating periodic diffeomorphisms f_n in (2.1) on the nth step of the construction is typically provided by taking an invariant under an S_{α_n} (and, hence, under S_{1/q_n}) collection of 'kernels', usually smooth balls, and redistributing them in the phase space in a prescribed fashion (also S_{1/q_n} invariant). In ergodic constructions, one requires the complement to the union of the kernels to have a small volume and, hence, most orbits of $\mathcal{S}$ (and, consequently, of any finite subgroup C_q for a sufficiently large q) to spend most of the time inside the kernels. In the topological versions, the kernels need to be chosen in such a way that *every* orbit of $\mathcal{S}$ spends most of its time inside the kernels. This requires more care and certain attention to the geometry of orbits. See, for example, Figures 1 and 2 in Section 3.4.

A natural way of selecting the kernels, their intended images and constructing a map h_n satisfying (2.3) is by taking a fundamental domain Δ for S_{α_n} (or, equivalently, for S_{1/q_n}) choosing kernels and images inside Δ, constructing a diffeomorphism of Δ to itself

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identical near its boundary which sends kernels into their intended images and extending the map to the images S_{k/q_n}, $k = 1, \dots, q_n - 1$ by commutativity. This method, in particular, is used in the construction of a diffeomorphism conjugate to a rotation, see Question 2.4, as well as later in this paper in the proofs of Theorems 3.3, 4.1 and 5.1. However, in certain cases, for example to achieve weak mixing, it is necessary to use more general constructions.

2.3. *Closure of conjugates.* By controlling the initial steps of the construction, one can keep the final diffeomorphism f within any given C^∞ neighborhood of a given element S_α of the action $\mathcal{S}$. On the other hand, by applying a fixed diffeomorphism g first, the construction can be carried out in a neighborhood of any diffeomorphism conjugate to an element of the action. Thus, a closure $\mathcal{A}$ of diffeomorphisms of the form gS_tg^{-1} in, say, C^∞ topology is a natural ambient space for considering what a construction of this type can produce. This was first noted in [**1**, §7] in connection with ergodic properties with respect to the invariant volume and was used in [**7**] to control the topological properties.

If the action $\mathcal{S}$ has fixed points, it is natural to consider a smaller space $\mathcal{A}_F \subset \mathcal{A}$, namely the closure of gS_tg^{-1} where the conjugacy g fixes every point of the fixed point set F of the action $\mathcal{S}$. Obviously, for $f \in \mathcal{A}_F$, every point of F is also fixed. There does not seem to be much difference between the dynamical properties which may appear in the whole space $\mathcal{A}$ and in $\mathcal{A}_F$. In §3.1 we will discuss the situation for the standard two-dimensional examples: the disc, the annulus and the sphere with the S^1 action by rotations. At this point, we would also like to introduce *restricted* spaces of conjugacy closures, namely, for $\alpha \in \mathbb{R}/\mathbb{Z}$, we define the spaces $\mathcal{A}_\alpha$ and $\mathcal{A}_{F,\alpha}$ as the closure in C^∞ topology of diffeomorphisms of the form $gS_\alpha g^{-1}$ where g is, correspondingly, an arbitrary C^∞ diffeomorphism or a C^∞ diffeomorphism fixing every point of the set F. Since, for any $p/q \in \mathbb{Q}$ and any $f \in A_{p/q}$, one has $f^q = \mathrm{Id}$, only spaces A_α for irrational α's are of interest.

Let us describe a classical example showing that the space $\mathcal{A}$ may be quite different from the union of $\mathcal{A}_\alpha$.

2.3.1. *The horocycle flow.* Consider the group $SL(2, \mathbb{R})$ and three one-parameter subgroups: the standard rotations

$$R_t = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix};$$

the diagonals

$$G_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix};$$

and the upper-triangular unipotents

$$H_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The commutation relation

$$G_t R_s G_{-t} = \begin{pmatrix} \cos 2\pi s & e^{2t} \sin 2\pi s \\ -e^{-2t} \sin 2\pi s & \cos 2\pi s \end{pmatrix}$$

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implies

$$\lim_{t\to\infty} G_t R_{e^{-2t}/2\pi} G_{-t} = H_1. \tag{2.5}$$

Now let Γ be a cocompact lattice in $SL(2,\mathbb{R})$. Since left translations on any group commute with right translations, we can consider the actions of the one-parameter subgroups described earlier on the compact manifold $M = SL(2,\mathbb{R})/\Gamma$. In the case when Γ has no elliptic elements, M can be identified with the unit tangent bundle of the compact surface $V = SO(2)\backslash SL(2,\mathbb{R})/\Gamma$ of constant negative curvature and the right actions of $\{G_t\}$ and $\{H_t\}$ are called the *geodesic flow* and *horocycle flow* on V correspondingly [**29**, §§5.4 and 17.5]. We will denote the actions of R_t, G_t and H_t by right translations on $SL(2,\mathbb{R})/\Gamma$ by r_t, g_t and h_t correspondingly. The first of these is, of course, an S^1 action, while every non-trivial element of the last one is uniquely ergodic, mixing of all orders and has a countable Lebesgue spectrum (see [**32**, §1.4e] and references thereof). It follows from (2.5) that

$$\lim_{t\to\infty} g_t r_{e^{-2t}/2\pi} g_{-t} = h_1.$$

Thus, $h_1 \in \mathcal{A}$: it is the limit of conjugates of certain elements of the circle action which converge to zero.

PROPOSITION 2.1. *$h_1 \notin \mathcal{A}_\alpha$ for any $\alpha \in S^1$.*

Proof. Obviously, $\mathcal{A}_0 = \{\mathrm{Id}\}$; thus, we need to prove the statement for $\alpha \neq 0$. For any volume preserving diffeomorphism f of a compact manifold homotopic to identity, one defines the *rotation class* $\rho(f)$ as an element of the first homology group of the direct product $M \times S^1$. It is constructed as the asymptotic cycle of the suspension flow of f [**29**, §14.7b]. The rotation class is continuous in C^0 topology and equivariant under the conjugacy. Thus, $\rho(h_1) = 0$. On the other hand, the projection of $\rho(r_\alpha)$ to $H_1(M,\mathbb{R})$ has the form $\alpha\rho_0$, where ρ_0 is the non-zero integer homology class realized by the embedding of $SO(2)$ into $SL(2,\mathbb{R})/\Gamma$. Since the action of the diffeomorphism group by conjugacies respects the integer structure in this projection, the rotation classes of elements of conjugates of r_α are separated from zero and, hence, for any $f \in \mathcal{A}_\alpha$, $\rho(f) \neq 0$. □

The following question remains open.

PROBLEM 2.2. *Does there exist a circle action $\mathcal{S}$ on a compact manifold and α such that the corresponding space $\mathcal{A}_\alpha$ contains a mixing transformation?*

In §6 we will give a positive answer for a version of this question with a $\mathbb{T}^3$ action instead of an S^1 action. We will also provide more examples of circle actions with mixing transformations in the unrestricted space $\mathcal{A}$.

2.4. *Generic and non-generic constructions.* The development of the approximation by conjugation method was motivated by several natural questions about the ergodic properties of volume-preserving diffeomorphisms. Two characteristic examples of such questions are given below, both answered positively in [**1**].

QUESTION 2.3. *Does there exist an ergodic area-preserving diffeomorphism of the two-disc $\mathbb{D}^2$?*

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QUESTION 2.4. *Does there exist a diffeomorphism of a compact manifold* M, $\dim M > 1$ *preserving a smooth volume and measurably conjugate to an irrational rotation of the circle?*

These positive answers looked somewhat surprising at the beginning: in the former case because, usually near an elliptic fixed point, there are many invariant curves making the map highly non-ergodic; and, in the latter, because the eigenfunctions of any such transformation must be highly discontinuous. While ergodic diffeomorphisms of a disc without genuine elliptic points which exhibit much stronger stochastic properties than ergodicity were found later [**26**], the original examples in [**1**] do have elliptic points and are, in fact, small perturbations of (non-ergodic) rotations of the disc.

Constructions answering Questions 2.3 and 2.4 are characteristic for two different versions of approximation by the conjugation method.

2.4.1. *Control at some time scales.* In constructions of the first kind, it is sufficient *to control the behavior of approximating and, hence, the resulting diffeomorphisms on a series of growing but unrelated time scales*. The approximate pictures may look quite whimsical (see, e.g., the original weak mixing construction in [**1**, §5] and a modern version in [**17**]) but as long as a diffeomorphism $f \in \mathcal{A}$ is close enough to conjugates of rotations appearing in such pictures, the property is guaranteed. In fact, for a proof of the genericity in $\mathcal{A}$ of a property exhibited by a construction of this sort, no actual inductive construction is needed. One just needs to show that an approximate picture at each scale appears for an open dense subset of conjugates of rotations. If appearance in an approximate picture at infinitely many growing scales guarantees the property, then by the Baire category theorem the property holds for a dense G_δ subset of $\mathcal{A}$.

Ergodicity is generic in $\mathcal{A}$: in the case of the disc with S^1 acting by rotations around the center, this, in particular, implies that invariant circles which are very hard to avoid from the classical standpoint (twists, frequency locking, etc.) can, in fact, be destroyed easily if one adopts a less conventional point of view. We will discuss disc diffeomorphisms in detail in §3.1.

Beyond ergodicity, the basic ergodic properties which are generic in $\mathcal{A}$ are the arbitrary fast cyclic approximation (which implies ergodicity and also simple singular spectrum) [**1**, Theorem 7.1] and weak mixing. Those properties are among the generic properties for measure-preserving automorphisms of Lebesgue space in weak topology [**30, 27**] which also possess some more refined generic properties such as the mutual singularity of the maximal spectral type and all its convolutions [**27**] or the existence of roots of all orders [**31**]. The first of these as well as many others follows from the genericity of the arbitrary fast periodic approximation of arbitrary type (not just cyclic) [**27**].

We will later discuss open questions and difficulties related with establishing the genericity of such properties in $\mathcal{A}$ (§7.2).

In this paper we present a new example of a generic construction in §3. A proper extension of this construction from the basic two-dimensional examples (the disc, the annulus and the sphere) to manifolds (with or without boundaries) with circle actions provides a common generalization of the genericity of ergodicity in the general case of

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non-trivial circle actions [**1**] and the genericity of minimality and unique ergodicity in the case of fixed-point free actions on closed manifolds [**7**] (see §3.6).

2.4.2. *Control at all times.* In the constructions of the second kind including the one answering Question 2.4, approximations at different steps of the construction are linked and, hence, in principle *the asymptotic behavior of the resulting diffeomorphism is controlled for all times*. Such a construction is never generic in $\mathcal{A}$ or in any of the other spaces introduced in §2.3.

Constructions of this kind appear most naturally when the resulting diffeomorphism is constructed to be measure-theoretically conjugate to a map of a particular kind such as a rotation in Question 2.4 or a translation on a torus of higher dimension [**1**, §6]. These constructions also appear in topological situations, for example in producing minimal diffeomorphisms with more than one ergodic measure [**40**]. Our construction in §4 is non-generic although quite natural in the sense described at the end of §2.2, and is similar, in some respects, to the construction of a diffeomorphism conjugate to a rotation. One may also extend the construction from [**40**] to obtain extra invariant measures for the case of actions with fixed points or on manifolds with boundaries (such as four or more ergodic measures on the disc; see §3.7). The construction in §5 is also non-generic and is of a similar type.

A new and quite different type of non-generic constructions are those which produce mixing diffeomorphisms. Presently we need an action of $\mathbb{T}^3$ to carry out such a construction. This construction is based on the combination of approximation by conjugation and the methods developed by the first author for constructing mixing time changes for linear flows on $\mathbb{T}^k$, $k \geq 3$ [**8, 9**]. The latter may be interpreted as a 'multiple frequencies' version of the approximation by conjugation method.

We would like to point out that Michel Herman was a great master in the use of Baire category theorem with results which were often striking (it was his 'favorite theorem' according to Albert Fathi) and, in particular, he very effectively used generic constructions in the context of the method of approximation by conjugation. In his published work though, he did not make any direct use of non-generic constructions in the context of the approximation by conjugation method. It was, however, his original insight which stimulated the first author, his last PhD student, to develop the non-generic constructions of mixing diffeomorphisms.

3. *Diffeomorphisms of the disc, the annulus, and the sphere with exactly three ergodic measures*

3.1. *Dynamics and invariant measures of disc diffeomorphisms.* The most basic fact about dynamics of diffeomorphisms of the disc $\mathbb{D}^2$ is the presence of a fixed point (the Brouwer Fixed Point Theorem). Furthermore, by the Brouwer translation theorem for an area-preserving map, there is a fixed point inside the disc.

As early as 1930, Shnirelman [**39**], in an attempt to prove a 'converse' to the Brouwer theorem, constructed an example of a topologically transitive homeomorphism h of $\mathbb{D}^2$

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which in somewhat modified polar coordinates (ρ, θ) has the form

$$h(\rho, \theta) = (\rho + g(\theta), \theta + \alpha). \tag{3.1}$$

(Here, for example, one can take $\rho = -(\tan(r/\pi))^{-1}$, where r is the usual radius.) Shnirelman was mistaken in his assertion that every orbit inside the disc other than the origin was dense. For maps of the form (3.1) the existence of non-dense orbits in the interior can be easily shown. (Besicovitch who repeated this mistake later corrected it in [**2**].) It is also not too difficult to see that, for any homeomorphism of the disc fixing the origin, there are non-dense *semiorbits* in the interior. The existence of non-dense *orbits* is much more difficult and it was proven only recently (45 years after the Besicovitch paper) by Le Calvez and Yoccoz [**5**].

It is true, however, that the only ergodic invariant measures in the Shnirelman example are the δ-measure at the origin and the Lebesgue measure on the boundary and that, for almost every point with respect to the two-dimensional Lebesgue measure, the asymptotic distribution exists and is exactly the average of these two measures. A careful study of the Shnirelman–Besicovitch examples was one of the sources which led Anosov and the second author to the invention of the approximation by conjugation method [**1**, Introduction].

Note that, unlike Shnirelman–Besicovitch maps, any area-preserving diffeomorphism of the disc has at least three ergodic invariant measures: the δ-measure at a fixed point inside the disc, a measure supported at the boundary plus any ergodic component of the area. Naturally, the only possibility to achieve this lower bound of three ergodic invariant measures appears when there is only one fixed point inside, only one measure supported by the boundary and the map is ergodic with respect to the area (this is not sufficient though).

Now consider the closure spaces $\mathcal{A}$ and $\mathcal{A}_F$ of conjugates of rotations described in §2.3. Every diffeomorphism $f \in \mathcal{A}_F$ preserves the origin, the boundary and the volume. In this section we will show that the minimal number of three ergodic invariant measures is achieved and is even generic in a properly defined sense. Moreover, in our examples (albeit not generically), any point in the interior whose orbit closure is not dense (such points exist by [**5**]) has an asymptotic distribution which is either the δ measure at the origin, or Lebesgue measure on the boundary, in contrast with the Shnirelman–Besicovitch maps of the form (3.1).

Any map $f \in \mathcal{A}$ preserves the boundary and, hence, has a rotation number on the boundary. By continuity, for $f \in \mathcal{A}_\alpha$ this rotation number is equal to α. Our examples belong to spaces $\mathcal{A}_\alpha$ with α irrational but very well approximated by rationals (Liouvillean numbers). This is the only possibility for ergodicity with respect to the area in a space $\mathcal{A}_\alpha$. For, if α is rational, then every $f \in \mathcal{A}_\alpha$ is periodic. If α is Diophantine, the restriction of every $f \in \mathcal{A}_\alpha$ to the boundary of $\mathbb{D}^2$ is C^∞ conjugate to the rotation R_α by the Herman–Yoccoz theorem [**20, 41**]. Hence, as Herman proved in his unpublished work, there are smooth f-invariant circles arbitrarily close to the boundary and, hence, ergodicity or topological transitivity is not possible.

The only other way ergodicity with respect to the area may appear in the whole space $\mathcal{A}$ would be for a map with a rational rotation number p/q on the boundary which is a limit of conjugates of rotations with variable rotation numbers converging to p/q. While we

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do not know whether such ergodic examples exist, their structure must be quite different since, due to recent results in Le Calvez [**6**] any such map has infinitely many periodic points (and, hence, infinitely many ergodic invariant measures) in the interior†.

The following problem is related to the previous discussion as well as to Problem 2.2.

