Math. Proc. Camb. Phil. Soc. (1996), **119**, 287
Printed in Great Britain

Projection theorems for box and packing dimensions

BY K. J. FALCONER AND J. D. HOWROYD
Mathematical Institute, University of St. Andrews, North Haugh, St. Andrews, Fife KY16 9SS, Scotland

(*Received* 21 *July* 1994)

Abstract

We show that if E is an analytic subset of $\mathbb{R}^n$ then

$$\dim_{\mathrm{P}} \mathrm{proj}_V E \geqslant \frac{\dim_{\mathrm{P}} E}{1+(1/m-1/n)\dim_{\mathrm{P}} E}$$

for almost all m-dimensional subspaces V of $\mathbb{R}^n$, where $\mathrm{proj}_V E$ is the orthogonal projection of E onto V and $\dim_{\mathrm{P}}$ denotes packing dimension. The same inequality holds for lower and upper box counting dimensions, and these inequalities are the best possible ones.

1. *Introduction and notation*

This paper concerns orthogonal projections of subsets of Euclidean n-space, $\mathbb{R}^n$, onto linear subspaces. We are particularly concerned with the relationship between the dimension of the projections and the dimension of the original set. With any reasonable definition of dimension, every orthogonal projection of a set E onto an m-dimensional subspace has dimension at most $\min\{m, \text{dimension } (E)\}$, so it is the lower bounds for dimensions of projections that are of interest.

The fundamental result on projections concerns Hausdorff dimensions: the projection of an analytic subset E of $\mathbb{R}^n$ onto almost all m-dimensional subspaces has Hausdorff dimension equal to $\min\{m, \dim_{\mathrm{H}} E\}$ where $\dim_{\mathrm{H}} E$ denotes the Hausdorff dimension of E. (Of course, the class of analytic sets include the Borel sets, compact sets and all sets normally encountered.) This was proved in the case $n = 2, m = 1$ by Marstrand [7], and generalized to higher dimensions by Mattila [8] using a potential theoretic method of Kaufman [6].

Although Hausdorff dimension is convenient for many mathematical purposes, other definitions of dimension, such as packing dimension and box-counting dimensions, are increasingly being used. It is natural to consider projection theorems that relate to these definitions of dimension. Surprisingly, the direct analogue of the Hausdorff dimension result is not true; for $1 \leqslant m \leqslant n-1$ and all $0 < s < n$ there are subsets of $\mathbb{R}^n$ of packing dimension s with projections onto all m-dimensional subspaces of packing dimension strictly less than $\min(m, s)$. Examples of such sets were given by Järvenpää in [5] (see Section 3), who also showed that $\dim_{\mathrm{P}} \mathrm{proj}_V E \geqslant (m/n)\dim_{\mathrm{P}} E$ for almost all subspaces V, where $\mathrm{proj}_V E$ is the orthogonal projection of E onto V and $\dim_{\mathrm{P}}$ denotes packing dimension.

https://doi.org/10.1017/S0305004100074168 Published online by Cambridge University Press

In this paper we improve this estimate to the best possible one:

$$\dim_{\mathrm{P}} \mathrm{proj}_V E \geqslant \frac{\dim_{\mathrm{P}} E}{1+(1/m-1/n)\dim_{\mathrm{P}} E}$$

for almost all m-dimensional subspaces V and all analytic subsets $E \subseteq \mathbb{R}^n$. We achieve this by first obtaining the corresponding results for upper and lower box-counting dimensions, using an estimate of a combinatorial geometric nature. We then use the relationship between upper box-counting dimension and packing dimension to get the packing dimension inequality.

We work in $\mathbb{R}^n$ with the usual norm $|\cdot|$. We denote by $G_{n,m}$ the Grassmann manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$. The Grassmann manifold carries a natural invariant measure which we denote by $\gamma_{n,m}$ which we assume to be normalized to have unit total measure. In particular this enables us to refer to a property pertaining to 'almost all' linear spaces V, meaning a set of V whose complement has $\gamma_{n,m}$-measure 0. For $V \in G_{n,m}$ we let $\mathrm{proj}_V : \mathbb{R}^n \to V$ denote orthogonal projection onto V.

