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On Σ_1-definable closed unbounded sets

Omer Ben-Neria and Philipp Lücke

Abstract. Definable stationary sets, and specifically, ordinal definable ones, play a significant role in the study of canonical inner models of set theory and the class HOD of hereditarily ordinal definable sets. Fixing a certain notion of definability and an uncountable cardinal, one can consider the associated family of definable closed unbounded sets. In this paper, we study the extent to which such families can approximate the full closed unbounded filter and their dependence on the defining complexity. Focusing on closed unbounded subsets of a cardinal κ which are Σ_1-definable in parameters from H_κ and ordinal parameters, we show that the ability of such closed unbounded sets to well approximate the closed unbounded filter on κ can highly vary and strongly depends on key properties of the underlying universe of set theory.

1 Introduction

The concepts of closed unbounded and stationary subsets capture many of the key aspects of the combinatorics of cardinals of uncountable cofinalities. Recent developments in the study of canonical inner models of set theory provide strong motivations for analyzing the definability – specifically, the ordinal definability – of these objects. In particular, the notion of a *ω-strongly measurable cardinal κ in* HOD, introduced by Woodin (see [21, Definition 189]) to measure local failures of the inner model HOD to approximate the set-theoretic universe V, is equivalent to the fact that the restriction of the closed unbounded filter on $\{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \omega\}$ to HOD identifies with the intersection of a small number of ordinal definable normal measures on κ in HOD (see [2, Lemma 2.4]). Since, in HOD, many of the sets in this intersection do not contain a closed unbounded subset, it follows that there are many subsets of the cardinal κ that are HOD-stationary (i.e., that meet every closed unbounded subset of κ that is an element of HOD) but are not stationary subsets of κ in V.

Motivated by this observation, the purpose of the work presented in this paper is to study the extent to which hierarchies of definable closed unbounded subsets of an uncountable cardinal κ can approximate the full closed unbounded filter on κ, where

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the different hierarchies are given in terms of the complexity of the defining formulas (e.g., Σ_n-formulas[1] for some natural number n) and the set of parameters allowed in these formulas (e.g., ordinal parameters or parameters contained in H_κ). Since ordinal definable sets of ordinals can be represented as the unique solutions of Σ_2-formulas with ordinal parameters (see the proof of [9, Lemma 13.25] for details) and closed unbounded subsets of uncountable cardinals κ that are definable by Σ_0-formulas with ordinals and elements of H_κ can be shown to have a very simple, eventually periodic structure,[2] the first place to expect different phenomena is with Σ_1-definable closed unbounded sets. In order to motivate the definitions and results contained in this paper, we start by presenting two settings in which Σ_1-definable closed unbounded sets give an optimal approximation to the collection of all closed unbounded sets:

- In Gödel's constructible universe L, every subset (and, in particular, every closed unbounded subset) of an infinite cardinal κ is the unique solution of a Σ_1-formula with parameters from κ^+. This follows directly from the fact that every such subset is the η-th element in the canonical well-ordering of L for some $\eta < \kappa^+$, and transitive models of "$\mathrm{ZFC}^- + V = L$" containing an ordinal η can uniformly compute the initial segment of order-type $\eta + 1$ of this well-ordering.
- *Martin's Maximum* MM implies that every closed unbounded subset of ω_1 contains a closed unbounded subset that is the unique solution of a Σ_1-formula that only uses the ordinal ω_1 and real numbers as parameters. This statement follows directly from results of Woodin that show that MM implies *admissible club guessing* (see [20, Theorems 3.16, 3.17 & 3.19]).

In addition, results that will be contained in a sequel to this paper show that for every uncountable cardinal κ satisfying $\kappa^{<\kappa} = \kappa$, there is cofinality-preserving forcing extension $V[G]$ of the ground model V in which there exists a subset E of κ such that every closed unbounded subset of κ contains a closed unbounded subset that is the unique solution of a Σ_1-formula that only uses the set E and ordinals less than κ^+ as parameters.