PROBLEM 3.1. *Does there exist an area-preserving mixing diffeomorphism of the disc* $\mathbb{D}^2$ *with zero metric (or topological) entropy?*

Remark 3.2. Note that such examples exist on any orientable surface of genus $g \geq 1$ with or without holes. They are obtained as time-one maps of mixing flows first constructed by Kočergin [**33**]. It is not known whether there are examples on the sphere or the cylinder.

3.2. *Statement and description of the construction.* Note that the pictures of rotations and conjugacies are essentially identical on the disc, the annulus $[0, 1] \times S^1$ and the sphere S^2. In all three cases, one has polar coordinates (ρ, θ) and the rotations are given by $S_t(\rho, \theta) = (\rho, \theta + t)$. Here $0 \leq \rho \leq 1$ and θ is the cyclic coordinate of period 1. All the conjugacies involved in our constructions will coincide with the identity near $r = 0$ and $r = 1$; hence, the difference between the three cases will be insignificant. Accordingly, our drawings will refer to the case of the annulus which is easier to represent graphically. Let us call the Lebesgue measure on the manifold, the δ-measures at the fixed points of the rotations and Lebesgue measures on the boundary components the *natural measures*.

THEOREM 3.3. *Let* M *be* $\mathbb{D}^2$, $[0, 1]\times S^1$ *or* S^2, *and* S_t *be the standard action by rotations. There exists a* C^∞ *diffeomorphism* $f : M \to M$ *that has exactly three ergodic invariant measures, namely the natural measures on* M. *Furthermore, diffeomorphisms with this property form a residual set in the space* $\mathcal{A}'$: *the closure in the* C^∞ *topology of the conjugates of rotations with conjugacies fixing the fixed points of* $\mathcal{S}$ *and every point of the boundary.*

We will call the boundary components and fixed points of the rotations *singularities*. At step $n - 1$, h_{n-1} is constructed and then α_n is chosen. We assume that, after step $n - 1$, there are two neighborhoods of the singularities B^l_{n-1} and B^r_{n-1} on which $H_{n-1} \equiv \mathrm{Id}$. At step n, we define three collections of closed sets $\delta^l_n = \{\delta^l_{n,i}, i = 1, \ldots, l_n\}$, $\delta^l_{n,i} \subset B^l_{n-1}$, $\delta^r_n = \{\delta^r_{n,i}, i = 1, \ldots, l_n\}$, $\delta^r_{n,i} \subset B^r_{n-1}$ and $\xi_n = \{\xi_{n,i}, i = 1, \ldots, l_n\}$, the collection of kernels. These collections will have the property that the union of their atoms capture an increasingly large proportion of *any* rotation orbit. The map h_n is then chosen so that the images of the kernels $H_n(\xi_{n,i})$ have small diameters, and such that $H_n \equiv \mathrm{Id}$ on two smaller neighborhoods of the singularities B^l_n and B^r_n that still contain all elements of δ^l_n and δ^r_n correspondingly. At the end of step n, α_{n+1} is chosen to insure convergence of the construction. We will use polar coordinates (ρ, θ) in our calculations.

3.3. *Properties of the collections* ξ_n, δ^r_n *and* δ^r_n. The following proposition gives a description of the configuration represented by Figure 1. Note that the parameter η_n

† This has been pointed out to us by the referee of the paper.

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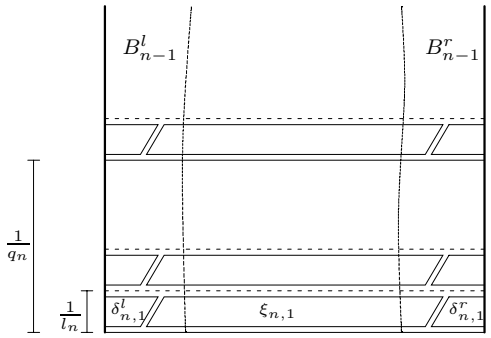


FIGURE 1. The collections ξ_n, δ^l_n, and δ^r_n.

describing the precision of the configuration can be made arbitrarily small since we are free to take l_n arbitrarily large.

PROPOSITION 3.4. *Assuming that neighborhoods B^l_{n-1} and B^r_{n-1} of the boundary components are given and given any $\eta_n > 0$, there exists an integer l_n multiple of q_n, and three collections of disjoint closed sets on M, δ^l_n, δ^r_n and ξ_n each one containing l_n elements and satisfying the following conditions.*

(1) *δ^l_n, δ^r_n and ξ_n are invariant by S_{1/l_n} and $\xi_{n,1} \subset (0,1) \times (0, 1/l_n)$, $\delta^l_{n,1} \subset [0, 1/n] \times (0, 1/l_n)$ and $\delta^r_{n,1} \subset [1 - 1/n, 1] \times (0, 1/l_n)$.*

(2) *$\mu(\xi_{n,1}) \le \eta_n$ and $\mu(\cup \xi_{n,i}) \ge 1 - \eta_n$.*

(3) *$\delta^l_{n,i} \subset B^l_{n-1}$ and $\delta^r_{n,i} \subset B^r_{n-1}$.*

(4) *Any circle $V_\rho = \{\rho\} \times S^1$ intersects the three collections in the following way: there exists a constant $c_n(\rho) \in [0, 1]$ with $c_n(1/2) = 1$, $c_n(0) = c_n(1) = 0$ and such that for any $1 \le i \le l_n$, $V_\rho \cap \xi_{n,i}$ is an empty set, a point or a segment and so are $V_\rho \cap \delta^l_{n,i}$ and $V_\rho \cap \delta^r_{n,i}$; moreover, we require that*

$$(1 - \eta_n)\frac{c_n(\rho)}{l_n} \le |V_\rho \cap \xi_{n,i}| \le (1 + \eta_n)\frac{c_n(\rho)}{l_n},$$

and if $\rho \in [0, 1/2]$, $V_\rho \cap \delta^r_n = \emptyset$ while

$$(1 - \eta_n)\frac{(1 - c_n(\rho))}{l_n} \le |V_\rho \cap \delta^l_{n,i}| \le (1 + \eta_n)\frac{(1 - c_n(\rho))}{l_n},$$

and if $\rho \in [1/2, 1]$, $V_\rho \cap \delta^l_n = \emptyset$ while

$$(1 - \eta_n)\frac{(1 - c_n(\rho))}{l_n} \le |V_\rho \cap \delta^r_{n,i}| \le (1 + \eta_n)\frac{(1 - c_n(\rho))}{l_n}.$$

3.4. *Properties of the successive conjugating diffeomorphisms H_n.* Using the collections of sets introduced in §3.3, we will state and prove some sufficient conditions on the sequences α_n and H_n that guarantee the convergence in (2.1) to a diffeomorphism having exactly three invariant ergodic measures as in Theorem 3.2. We require that the collections in Proposition 3.4 be constructed with $\eta_n \to 0$ and we will extract later more information on how fast η_n should converge to 0.

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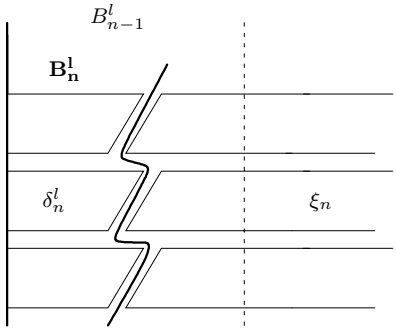


FIGURE 2. $B^l_n \subset B^l_{n-1}$.

LEMMA 3.5. *If $\alpha_n = p_n/q_n$ is a sequence of rationals and H_n is a sequence of diffeomorphisms of M satisfying (2.2) and (2.3) and if there exists a sequence $\epsilon_n \to 0$ such that, for every n large enough, we have the following:*

(1) *$H_n \equiv \mathrm{Id}$ on two open neighborhoods B^l_n of $\{0\} \times S^1$ and B^r_n of $\{1\} \times S^1$ satisfying*

$$\delta^j_{n,i} \subset B^j_n \subset B^j_{n-1}, \quad j = r, l,\ i = 1, \ldots, l_n;$$

(2) *for any $i = 1, \ldots, l_n$,*

$$\mathrm{diam}(H_n(\xi_{n,i})) \le \epsilon_n;$$

(3) *$|\alpha_{n+1} - \alpha_n| \le 1/2^n q_n \|H_n\|^n_{C^n}$ and $q_{n+1} \ge n l_n$;*

then $H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$ converges in the C^∞ topology to a diffeomorphism f having exactly three invariant ergodic measures: the Lebesgue measure μ on M and the measures δ^l and δ^r supported on the singularities corresponding to Lebesgue measures on $\{0\} \times S^1$ and $\{1\} \times S^1$, respectively.

Remark 3.6. In §3.5, we will have to construct h_n so that H_n fulfills conditions (1) and (2). Condition (1) is consistent with (3) in Proposition 3.4 and condition (2) will be consistent with (2) in Proposition 3.4 if, at step n, we choose $\eta_n \le (\epsilon_n / \|H_{n-1}\|_{C^1})^3$.

3.4.1. *Convergence.* Condition (3) of Lemma 3.5 insures the convergence of $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$ in the C^∞ topology to a volume-preserving diffeomorphism f and the choice of α_{n+1} is made only after h_n is constructed. It also insures the closeness between f^m and f^m_n as $m \le q_{n+1}$ (cf. §3.4.3). The last condition, $q_{n+1} \ge n l_n$, guarantees that, for any point in M, a $(1 - 1/\eta_n)$-proportion of its q_{n+1} (vertical) orbit under $S_{\alpha_{n+1}}$ is captured by the atoms of ξ_n, δ^l_n and δ^r_n which is crucial in the estimations of ergodic averages (cf. (3.12) and (3.13)).

Proof of Lemma 3.5. For $\varphi \in C^\infty(M, \mathbb{C})$, we use the notation

$$\hat{\varphi} = \int_M \varphi(u, v)\, du\, dv, \quad \hat{\varphi}^l = \int_{S^1} \varphi(0, v)\, dv, \quad \hat{\varphi}^r = \int_{S^1} \varphi(1, v)\, dv.$$

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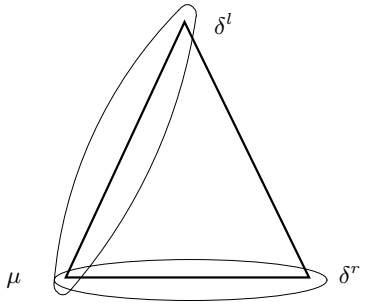


FIGURE 3. The simplex of invariant measures and possible averages for a large m_n.

3.4.2. *Criterion for the existence of exactly three ergodic invariant measures.* Fix $\varphi \in C^\infty(M, \mathbb{C})$ and $z \in M$ and denote by $S_{f,m}\varphi(z)$ the Birkhoff sums

$$S_{f,m}\varphi(z) = \varphi(z) + \varphi(f(z)) + \cdots + \varphi(f^{m-1}(z)).$$

Lemma 3.5 clearly follows if we prove that there exists a sequence of functions $\varepsilon_n \to 0$, a sequence $m_n \to \infty$ and a sequence $c_n(z) \in [0, 1]$ such that, for either $j = l$ or $j = r$, we have

$$\begin{aligned} c_n(z)\hat{\varphi} + (1 - c_n(z))\hat{\varphi}^j - \varepsilon_n \le \frac{S_{f,m_n}\varphi(z)}{m_n} \\ \le c_n(z)\hat{\varphi} + (1 - c_n(z))\hat{\varphi}^j + \varepsilon_n. \end{aligned} \tag{3.2}$$

3.4.3. *The diffeomorphisms f and $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$.* Lemma 3.7 follows from condition (3) of Lemma 3.5.

LEMMA 3.7. *For any $m \le q_{n+1}$, we have*

$$\|f^m - f_n^m\|_{C^1} \le \frac{1}{2^n}.$$

Proof. Since h_{n+1} commutes with $S_{\alpha_{n+1}}$, we have that $f_n = H_{n+1} \circ S_{\alpha_{n+1}} \circ H_{n+1}^{-1}$; hence,

$$\begin{aligned} \|f_{n+1}^m - f_n^m\|_{C^1} &\le \|H_{n+1}\|_{C^1}\|H_{n+1}\|_{C^2} m|\alpha_{n+2} - \alpha_{n+1}| \\ &\le \frac{m}{2^{n+1}q_{n+1}} \end{aligned}$$

by Lemma 3.5 (3) of which yields the conclusion of the lemma. □

As a consequence, (3.2) will follow (with some $\varepsilon_n' \to 0$) if we pick $m_n = q_{n+1}$ and prove that, for a sequence $\varepsilon_n \to 0$, we have

$$c_n\hat{\varphi} + (1 - c_n)\hat{\varphi}^j - \varepsilon_n \le \frac{S_{f_n,q_{n+1}}\varphi(z)}{q_{n+1}} \le c_n\hat{\varphi} + (1 - c_n)\hat{\varphi}^j + \varepsilon_n. \tag{3.3}$$

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3.4.4. *Control of a large proportion of any orbit.* The point $z \in M$ being fixed, we write it as $H_n(\rho_0, \theta_0)$ so that

$$f_n^m(z) = H_n \circ S_{\alpha_{n+1}}^m(\rho_0, \theta_0).$$

For a set $A \subset M$, we define

$$L_{q_{n+1}}(A) := \{0 \le m \le q_{n+1} - 1 / S_{\alpha_{n+1}}^m(\rho_0, \theta_0) \in A\}.$$

The following lemma is a straightforward consequence of Proposition 3.4 (4).

LEMMA 3.8. *We have, for every* $1 \le i \le l_n$,

$$(1 - \eta_n)\frac{q_{n+1}c_n(\rho_0)}{l_n} - 1 \le \#L_{q_{n+1}}(\xi_{n,i}) \le (1 + \eta_n)\frac{q_{n+1}c_n(\rho_0)}{l_n} + 1, \tag{3.4}$$

and

$$(1 - \eta_n)\frac{q_{n+1}(1 - c_n(\rho_0))}{l_n} - 1 \le \#L_{q_{n+1}}(\delta_{n,i}^j) \le (1 + \eta_n)\frac{q_{n+1}(1 - c_n(\rho_0))}{l_n} + 1, \tag{3.5}$$

where $j = l$ *if* $\rho_0 \le \frac{1}{2}$ *and* $j = r$ *if* $\rho_0 \ge \frac{1}{2}$.

In particular, the number of points among the first q_{n+1} *iterates of* (ρ_0, θ_0) *under* $S_{\alpha_{n+1}}$ *that are not considered in the lemma is less than* $2\eta_n q_{n+1} + 4l_n$.

3.4.5. *Estimating* φ *on* $H_n(\xi_n)$. Proposition 3.4 and the fact that S_t and H_n preserve the area μ imply that $1 - \eta_n \le l_n \mu(H_n(\xi_{n,i})) < 1$ for any $i \le l_n$. Hence, as a consequence of condition (2) in Lemma 3.5, there exists a sequence $\epsilon_{1,n} \to 0$ such that, for any $(\rho, \theta) \in H_n(\xi_{n,i})$, we have

$$(1 - \epsilon_{1,n}) \int_{H_n(\xi_{n,i})} \varphi(u, v)\, du\, dv \le \frac{1}{l_n}\varphi(\rho, \theta) \le (1 + \epsilon_{1,n}) \int_{H_n(\xi_{n,i})} \varphi(u, v)\, du\, dv. \tag{3.6}$$

On the other hand property (2) in Proposition 3.4 and the fact that H_n preserves μ also implies that there exists $\epsilon_{2,n} \to 0$ such that

$$(1 - \epsilon_{2,n})\hat{\varphi} \le \sum_{i=1}^{l_n} \int_{H_n(\xi_{n,i})} \varphi(u, v)\, du\, dv \le (1 + \epsilon_{2,n})\hat{\varphi}, \tag{3.7}$$

where we recall that $\hat{\varphi} = \int_M \varphi(u, v)\, du\, dv$.