Box-counting dimension, or Minkowski dimension, of subsets of $\mathbb{R}^n$ may be defined in several equivalent ways, see for example [**1**, **10**, **11**]. We make use of two such equivalent definitions here. For $E \subseteq \mathbb{R}^n$ we define $\mathrm{N}_r(E)$ to be the minimum number of subsets of $\mathbb{R}^n$ of diameter at most r that cover E and $\mathrm{N}_r^*(E)$ to be the greatest possible number of disjoint closed balls of radius r that may be found with centres in E. The *upper box-counting dimension* $\overline{\dim}_{\mathrm{B}} E$ of E is given by

$$\overline{\dim}_{\mathrm{B}} E = \limsup_{r \searrow 0} \frac{\log \mathrm{N}_r(E)}{-\log r} = \limsup_{r \searrow 0} \frac{\log \mathrm{N}_r^*(E)}{-\log r}.$$

The *lower-box counting dimension* $\underline{\dim}_{\mathrm{B}} E$ of E is given by

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{r \searrow 0} \frac{\log \mathrm{N}_r(E)}{-\log r} = \liminf_{r \searrow 0} \frac{\log \mathrm{N}_r^*(E)}{-\log r}.$$

(For the equivalence of the N_r and N_r^* definitions, see section 3·1 of [**1**].) If $\overline{\dim}_{\mathrm{B}} E = \underline{\dim}_{\mathrm{B}} E$ the common value is referred to as the *box-counting* or *box dimension* of E.

For $E \subseteq \mathbb{R}^n$ we call a disjoint collection of closed balls $(B_i)_{i=1}^{\infty}$ of radii at most δ and with centres in E a *δ-packing* of E. For $s \geqslant 0$ the *packing s-measure* Π^s, which was introduced by Tricot [**10**], is defined using the following steps:

$$\Pi_\delta^s(E) = \sup\left\{ \sum_{i=1}^{\infty} (\mathrm{diam}\, B_i)^s \,\middle|\, (B_i)_{i=1}^{\infty} \text{ is a } \delta\text{-packing of } E \right\},$$

$$\Pi_0^s(E) = \lim_{\delta \searrow 0} \Pi_\delta^s(E) = \inf_{\delta > 0} \Pi_\delta^s(E),$$

$$\Pi^s(E) = \inf\left\{ \sum_{i=1}^{\infty} \Pi_0^s(E_i) \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}.$$

The packing dimension $\dim_{\mathrm{P}} E$ of E is then defined as

$$\dim_{\mathrm{P}} E = \inf\{s > 0 \mid \Pi^s(E) = 0\}.$$

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There is a very useful characterization of the packing dimension of a subset of $\mathbb{R}^n$ in terms of upper box dimension:

$$\dim_{\mathrm{P}} E = \inf\left\{ \sup_{1 \leqslant i < \infty} \overline{\dim}_{\mathrm{B}} E_i \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} E_i \right\},$$

see Proposition 3·8 of [**1**] for details.

2. *Projection theorems*

The following proposition, which is essentially a result in combinatorial geometry, is central to our dimension estimates for projections.

PROPOSITION 1. *Let K be a k point subset of $\mathbb{R}^n$ with the property that any pair of points of K are a distance at least $2r$ apart. Then for all h with $0 < h \leqslant r$*

$$\int_{G_{n,m}} (\mathrm{N}_h(\mathrm{proj}_V K))^{-1} \,\mathrm{d}\gamma_{n,m}(V) \leqslant c h^m k^{-m/n} r^{-m} + k^{-1}, \tag{1}$$

where c depends only on n and m.