In contrast to these settings, the results of this paper will isolate several canonical contexts in which collections of Σ_1-definable closed unbounded sets fail to approximate the closed unbounded filter. In order to formulate these results, we now specify the notions of definability used in this paper.

Definition 1.1 A class Γ is *definable* by a formula $\varphi(v_0, \dots, v_n)$ and parameters $y_0, \dots, y_{n-1}$ if

$$\Gamma = \{x \mid \varphi(x, y_0, \dots, y_{n-1})\}.$$

Throughout this paper, we study the definability of certain sets of ordinals. Given such a set E, we will mostly consider the questions that ask whether the set $\{E\}$ is definable (in the sense of Definition 1.1) by certain formulas with parameters coming from a given class. Note that this form of definability differs from the statement that

[1] See [9, p. 183].

[2] More precisely, it is possible to use [14, Lemma 2.3] to show that if $\varphi(v_0, \dots, v_n)$ is a Σ_0-formula, κ is an uncountable cardinal, and $z_0, \dots, z_{n-1} \in H_\kappa \cup \mathrm{Ord}$ with the property that $C = \{\alpha < \kappa \mid \varphi(\alpha, z_0, \dots, z_{n-1})\}$ is a closed unbounded subset of κ, then there exists a set N of natural numbers such that $C \cap [\lambda, \lambda + \omega) = \{\lambda + n \mid n \in N\}$ holds for coboundedly many limit ordinals $\lambda < \kappa$.

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the set E itself is definable (again in the sense of Definition 1.1) in the given way. While the former implies the latter, it is easy to see that the converse implication fails for the set ω_1 of all countable ordinals and definability by Σ_1-formulas without parameters.

We now introduce the main concepts studied in this paper:

Definition 1.2 Let n be a natural number, let κ be a uncountable cardinal, and let S be a subset of κ.

(1) Given a class R, the subset S is $\Sigma_n(R)$*-stationary in* κ if $C \cap S \neq \varnothing$ holds for every closed unbounded subset C of κ with the property that the set $\{C\}$ is definable by a Σ_n-formula with parameters in $R \cup \{\kappa\}$.
(2) The subset S is Σ_n*-stationary in* κ if it is $\Sigma_n(\varnothing)$-stationary in κ.
(3) Given a class R, the subset S is $\mathbf{\Sigma}_n(R)$*-stationary in* κ if it is $\Sigma_n(R \cup \mathrm{H}_\kappa)$-stationary in κ.
(4) The subset S is $\mathbf{\Sigma}_n$*-stationary in* κ if it is $\mathbf{\Sigma}_n(\varnothing)$-stationary in κ.

Trivially, all stationary subsets of a given uncountable cardinal κ are $\Sigma_n(R)$-stationary in κ for every parameter class R and every natural number n. As indicated by Definition 1.2, our focus in this paper will be on natural classes R such as Ord, H_κ, H_μ for some $\mu < \kappa$, and their combinations. We therefore view the existence of nonstationary $\Sigma_n(R)$-stationary sets as a measure for the discrepancy between the collection of $\Sigma_n(R)$-definable closed unbounded subsets and the collection of all closed unbounded subsets. Our results will isolate several settings in which highly nonstationary sets (such as singletons, or sets of successor cardinals below a limit cardinal) are stationary for rich collections of definable sets. In particular, even in the case of singular cardinals of countable cofinality, where stationarity coincides with coboundedness, we will present canonical examples of such cardinals in which sparse subsets are $\Sigma_n(R)$-stationary. The proofs of all these results reveal that, in the given settings, the considered collections of $\Sigma_n(R)$-definable closed unbounded subsets and $\Sigma_n(R)$-stationary sets possess many of the structural features provable for the collections of closed unbounded and stationary subsets of regular uncountable cardinals. Below, we list six of our main results that best illustrate the phenomenon described above:

- If κ is a Ramsey cardinal, then every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1$-stationary (Theorem 4.1).
- If κ is a (possibly singular) limit of measurable cardinals, then every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary (Theorem 4.6).
- If ω_ω is a Jónsson cardinal, then every unbounded subset of $\{\omega_n \mid n < \omega\}$ is $\mathbf{\Sigma}_1$-stationary in ω_ω (Theorem 4.13).
- It is equiconsistent with the existence of a measurable cardinal that every unbounded subset of $\{\omega_n \mid n < \omega\}$ is $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary in ω_ω (Theorem 5.3).
- It is equiconsistent with the existence of a Mahlo cardinal that there is a regular cardinal μ for which the singleton $\{\mu\}$ is $\Sigma_1(\mathrm{H}_\mu)$-stationary in μ^+ (Theorem 5.11).
- While it is provable that for every set A of cardinality less than the *reaping number* $\mathfrak{r}$[3] and every singular cardinal κ of countable cofinality, there are disjoint

[3] See Definition 2.1 below.

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$\Sigma_1(A)$-stationary subsets of κ (Proposition 2.8), it is equiconsistent with the existence of a measurable cardinal that there is a singular cardinal κ of countable cofinality such that for every subset A of H_κ of cardinality $\mathfrak{r}$, there are are disjoint $\Sigma_1(A)$-stationary subsets of κ (Corollary 5.10).

We now briefly outline the structure of this paper: In Section 2, we prove preliminary results about the theory of $\Sigma_n(R)$-stationary sets using mostly combinatorial arguments. In addition, we introduce an auxiliary property of Σ_n-undefinability of an ordinal, which plays a key role in the result of the paper. In Section 3, we examine Σ_1-definability in the Dodd-Jensen core model and prove several results regarding Σ_1-stationary sets in K^{DJ}. Section 4 is devoted to showing that Σ_1-stationarity is weak in the presence of sufficiently large cardinals (e.g., measurable cardinals or stably measurable cardinals) or at small cardinals with strong partition properties (e.g., when ω_ω is Jónsson). In Section 5, we build on the results about K^{DJ} as well as on forcing results with large cardinals to prove several equiconsistency results about weakness of $\Sigma_1(\mathrm{Ord})$-definable closed unbounded sets. In Section 6, we conclude the paper by listing some problems left open by our results.

2 Preliminaries about Σ_n-Stationary sets

In this section, we start to develop the theory of $\Sigma_n(R)$-stationary sets for natural classes R.

2.1 A proper hierarchy

In order to motivate the below results, we start our investigations of $\Sigma_n(R)$-stationary sets by presenting a setting in which these sets form a properly descending hierarchy in the parameter n.

Theorem 2.1 *Assume that the* GCH *holds. Then, for every uncountable regular cardinal κ and every cardinal θ satisfying $\theta = \theta^\kappa$, there is a uniformly definable partial order $\mathbb{P}_{\kappa,\theta}$ such that forcing with $\mathbb{P}_{\kappa,\theta}$ preserves cofinalities, and if G is $\mathbb{P}_{\kappa,\theta}$-generic over V, then, in $V[G]$, for every natural number $n > 0$ and every set A of cardinality less than θ with the property that the set $\{A\}$ is definable by a Σ_2-formula with parameters in A, there is a subset of κ that is $\Sigma_n(A)$-stationary in κ and not $\Sigma_{n+1}(A)$-stationary in κ.*

Note that the definability assumptions on the parameter set A stated in the above theorem are satisfied in the case where $\theta = \kappa^{++}$ and $A = \mathrm{H}_\kappa \cup \kappa^+$. We therefore directly get the following corollary whose statement should be compared with the two settings discussed in Section 1, in which all $\mathbf{\Sigma}_1(\kappa^+)$-stationary subsets of an uncountable cardinal κ are stationary.