3.4.6. *Estimating* φ *on* $H_n(\delta_n^l)$ *and* $H_n(\delta_n^r)$. As a consequence of Proposition 3.4 (1), there exists a sequence $\epsilon_{3,n}$ such that, for any $(\rho, \theta) \in \delta_{n,i}^l$, we have that

$$(1 - \epsilon_{3,n})\varphi\left(0, \frac{i}{l_n}\right) \le \varphi(\rho, \theta) \le (1 + \epsilon_{3,n})\varphi\left(0, \frac{i}{l_n}\right). \tag{3.8}$$

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On the other hand there exists a sequence $\epsilon_{4,n} \to 0$ such that

$$(1-\epsilon_{4,n})\hat{\varphi}^l \le \frac{1}{l_n}\sum_{i=1}^{l_n}\varphi\left(0,\frac{i}{l_n}\right) \le (1+\epsilon_{4,n})\hat{\varphi}^l, \tag{3.9}$$

where, as before, $\hat{\varphi}^l = \int_{S^1}\varphi(0,v)\,dv$. Similar equations as (3.8) and (3.9) hold for $(\rho,\theta)\in\delta^r_{n,i}$ and $\hat{\varphi}^r = \int_{S^1}\varphi(1,v)\,dv$.

3.4.7. *Proof of (3.3).* We will assume, for definiteness, that $z = H_n(\rho_0,\theta_0)$ with $\rho_0 \le \frac{1}{2}$, the case $\rho_0 \ge \frac{1}{2}$ being similar. In light of Lemma 3.8 and (3.6)–(3.9), we get the following estimations of

$$\begin{gathered} U_n := \frac{1}{q_{n+1}} S_{f_n,q_{n+1}}\varphi(\rho_0,\theta_0) \\ V_n^1 + V_n^2 + V_n^3 \le U_n \le U_n^1 + U_n^2 + U_n^3 \end{gathered} \tag{3.10}$$

where U_n^3 is a bound on the contribution of the non-controlled points

$$U_n^3 \le 2\left(\epsilon_n + \frac{2l_n}{q_{n+1}}\right)\|\varphi\|, \tag{3.11}$$

and U_n^1 is relative to the points that fall in $H_n(\xi_n)$ for which we use (3.4) of Lemma 3.8 and (3.6) and (3.7) to obtain the following upper bound on their contribution:

$$\begin{aligned} U_n^1 &= \frac{1}{q_{n+1}}\sum_{i=1}^{l_n}[(1+\eta_n)c_n(\rho_0)q_{n+1}+l_n](1+\epsilon_{1,n})\int_{H_n(\xi_{n,i})}\varphi(u,v)\,du\,dv \\ &\le (1+\eta_n)(1+\epsilon_{1,n})(1+\epsilon_{2,n})c_n(\rho_0)\hat{\varphi} + \frac{l_n}{q_{n+1}}\|\varphi\|, \end{aligned} \tag{3.12}$$

and U_n^2 is a bound on the contribution of the points in δ_n^l for which (3.5) of Lemma 3.8 and (3.8) and (3.9) lead to

$$\begin{aligned} U_n^2 &= \frac{1}{q_{n+1}}\sum_{i=1}^{l_n}\left[(1+\eta_n)\frac{(1-c_n(\rho_0))q_{n+1}}{l_n}+1\right](1+\epsilon_{3,n})\varphi\left(0,\frac{i}{l_n}\right) \\ &\le (1+\eta_n)(1+\epsilon_{3,n})(1+\epsilon_{4,n})(1-c_n(\rho_0))\hat{\varphi}^l + \frac{l_n}{q_{n+1}}\|\varphi\|. \end{aligned} \tag{3.13}$$

Recall that α_{n+1} is chosen such that $l_n/q_{n+1} \le 1/n$, hence (3.11)–(3.13) with similar lower bounds for V_n^1, V_n^2 and V_n^3 yield (3.3). This finishes the proof of Lemma 3.5. □

3.5. *Construction of h_n and choice of α_{n+1}.* In this section, we construct h_n and choose α_{n+1} so that the conditions of Lemma 3.5 hold for $H_n = h_1\circ\cdots\circ h_n$.

We fix a sequence $\epsilon_n \to 0$. We assume that the conditions of Lemma 3.5 hold up to step $n-1$ and consider a configuration of collections as in Proposition 3.4 with

$$\eta_n = (\epsilon_n/\|H_{n-1}\|_{C^1})^3. \tag{3.14}$$

Then we have the following proposition.

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PROPOSITION 3.9. *There exists a diffeomorphism h_n of class C^∞ on M and two neighborhoods B_n^l and B_n^r of the singularities with $\delta_n^j \subset B_n^j \subset B_{n-1}^j$, $j = l, r$, such that*

(1) *h_n is volume preserving,*
(2) *$h_n \circ S_{\alpha_n} = S_{\alpha_n} \circ h_n$,*
(3) *$h_n \equiv \mathrm{Id}$ on $B_n^l \cup B_n^r$ and*
(4) *for every $\xi_{n,i} \in \xi_n$, $\mathrm{diam}(h_n(\xi_{n,i})) \leq \epsilon_n / \|H_{n-1}\|_{C^1}$.*

Proof. As explained in Remark 3.6, property (3) is possible due to (3) in Proposition 3.4 and the fact that the unions of the sets in the collections δ_n^j, $j = l, r$ are closed while B_{n-1}^j, $j = l, r$ are open. Condition (2) can be realized in particular with $h_n \equiv \mathrm{Id}$ on the singularities of the fundamental domain $[0, 1] \times [0, 1/q_n]$ due to (1) in Proposition 3.4 and to the fact that l_n is a multiple of q_n. Condition (4) can be realized since $\mu(\xi_{n,i}) \leq \eta_n$ and that we chose in (3.14) $\eta_n = (\epsilon_n / \|H_{n-1}\|_{C^1})^3 = o((\epsilon_n / \|H_{n-1}\|_{C^1})^2)$. □

Checking the conditions of Lemma 3.5. Clearly, conditions (1) and (2) on H_n in Lemma 3.5 follow, respectively, from (3) and (4) of Proposition 3.9. Finally, the choice of α_{n+1} is done so that condition (3) of Lemma 3.5 holds. The construction is thus completed. □

Proof of the genericity of diffeomorphisms with three ergodic measures in $\mathcal{A}'$. Fix a function φ and $\epsilon > 0$, and define, for an integer m and a function c, the set $S(\varphi, \epsilon, m, c)$ of maps $f \in \mathcal{A}'$ such that the averages at time m are close to $c\mu + (1 - c)\delta$. This set is open by definition. We have, in fact, proved that the union over c and over m is dense, hence the intersection over a countable base φ_i and a sequence $\epsilon_j \to 0$ is a dense G_δ. □

3.6. *Minimal number of invariant measures in $\mathcal{A}$.* Consider a general setting for approximation by conjugation construction, namely a smooth action $\mathcal{S}$ of the circle on a compact manifold M, possibly with a boundary. For the special case of fixed-point free actions on closed manifolds, Fathi and Herman [**7**] showed that there is a residual subset of $\mathcal{A}$ which consists of uniquely ergodic diffeomorphisms.

A proper fairly straightforward generalization of the construction described earlier produces in the case of a general S^1 action on a closed manifold a residual subset of $\mathcal{A}_F$ which consists of diffeomorphisms with only one invariant measure (the smooth volume) not supported on the set F. In other words, ergodic invariant measures for such a diffeomorphism are δ-measures at the points of F and the smooth volume.

The only essential extra observation concerns the structure of orbits of a circle action near the set F. In order to obtain a proper counterpart of the picture presented on Figure 2, one needs to take an $\mathcal{S}$ invariant neighborhood of a connected component of F in place of a thin vertical strip and perturb it in a $S_{1/q}$ invariant way to make it mostly transversal to the orbits.

The general case of a manifold with a boundary in dimension greater than two is slightly more complicated. Unlike the disc and the annulus where the action $\mathcal{S}$ on the boundary components is transitive and, hence, has ergodic elements, in this case in order to minimize the number of ergodic measures, one needs to carry out the construction to the boundary. This can be done with a certain care, the result being a diffeomorphism in $\mathcal{A}_F$

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whose only non-atomic ergodic measures are the smooth volume and smooth measures on connected components of the boundary. A good test case to understand this picture is the three-dimensional closed ball $\mathbb{D}^3$ with the circle action by rotations around a fixed axis. The perturbation to which we are referring is ergodic volume preserving, ergodic area preserving on the boundary, fixes every point of the rotation axis and has no other ergodic invariant measures. Another example with a finite number of invariant measures is a rotationally symmetric solid torus ('the doughnut'). Here a small perturbation of a rotation may produce a volume-preserving diffeomorphism with only two ergodic measures, the volume and a smooth measure on the boundary.

3.7. *More measures.* One may ask how the number of ergodic invariant measures can be increased in a controlled way. We will briefly discuss the standard cases of area-preserving diffeomorphisms of the annulus, the disc and the sphere. Higher-dimensional cases are actually easier to handle.

An obvious way to produce an example with any odd number $2m + 1, m \geq 2$, of invariant measures is by pasting m annuli, $m - 1$ annuli capped on one side by a disc or $m - 2$ annuli capped on each side by a disc, with diffeomorphisms described earlier in this section. One picks the same rotation number in each copy and, since our examples are C^∞ tangent to the corresponding rotation on the boundary, the resulting diffeomorphisms will be C^∞. Naturally these examples are neither ergodic nor topologically transitive.

A modification of our method allows us to produce in the spaces $\mathcal{A}'$ topologically transitive diffeomorphisms with exactly $m \geq 4$ ergodic invariant measures all but two of which are absolutely continuous. These diffeomorphisms are naturally not ergodic with respect to the area measure. The construction remains the same near the singularities while inside instead of a single parallelogram $\xi_{n,1}$ and its shifts, one takes $m - 2$ next to each other in such a way that their union still intersects every vertical line by a set of large conditional measure to guarantee a counterpart of condition (4) of Proposition 3.4. The images of those parallelograms have small diameters (cf. Lemma 3.5(3)). At this step, however, there is a new element in the construction which makes it non-generic. There are $m - 2$ invariant cylinders whose images will represent distributions which we intend to have as approximating ergodic absolutely continuous invariant measures for the limit diffeomorphism. These distributions must converge at least weakly. We may, in fact, make their densities converge in L_1 by keeping most of each cylinder inside itself. However, extra steps are needed to provide topological transitivity. This is done by sending a small number of parallelograms in each of $m - 2$ collection to other cylinders. A construction of this kind in a more general and considerably more complicated setting is described in detail in [**40**].

Finally, it is possible to increase the number of ergodic invariant measures in a controlled way and still keep the volume ergodic; furthermore, the additional singular ergodic measures may be made supported by the whole manifold. The corresponding construction involves conjugacies which are not volume preserving but which, at the end, produce a volume-preserving diffeomorphism. For reasons of space, we will not even outline this construction here.

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4. *Minimal sets with arbitrary measure*

4.1. *Preliminaries.* In this section we describe an answer to a question posed in 1977 in the paper by Fathi and Herman [**7**, Problem 1.6]. We do not claim any particular novelty or originality: rather our goal is to put the question into the general context of the approximation by conjugation method and indicate a variety of possible topological and ergodic properties for exotic minimal sets. A similar construction for the disc case can be found in Handel's paper [**18**]. Handel shows that his minimal set is nowhere locally connected (in fact, it is a pseudo-circle) although he does not address the question about its Lebesgue measure. Herman himself produced similar examples for rational maps of the Riemann sphere [**22**].

THEOREM 4.1. *Let M be an m-dimensional differentiable manifold $m \geq 2$, μ a normalized smooth volume on M. Given any number $s \in [0, 1)$, there exists a compactly supported C^∞ diffeomorphism f of M preserving μ and a compact invariant minimal set C of f such that:*

(1) *$\mu(C) = s$; and*

(2) *the set C is connected, has dimension $m - 1$ and is not homeomorphic locally to a product of $\mathbb{R}^{m-1}$ with a Cantor set.*

Furthermore, f can be chosen to be arbitrary close to identity in the C^∞ topology.

As in the previous section, we use an appropriately adjusted version of the approximation by conjugation method of §2.1. By specifying the parameters of the construction, extra information can be provided both about topology of the set C and about topological and measure-theoretic (ergodic) properties of f restricted to that set. We will discuss this after the proof of the theorem in its basic form.

Remark 4.2. If the manifold M admits a non-trivial smooth action of the circle a proper modification of our construction allows us to construct a diffeomorphism with a minimal set with desired properties in any C^∞ neighborhood of any element of the circle action.

4.2. *Reduction to $\mathbb{D}^{m-1} \times S^1$.* Consider the product $P = \mathbb{B}^{m-1} \times S^1$ of the $(m-1)$-dimensional open unit ball with the circle. Let λ be the product of normalized Lebesgue measures. We will call the $(m-1)$-dimensional direction 'horizontal' and draw it accordingly and the S^1 direction 'vertical'. The action of the circle on P by vertical translations will be denoted $\mathcal{S} = \{S_t\}$ as usual. In the natural system of coordinates (ρ, θ), one obviously has $S^t(\rho, \theta) = (\rho, \theta + t)$.

The topology of M is essentially eliminated from our subsequent considerations due to the following fact.

PROPOSITION 4.3. *Given any $\epsilon > 0$, there exists a C^∞ diffeomorphic embedding $E : P \to M$ such that $\mu(E(P)) > 1 - \epsilon$ and $E_*(\lambda)$ is a scalar multiple of μ.*

Proof. Let $E_0 : P \to M$ be an arbitrary C^∞ diffeomorphic embedding. Then $(E_0)_*(\lambda) = \rho\mu$, where ρ is a positive C^∞ function defined on the manifold with boundary $E_0(P) \subset M$ and, by definition, $\int_{E_0(P)} \rho \, d\mu = 1$. Extend ρ to a positive C^∞ function on the whole manifold M, which we will still denote by ρ, in such a way that $\int_M \rho \, d\mu < (1 - \epsilon)^{-1}$.

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By [**35**], there exists a diffeomorphism $h : M \to M$ such that $h_*(\rho\mu) = (\int_M \rho\, d\mu)\mu$. Now take $E = h \circ E_0$. □

Thus, it is enough to prove the theorem for the particular case of the open manifold P with measure λ. A compactly supported diffeomorphism of P is translated to $E(P) \subset M$ via the embedding E and is extended by identity to the rest of M.

In fact, we will deal with the product $\mathbb{D}^{m-1} \times S^1$ of the closed ball with the circle and we will construct a diffeomorphism with the desired minimal set which, on the boundary $S^{m-2} \times S^1$, coincides with a certain element S_α of the action $\mathcal{S}$. By slightly shrinking the horizontal direction by a homothety and extending the diffeomorphism to P by vertical rotations with the angle depending smoothly on the radius and decreasing from α to identical zero near the boundary, we produce the desired compactly supported diffeomorphism of P.

4.3. *Statement and description of the construction.*

PROPOSITION 4.4. *Given any $s \in [0, 1)$, there exists a sequence α_n of rationals and a sequence H_n of diffeomorphisms preserving λ and constructed as in (2.2) and (2.3), such that the sequence of diffeomorphisms $H_n S_{\alpha_{n+1}} H_n^{-1}$ converges in the C^∞ topology to a diffeomorphism of $\mathbb{D}^{m-1} \times S^1$ preserving λ and a compact invariant minimal set of measure s.*

Moreover, f can be made arbitrary close to identity in the C^∞ topology.

4.3.1. *Criterion of minimality.*

Definition 4.5. Let f be a map on a complete separable metric space M. Given $\epsilon > 0$ and a subset $A \subset M$, we say that f is *ϵ-minimal on A* if, given any two points $x, y \in A$, there exists $n = n(x, y) \in \mathbb{N}$ such that $d(f^n(x), y) < \epsilon$.

If m is such that n as before can be chosen less than m for any pair $x, y \in A$, we say that f is *(ϵ, m)-minimal on A*.

LEMMA 4.6. *If $A_n \subset M$ is a sequence of closed sets such that:*
(1) *for every $n \in \mathbb{N}$, $f(A_n) = A_n$; and*
(2) *there exists a sequence $\epsilon_n \to 0$ such that f is ϵ_n-minimal on A_n,*
then $\bigcap_{n\in\mathbb{N}} A_n$ is a minimal set for f.

Proof. We have that $A = \cap A_n$ is a closed set and that $f(A) \subset A$. On the other hand, given any two points $x, y \in A$, there exists a sequence k_n such that $d(f^{k_n}(x), y) < \epsilon_n$; hence, A is a minimal set for f. □

Remark 4.7. In our construction, we will have A_n decreasing and $\mu(A_n) \to s$, where s is arbitrarily chosen in $(0, 1)$.