Proof. We define μ to be the (discrete) probability measure on K that assigns a mass of k^{-1} to each point of K. For $V \in G_{n,m}$ we denote the projection of μ onto V by μ_V, so

$$\mu_V(A) = \mu(\{x \in \mathbb{R}^n \,|\, \mathrm{proj}_V(x) \in A\}) \quad (\text{for all } A \subseteq V).$$

Let $U_1, \ldots, U_q$ be a cover of $\mathrm{proj}_V K$ such that for every i the set U_i has diameter at most h. In estimating $\mathrm{N}_h(\mathrm{proj}_V K)$ we may assume that the sets U_i are disjoint. Using Cauchy's inequality

$$\begin{aligned} 1 = \left(\mu_V\left(\bigcup_{i=1}^{q} U_i\right)\right)^2 &\leqslant \left(\sum_{i=1}^{q} \mu_V(U_i)\right)^2 \leqslant q \sum_{i=1}^{q} (\mu_V(U_i))^2 \\ &\leqslant q(\mu_V \times \mu_V)\{(u,v) \in V \times V \,|\, |u-v| \leqslant h\}. \end{aligned}$$

Hence

$$(\mathrm{N}_h(\mathrm{proj}_V K))^{-1} \leqslant (\mu_V \times \mu_V)\{(u,v) \in V \times V \,|\, |u-v| \leqslant h\}.$$

Integrating over $G_{n,m}$ and using Fubini's Theorem we have

$$\begin{aligned} &\int_{G_{n,m}} (\mathrm{N}_h(\mathrm{proj}_V K))^{-1} \,\mathrm{d}\gamma_{n,m}(V) \\ &\leqslant \int_{G_{n,m}} \iint_{|u-v| \leqslant h} \mathrm{d}\mu_V(u) \,\mathrm{d}\mu_V(v) \,\mathrm{d}\gamma_{n,m}(V) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \gamma_{n,m}\{V \in G_{n,m} \,|\, |\mathrm{proj}_V(x) - \mathrm{proj}_V(y)| \leqslant h\} \,\mathrm{d}\mu(y) \,\mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_{n,m}\left(\frac{x-y}{h}\right) \mathrm{d}\mu(y) \,\mathrm{d}\mu(x) \end{aligned} \tag{2}$$

where

$$\phi_{n,m}(z) = \gamma_{n,m}\{V \in G_{n,m} \,|\, |\mathrm{proj}_V(z)| \leqslant 1\}$$

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(inequality (2) follows by the linearity of proj_V). Inequality (1) will follow if we can show

$$\int_{\mathbb{R}^n} \phi_{n,m}\left(\frac{x-y}{h}\right) \mathrm{d}\mu(y) \leqslant ch^m k^{-m/n} r^{-m} + k^{-1}, \tag{3}$$

where c depends only on n and m.

We note that $\phi_{n,m}\colon \mathbb{R}^n \to \mathbb{R}$ is spherically symmetric and $\phi_{n,m}(z)$ is non-increasing as $|z|$ increases. (Thus $\phi_{n,m}$ may be viewed as a convolution kernel.) Further, for all $z \in \mathbb{R}^n$,

$$\phi_{n,m}(z) \leqslant \min\{1, c_1 |z|^{-m}\}$$

where c_1 depends only on n and m. The first estimate here is clear from the definition of $\phi_{n,m}$, the second estimate is a routine calculation, see for example section 2 of [8]. It follows immediately that

$$\int_{\mathbb{R}^n} \phi_{n,m}(z)\, \mathrm{d}\mu(z) \leqslant c_1 \int_{\mathbb{R}^n \setminus \mathrm{B}(0,1)} |z|^{-m}\, \mathrm{d}\mu(z) + \mu(\mathrm{B}(0,1))$$

and hence by scaling

$$\int_{\mathbb{R}^n} \phi_{n,m}\left(\frac{x-y}{h}\right) \mathrm{d}\mu(y) \leqslant c_1 h^m \int_{\mathbb{R}^n \setminus \mathrm{B}(x,h)} \frac{\mathrm{d}\mu(y)}{|x-y|^m} + \mu(\mathrm{B}(x,h)).$$

(We write $\mathrm{B}(x,h)$ for the closed ball of centre x and radius h.)