Corollary 2.2 *If the* GCH *holds, then for every uncountable regular cardinal κ, there is a uniformly definable cofinality preserving partial order $\mathbb{P}_\kappa$ with the property that whenever G is $\mathbb{P}_\kappa$-generic over V and $n > 0$ is a natural number, then, in $V[G]$, there is a subset of κ that is $\mathbf{\Sigma}_n(\kappa^+)$-stationary in κ and not $\mathbf{\Sigma}_{n+1}(\kappa^+)$-stationary in κ.*

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The above result is a consequence of the following two observations that might be of independent interest:

Proposition 2.3 *Let $n > 0$ be a natural number, let κ be an uncountable cardinal with $\mathcal{P}(\kappa) \subseteq \mathrm{HOD}$, and let A be a set with the property that the set $\{A\}$ is definable by a Σ_{n+1}-formula with parameters in A. If there is a nonstationary subset of κ that is $\Sigma_n(A)$-stationary in κ, then there is a subset of κ that is $\Sigma_n(A)$-stationary in κ and not $\Sigma_{n+1}(A)$-stationary in κ.*

Proof Let E be the least nonstationary subset of κ in the canonical well-ordering of HOD that is $\Sigma_n(A)$-stationary in κ. Since the collection of all initial segments of the canonical well-ordering of HOD is definable by a Σ_2-formula without parameters, and the collection $C^{(n)}$ of all ordinals λ with $V_\lambda \prec_{\Sigma_n} V$ is definable by a Π_n-formula without parameters (see [1, Section 1]), we know that the set $\{E\}$ is definable by a Σ_{n+1}-formula with parameters in A. Let C denote the least closed unbounded subset of κ in the canonical well-ordering of HOD that is disjoint from E. But then we know that the set $\{C\}$ is also definable by a Σ_{n+1}-formula with parameters in A, and therefore, C witnesses that E is not $\Sigma_{n+1}(A)$-stationary in κ. ∎

Proposition 2.4 *Let κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, let $\theta > \kappa$ be a cardinal with $\theta^\kappa = \theta$, let G be $\mathrm{Add}(\kappa, \theta)$-generic over V, and let $\mathcal{C}$ be a collection of closed unbounded subsets of κ of cardinality less than θ in $\mathrm{V}[G]$. Then, in $\mathrm{V}[G]$, there exists a nonstationary subset E of κ with the property that $C \cap E \neq \emptyset$ holds for all $C \in \mathcal{C}$.*

Proof By our assumptions, we can find a generic extension M of V such that $\mathcal{C} \in M \subsetneq \mathrm{V}[G]$ and $\mathrm{V}[G]$ is a nontrivial $\mathrm{Add}(\kappa, \theta)$-generic extension of M. Let $H \in \mathrm{V}[G]$ be $\mathrm{Add}(\kappa, 1)$-generic over M, let X denote the subset of κ corresponding to H, and set $E = X \backslash \mathrm{Lim}(X)$. Genericity then ensures that E intersects every unbounded subset of κ in M. In particular, we know that $C \cap E \neq \emptyset$ holds for every $C \in \mathcal{C}$. Finally, since E is disjoint from $\mathrm{Lim}(X)$, we know that E is a nonstationary subset of κ in $\mathrm{V}[G]$. ∎

A combination of these two observations now directly yields the desired consistency proof:

Proof of Theorem 2.1 Assume that the GCH holds. Given an uncountable regular cardinal κ and a cardinal θ satisfying $\theta = \theta^\kappa$, we define $\mathbb{P}_{\kappa,\theta}$ to be the two-step iteration $\mathrm{Add}(\kappa, \theta) * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is the canonical $\mathrm{Add}(\kappa, \theta)$-name for the $<\theta^+$-closed partial order that codes $\mathcal{P}(\kappa)$ into the GCH-pattern above θ^+ (see, for example, [7]). Let $G * H$ be $(\mathrm{Add}(\kappa, \theta) * \dot{\mathbb{Q}})$-generic over V. Fix a natural number $n > 0$ and a set $A \in V[G, H]$ of cardinality less than θ with the property that, in $V[G, H]$, the set $\{A\}$ is definable by a Σ_2-formula with parameters in A. Since $\mathcal{P}(\mathcal{P}(\kappa))^{V[G]} = \mathcal{P}(\mathcal{P}(\kappa))^{V[G,H]}$ and, in $V[G, H]$, there are less than θ-many closed unbounded subsets C of κ with the property that the set $\{C\}$ is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$, an application of Proposition 2.4 shows that, in $V[G, H]$, there is a nonstationary subset of κ that is $\Sigma_n(A)$-stationary in κ. But our setup ensures that $\mathcal{P}(\kappa)^{V[G]} = \mathcal{P}(\kappa)^{V[G,H]} \subseteq \mathrm{HOD}^{V[G,H]}$, and hence, Proposition 2.3 allows us to conclude that, in $V[G, H]$, there is a subset of κ that is $\Sigma_n(A)$-stationary in κ and not $\Sigma_{n+1}(A)$-stationary in κ. ∎

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In a sequel to this paper, we will provide analogous results about the properness of the hierarchy of $\Sigma_n(A)$-stationary sets for singular cardinals.

2.2 Combinatorial arguments

We now continue by analyzing basic structural features of the collection of all $\Sigma_n(A)$-stationary sets. All results presented in this section are purely combinatorial in the sense that they only rely on counting arguments. We nevertheless decided to phrase them in such a way that they make statements about collections of definable subsets. These formulations motivate several of our later results that will show that, in general, we cannot relax the stated assumptions on the sizes of parameter sets. We start our analysis by comparing $\Sigma_n(A)$-stationarity with standard stationarity and isolating settings in which counting arguments ensure the existence of $\Sigma_n(A)$-stationary sets that are not stationary.

Proposition 2.5 *If κ is a cardinal of uncountable cofinality, A is a set of cardinality at most $\mathrm{cof}(\kappa)$, and n is a natural number, then there is an unbounded, nonstationary subset of κ that is $\Sigma_n(A)$-stationary in κ.*

Proof Let $\langle C_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ be an enumeration of all closed unbounded subsets C of κ with the property that the set $\{C\}$ is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$. In addition, define C to be a diagonal intersection of these sets with respect to some strictly increasing continuous sequence of order type $\mathrm{cof}(\kappa)$ in κ. Define $S = C \backslash \mathrm{Lim}(C)$. Then S is an unbounded and nonstationary subset of κ. Moreover, we have $C \cap S \neq \emptyset$ whenever C is a closed unbounded subset of κ with the property that the set $\{C\}$ is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$. ■

The results of this paper show that the implication in the above proposition can fail if we consider sets of parameters of cardinality $\mathrm{cof}(\kappa)^+$. In the case of regular cardinals, the examples given in Section 1 about L and under MM provide examples of such failures. For singular cardinals κ of uncountable cofinality, Corollary 3.4 below will show that, in the Dodd-Jensen core model, stationarity coincides with $\Sigma_1(\mathcal{P}(\mathrm{cof}(\kappa)))$-stationarity and therefore also with $\mathbf{\Sigma}_1$-stationarity in κ. Moreover, 5.2 shows that when κ is singular of uncountable cofinality, the existence of $\mathbf{\Sigma}_1$-stationarity subset of κ that is not stationary is equiconsistent with the existence of $\mathrm{cof}(\kappa)$ many measurable cardinals. In the case of singular cardinals of countable cofinalities, Theorem 5.3 will give an analogous equiconsistency result. We start with a quick combinatorial argument.