4.3.2. *Properties of the successive conjugating diffeomorphisms H_n.*

In addition to condition (2.4) that yields convergence in (2.1), we will now list certain extra conditions on the sequences α_n and H_n under which the limiting map will possess a minimal invariant

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set of the given measure $s \in [0, 1)$. In the next section, we will show how at step n h_n is constructed such that the required conditions on H_n hold, then choose α_{n+1} to satisfy the remaining requirements.

Denote by $B(r)$ the ball in $\mathbb{R}^{m-1}$ of radius r around the origin and let r_s be such that $\lambda(B(r_s)) = s$. For n large enough, we consider in M the set $C_n = B(r_s + (1/n)) \times S^1$.

LEMMA 4.8. *If α_n is a sequence of rationals and H_n is a sequence of diffeomorphisms of M satisfying (2.2) and (2.3) and if, for some sequence $\epsilon_n \to 0$, one can find a sequence m_n such that*

(1) $H_n \equiv H_{n-1}$ *outside the interior of* C_{n-1},

(2) $H_n S_{\alpha_{n+1}} H_n^{-1}$ *is* (ϵ_n, m_n)*-minimal on* $H_n(C_n)$ *and*

(3) $|\alpha_{n+1} - \alpha_n| < 1/2^n m_{n-1} \|H_n\|^n_{C^n}$,

then the sequence $H_n S_{\alpha_{n+1}} H_n^{-1}$ converges in the C^∞ topology to a diffeomorphism f that preserves λ and is minimal on the set $\cap H_n(C_n)$ whose measure is equal to s.

Proof. Condition (3) implies convergence to a volume-preserving C^∞ diffeomorphism f (cf. (2.4) in §2.1). We claim that (l) implies that $f(H_n(C_n)) = H_n(C_n)$: indeed since H_{n+1} and H_n coincide on the boundaries of C_n, we have that $H_{n+1}(C_n) = H_n(C_n)$ and similarly, because the sequence C_n is nested, $H_l(C_n) = H_n(C_n)$ for any $l \geq n$. Hence, $H_l S_{\alpha_{l+1}} H_l^{-1} H_n(C_n) = H_l S_{\alpha_{l+1}} C_n = H_l C_n = H_n(C_n)$ and the claim follows as $l \to \infty$. The sets $H_n(C_n)$ are nested and $\lambda(H_n(C_n)) = \lambda(C_n) \to s$, hence $\lambda(\cap H_n(C_n)) = s$.

Condition (3) implies, as in Lemma 3.7 of §3, that $\| f^m - H_n \circ S^m_{\alpha_{n+1}} \circ H_n^{-1} \| \leq m/2^n m_n$; hence, (2) implies that f itself is $(\epsilon_n + 1/2^n, m_n)$-minimal on $H_n(C_n)$. The fact that the set $\cap H_n(C_n)$ is a minimal set for f then follows from Lemma 4.6. □

In the next lemma, we will show how to deduce conditions (2) and (3) of the previous lemma from certain conditions on H_n and α_n that, unlike (2), do not involve explicit iterations.

Definition 4.9. We will say that a collection ξ of subsets of the set $C = B(r) \times S^1$ is *a* ν-collection of horizontal strips if the elements ξ_i of ξ are disjoint and have the form $\xi_i = B(r) \times J_i$, where J_i is a segment of S^1 of length $|J_i| \geq \nu$.

Definition 4.10. A set $A \subset B \subset M$ is said to be ϵ-dense in B if, given any $y \in B$, there exists $x \in A$ such that $d(x, y) \leq \epsilon$.

Remark 4.11. Let $\alpha_{n+1} = p_{n+1}/q_{n+1}$ with p_{n+1} and q_{n+1} relatively prime. Then, given any $(\rho, \theta) \in C_n$ and any horizontal strip ξ in C_n of width larger than $1/q_{n+1}$, there exists $m \leq q_{n+1}$ such that

$$S^m_{\alpha_{n+1}}(\rho, \theta) = (\rho, \theta + m\alpha_{n+1} \bmod (1)) \in \xi.$$

The following lemma implies Lemma 4.8 with $m_n = q_{n+1}$.

LEMMA 4.12. *If $\alpha_n = p_n/q_n$ is a sequence of rationals and H_n is a sequence of diffeomorphisms of M satisfying (2.2) and (2.3) and if we have, for some sequence $\epsilon_n \to 0$, a sequence $\xi_n = \{\xi_{n,i}\}$ of $1/(q_{n+1})$-collections of horizontal strips in C_n such that*

(1) $H_n \equiv H_{n-1}$ *is outside the interior of* C_{n-1};

(2) *the set* $\cup_i H_n(\xi_{n,i})$ *is* ϵ_n*-dense in* $H_n(C_n)$;

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(3) *for any element* $\xi_{n,i} \in \xi_n$, $\operatorname{diam}(H_n(\xi_{n,i})) \le \epsilon_n$*; and*
(4) $|\alpha_{n+1} - \alpha_n| < 1/2^n q_n \|H_n\|_{C^n}^n$,
then the conclusions of Lemma 4.8 hold.

Proof. Fix an arbitrary $z \in H_n(C_n)$. We have, from Remark 4.11, that, for any element $\xi_{n,i} \in \xi_n$, there exists $m \le q_{n+1}$ such that $H_n S_{\alpha_{n+1}}^m H_n^{-1}(z) \in H_n(\xi_{n,i})$. It follows then from (2) and (3) that $H_n S_{\alpha_{n+1}}^m H_n^{-1}$ is $(3\epsilon_n, q_{n+1})$-minimal on $H_n(C_n)$ which is (2) of Lemma 4.8. Finally, with our choice of m_n Condition (3) matches with the corresponding condition in Lemma 4.8. □

4.4. *Construction of h_n and choice of α_{n+1}.* We fix a sequence $\epsilon_n < 1/n^3$ for all $n \in \mathbb{N}$. Now we assume that the sequence $H_l = h_l \circ \cdots \circ h_1$ and α_{l+1} are constructed up to $l = n-1$ and satisfy the conditions of Lemma 4.12 and proceed to construct h_n and choose α_{n+1} such that the conditions of the lemma hold for $l = n$.

4.4.1. *Construction of h_n.* Define

$$\epsilon_n' := \frac{\epsilon_n}{\|H_{n-1}\|_{C^1}}. \tag{4.1}$$

Define

$$\overline{C}_n := B\left(r_s + \frac{1}{n} + \epsilon_n'^3\right) \times S^1.$$

Clearly, $C_n \subset \overline{C}_n \subset C_{n-1}$ and both inclusions are strict (cf. Figure 4).

We pick ν_n sufficiently small and consider a ν_n-collection of horizontal strips ξ_n of C_n such that:

(i) ξ_n is invariant by S_{1/q_n};
(ii) For every $\xi_{n,i} \in \xi_n$, $\lambda(\xi_{n,i}) \le \lambda(\tilde{B}(\epsilon_n'/4))$ (where $\tilde{B}(r)$ denotes the ball in $\mathbb{R}^m$ of radius r centered at the origin); and
(iii) $\lambda(\cup_i(\xi_{n,i})) \ge \lambda(\overline{C}_n) - \lambda(\tilde{B}(\epsilon_n'/4))$.

Condition (iii) is possible since $\lambda(\overline{C}_n) \le \lambda(C_n) + \lambda(\tilde{B}(\epsilon_n'))$. Due to (ii), in the definition of ξ_n, it is possible to ask from a volume-preserving diffeomorphism h_n to send each strip $\xi_{n,i}$ inside a ball of radius ϵ_n' which, in turn, is satisfied as we choose ν_n sufficiently small. More precisely, we have the following proposition.

PROPOSITION 4.13. *There exists a diffeomorphism h_n of class C^∞ on $\mathbb{D}^{m-1} \times S^1$ such that*

(1) *h_n is volume preserving,*
(2) $h_n \circ S_{\alpha_n} = S_{\alpha_n} \circ h_n$,
(3) $h_n \equiv \mathrm{Id}$ *outside* $\overline{C}_n$,
(4) *for any strip* $\xi_{n,i} \in \xi_n$, $\operatorname{diam}(h_n(\xi_{n,i})) \le \epsilon_n'$.

Condition (2) can be satisfied since the collection ξ_n is invariant by S_{α_n} and (1) and (4) are compatible since we have imposed that $\lambda(\xi_{n,i}) \le \lambda(B(\epsilon_n'/2))$.

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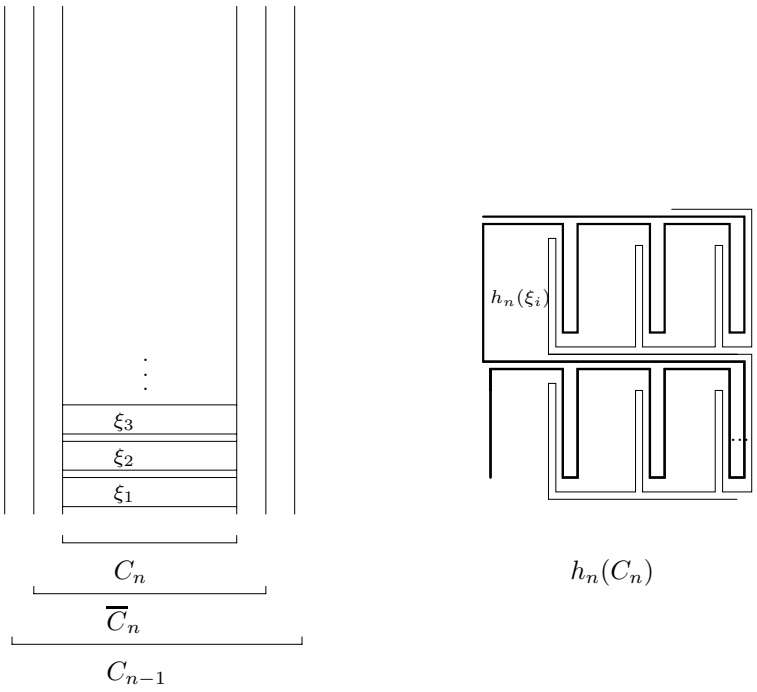


FIGURE 4. Construction of h_n

4.4.2. *Checking the conditions of Lemma 4.12 and choice of* α_{n+1}. By (1) and (2), we can define the conjugating diffeomorphisms $H_n = h_1 \circ \cdots \circ h_n$ as in (2.2) and (2.3). By definition of $\overline{C}_n$, (3) implies (1) of Lemma 4.12. Due to our choice of ϵ_n' in (4.1), condition (4) implies (3) of Lemma 4.12. Next, due to (iii) in the definition of ξ_n, we have that any ball of radius ϵ_n' centered at a point in $\overline{C}_n$ intersects some $h_n(\xi_{n,i}) \subset \overline{C}_n$. Hence, $\cup_i h_n(\xi_{n,i})$ is ϵ_n'-dense in $\overline{C}_n = h_n(\overline{C}_n)$. Again, from our choice of ϵ_n', (2) of Lemma 4.12 follows.

Finally, we choose $\alpha_{n+1} = p_{n+1}/q_{n+1}$ such that (4) in Lemma 4.12 is satisfied and such that $q_{n+1} \geq 1/\nu_n$ in order for the ν_n-collection ξ_n considered to be a $1/q_{n+1}$-collection as required in Lemma 4.12.

All the conditions of Lemma 4.12 being satisfied by the sequences α_n and H_n thus constructed, the diffeomorphisms $H_n S_{\alpha_{n+1}} H_n^{-1}$ converge in the C^∞ topology to a diffeomorphism with a minimal set of measure s. □

4.4.3. *Connectedness and rigidity.* The set C is connected since it is the intersection of the nested sequence of connected sets $H_n(C_n)$. Furthermore, in the two-dimensional case $m = 2$, the complement to each set $H_n(C_n)$ to $\mathbb{D} \times S^1$ consists of two open connected components and this remains true for C. Each connected component of the complement is dense in C. For $m > 2$, the set $\mathbb{D}^{m-1} \times S^1 \setminus C$ is connected.

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It follows immediately from our construction that the diffeomorphism f is C^∞ *rigid*, namely $f^{q_n} \to \mathrm{Id}$ as $n \to \infty$ in the C^∞ topology.

4.4.4. *Special choice of h_n and α_{n+1}.* As was mentioned in §2.2, the most natural way to ensure commutativity relation (2) in Proposition 4.13 is to take the fundamental domain $\Delta_n = \mathbb{D}^{m-1} \times [0, 1/q_n]$, construct a volume-preserving diffeomorphism g of Δ onto itself, identical near its boundary and satisfying conditions (3) and (4) and extend g by periodicity. It is also natural to choose q_{n+1} to be a multiple of q_n.

For the purposes of the further discussion, we will simply call the choice of h_n's and q_n's described here *special*.

With these special choices one produces a nested sequence of partitions invariant under $H_n S_{\alpha_{n+1}} H_n^{-1}$. Those partitions are $H_n(\zeta_n)$ where ζ_n is simply the partition into the 'layers' $\mathbb{D}^{m-1} \times [k/q_n, (k+1)/q_n]$, $k = 0, \dots, q_n - 1$. These layers are simply images of the fundamental domain Δ_n under iterates of the rotation S_{α_n}. Restricted to the minimal set C, this sequence of partitions becomes exhaustive in the measurable sense (converges to ϵ in the more customary terminology of ergodic theory) due to Lemmas 4.12 and 4.13(4). Note also that on the boundaries of the partition elements the conjugacies H_n stabilize: for the boundary of ζ_n, all h_m with $m > n$ are identities. This implies, in particular, that the set C contains a dense set of smooth $(m-1)$-dimensional discs

$$D_{n,k} =: H_n\left(B(r_s) \times \left\{\frac{k}{q_n}\right\}\right), \quad n = 1, 2, \dots ; k = 0, 1, \dots, q_n - 1. \tag{4.2}$$

The limit partition $\xi =: \lim_{n\to\infty} H_n(\zeta_n)$ is invariant under f. Note that the elements of this partition are closed connected sets. Furthermore, by our construction, not only f but every power of it is minimal.

LEMMA 4.14. *Every element $c \in \xi$ is nowhere dense in C and $f^n c \neq c$.*

Proof. Suppose $f^n c = c$. Then the boundary of c is nowhere dense and the union of its images is a nowhere dense f invariant subset of C contradicting minimality. If c has an interior, then all images of it are disjoint and their union is an invariant open set. By minimality, it coincides with C and, hence, c is both open and closed, contradicting the connectedness of C. □

LEMMA 4.15. *The discs $D_{n,k}$ (and, hence, their images) are elements of the partition ξ.*

Proof. These discs are parts of larger discs $H_n(\mathbb{D}^{m-1} \times S^1)$; hence, the discs $D_{n,k}$ are ordered cyclically: in other words, in the same way as the numbers k/q_n, (mod 1). Furthermore, one can consider more discs of the form $f^{q_m} D_{n,k}$, $m \in \mathbb{Z}$, and all these discs are ordered the same way as n, k, such that $k/q_n \leq y \leq (k+1)/q_n$. Obviously every disc $D_{n,k}$ lies within an element of ξ which we will denote $C_{n,k}$ and every image $f^l D_{n,k}$ for $l \neq 0$ belongs to a different element. But, since both f^{q_n} and f^{-q_n} converge to identity as $n \to \infty$, one concludes that $C_{n,k} = D_{n,k}$ because the whole element $C_{n,k}$ must lie between the discs $f^{q_n} D_{n,k}$ and $f^{-q_n} D_{n,k}$. □

To finish the proof of Theorem 4.1, it is sufficient to prove the following statement.