We fix $x \in K$ and consider the expression

$$\int_{\mathbb{R}^n} \phi_{n,m}\left(\frac{x-y}{h}\right) \mathrm{d}\mu(y). \tag{4}$$

By a simple packing argument the ball $\mathrm{B}(x, 4n^{1/2}k^{1/n}r)$ contains a ball of radius r, say $\mathrm{B}(w,r)$, that contains no point of K. If K contains any point z outside $\mathrm{B}(x, 4n^{1/2}k^{1/n}r)$ then replacing z by w does not decrease the value of (4), by the monotonicity of $\phi_{n,m}(z)$ in $|z|$. Thus when seeking an upper estimate for (4) we may assume that $K \subseteq \mathrm{B}(x, ak^{1/n}r)$ where a depends only on n. By the hypothesis that $0 < h \leqslant r$ and comparing μ with a multiple of Lebesgue measure $\mathscr{L}$ restricted to suitable balls of radius $\frac{1}{2}r$ with the points of K as their centres it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi_{n,m}\left(\frac{x-y}{h}\right) \mathrm{d}\mu(y) &\leqslant c_1 h^m \int_{\mathbb{R}^n \setminus \mathrm{B}(x,h)} \frac{\mathrm{d}\mu(y)}{|x-y|^m} + \mu(\mathrm{B}(x,h)) \\
&\leqslant c_2 h^m \int_{\mathrm{B}(x, ak^{1/n}r)} \frac{r^{-n}k^{-1}\, \mathrm{d}\mathscr{L}(y)}{|y-x|^m} + \mu(\mathrm{B}(x,r)) \\
&\leqslant c_3 h^m r^{-n} k^{-1} \int_{\rho=0}^{ak^{1/n}r} \rho^{-m} \rho^{n-1}\, \mathrm{d}\rho + k^{-1} \\
&\leqslant ch^m k^{-1+(n-m)/n} r^{-m} + k^{-1} \\
&= ch^m k^{-m/n} r^{-m} + k^{-1},
\end{aligned}$$

where c_2, c_3 and c depend only on n and m; the k^{-1} term comes from considering μ near x. This is inequality (3) and hence the result follows.

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COROLLARY 2. *Suppose that for some $E \subseteq \mathbb{R}^n$ and some $r > 0$ we have $\mathrm{N}_r^*(E) \geqslant r^{-s}$. Then*

$$\int_{G_{n,m}} (\mathrm{N}_h(\mathrm{proj}_V E))^{-1} \, \mathrm{d}\gamma_{n,m}(V) \leqslant (c+1)\, r^s$$

where $h = r^{1+s(1/m-1/n)}$ and c is as in Proposition 1.

Proof. In Proposition 1, take K to be the set of centres of $\mathrm{N}_r^*(E)$ disjoint balls with centres in E, so that $k = \mathrm{N}_r^*(E)$, to get

$$\begin{aligned}\int_{G_{n,m}} (\mathrm{N}_h(\mathrm{proj}_V E))^{-1} \, \mathrm{d}\gamma_{n,m}(V) &\leqslant \int_{G_{n,m}} (\mathrm{N}_h(\mathrm{proj}_V K))^{-1} \, \mathrm{d}\gamma_{n,m}(V) \\ &\leqslant c r^{m+s-sm/n} r^{sm/n-m} + r^s \\ &\leqslant (c+1)\, r^s.\end{aligned}$$

We can now deduce the upper and lower box-dimension inequalities.

THEOREM 3. *For $E \subseteq \mathbb{R}^n$ we have*

$$(a) \qquad \overline{\dim}_{\mathrm{B}} \mathrm{proj}_V E \geqslant \frac{\overline{\dim}_{\mathrm{B}} E}{1+(1/m-1/n)\,\overline{\dim}_{\mathrm{B}} E}$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$, and

$$(b) \qquad \underline{\dim}_{\mathrm{B}} \mathrm{proj}_V E \geqslant \frac{\underline{\dim}_{\mathrm{B}} E}{1+(1/m-1/n)\,\underline{\dim}_{\mathrm{B}} E}$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$.