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PROPOSITION 4.16. *With any special choice of diffeomorphisms h_n and α_n for any $x \in C$ and any sufficiently small open neighborhood V of x, the intersection $C \cap V$ has topological dimension $m-1$ and is not homeomorphic to the product of a disc with a Cantor set.*

Proof. The union of $(m-1)$-dimensional discs $D_{n,k} \subset C$ (4.2) is dense in C. Now assume that, for some $x \in C$, there is an open neighborhood V such that $C \cap V$ is homeomorphic to the product of $\mathbb{R}^{m-1}$ with a Cantor set K. Since elements of ξ are connected for any $x \in C \cap V$, the fiber F_x in the product containing x belongs to a single element of ξ. By rigidity, $f^{q_n}(F_x)$ converge to F_x and, hence, lie on different fibers. Now take the fiber F_x corresponding to an endpoint of the complementary interval to the Cantor set. Replacing V by a slightly smaller compact set which we still denote by V, we thus obtain the image V of $[-1, 1] \times [0, 1]$ under a homeomorphism which we will denote $\mathcal{H}$ such that $F_x = \mathcal{H}(\{0\} \times [0, 1])$ and $\mathcal{H}((0, 1] \times [0, 1])$ does not intersect C while there is a sequence of negative numbers $x_n \to 0$ such that $\mathcal{H}(F_{f^{q_n}x}) \subset \{x_n\} \times [0, 1]$. Since $f^{q_n} \to \mathrm{Id}$, we may assume, without loss of generality, that $\mathcal{H}(\{x_n\} \times [0, 1]) \subset C$. Now consider the sets $f^{q_n}\mathcal{H}((0, 1] \times [\epsilon, 1-\epsilon])$. By the invariance of C, these sets are disjoint from C. Since $f^{q_n} \to \mathrm{Id}$ for a sufficiently small $\epsilon > 0$ and any large enough n, this set lies in V and, hence, $\mathcal{H}^{-1} f^{q_n}\mathcal{H}((0, 1] \times [\epsilon, 1-\epsilon]) \cap (\{x_m\} \times [0, 1]) = \emptyset$ for any sufficiently large m. Consequently $\mathcal{H}^{-1} f^{q_n}\mathcal{H}([0, 1] \times [\epsilon, 1-\epsilon]) \subset [0, 1] \times [0, 1]$. But this implies that $F_{f^{q_n}} f \subset F_x$, a contradiction. □

PROPOSITION 4.17. *In dimension two for the special choice of parameters, the set C is never locally connected.*

Proof. Every complete locally connected space is locally path connected. As is shown in [**3**], such a set must be a simple closed curve. This is impossible since, for example, the closed arcs $f^{q_n} D_{1,0}$ are disjoint due to the ordering and, due to rigidity, their diameters are bounded away from zero. □

4.5. *Other versions of the construction.* Note that, in Handel's construction [**18**] for the two-dimensional case, the minimal set is a pseudo-circle. Since a pseudo-circle does not contain any embedded topological arcs, any set C obtained with a special choice of parameters as in §4.4.4 is different from those obtained by Handel [**18**] which are pseudo-circles.

Handel's construction may be put into our framework to make the set of positive (and, in fact, given) measure. The main mechanism that makes the resulting set non-trivial in that construction is a twist of the standard fundamental domains along the orbits of the rotation rather than spreading the orbits within those domains as for special choices of parameters.

For another special choice of parameters, one can actually make C a simple closed curve of positive area. In this case the diffeomorphisms h_n converge to the identity in C^0 topology and their products H_n converge to an area-preserving continuous map which is not injective.

PROBLEM 4.18. *Let C be a minimal set for a diffeomorphism f of a two-dimensional manifold M.*

(1) *Can C be nowhere dense and path-connected?*

(2) *Can C be locally homeomorphic to the product of the line and the Cantor set?*

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4.6. *Ergodic properties of the minimal set.* The restriction of the diffeomorphism f to the minimal set C preserves the Lebesgue measure, or, more precisely, the absolutely continuous probability measure λ_C obtained by normalizing the restriction of the measure λ to C. Our construction guarantees the unique ergodicity of f restricted to the set, C. Furthermore, the special choice of parameters described in §4.4.4 makes (f, λ_C) metrically isomorphic to the rotation of the circle by the angle $\alpha = \lim_{n\to\infty} \alpha_n$ as in the construction of [**1**, §4]. On the other hand, one can choose parameters in a different way to make f weakly mixing with respect to λ_C. Other modifications are also possible, see [**1**].

On the other hand, by modifying our construction along the lines described in [**40**], one may violate unique ergodicity; for example, make λ_C decompose into any given finite or countable number of ergodic components. Furthermore, one may modify our construction by making the transformations h_n not volume preserving in such a way that the minimal set will still have positive (and prescribed) Lebesgue measure but invariant measure(s) of f supported by the set C will be all singular or some of them will be absolutely continuous and some singular. Presenting all these variations in detail will take too much space and will look rather tedious.

PROBLEM 4.19. *Is it possible for a minimal set of a diffeomorphism of a two-dimensional manifold to carry an invariant mixing measure?*

Note that this question is open both for nowhere dense invariant minimal sets and for minimal diffeomorphisms of $\mathbb{T}^2$.

5. *Transitive non-ergodic diffeomorphisms*

Let f be a map on a complete separable metric space M. Denote by $\phi_f \subset M$ the set of points with dense orbit, i.e. $x \in \phi_f$ if and only if $\overline{\{f^n(x)\}_{n\in\mathbf{Z}}} = M$. Recall that a map f is said to be topologically transitive (or simply transitive) if ϕ_f is not empty and, in this case, it is easy to see that ϕ_f is a G_δ dense set in M. When M is a Riemannian manifold, it is natural to ask how small can the measure of ϕ_f be for a topologically transitive map f, especially in the case of a volume-preserving map f. If such a map is ergodic, then the set ϕ_f has full measure. The same may be true for non-ergodic maps, see, e.g., [**16**]. We show, in this section, that the opposite can also be true. To build examples, we again use an appropriately adjusted version of the approximation by conjugation method of §2.1.

THEOREM 5.1. *Let M be an m-dimensional differentiable manifold with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$ preserving a smooth volume μ. There exists a C^∞ diffeomorphism f of M preserving μ such that f is transitive and $\mu(\phi_f) = 0$. Furthermore, f can be chosen to be arbitrary close to identity in the C^∞ topology.*

5.1. *Reduction to the case of $\mathbb{D}^{m-1} \times S^1$.* We need a statement somewhat similar to but stronger than Proposition 4.3. Its proof goes along the same lines as proofs of similar statements in [**1, 40**]. Let λ be the product of Lebesgue measures on $\mathbb{D}^{m-1}$ and S^1. Denote by $\mathcal{R}$ the standard 'vertical' action of S^1 on $\mathbb{D}^{m-1} \times S^1$.

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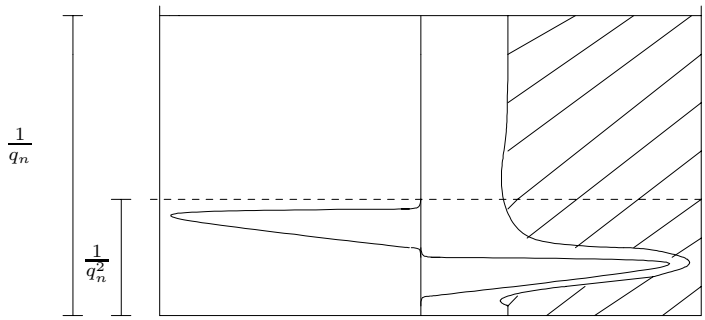


FIGURE 5. $h_n(\{1/2\} \times [0, 1/q_n])$ and $h_n(D^r_{1/2+\rho_0})$.

PROPOSITION 5.2. *Let M be an m-dimensional differentiable manifold with an effective circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$ preserving a smooth volume μ. Let $B =: \partial M \cup F \cup (\bigcup_q F_q)$ (see §2.1). There exists a continuous surjective map $F : \mathbb{D}^{m-1} \times S^1 \to M$ with the following properties:*

(1) *the restriction of F to the interior $\mathbb{B}^{m-1} \times S^1$ is a C^∞ diffeomorphic embedding;*
(2) $\mu F(\partial(\mathbb{D}^{m-1} \times S^1)) = 0$;
(3) $F\partial(\mathbb{D}^{m-1} \times S^1) \supset B$;
(4) $F_*(\lambda) = \mu$; *and*
(5) $\mathcal{S} \circ F = F \circ \mathcal{R}$.

Now suppose $f : \mathbb{D}^{m-1} \times S^1 \to \mathbb{D}^{m-1} \times S^1$ is a topologically transitive diffeomorphism such that $\mu(\phi_f) = 0$, suppose furthermore that $f|\partial\mathbb{D}^{m-1} \times S^1 = \mathbb{R}_\alpha$ for some α. Then one can define a homeomorphism $g : M \to M$ as

$$g(x) = F^{-1}(f(Fx)) \quad \text{for } x \in F(\mathbb{B}^{m-1} \times S^1),$$

and

$$g(x) = \mathbb{R}_\alpha(x) \quad \text{for } \in F(\partial(\mathbb{D}^{m-1}) \times S^1).$$

If, moreover, f has the same jets of all orders on as $\mathbb{R}_\alpha(x)$ on $\partial(\mathbb{D}^{m-1}) \times S^1$ and the difference is sufficiently flat (i.e. the jets of the difference decay fast enough near to the boundary—this condition depends on F), then g, constructed as before, is a C^∞ diffeomorphism satisfying the assertions of Theorem 5.1.

5.2. *Construction of the diffeomorphism h_n.* We will describe the construction only for the case $m = 2$. This makes it easier to visualize (see Figure 5). In the general case, the two 'bumps' have to be replaced by a curve whose projection to the horizontal $\mathbb{D}^{m-1}$ direction is almost dense. For example, for $m = 3$, one may take a spiral beginning at the center of the disc, going outside almost to the boundary, then changing direction and returning to the center. Closeness to the rotation with all derivatives near the boundary required for the reduction described earlier can be guaranteed by taking the successive conjugating diffeomorphisms sufficiently flat near the boundary.

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Assume that the diffeomorphisms h_l have been constructed up to $l = n-1$ and that the rational $\alpha_n = p_n/q_n$ has been chosen.

Definition 5.3. For every integer $n \geq 1$ and every integer $k \leq q_n - 1$, we define

$$B^r_{n,k} = \left[\frac{1}{2} + \frac{k}{2q_n}, \frac{1}{2} + \frac{k+1}{2q_n}\right] \times \left[0, \frac{1}{q_n}\right],$$
$$B^l_{n,k} = \left[\frac{1}{2} - \frac{k+1}{2q_n}, \frac{1}{2} - \frac{k}{2q_n}\right] \times \left[0, \frac{1}{q_n}\right],$$
$$I^r_{n,k} = \left\{\frac{1}{2}\right\} \times \left[\frac{k}{q_n^3} + \frac{1}{4q_n^3}, \frac{k}{q_n^3} + \frac{3}{4q_n^3}\right] \subset \left\{\frac{1}{2}\right\} \times \left[0, \frac{1}{q_n^2}\right],$$
$$I^l_{n,k} = \left\{\frac{1}{2}\right\} \times \left[\frac{1}{q_n^2} + \frac{k}{q_n^3} + \frac{1}{4q_n^3}, \frac{1}{q_n^2} + \frac{k}{q_n^3} + \frac{3}{4q_n^3}\right] \subset \left\{\frac{1}{2}\right\} \times \left[\frac{1}{q_n^2}, \frac{2}{q_n^2}\right].$$

For every integer $n \geq 1$ and for any $\rho_0 \in [0, 1]$, define

$$D^r_{\rho_0} = \{(\rho, \theta) \in M \mid \rho_0 \leq \rho \leq 1,\ \theta \in \mathbb{T}\},$$
$$D^l_{\rho_0} = \{(\rho, \theta) \in M \mid 0 \leq \rho \leq \rho_0,\ \theta \in \mathbb{T}\}.$$

Finally, for any $n \geq 1$, define

$$M_n = [0, 1] \times \left[\frac{1}{q_n^{3/2}}, \frac{1}{q_n}\right].$$

PROPOSITION 5.4. (Properties of h_n) *For every integer $n \geq 1$ there exists a diffeomorphism h_n with the following properties:*

(1) *h_n preserves μ;*
(2) *$h_n S_{1/q_n} = S_{1/q_n} h_n$;*
(3) *$h_n \equiv \mathrm{Id}$ on the set M_n, in particular $h_n(\frac{1}{2}, 0) = (\frac{1}{2}, 0)$;*
(4) *for any $k \leq q_n - 1$, $h_n(I^j_{n,k}) \subset B^j_{n,k}$, $j = r, l$;*
(5) *for any $\rho_0 > 1/n$, we have*

$$h_n(D^r_{1/2+\rho_0}) \subset D^r_{1/2+\rho_0-1/2^n},$$
$$h_n(D^l_{1/2-\rho_0}) \subset D^l_{1/2-\rho_0+1/2^n}.$$

5.3. *Choice of α_{n+1} and convergence.* As in the other sections, we choose α_{n+1} to insure the convergence of $f_n = H_n S_{\alpha_{n+1}} H_n^{-1}$ to a diffeomorphism f and have, in addition, that, for any $m \leq q_{n+1}$,

$$\|f_n^m - f^m\| \leq \frac{1}{n}. \tag{5.1}$$

5.4. *Transitivity.* In this section, we will prove that the limit diffeomorphism f is transitive. Specifically, we will prove that the orbit of the point $(\frac{1}{2}, 0)$ is dense.

Definition 5.5. Given a point $(\rho, \theta) \in M$, we define

$$O_{f,m}(\rho, \theta) = \{(\rho, \theta), \ldots, f^{m-1}(\rho, \theta)\}.$$

We recall that a set $O \subset M$ is said to be ϵ-dense in M if, given any $y \in M$, there exists $x \in O$ such that $d(x, y) \leq \epsilon$.

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5.4.1. *The orbit of* $(\frac{1}{2}, 0)$ *under* f_n. Due to (5.1), the density of the orbit of $(\frac{1}{2}, 0)$ under f will follow if we prove that the set $O_{f_n, q_{n+1}}(\frac{1}{2}, 0)$ is $1/n$-dense.

The following is evident if, in our choice of α_{n+1}, we require that $q_{n+1} \geq q_n^4$: for any $0 \leq p \leq q_n - 1$ and any $0 \leq k \leq q_n - 1$ and for $j = l$ and $j = r$, there exists an integer $m(p, k, j) \leq q_{n+1}$ such that

$$\{m\alpha_{n+1}\} \in S_{p/q_n}(I^j_{n,k}).$$

Since $H_n(\frac{1}{2}, 0) = (\frac{1}{2}, 0)$, we have that

$$\begin{aligned} f_n^{m(p,k,j)}(\tfrac{1}{2}, 0) &\in H_n S_{p/q_n}(I^j_{n,k}), \\ &= H_{n-1} h_n S_{p/q_n}(I^j_{n,k}) \\ &= H_{n-1} S_{p/q_n} h_n(I^j_{n,k}) \quad \text{from Lemma 5.4(2),} \\ &\subset H_{n-1} S_{p/q_n}(B^j_{n,k}) \quad \text{from Lemma 5.4(4).} \end{aligned}$$

In conclusion, we proved that, for any $0 \leq p \leq q_n - 1$ and any $0 \leq k \leq q_n - 1$, for $j = l$ and $j = r$, there exists an integer $m(p, k, j) \leq q_{n+1}$ such that

$$f_n^{m(p,k,j)}(\tfrac{1}{2}, 0) \in H_{n-1} S_{p/q_n}(B^j_{n,k}).$$

Assuming that $\|H_{n-1}\|_{C^1} \leq q_n/n$, we have that $\operatorname{diam}(H_{n-1} S_{p/q_n}(B^j_{n,k})) \leq 1/n$, hence the partition $\{H_{n-1} S_{p/q_n}(B^j_{n,k})\}$, $0 \leq p \leq q_n - 1$, $0 \leq k \leq q_n - 1$, $j = l, r$, is a $1/n$ grid of M. Therefore, the set $O_{f_n, q_{n+1}}(\frac{1}{2}, 0)$ is indeed $(1/n)$-dense in M and the proof of transitivity is complete. □

5.5. *Non-ergodicity:* $\mu(\phi_f) = 0$.

5.5.1. *Preliminary.* As a consequence of property (5) of Lemma 5.4, we get the following properties on the diffeomorphism $H_n = h_1 \circ \cdots \circ h_n$.

COROLLARY 5.6. *For any* $\rho_0 > 0$, *there exists* $n(\rho_0)$ *such that, for any* $n \geq n(\rho_0)$, *we have*

$$\begin{aligned} H_n(D^r_{1/2+\rho_0}) &\subset H_{n(\rho_0)}(D^r_{1/2+\rho_0/2}), \\ H_n(D^l_{1/2-\rho_0}) &\subset H_{n(\rho_0)}(D^l_{1/2-\rho_0/2}). \end{aligned}$$

Proof. Fix $\rho_0 > 0$ and let $n(\rho_0) \geq 4$ be such that $\rho_0 \geq 2/n(\rho_0)$.