Proof. (*a*) Let $s < \overline{\dim}_{\mathrm{B}} E$; then there exists a sequence $(r_k)_{k=1}^{\infty}$ with $r_k \searrow 0$ such that $\mathrm{N}_{r_k}^*(E) \geqslant r_k^{-s}$. Setting $h_k = r_k^{1+s(1/m-1/n)}$ we see from Corollary 2 that

$$\int_{G_{n,m}} (\mathrm{N}_{h_k}(\mathrm{proj}_V E))^{-1} \, \mathrm{d}\gamma_{n,m}(V) \leqslant (c+1)\, r_k^s.$$

Hence for all $\epsilon > 0$ we have

$$\gamma_{n,m}\{V \in G_{n,m} \mid (\mathrm{N}_{h_k}(\mathrm{proj}_V E))^{-1} \geqslant r_k^{s-\epsilon}\} \leqslant (c+1)\, r_k^{\epsilon}.$$

Now

$$\begin{aligned}&\{V \in G_{n,m} \mid \overline{\dim}_{\mathrm{B}} \mathrm{proj}_V E < (s-\epsilon)/(1+s(1/m-1/n))\} \\ &\quad \subseteq \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} \{V \in G_{n,m} \mid (\mathrm{N}_{h_k}(\mathrm{proj}_V E))^{-1} \geqslant r_k^{s-\epsilon}\}\end{aligned}$$

and hence this set has $\gamma_{n,m}$-measure 0. Since ϵ may be chosen as small as we wish, it follows that $\overline{\dim}_{\mathrm{B}} \mathrm{proj}_V E \geqslant s/(1+s(1/m-1/n))$ for $\gamma_{n,m}$-almost all $V \in G_{n,m}$. The result now follows since s may be taken arbitrarily close to $\overline{\dim}_{\mathrm{B}} E$.

(*b*) Let $s < \underline{\dim}_{\mathrm{B}} E$; then $\mathrm{N}_{2^{-k}}^*(E) \geqslant 2^{ks}$ for all sufficiently large k, say for $k \geqslant k_0$. Thus for all $\epsilon > 0$ we have by Corollary 2

$$\int_{G_{n,m}} \sum_{k=k_0}^{\infty} 2^{k(s-\epsilon)} (\mathrm{N}_{h_k}(\mathrm{proj}_V E))^{-1} \, \mathrm{d}\gamma_{n,m}(V) \leqslant \sum_{k=k_0}^{\infty} 2^{k(s-\epsilon)} (c+1)\, 2^{-ks} < \infty,$$

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where $h_k = 2^{-k(1+s(1/m-1/n))}$. Hence for $\gamma_{n,m}$-almost all $V \in G_{n,m}$ the numbers

$$h^{(s-\epsilon)/(1+s(1/m-1/n))}\, \mathrm{N}_h(\mathrm{proj}_V E) \tag{5}$$

are bounded away from 0 for $h = h_k$ with $k \geqslant k_0$. But since h_k converges to 0 at a geometric rate and $\mathrm{N}_h(\mathrm{proj}_V E)$ is non-decreasing as h decreases, the expression (5) is bounded away from 0 for all h with $0 < h < 1$. It follows that

$$\underline{\dim}_{\mathrm{B}} \mathrm{proj}_V E \geqslant (s-\epsilon)/(1+s(1/m-1/n))$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$. Since ϵ may be chosen as small as we wish it follows that $\underline{\dim}_{\mathrm{B}} \mathrm{proj}_V E \geqslant s/(1+s(1/m-1/n))$ for $\gamma_{n,m}$-almost all $V \in G_{n,m}$. The inequality now follows since s may be chosen arbitrarily close to $\underline{\dim}_{\mathrm{B}} E$.

We next deduce the projection theorem for packing dimension from the result for upper box dimension. We require the following two simple lemmas on packing dimension. Various versions of these may be found in basic treatments of packing dimension see [1, 11]; for convenience we give short proofs of forms suitable for our purposes.

LEMMA 4. *Let E be an analytic subset of $\mathbb{R}^n$. For all s with $0 < s < \dim_{\mathrm{P}} E$, there exists a compact set $F \subseteq E$ with $\dim_{\mathrm{P}}(F \cap G) \geqslant s$ for every open set $G \subseteq \mathbb{R}^n$ that intersects F.*

Proof. Using the regularity of packing measures (see [3]) there exists a compact set $H \subseteq E$ of positive Π^s-measure. Letting F be the support of the measure Π^s restricted to H gives a set with the desired properties.

LEMMA 5. *Let F be a compact subset of $\mathbb{R}^n$ with $\overline{\dim}_{\mathrm{B}}(F \cap G) \geqslant s$ for all open sets $G \subseteq \mathbb{R}^n$ that intersect F. Then $\dim_{\mathrm{P}} F \geqslant s$.*

Proof. Let $F \in \bigcup_{i=1}^{\infty} F_i$. By Baire's category theorem there exists i and an open set $G \subseteq \mathbb{R}^n$ such that $F \cap G \subseteq F_i$, so $\overline{\dim}_{\mathrm{B}} F_i \geqslant s$. Therefore

$$\dim_{\mathrm{P}} F = \inf\left\{ \sup_{1 \leqslant i < \infty} \overline{\dim}_{\mathrm{B}} F_i \,\middle|\, F \subseteq \bigcup_{i=1}^{\infty} F_i \right\} \geqslant s.$$

THEOREM 6. *Let E be an analytic subset of $\mathbb{R}^n$. Then*

$$\dim_{\mathrm{P}} \mathrm{proj}_V E \geqslant \frac{\dim_{\mathrm{P}} E}{1+(1/m-1/n)\dim_{\mathrm{P}} E}$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$.

Proof. For $s < \dim_{\mathrm{P}} E$ we may, by Lemma 4, find a compact set $F \subseteq E$ with

$$\overline{\dim}_{\mathrm{B}} F \cap G \geqslant \dim_{\mathrm{P}} F \cap G \geqslant s$$

for all open sets $G \subseteq \mathbb{R}^n$ that intersect F. By the separability of $\mathbb{R}^n$ there exists a countable basis of the usual topology on $\mathbb{R}^n$ (open balls with rational radii and centres with rational coordinates, say). We let $(G_i)_{i=1}^{\infty}$ be an enumeration of those sets of such a basis that intersect F. By Theorem 3(*a*), there exists for each i $(1 \leqslant i < \infty)$ a set $S_i \subseteq G_{n,m}$ of $\gamma_{n,m}$-measure 0 such that

$$\overline{\dim}_{\mathrm{B}} \mathrm{proj}_V(F \cap G_i) \geqslant \frac{s}{1+(1/m-1/n)\,s} \qquad (\text{for all } V \in G_{n,m} \backslash S_i).$$

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Then $S = \bigcup_{i=1}^{\infty} S_i$ has $\gamma_{n,m}$-measure 0. Let $V \in G_{n,m} \backslash S$. For any open subset U of V that intersects $\mathrm{proj}_V F$ we may choose an integer i such that $G_i \subseteq \mathrm{proj}_V^{-1}(U)$. Then

$$\mathrm{proj}_V(F \cap G_i) \subseteq (\mathrm{proj}_V F) \cap U$$

and hence

$$\overline{\dim}_{\mathrm{B}}(\mathrm{proj}_V F) \cap U \geqslant \overline{\dim}_{\mathrm{B}} F \cap G_i \geqslant \frac{s}{1+(1/m-1/n)s}.$$

Applying Lemma 5 to $\mathrm{proj}_V F$ we get

$$\dim_{\mathrm{P}} \mathrm{proj}_V E \geqslant \dim_{\mathrm{P}} \mathrm{proj}_V F \geqslant \frac{s}{1+(1/m-1/n)s}. \tag{6}$$

Inequality (6) holds for all $s < \dim_{\mathrm{P}} E$ and $\gamma_{n,m}$-almost all $V \in G_{n,m}$, so by taking a sequence of s tending to $\dim_{\mathrm{P}} E$ the result follows.

3. *Examples*

We give some examples to show that the results of Section 2 are best possible.

Example 1. For all $n \geqslant 2$ and $0 < s < n$ there exists a compact set $E \subseteq \mathbb{R}^n$ such that

$$\underline{\dim}_{\mathrm{B}} E = \overline{\dim}_{\mathrm{B}} E = s$$

and, for $m = 1, \ldots, n-1$,

$$\underline{\dim}_{\mathrm{B}} \mathrm{proj}_V E = \overline{\dim}_{\mathrm{B}} \mathrm{proj}_V E = \frac{s}{1+(1/m-1/n)s}$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$.