For $n \geq n(\rho_0)$, we have that $1/n < \rho_0$; hence, by property (5) of Lemma 5.4,

$$h_n(D^r_{1/2+\rho_0}) \subset D^r_{1/2+\rho_0-1/2^n},$$

and if $n - 1 \geq n(\rho_0)$, we have $1/(n-1) \leq 1/n(\rho_0) < \rho_0 - 1/2^n$. Hence,

$$h_{n-1} h_n(D^r_{1/2+\rho_0}) \subset D^r_{1/2+\rho_0-(1/2^n)-1/(2^{n-1})}.$$

Since, for any $l \geq n(\rho_0)$, we have $\rho_0 - 1/2^n - \cdots - 1/2^l \geq \rho_0 - 1/2^{n(\rho_0)-1} \geq \rho_0/2 \geq 1/n(\rho_0)$, we can continue inductively and obtain

$$h_{n(\rho_0)+1} \circ \cdots \circ h_n(D^r_{1/2+\rho_0}) \subset D^r_{1/2+\rho_0/2}$$

which implies the conclusion of the corollary, the case of $D^l_{1/2-\rho_0}$ being similar. □

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5.5.2. *Proof of* $\mu(\phi_f) = 0$. We introduce the following set:

$$E_l = \{(\rho, \theta) \in M/\rho \neq \tfrac{1}{2} \text{ and for any } n \geq l+1, h_n(x) = x\}.$$

From property (3) in Lemma 5.4, we have that

$$\begin{aligned}\mu(E_l) &\geq 1 - \frac{1}{\sqrt{q_{l+1}}} - \frac{1}{\sqrt{q_{l+2}}} - \cdots \\ &\geq 1 - \frac{2}{\sqrt{q_{l+1}}}.\end{aligned}$$

Since H_l is measure preserving, we also have $\mu(H_l(E_l)) \geq 1 - 2/\sqrt{q_{l+1}}$. We will, hence, finish if we show that, for any point $z \in H_l(E_l)$, the orbit of z under f is not dense in M. From the convergence condition (5.1) in §5.3, it suffices to prove that there exists a fixed closed set $D(z)$ strictly included in M such that, for any n, $O_{f_n, q_{n+1}}(z) \in D(z)$.

Fix $z \in H_l(E_l)$ and write $z = H_l(z')$. We can assume that, for some $\rho_0 > 0$, $z' \in E_l \cap D^r_{1/2+\rho_0}$, the other case $z' \in E_l \cap D^l_{1/2-\rho_0}$ being similar.

For any $n \geq l$ and any $m \leq q_{n+1}$, we have that $f_n^m(z) = H_n S^m_{\alpha_{n+1}}(z')$. Since $S^m_{\alpha_{n+1}}(z') \in D^r_{1/2+\rho_0}$, Corollary 5.6 implies that, for any $n \geq n(\rho_0)$, we have $f_n^m(z) \in H_{n(\rho_0)} D^r_{1/2+\rho_0/2}$. The latter being a fixed closed set strictly included in M, the proof is complete. □

6. *Mixing flows on manifolds admitting a non-trivial* $\mathbb{T}^3$ *action*

All the constructions we have presented up to now share the property of being C^∞ rigid, namely that the limit diffeomorphism f satisfies $f^{q_n} \to \mathrm{Id}$ as $n \to \infty$ in the C^∞ topology for some sequence of integers q_n. This is also the case for all the available constructions following the approximation by conjugation method introduced in [**1**]. It is an open problem whether there exists a circle action $\mathcal{S}$ on a compact manifold and α such that the corresponding space $\mathcal{A}_\alpha$ contains a non-rigid transformation. This would be a first step towards proving the existence of mixing maps in $\mathcal{A}_\alpha$ (cf. Problem 2.2).

6.1. *Time changes as limits of conjugacies.* The situation is different if we consider higher-dimensional actions. Indeed, it is proved in [**8**] that there are linear flows on $\mathbb{T}^3$ that can be made mixing via smooth (real analytic) reparametrization. Note that reparametrizations of linear flows lie in the closure of conjugacies of periodic linear flows (and even of the standard flow $\mathcal{S}$; $S_t(x_1, x_2, x_3) = (x_1, x_2, x_3 + t) \pmod 1$ on $\mathbb{T}^3$ by the following reasoning).

Diophantine frequencies are dense and any smooth reparametrization of a Diophantine linear flow is smoothly conjugated to a linear flow. Hence, taking a very close rational approximation of the frequency vector and applying the same conjugacy produces a periodic flow approximating the given one. Moreover, every periodic linear flow on $\mathbb{T}^n$ is conjugate to the standard 'vertical' flow $\mathcal{S}$ along the last coordinate on the torus albeit with a constant reparametrization to normalize the period to one. The conjugacy is an automorphism of the torus and, thus, is not homotopic to identity. Thus, every diffeomorphism from the mixing flow obtained by a reparametrization belongs in the space $\mathcal{A}$ of limits of conjugates of elements of the action $\mathcal{S}$.

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It is not clear, however, whether these or any other mixing maps in $\mathcal{A}$ belong to any of the spaces $\mathcal{A}_\alpha$ (cf. discussion in §2.3 and, in particular, Problem 2.2). Another open question more specific to the torus is the following.

PROBLEM 6.1. *Consider the space $\tilde{A}$, the closure of conjugacies of the elements of the vertical action $\mathcal{S}$ on $\mathbb{T}^n$ by volume-preserving diffeomorphisms homotopic to the identity. Does the space $\tilde{A}$ contain a mixing diffeomorphism?*

6.2. *Formulation of results.* In this section, given a manifold M admitting a non-trivial smooth $\mathbb{T}^3$ action preserving a smooth volume μ, we construct μ-mixing diffeomorphisms in the closure of conjugacies of elements of the action. To this end, we combine the techniques of reparametrization of linear flows on $\mathbb{T}^3$ with the explicit approximation by conjugation methods. Besides working with higher-dimensional actions and requiring a control for *all iterates* of the limit diffeomorphism, the main difference from former sections is that we need to work in restricted spaces $\mathcal{A}_\alpha$ with special Liouvillean vectors $\alpha \in \mathbb{R}^3$ chosen *a priori* such that the linear flow $R_{t\alpha}$ admits smooth mixing reparametrizations. Conditions of the type (2.4) become more restrictive since we do not have complete freedom in the choice of the periodic approximations of α and explicit bounds on the conjugations used are required. This corresponds to quantitative versions of the approximation by conjugation method similar to the approach in [**13**].

The comments about mixing flows on the torus and the difference between the spaces $\mathcal{A}$ and $\mathcal{A}_\alpha$, of course, apply to this more general situation.

THEOREM 6.2. *Assume M is a differentiable manifold admitting a non-trivial smooth $\mathbb{T}^3$ action $\mathcal{S} = \{S_v\}_{v\in\mathbb{R}^3}$, $S_{v+k} = S_v$ if $k \in \mathbb{Z}^3$ and let μ be a smooth volume preserved by $\mathcal{S}$. Then there exists a sequence $\gamma_n \in \mathbb{Q}^3$ and a sequence H_n of diffeomorphisms preserving μ such that the sequence $H_n S_{t\gamma_n} H_n^{-1}$ converges in the C^∞ topology to a flow preserving μ and mixing for this measure.*

Remark 6.3. This is the optimal result for flows, in the sense that a $\mathbb{T}^2$ action is not sufficient: for example, there are no mixing fixed-point free flows on $\mathbb{T}^2$. One may expect that, on a manifold with a non-trivial $\mathbb{T}^2$ action, a discrete-time version of our construction should work. Namely, considering successive conjugations of some sequence of rational elements of the action, it should be possible to obtain mixing diffeomorphisms in the limit. However, at the moment we are not able to provide such a construction even on $\mathbb{T}^2$.

6.3. *Reduction to $\mathbb{T}^3 \times \mathbb{D}^m$.* Theorem 6.2 will follow if we can prove it in the particular case of a direct product $\mathbb{T}^3 \times \mathbb{D}^m$ with successive conjugacies H_n having compact support in the interior of $\mathbb{T}^3 \times \mathbb{D}^m$.

This follows from the modification of Proposition 5.2 for the case of the torus action.

PROPOSITION 6.4. *Let M be an $(m + k)$-dimensional differentiable manifold with an effective action $\mathcal{T}$ of $\mathbb{T}^k$ preserving a smooth volume, μ. Let B be the union of the boundary of M and the set of points with a non-trivial stationary stabilizer. Then, there exists a continuous surjective map $F : \mathbb{D}^m \times \mathbb{T}^k \to M$ with the following properties:*

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(1) *the restriction of F to the interior $\mathbb{B}^m \times \mathbb{T}^k$ is a C^∞ diffeomorphic embedding;*
(2) $\mu F(\partial(\mathbb{D}^m \times \mathbb{T}^k)) = 0$*;*
(3) $F\partial(\mathbb{D}^{m-1} \times S^1) \supset B$*;*
(4) $F_*(\lambda) = \mu$*;*
(5) $\mathcal{T} \circ F = F \circ \mathcal{S}$*, where $\mathcal{S}$ is the standard vertical action on the direct product.*

6.4. *Plan of the construction.* We will use the natural $\mathbb{T}^3$ action on $\mathbb{T}^3 \times \mathbb{D}^m$, $S_v(\theta, r) = (\theta + v, r)$. We will not directly construct the mixing flow as a limit of conjugacies of periodic flows $H_n S_{t\gamma_n} H_n^{-1}$. Instead we will use successive conjugacies of a flow $T^t \times \mathrm{Id}_{\mathbb{D}^m}$ where T^t is the mixing flow on $\mathbb{T}^3$ obtained in [**8**] via reparametrization of a Liouvillean linear flow on $\mathbb{T}^3$. Theorem 6.2 then follows since reparametrizations of linear flows are in the closure of conjugacies of periodic linear flows on $\mathbb{T}^3$ as has been explained in §6.1.

6.5. *Mixing reparametrizations of linear flows.* Since some specific features of the mixing reparametrized flow constructed in [**8**] will be needed in the proof of Theorem 6.2, we will summarize, in this section, the construction and mention the properties that will be useful in the following.

6.5.1. *Notation and definitions.* For $\alpha \in \mathbb{R}^3$, we denote by R_α^t the linear flow on the torus $\mathbb{T}^3$ given by $d\theta/dt = \alpha$, where $\theta \in \mathbb{T}^3$. For a continuous function $\phi : \mathbb{T}^3 \to \mathbb{R}_+^*$, we denote by $R_{\alpha,\phi}^t$ the reparametrized flow given by $d\theta/dt = \phi(\theta)\alpha$. If the coordinates of α are rationally independent, then the linear flow R_α^t is uniquely ergodic and so is the reparametrized flow that preserves the measure with density $1/\phi$. In contrast, other properties of the linear flow, such as mixing, may be very sensitive to reparametrizations. The reparametrized flow $R_{\alpha,\phi}^t$ can be viewed as a special flow $T_{\alpha,\varphi}^t$ over R_α and under φ where α this time is an $\mathbb{R}^2$ vector and φ is a strictly positive function over $\mathbb{T}^2$. We denote by $M_{\alpha,\varphi}$ the space where the special flow acts which is obtained as a quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation $(x, s + \varphi(x)) \sim (x + \alpha, s)$ and that can be identified with $\mathbb{T}^3$. Conversely, any smooth special flow $T_{\alpha,\phi}$ can be viewed as a reparametrization of some linear flow on $\mathbb{T}^3$. Henceforth, we will only work with special flows that are easier to manipulate than reparametrizations. The unique probability measure preserved by the special flow is given by the product of the Haar measure on the base $\mathbb{T}^2$ and the Lebesgue measure on the fibers normalized by the mean value of φ. The k-dimensional Lebesgue measures will be denoted by $\lambda^{(k)}$.

6.5.2. *Mixing special flows over a translation of $\mathbb{T}^2$.* Our choice of the translation $R_{\alpha,\alpha'}$ and the ceiling function φ follows [**42**]. We take $(\alpha, \alpha') \in (\mathbb{R} \setminus \mathbb{Q})^2$, such that their sequences of best approximations q_n and q_n' satisfy, for any $n \geq n_0(\alpha, \alpha')$,

$$q_n' \geq e^{4q_n}, \tag{6.1}$$

$$q_{n+1} \geq e^{4q_n'}. \tag{6.2}$$

The ceiling function we will consider is the following strictly positive real-analytic function on $\mathbb{T}^2$:

$$\varphi(x, y) = 1 + \mathrm{Re}\left(\sum_{k=2}^{\infty} \frac{e^{i2\pi kx}}{e^k} + \sum_{k=2}^{\infty} \frac{e^{i2\pi ky}}{e^k} \right). \tag{6.3}$$

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In [**8**], the following was proved.

THEOREM 6.5. *Under conditions (6.1)–(6.3), the special flow $T^t_{\alpha,\alpha',\varphi}$ is mixing for its unique invariant probability measure.*

Since α, α' and φ are fixed, to alleviate the notation we will denote by T^t the special flow $T^t_{\alpha,\alpha',\varphi}$ and by M the space $M_{\alpha,\alpha',\varphi}$. We will need the following properties of T^t, the first one of which is an obvious approximation of the flow by special flows over periodic translations and the other being a refinement of the mixing property that can be derived from the proof of mixing given in [**8**].

We will denote by α_n and α'_n the nth convergents of α and α', i.e. $\alpha_n = p_n/q_n, \alpha'_n = p'_n/q'_n$. By T^t_n we denote the special flow over the periodic translation R_{α_n,α'_n} and under the function φ. Since $|\alpha - \alpha_n| \le 1/q_n q_{n+1}$ and $|\alpha' - \alpha'_n| \le 1/q'_n q'_{n+1}$, we deduce that

$$\|T^t - T^t_n\| = O\left(\frac{t}{q_n q_{n+1}}\right). \tag{6.4}$$

6.5.3. *The specialized mixing property of T^t.* Let $\delta_n = 1/q_n$. Consider a collection $\mathcal{C}_n = \{C_{n,j}\}_{j \le M_n}$ of disjoint sets in M of the form

$$C_{n,j} = \left[\frac{i}{q_n}, \frac{i+1}{q_n}\right] \times \left[\frac{i'}{q'_n}, \frac{i'+1}{q'_n}\right] \times [s, s + \delta_n]$$

for some $i = i_j \le q_n - 1$ and $i' = i'_j \le q'_n - 1$ and $s = s_j$ satisfying

$$s + \delta_n \le \min_{(x,x') \in [i_j/q_n, (i_j+1)/q_n] \times [i'_j/q'_n, (i'_j+1)/q'_n]} \varphi(x, x').$$

We take the collection $\mathcal{C}_n$ such that $\mu(\cup C_{n,j}) \ge 1 - \epsilon_n$ with $\epsilon_n = O(1/q_n)$.

Definition 6.6. An *ϵ-grid of a set* (A, μ) is any partition of A into disjoint measurable sets of diameter less than ϵ. An *ϵ-grid of a collection of disjoint similar sets* $\{C_j\}_{j \le M}$ is given by some ϵ-grid of C_1 identically repeated inside each C_j, $j \le M$.

Definition 6.7. For a pair of strictly positive numbers (ϵ, η), a set $B \subset A$ is said to be *(ϵ, η)-uniformly-distributed inside A* if there exists an ϵ-grid of A such that, for each atom A_k of the grid, we have

$$\left|\frac{\mu(B \cap A_k)}{\mu(B)} - \frac{\mu(A_k)}{\mu(A)}\right| \le \eta \frac{\mu(A_k)}{\mu(A)}.$$

Definition 6.8. For a pair of strictly positive numbers (u, v), a set B is said to be *(u, v)-identically-distributed in a collection of disjoint similar sets* $\{C_j\}_{j \le M}$ if there exists a u-grid of the collection $\{C_j\}_{j \le M}$ such that, for every pair $j, j' \le M$ and for any $k \le K$ (K referring to the number of atoms inside each C_j), we have

$$|\mu(B \cap C_{j,k}) - \mu(B \cap C_{j',k})| \le v \mu(B \cap C_{j,k}).$$

Remark 6.9. In this definition, the measurable sets B, A, A_k, $C_{j,k}$ might be of different dimensions and μ should be replaced when necessary by its appropriate conditionals.

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Remark 6.10. A set B that is (u, v)-identically-distributed inside the collection $\mathcal{C}_n = \{C_{n,j}\}_{j\le M_n}$ is $(2/q_n, v)$-uniformly-distributed in M (this is due to the fact that the diameter of the sets $C_{n,j}$ is less than $2/q_n$). Moreover, because each set $C_{n-1,j}$ is almost a union of sets of $\mathcal{C}_n$ then $B \cap C_{n-1,j}$ is $(2/q_n, 2v)$-uniformly distributed inside $C_{n-1,j}$.