Proof. For $i = 1, 2, \ldots$, let $E_i \subseteq \mathbb{R}^n$ be a subset of the sphere centre the origin and radius $i^{1-n/s}$ consisting of i^{n-1} points distributed 'as evenly as possible' over the sphere (in fact all we require is that the mutual separation of the points in E_i is at least $ci^{-n/s}$ for some c independent of i). Let E be the compact set $\bigcup_{i=1}^{\infty} E_i \cup \{0\}$.

For $0 < r < 1$ let k be the positive integer such that

$$k^{-n/s} \leqslant r < (k-1)^{-n/s}.$$

Then
$$E = \left(\bigcup_{i=1}^{k-1} E_i\right) \cup \left(\bigcup_{i=k}^{\infty} E_i\right) \cup \{0\} \subseteq \left(\bigcup_{i=1}^{k-1} E_i\right) \cup \mathrm{B}(0, k^{1-n/s}).$$

But $\mathrm{B}(0, k^{1-n/s})$ may be covered by $c_1(k^{1-n/s}/k^{-n/s})^n = c_1 k^n$ balls of radius $k^{-n/s} \leqslant r$, where c_1 is independent of r. Also $\bigcup_{i=1}^{k-1} E_i$ consists of $\sum_{i=1}^{k} i^{n-1} \leqslant k^n$ points. Thus $\mathrm{N}_r(E) \leqslant (c_1+1)k^n \leqslant c_2 r^{-s}$ where c_2 is independent of r. It follows that $\overline{\dim}_{\mathrm{B}} E \leqslant s$.

For the opposite inequality we note that the points of $\bigcup_{i=1}^{k} E_i$ have mutual separation at least

$$\min\{k^{1-n/s} - (k-1)^{1-n/s}, ck^{-n/s}\} \geqslant c_3 k^{-n/s}.$$

For all sufficiently small r we may find an integer k such that $\frac{1}{2}c_3(k+1)^{-n/s} \leqslant r < \frac{1}{2}c_3 k^{-n/s}$ so

$$\mathrm{N}_r(E) \geqslant \sum_{i=1}^{k} i^{n-1} \geqslant c_4 k^n$$

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where c_4 is independent of k. Thus $N_r(E) \geqslant c_5 r^{-s}$ for $0 < r < 1$ where c_5 is independent of r. Therefore $\underline{\dim}_B E \geqslant s$.

For $V \in G_{n,m}$ we consider the set $\mathrm{proj}_V E$. For $0 < r < 1$, let k be the positive integer such that $k^{1-n/s-n/m} \leqslant r < (k-1)^{1-n/s-m/m}$. Then as before we have

$$\mathrm{proj}_V E \subseteq \left(\bigcup_{i=1}^{k-1} \mathrm{proj}_V E_i\right) \cup \mathrm{proj}_V \mathrm{B}(0, k^{1-n/s}).$$

Now $\mathrm{proj}_V \mathrm{B}(0, k^{1-n/s})$ may be covered by $c_6(k^{1-n/s}/k^{1-n/s-n/m})^m = c_6 k^n$ balls of radius r where c_6 is independent of k. Also $\bigcup_{i=1}^{k-1} \mathrm{proj}_V E_i$ consists of at most $\sum_{i=1}^{k-1} i^{n-1} \leqslant k^n$ points. Hence

$$N_r(\mathrm{proj}_V E) \leqslant (c_6 + 1) k^n \leqslant c_7 r^{n/(1-n/s-n/m)}.$$

Thus
$$\overline{\dim}_B \mathrm{proj}_V E \leqslant n/(n/s + n/m - 1) = \frac{s}{1 + (1/m - 1/n)s}$$

and this is true for all $V \in G_{n,m}$. The result now follows by Theorem 3.

For a nice variation of this example in the case $n = 2$, $m = 1$, the spiral of points given in polar coordinates by $r = \theta^{1-2/s}$ for $\theta = k^{\frac{1}{2}}$ $(k = 1, 2, \ldots)$ has lower and upper box dimensions s and projections in almost all directions with the upper and lower box dimensions $s/(1 + \frac{1}{2}s)$, by a similar argument to the above.