The specific mixing property of T^t that will be crucial later is the following one.

PROPOSITION 6.11. *Let ξ_n be a set in $M_{\alpha,\varphi}$ of the form $I \times J \times \{s\}$ where I and J are intervals of size $1/e^{q'_n}$. Then, for any $t \ge e^{3q'_n}$, we have that $T^t\xi_n$ is $(1/e^{q'_n}, 1/n^2)$-identically-distributed in the collection $\mathcal{C}_n = \{C_{n,j}\}_{j\le M_n}$.*

6.6. *The successive conjugations scheme.* We will use the same notation T^t and T^t_n to indicate the special flows over $T^t_{\alpha,\alpha',\varphi}$ and $T^t_{\alpha_n,\alpha'_n,\varphi}$ as well as the flows $T^t \times \mathrm{Id}_{\mathbb{D}^m}$ and $T^t_n \times \mathrm{Id}_{\mathbb{D}^m}$. For $r \in (0, 1]$, we denote by $D(0, r)$ the closed disc of radius r inside $\mathbb{D}^m$. The general scheme of the construction follows. We construct volume-preserving diffeomorphisms h_n compactly supported on $M_{\alpha_n,\alpha'_n,\varphi} \times \mathbb{D}^m$ of the form

$$h_n(x, x', s, r) = (x, x', s, g_{n,x,x'}(r)) \tag{6.5}$$

where, for each $(x, x') \in \mathbb{T}^2$, $g_{n,x,x'}$ is a compactly supported volume-preserving diffeomorphism of $\mathbb{D}^m$ satisfying

$$g_{n,x,x'}(r) = g_{n,x+\alpha_n,x'+\alpha'_n}(r) \tag{6.6}$$

for all (x, x', r). This implies that h_n commutes with T^t_n. We can choose $g_{n,x,x'}$ increasingly sensitive with respect to x and x' but in a controlled way as follows.

PROPOSITION 6.12. *It is possible to construct the diffeomorphism h_n as in (6.5)–(6.6) such that the following holds.*

(1) *h_n is volume preserving and compactly supported.*
(2) *$h_n T^t_n = T^t_n h_n$.*
(3) *For any $l \in \mathbb{N}$, for n large enough $\|h_n\|_{C^l} \le e^{\frac{1}{2}\sqrt{q'_n}}$.*
(4) *For any $r \in D(0, 1 - e^{-n}) \subset \mathbb{D}^m$, for any $j \le M_n$, we have that $h_n(C_{n,j} \times \{r\})$ is $(e^{-q'_{n-1}}, 1/n)$-uniformly-distributed inside $C_{n,j} \times \mathbb{D}^m$ in the sense that there exists a partition of $\mathbb{D}^m$ in sets having diameter less than $e^{-q'_{n-1}}$ such that, for any atom D of the partition, we have*

$$|\lambda^{(3)}(h_n(C_{n,j} \times \{z\}) \cap (C_{n,j} \times D)) - \lambda(3)(C_{n,j})\lambda^{(m)}(D)| \le \frac{1}{n}\lambda^{(3)}(C_{n,j})\lambda^{(m)}(D).$$

We now consider the flows T^t, T^t_n and the diffeomorphisms h_n as flows and diffeomorphisms on $M \times \mathbb{D}^m$ and define $H_n = h_1 \circ \cdots \circ h_n$ and consider the sequence $H_n T^t H_n^{-1}$.

From the previous proposition, we can easily deduce the following properties of H_n.

PROPOSITION 6.13. *The diffeomorphisms H_n are compactly supported in the interior of $M \times \mathbb{D}^m$ and satisfy the following conditions:*

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(1) *H_n is volume preserving;*
(2) *for any $l \in \mathbb{N}$, for n large enough $\|H_n\|_{C^l} \leq e^{\sqrt{q'_n}}$; and*
(3) *for any $z \in D(0, 1 - e^{-n}) \subset \mathbb{D}^m$, for any $j \leq M_n$, we have that $H_n(C_{n,j} \times \{z\})$ is $(e^{-(1/2)q'_{n-1}}, 1/n)$-uniformly-distributed inside $C_{n,j} \times \mathbb{D}^m$.*

Proof. Property (2) follows from (3) of Proposition 6.12. Property (3) then follows from (4) of Proposition 6.12 applied to h_n and from (1) and (2) of Proposition 6.13 applied to H_{n-1}. □

6.7. *Convergence and approximation.*

PROPOSITION 6.14. *The sequence of flows $H_n T^t H_n^{-1}$ converges in the C^∞ topology to a volume-preserving flow $\overline{T}^t$ of $M \times \mathbb{D}^m$. Moreover, for $t \in [e^{3q'_n}, e^{3q'_{n+1}}]$ and $l \in \mathbb{N}$, we have, for n large enough,*

$$\|H_n T^t H_n^{-1} - \overline{T}^t\|_{C^l} \leq \frac{1}{2^n}.$$

Proof. From (2) of Proposition 6.12, we have that

$$H_{n+1} T_{n+1}^t H_{n+1}^{-1} = H_n T_{n+1}^t H_n^{-1};$$

hence,

$$\|H_n T^t H_n^{-1} - H_{n+1} T^t H_{n+1}^{-1}\|_{C^l} \leq \psi_n + \psi'_n$$

where

$$\psi_n = \|H_n T^t H_n^{-1} - H_n T_{n+1}^t H_n^{-1}\|_{C^l},$$
$$\psi'_n = \|H_{n+1} T_{n+1}^t H_{n+1}^{-1} - H_{n+1} T^t H_{n+1}^{-1}\|_{C^l}.$$

Hence, from (6.4) and (2) of Proposition 6.13, we conclude that

$$\|H_n T^t H_n^{-1} - H_{n+1} T^t H_{n+1}^{-1}\|_{C^l} = o\left(\frac{e^{q'_{n+1} t}}{q_{n+1} q_{n+2}}\right).$$

Since $q_{n+2} \geq e^{4q'_{n+1}}$ the proposition follows. □

6.8. *Proof of mixing.* It remains to prove the following proposition.

PROPOSITION 6.15. *The volume-preserving flow of $M \times \mathbb{D}^m$*

$$\overline{T}^t = \lim_{n \to \infty} H_n T^t H_n^{-1}$$

is mixing.

6.8.1. The following is an almost straightforward Fubini argument.

LEMMA 6.16. *If for an arbitrary set $C = B \times D \subset M \times \mathbb{D}^m$, where B and R are open balls in $\mathbb{D}^m$ and $\mathbb{T}^2$ and $\delta \leq c$.*

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If there is a sequence of measurable collections of subsets of $M \times \mathbb{D}^m$, $\{\mathcal{P}_t\}$ satisfying the following

$$\{\mathcal{P}_t\} \xrightarrow[t\to\infty]{} \varepsilon \text{ (partition into points)},$$

and such that, for any open set $A \subset M \times \mathbb{D}^m$ and any $\varepsilon > 0$, there exists t_0 such that, for $t > t_0$, we have, for every atom $P_t \in \{\mathcal{P}_t\}$,

$$|\mu(P_t \cap \overline{T}^{-t}(A)) - \mu(P_t)\mu(A)| \le \epsilon\mu(P_t),$$

then the flow $\overline{T}^t$ is mixing.

Remark 6.17. The atoms of the collection $\{\mathcal{P}_t\}$ might have positive codimension and the lemma remains valid if appropriate conditional measures are used in (6.8.1).

6.8.2. *Choice of the collection $\mathcal{P}_t$.* Let $t \in \mathbb{R}_+$ and take n such that $t \in [e^{3q'_n}, e^{3q'_{n+1}}]$. We first consider a collection $\mathcal{U}_n$ converging to the point partition in $M_{\alpha,\varphi} \times \mathbb{D}^3$ consisting of sets of the form $\xi \times \{s, r\}$ where ξ is a square on $\mathbb{T}^2$ with side $1/e^{q'_n}$. Since the diffeomorphism $h_n T^t$ is volume preserving, we remain with a collection converging to the point partition if we discard the elements $\xi \times \{s, r\}$ that do not satisfy

$$\lambda^{(2)}[\xi \times \{s, r\} \cap T^{-t} h_n^{-1}(M_{\alpha,\varphi} \times D(0, 1 - e^{-(n-1)}))] \ge \left(1 - \frac{1}{n}\right)\lambda^{(2)}(\xi).$$

Finally, we consider the collection $\mathcal{P}_t$ whose atoms are the sets $H_n(\xi \times \{s, r\})$ where $\xi \times \{s, z\}$ is as before. Due to the fact that $\|H_n\|_{C^1} \le e^{\sqrt{q'_n}}$, we have that $\mathcal{P}_t$ converges to the partition into points of $M_{\alpha,\varphi} \times \mathbb{D}^m$ as t goes to infinity.

6.8.3. Fix an open set $A \subset M \times \mathbb{D}^m$ and $\varepsilon > 0$. We finish if we check (6.8.1) of Lemma 6.16 for the elements of $\mathcal{P}_t$ as $t \in [e^{3q'_n}, e^{3q'_{n+1}}]$ goes to infinity or, equivalently, as $n \to \infty$. To have (6.8.1) for this range of t, we are reduced in the light of Proposition 6.14 to proving that, for n large enough,

$$|\lambda^{(2)}(\xi \times \{s, r\} \cap T^{-t} H_n^{-1}(A)) - \lambda^{(2)}(\xi)\mu(A)| \le \varepsilon\lambda^{(2)}(\xi)$$

where ξ is a square in $\mathbb{T}^2$ of side $1/e^{q'_n}$ and $\xi \times \{s, r\}$ satisfies (6.8.2).

6.8.4. Since the collection $\mathcal{C}_{n-1} = \{C_{n-1,j}\}_{j \le M_{n-1}}$ converges to the point partition, (6.8.3) will follow if, for an arbitrary measurable set $\Delta \subset \mathbb{D}^m$, we prove that, for n large enough, we have, for any $j \le M_{n-1}$,

$$\begin{aligned}|\lambda^{(2)}(\xi \times \{s, r\} \cap T^{-t} H_n^{-1}(C_{n-1,j} \times \Delta)) - \lambda^{(2)}(\xi)\lambda^{(3)}(C_{n-1,j})\lambda^{(m)}(\Delta)| \\ \le \varepsilon\lambda^{(2)}(\xi)\lambda^{(3)}(C_{n-1,j}).\end{aligned}$$

6.8.5. Having fixed t and the atom $\xi \times \{s, r\} \in \mathcal{P}_t$, we set

$$U := h_n(T^t(\xi \times \{s, r\}))$$

and denote by $\tilde{\lambda}^{(2)}$ the measure $(h_n T^t)_* \lambda^{(2)}$.

The equation we have to establish (6.8.4) becomes

$$|\tilde{\lambda}^{(2)}(U \cap H_{n-1}^{-1}(C_{n-1,j} \times \Delta)) - \tilde{\lambda}^{(2)}(U)\lambda^{(3)}(C_{n-1,j})\lambda^{(m)}(\Delta)| \le \varepsilon\tilde{\lambda}^{(2)}(U)\lambda^{(3)}(C_{n-1,j}).$$

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6.8.6. From Proposition 6.11, we recall that since $t \geq e^{3q'_n}$, then the set $T^t(\xi \times \{s\})$ is $(1/e^{q'_n}, 1/n^2)$-identically-distributed in $\mathcal{C}_n$. This means that $T^t(\xi \times \{s\})$ has essentially the same trace inside each $C_{n,j}$ with precision (or resolution) $e^{-q'_n}$. The diffeomorphism h_n introduced in Proposition 6.12 has an important deformation effect on the scale of the sets $C_{n,j}$ (Property (4)) but negligible on the scale $e^{-q'_n}$ (Property (3) asserting that $\| h_n \|_{C^1} \leq e^{\sqrt{q'_n}}$). On the other hand, by (6.5) and (6.6), we have that $h_n(C_{n,j} \times \mathbb{D}^m) = C_{n,j} \times \mathbb{D}^m$ for any $j, j' \leq M_n$ and that $h_{n|C_{n,j} \times \mathbb{D}^m} \equiv h_{n|C_{n,j'} \times \mathbb{D}^m}$. We obtain, therefore, the following.

CLAIM. *There exists a partition of $\mathbb{D}^m$ in open sets of diameter less than $1/e^{\frac{1}{2}q'_n}$, $\mathbb{D}^m = \bigcup_{l=1}^{l_n} D_l$, such that, for each $l \leq l_n$, the set $U_l := U \cap (M \times D_l)$ is $(1/e^{\frac{1}{2}q'_n}, 1/n)$-identically-distributed in the collection $\{C_{n,j} \times D_l\}_{j \leq M_n} \subset M \times \mathbb{D}^m$.*

6.8.7. Since the collection $\mathcal{C}_n$ is almost a partition of M that is much finer than $\mathcal{C}_{n-1}$ (cf. §6.5.3), it follows from the claim that, for any $j \leq M_{n-1}$,

$$|\tilde{\lambda}^{(2)}(U_l \cap (C_{n-1,j} \times D_l)) - \tilde{\lambda}^{(2)}(U_l)\lambda^{(3)}(C_{n-1,j})| \leq \frac{3}{n}\tilde{\lambda}^{(2)}(U_l)\lambda^{(3)}(C_{n-1,j}).$$

6.8.8. From Remark 6.10, Claim 6.8.6 also implies that the set U_l is $(2/q_n, 1/n)$-uniformly-distributed in $M \times D_l$ and, more specifically, that, for any $j \leq M_{n-1}$, the set $U_l \cap (C_{n-1,j} \times D_l)$ is $(2/q_n, 2/n)$-uniformly-distributed in $C_{n-1,j} \times D_l$.

6.8.9. From our choice of the collection $\mathcal{P}_t$ in (6.8.2), we have that

$$U \sim \bigcup_{D_l \subset D(0,1-e^{-(n-1)})} U_l \tag{6.7}$$

in the sense that the ratio between the measures of the left-hand side set and the right-hand side set converges to 1 as n goes to ∞. So we assume from now on that l is such that $D_l \subset D(0, 1 - e^{-(n-1)})$. We fix a measurable set $\Delta \in \mathbb{D}^m$ and $j \leq M_{n-1}$ and proceed to estimate $\tilde{\lambda}^{(2)}(U_l \cap H_{n-1}^{-1}(C_{n-1,j} \times \Delta))$ in the scope of proving (6.8.5).

6.8.10. Since $h_n(C_{n,j} \times \mathbb{D}^m) = C_{n,j} \times \mathbb{D}^m$ (from (6.5)) and $U_l = U \cap (M \times D_l)$, we have that

$$U_l \cap H_{n-1}^{-1}(C_{n-1,j} \times \Delta) = (U_l \cap (C_{n-1,j} \times D_l)) \cap H_{n-1}^{-1}(C_{n-1,j} \times \Delta).$$

6.8.11. Because $D_l \subset D(0, 1 - e^{-(n-1)})$, property (3) in Proposition 6.13 asserts that $H_{n-1}(C_{n-1,j} \times D_l)$ is $(e^{-(1/2)q'_{n-1}}, 1/(n-1))$-uniformly-distributed inside $C_{n-1,j} \times \mathbb{D}^m$. From (6.8.8) and the fact that $\|H_{n-1}\|_{C^1} \leq e^{\sqrt{q'_{n-1}}}$, we deduce that $H_{n-1}(U_l \cap (C_{n-1,j} \times D_l))$ is $(e^{-(1/4)q'_{n-1}}, 4/n)$-uniformly-distributed inside $C_{n-1,j} \times \mathbb{D}^m$ and get, with (6.8.10),

$$\tilde{\lambda}^{(2)}[U_l \cap H_{n-1}^{-1}(C_{n-1,j} \times \Delta)] \sim \tilde{\lambda}^{(2)}(U_l \cap (C_{n-1,j} \times D_l))\lambda^{(m)}(\Delta)$$

which, with (6.8.7), becomes

$$\tilde{\lambda}^{(2)}[U_l \cap H_{n-1}^{-1}(C_{n-1,j} \times \Delta)] \sim \tilde{\lambda}^{(2)}(U_l)\lambda^{(3)}(C_{n-1,j})\lambda^{(m)}(\Delta).$$

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6.8.12. Having (6.7) in mind, we can sum in (6.8.11) over l such that $D_l \subset D(0, 1 - e^{-(n-1)})$ and get

$$\tilde{\lambda}^{(2)}(H_{n-1}U \cap (C_{n-1,j} \times \Delta)) \sim \tilde{\lambda}^{(2)}(U)\lambda^{(3)}(C_{n-1,j})\lambda^{(m)}(\Delta).$$

This gives equation (6.8.5) and the proof of mixing is, hence, accomplished. □

7. *Other directions and more open problems*

We discuss two areas where some of the most interesting and challenging problems related to the approximation by conjugation method appear: analytic constructions and the problem of smooth realization.

7.1. *Real-analytic constructions.* The approximation by conjugation method is a major source constructing examples of smooth dynamical systems with prescribed ergodic or topological properties: other methods are skew products and time-changes. In the latter two cases, and except for some special exotic constructions [**12**], there is usually not much difference between the cases of sufficiently high finite differentiability and real analyticity. Constructions in [**16**], [**27**, §12.3, 12.4, 13.3, 13.4], [**8–11**] provide typical examples. In a somewhat crude way, this can be explained by the essential linearity of the time-change and skew-product constructions. The conjugation by approximation construction is essentially nonlinear and it is based on the convergence of maps obtained from certain standard maps by wildly diverging conjugacies.

Here a great difference between the differentiable and real-analytic maps becomes apparent. In the former case, as long as the conjugating diffeomorphisms are sufficiently smooth albeit with huge derivatives, fast convergence of the approximations is achieved by conditions like (2.4). Since no control over C^r norms is required, it is usually easy to construct C^∞ diffeomorphisms creating an approximate picture as required. Of course, usually smooth maps can be approximated by real-analytic ones. Such maps if they have very large derivatives in the real domain will usually have singularities in a fairly small complex neighborhood of it. But if either a map h or its inverse has complex singularities close to the real domain and $S_t,\ 0 \le t \le t_0,\ S_0 = \mathrm{Id}$ is an analytic family, then the family $h \circ S_t \circ h^{-1}$ is expected to have singularities close to the real domain for *any* $t > 0$. Thus, the domain of analyticity for maps f_n in (2.1) will shrink considerably at any step of the constructions and the limit map f will not be analytic.

There are two possible strategies for overcoming this problem in order to make a version of the approximation by conjugation method work in the real-analytic category.

(1) Find conjugacies h_n which are analytic with their inverses in a uniform complex neighborhood of the real domain.

(2) Compensate singularities of the conjugacies h_n and their inverses in such a way that the maps f_n have a much larger domain of analyticity than the conjugacies.

7.1.1. *Uniquely ergodic diffeomorphisms of spheres.* The first method has a good chance of working when there is a sufficiently large supply of maps commuting with the action $\mathcal{S}$ which can be modified in a controlled way maintaining analyticity in the large

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domain. This happens for skew products where the S^1 acts in the fibers and in the time changes of linear flows on the torus. A less obvious case of this situation was found by the second author [**28**]. We describe one of the results of that work in a special case.

Consider the standard embedding of the sphere S^{2n-1} into $\mathbb{R}^{2n}$ and the standard complexification $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$. The vector field defined in Euclidean coordinates as $v_0(x_1, \dots, x_{2n}) = 2\pi(x_2, -x_1, \dots, x_{2n}, -x_{2n-1})$ defines a linear action $\Phi = \{\varphi^t,\ t \in \mathbb{R},\ \varphi^1 = \mathrm{Id}\}$ of the circle S^1. In Euclidean coordinates,

$$\varphi^t(x_1, \dots, x_{2n}) = (\cos 2\pi t x_1 + \sin 2\pi t x_2, -\sin 2\pi t x_1 + \cos 2\pi t x_2, \dots, \cos 2\pi t x_{2n-1} + \sin 2\pi t x_{2n}, -\sin 2\pi t x_{2n-1} + \cos 2\pi t x_{2n}).$$

We will use the same notation v_0 and φ^t for extensions to $\mathbb{C}^{2n}$ or its subsets.

We will call a function on S^{2n-1} *entire* if it extends to a holomorphic function in $\mathbb{C}^{2n}$. A map $f : S^{2n-1} \to S^{2n-1}$ is called *entire* if its coordinate functions are entire. A diffeomorphism $f : S^{2n-1} \to S^{2n-1}$ is entire if both f and f^{-1} are entire maps. Invertible linear maps are obviously entire diffeomorphisms. Note that the product of entire diffeomorphisms is an entire diffeomorphism.

THEOREM 7.1. *For any t_0, $0 \le t_0 \le 1$, $\epsilon > 0$ and any compact set $K \subset \mathbb{C}^n$ there exists an entire diffeomorphism $f : S^{2n-1} \to S^{2n-1}$, preserving the Lebesgue measure and uniquely ergodic whose extension to $\mathbb{C}^{2n}$ satisfies*

$$\max\left\{\max_{z\in K}|f(z) - \varphi^{t_0}(z)|, \max_{z\in K}|f^{-1}(z) - \varphi^{-t_0}(z)|\right\} \le \epsilon. \tag{7.1}$$

Remark 7.2. Note that, while in the C^∞ category the approximation by conjugation method as outlined in §2.1 allows us to produce a great variety of perturbations of a rotation φ_t with interesting ergodic and topological properties, even minimal real-analytic diffeomorphisms of S^3 were not known before.

This result extends to compact Lie groups and some of their homogeneous spaces with an action of S^1 by left translations.

Let us sketch some of the ideas of the proof of Theorem 7.1 in the case of S^3. There is a natural identification of S^3 with the group $SU(2)$ which identifies Φ with an action by left translations. The space of orbits is naturally identified with S^2 (the Hopf fibration). Right translations on $SU(2)$ act on the factor space as rotations with respect to a natural metric which comes from a bi-invariant metric on $SU(2)$. Neither an individual translation nor a one-parameter group is transitive but, of course, the whole right action is transitive. Moreover, every right translation also extends to a linear map of $\mathbb{C}^4$ and is, hence, entire.

At nth iterative step of the construction, one produces an entire diffeomorphism h_n of $S^3 = SU(2)$ which commutes with φ_{1/q_n}. To imitate the uniform distribution in the factor space, one uses twisted right translations. First, one fixes a one-parameter subgroup r_t of right translations of $SU(2)$ or, equivalently, an axis of rotation on S^2. On each orbit $\mathcal{O}$ of this one-parameter group, the map h_n is a certain translation $r_{f(\mathcal{O})}$ where the 'twisting function' f is entire and is invariant under the left translation φ_{1/q_n}. By making this function sufficiently uniformly distributed mod 1, one achieves a distribution of orbits of the conjugate of Φ (and, hence, of any of its sufficiently large finite subgroups)

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which projects to S^2 into a distribution close to the distribution uniform on orbits of r_t. This distribution, however, is far from uniform. Further approximation is achieved by taking another one-parameter subgroup and averaging the distribution along its orbits. In order to implement this, one takes a number $q_n^{'} \gg q_n$ depending on the geometry of the twisting function and constructs another twisting function along the orbits of the new subgroup and invariant with respect to $\varphi_{1/q_n'}$, taking the conjugacy as the product of two twisted right translations. This product is still entire and it takes the distribution of orbits of Φ and, hence, any of its sufficiently large finite subgroups closer to uniform. In fact, in order to achieve an almost uniform distribution of *every* orbit, one need to either take more than two steps within the inductive step or choose one-parameters subgroups of right translations at different inductive steps carefully.

Remark 7.3. This construction produces conjugacies which are far from preserving natural fundamental domains of φ_{1/q_n}. In other words, the choice of h_n is not special in the sense of §4.4.4 (see also §2.2). This makes controlling subtle ergodic properties of the resulting uniquely ergodic diffeomorphism problematic.

A typical open problem related with this difficulty is the following.

PROBLEM 7.4. *Does there exist a real-analytic volume-preserving diffeomorphism of* S^{2n-1}, $n \geq 2$ *which is measurably conjugate to an irrational rotation of the circle?*

7.1.2. *Exotic holomorphic maps in one complex variable.* The second method requires a good understanding of the geometry of special holomorphic maps in the complex domain since just approximating the conjugacies by some standard regular objects even if successful usually produces singularities close to the real domain for inverse maps. In one complex variable, there is a sufficient understanding of such special geometry. This allowed Perez-Marco and Yoccoz [**35–37, 43**], developing ideas which originally appeared in Yoccoz' thesis to apply the approximation by conjugation method and construct analytic circle diffeomorphisms with Liouvillean rotation numbers and exotic properties (e.g. C^∞ but not analytic conjugacy to a rotation).

7.1.3. *Other situations.* Beyond the two cases described here the problem of the applicability of the approximation by conjugation scheme of §2.1 remains open. This is the case already in the setting of volume-preserving diffeomorphisms in dimension two where the majority of examples described in this paper appear. Certain characteristic open problems which had appeared already at the time of writing [**1**] can be summarized as follows.

PROBLEM 7.5. *Does there exist an area-preserving topologically transitive real-analytic diffeomorphism* f *of the disc* $\mathbb{D}^2$ *with any of the following properties:*

(1) *the restriction of* f *to the boundary is an irrational rotation;*
(2) f *has zero topological entropy; or*
(3) f *is* C^2 *close to the identity?*

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We finish our discussion of analytic constructions with several comments concerning this problem.

In case (1) the rotation number must be Liouvillean: otherwise, there are invariant circles as was explained in §3.1.

The applicability of the approximation by conjugation method in the analytic setting in the case of manifolds with a boundary such as $\mathbb{D}^2$ depends on our ability to construct real-analytic diffeomorphisms of such manifolds with very large derivatives which can be extended together with their inverses to fixed complex neighborhoods. Here is a characteristic question of this kind.

PROBLEM 7.6. *Does there exist an $r > 0$ such that, for any point $p = (x, y) \in \operatorname{Int} \mathbb{D}^2$, there exists a real-analytic area-preserving diffeomorphism $f_p : \mathbb{D}^2 \to \mathbb{D}^2$ which extends together with its inverse to the r-neighborhood of $\mathbb{D}^2$ in the complex domain and such that $f_p(0, 0) = (x, y)$?*

Note that, for p close to the boundary, such a diffeomorphism f_p must have very large derivatives.

Known examples of real-analytic area-preserving ergodic (or just topologically transitive) diffeomorphisms of $\mathbb{D}^2$ are based on modifications on the C^∞ construction first introduced in [**26**]; see [**15**] and [**34**]. Dynamically, these examples are non-uniformly hyperbolic, they have positive topological entropy and, with respect to the smooth measure, they are Bernoulli (or direct products of Bernoulli with finite permutations). On the boundary, these diffeomorphisms are either the identity or have rational rotation number with finitely many periodic orbits. Such examples can be made C^1 close to the identity (first observed by Michel Herman—private communication) but constructions of this kind are not compatible with C^2 closeness to the identity.

7.2. *The problem of smooth realization.* In its most basic and at the same time general form, the smooth realization problem may be formulated as follows.

PROBLEM 7.7. (Smooth realization) *Given a measure-preserving transformation T of a Lebesgue space (X, μ), does there exist a C^∞ diffeomorphism f of a compact manifold M preserving a smooth volume ν such that (f, ν) is measurably isomorphic to (T, μ)?*

There are numerous variations and specializations of the problem of which the following is among the most interesting.

PROBLEM 7.8. (Non-standard smooth realization) *Given a diffeomorphism g of a compact manifold N preserving a smooth volume μ and ergodic with respect to it, does there exist a diffeomorphism f of a compact manifold M preserving a smooth volume ν such that (f, ν) is measurably isomorphic to (g, μ) but not smoothly conjugate to it?*

The only known unconditional restriction to smooth realization is finiteness of entropy (see [**29**, Corollary 3.2.10, Theorem 4.5.3]). In other words, no measure-preserving transformation with finite entropy is known not to be realizable on *any* manifold. There are restrictions in low dimension, e.g. any weakly mixing surface diffeomorphism is Bernoulli.

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The approximation by conjugation method provides instances of both non-standard smooth realization of certain diffeomorphisms and smooth realization of maps whose original description is not smooth. This includes non-standard smooth realization of rotations with some Liouvillean rotation numbers previously discussed in Question 2.4 (see [**1**, §4]: this was recently improved to any Liouvillean rotation number [**14**]). Very likely these methods should allow non-standard realization of some Diophantine rotations by diffeomorphisms of finite differentiability. It looks though that the following problem tests the limits of the power of the approximation by conjugation method and most likely requires new ideas.

PROBLEM 7.9. *Does there exist a non-standard smooth realization of a Diophantine rotation of the circle by a C^∞ diffeomorphism?*

Such a diffeomorphism must act on a manifold of dimension at least two.

In [**1**, §6], non-standard smooth realizations of certain translations on tori were constructed, as well as smooth realizations of certain rotations on the infinite dimensional torus. The latter was the earliest example of smooth realization of a transformation whose natural 'habitat' is not a finite dimensional manifold. In [**27**, §8], a general setting is described which allows (with further specification of parameters) the smooth realization of transformations belonging to measurable combinatorial constructions. Let us list several characteristic problems which, in our view, can be approached by a version of the approximation by conjugation method or its modification with decreasing chances of success.

PROBLEM 7.10. *Find a smooth realization of*

(1) *a Gaussian dynamical system with a simple (Kronecker) spectrum;*
(2) *a dense G_δ set of minimal interval exchange transformations;*
(3) *an adding machine;*
(4) *the time-one map of the horocycle flow 2.3.1 on the modular surface $SO(2)\backslash SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ (which is not compact, so the standard realization cannot be used).*

A version of the smooth realization problem addresses the possibility of obtaining certain properties invariant under measurable conjugacy for volume-preserving diffeomorphisms of compact manifolds. In particular, one may ask about properties which are generic for measure-preserving transformations in weak topology (which are also then generic for volume-preserving homeomorphisms in the C^0 topology). While such properties cannot be generic in C^r smooth topology by reasons quite different for $r = 1$ and $r \geq 2$, one can consider spaces of closures of conjugacies as in §2.3 and try to determine whether the generic properties of measure-preserving diffeomorphisms can be realized there or are generic in those spaces which we denote by $\mathcal{A}$ with various indexes. It seems that the notion of the periodic approximation of a given type in the sense of [**27**, Definition 1.9] is quite useful in this respect. Since approximation of any type with any (arbitrary fast) speed is generic in weak topology [**27**, Theorem 2.1], any property which follows from any finite or countable combination of approximation type properties

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is generic too. Cyclic approximation where approximating periodic processes have a single tower is generic in the space $\mathcal{A}$ and certain types of homogeneous approximation where there may be many towers but their heights are all equal [**27**, §5] can be shown to be generic too. However, periodic processes with towers of unequal height such as type $(n, n+1)$ approximations with two towers whose heights differ by one do not naturally appear in the conjugacies of elements of S^1 actions. Those approximations and their generalizations are useful in establishing the genericity of various properties (see, e.g. [**27**, §3.3]). It looks likely that, in the case of actions of $\mathbb{T}^k$, $k \geq 2$, it may be easier to produce these types of approximation in the closure of conjugacies.

PROBLEM 7.11. *Given a circle action $\mathcal{S}$ and the corresponding space $\mathcal{A}$, is there a diffeomorphism $f \in \mathcal{A}$ with any of the following properties:*
(1) *a good approximation of type $(n, n+1)$;*
(2) *a maximal spectral type disjoint with its convolutions;*
(3) *a homogeneous spectrum of multiplicity two for the Carthesian square $f \times f$?*

See [**27**, §§3,4] for the relevant definitions and results.

PROBLEM 7.12. *Does a measurable square root (or roots of all orders) generic in $\mathcal{A}$ exist?* (cf. [**31**]).

Remark 7.13. Note that, unlike the previous problem, the existence is known here because, for example, any rotation of the circle has roots of all orders and there are diffeomorphisms measurably isomorphic to rotations [**1**, §4].

Given an action $\mathcal{T}$ of $\mathbb{T}^2$, denote the closure of conjugacies of elements of this action in the C^∞ topology by $\mathcal{B}$.

PROBLEM 7.14. *Is there a diffeomorphism $f \in \mathcal{B}$ with any of the following properties:*
(1) *a good approximation of type $(n, n+1)$;*
(2) *a maximal spectral type disjoint with its convolutions;*
(3) *a homogeneous spectrum of multiplicity two for the Carthesian square $f \times f$;*
(4) *mixing?*

Positive answer to the last question will be, of course, a discrete-time version of Theorem 6.2. See Remark 6.3.

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