For completeness we mention the following example that shows the packing dimension estimate in Theorem 6 is best possible. Such an example was discussed by Järvenpää in [5].

Example 2. For all $n \geqslant 2$ and $0 < s < n$ there exists a compact set $E \subseteq \mathbb{R}^n$ such that $\dim_P E = s$ and for $m = 1, \ldots, n-1$

$$\dim_P \mathrm{proj}_V E = \frac{s}{1 + (1/m - 1/n)s}$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$.

Proof. Let $(r_i)_{i=0}^{\infty}$ be a sequence of positive real numbers and $(k_i)_{i=0}^{\infty}$ be a sequence of integers such that the following conditions are satisfied:

1. $r_0 = 1$ and $(r_i)_{i=0}^{\infty}$ is rapidly decreasing with $\log r_{i+1}/\log r_i \to \infty$,
2. $k_0 = 1$ and $k_{i+1}^n r_{i+1}^s = r_i^s$ for $i = 0, 1, 2, \ldots$.

Consider the following nested construction of closed cubes. We start with E_0 as the unit cube in $\mathbb{R}^n$. Let E_1 be a block of $k_1 \times \ldots \times k_1 = k_1^n$ closed cubes of side length r_1 placed in a cubic lattice arrangement at the centre of E_0. Let E_2 be the union of blocks of $k_2 \times \ldots \times k_2 = k_2^n$ cubes of side length r_2 placed in a lattice arrangement at the centre of each of the cubes of E_1, etc. Let $E = \bigcap_{i=0}^{\infty} E_i$. Then E has the properties specified.

The dimension of E may be checked by showing that

$$0 < \liminf_{r \searrow 0} \mu(\mathrm{B}(x, r)) r^{-s} < \infty \quad (\forall x \in E)$$

where μ is the natural mass distribution on E obtained by repeated subdivision on the hierarchy of sets $(E_i)_{i=0}^{\infty}$. A box counting argument shows that

$$\dim_P \mathrm{proj}_V E \leqslant \overline{\dim}_B \mathrm{proj}_V E \leqslant \frac{s}{1 + (1/m - 1/n)s}$$

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for all $V \in G_{n,m}$ and the opposite inequality follows form Theorem 6. Full details are given in [5].

4. *Remarks*

For $E \subseteq \mathbb{R}^n$ one may consider the Hausdorff dimension of the exceptional set of subspaces

$$\{V \in G_{n,m} \mid \text{`dim' proj}_V E < t\}$$

where t is best possible lower bound or, more generally, as t varies. The case when 'dim' denotes Hausdorff dimension has been studied by various authors; see [2, 6, 8]. Methods akin to those used in [8] could with some effort give information about the Hausdorff dimension of the exceptional set of subspaces with 'dim' representing the packing dimension. To do this the integral in (1) would need to be taken with respect to measures ν supported on a small subset of $G_{n,m}$ rather than the measure $\gamma_{n,m}$. However, it may be more natural to seek the packing dimension of this exceptional set.

We note that the set E of Example 2 has Hausdorff dimension zero. On the other hand if the Hausdorff dimension and the packing dimension of a subset $E \subseteq \mathbb{R}^n$ are equal then by the result on the Hausdorff dimension of projections it immediately follows that

$$\dim_{\mathrm{P}} \mathrm{proj}_V E = \min\{m, \dim_{\mathrm{P}} E\}$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$. Thus a natural question is to find the best possible lower bound for $\dim_{\mathrm{P}} \mathrm{proj}_V E$ (for $\gamma_{n,m}$-almost all $V \in G_{n,m}$) given both the Hausdorff dimension and the packing dimension of E.

In a recent paper Hu and Taylor [4] obtained some results on projections of Borel measures which have equal local Hausdorff and local packing dimensions. Without such equality behaviour such as that shown here comes into play.

Note added in proof: Very recently, Falconer and Mattila have refined Theorem 6 to take account of $\dim_{\mathrm{H}} E$, and Falconer and Howroyd have obtained results on the exceptional set of projections and shown that the packing dimensions of the projections in almost all directions are equal.

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