Proposition 2.6 *If κ is a singular cardinal of countable cofinality, A is a set of cardinality less than κ^ω, and n is a natural number, then there is an unbounded subset of κ that is $\Sigma_n(A)$-stationary in κ and whose complement in κ is unbounded in κ.*

Proof Since the set $[\kappa]^\omega$ contains an almost disjoint family of cardinality κ^ω that consists of unbounded subsets of κ, we can find an element $b \in [\kappa]^\omega$ that is unbounded in κ and has the property that no infinite subset of b is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$. Then, $\kappa \backslash b$ is $\Sigma_n(A)$-stationary in κ and the complement of this set in κ is unbounded in κ. ■

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We continue by comparing the structural properties of $\Sigma_n(A)$-stationary sets with those of standard stationary sets. Our first focus in this comparison will be the question of the existence of disjoint $\Sigma_n(A)$-stationary sets at various uncountable cardinals. In the case of cardinals of uncountable cofinality, the fact that all stationary sets are $\Sigma_n(A)$-stationary already ensures the existence of such sets. The next result strengthens this conclusion by showing that a version of *Solovay's Theorem* on the splitting of stationary sets holds for $\Sigma_n(A)$-stationary sets.

Proposition 2.7 *Suppose that κ is a cardinal of uncountable cofinality, A is a set of cardinality $\mathrm{cof}(\kappa)$ with $A \cap \kappa$ cofinal in κ, $n < \omega$, and S is a $\Sigma_n(A)$-stationary subset of κ. Then, there exists a partition $\langle S_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ of S into $\Sigma_n(A)$-stationary subsets.*

Proof Since $A \cap \kappa$ is unbounded in κ, we know that there are $\mathrm{cof}(\kappa)$-many $\Sigma_n(A)$-definable closed unbounded subsets of κ. Let $\langle C_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ be an enumeration of all closed unbounded subsets C of κ with the property that the set $\{C\}$ is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$. Since the assumption that $A \cap \kappa$ is unbounded in κ implies that $C_\alpha \cap S$ is unbounded in κ for all $\alpha < \mathrm{cof}(\kappa)$, we can find a strictly increasing sequence $\langle \sigma_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ of elements of S with the property that $\sigma_{<\alpha_0,\alpha_1>} \in C_{\alpha_1}$ for all $\alpha_0, \alpha_1 < \mathrm{cof}(\kappa)$, where $<\cdot,\cdot> : \mathrm{Ord} \times \mathrm{Ord} \longrightarrow \mathrm{Ord}$ denotes the Gödel pairing function. If we now pick a partition $\langle S_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ of S with the property that $\{\sigma_{\alpha,\beta} \mid \beta < \mathrm{cof}(\kappa)\} \subseteq S_\alpha$ holds for all $\alpha < \mathrm{cof}(\kappa)$, then each S_α is $\Sigma_n(A)$-stationary in κ. ■

In the case of cardinals of countable cofinality, the existence of disjoint $\Sigma_n(A)$-stationary sets turns out to be closely connected to the *reaping number* $\mathfrak{r}$.

Definition 2.1 $\mathfrak{r}$ is the least cardinality of a subset A of $[\omega]^\omega$ with the property that for every $b \in [\omega]^\omega$, there is $a \in A$ such that either $a \backslash b$ or $a \cap b$ is finite.

We will later show that, in general, the conclusion of the following proposition cannot be extended to sets of parameters of cardinality $\mathfrak{r}$ (see Corollaries 3.6 and 5.10 below).

Proposition 2.8 *Let κ be a singular cardinal of countable cofinality, let A be a set of cardinality less than $\mathfrak{r}$, and let n be a natural number. Then, there exists a subset E of κ with the property that both E and $\kappa \backslash E$ are $\Sigma_n(A)$-stationary in κ.*

The proof of this proposition relies on the equivalence provided by the next lemma:

Lemma 2.9 *The following statements are equivalent for every infinite cardinal θ:*

(1) $\theta < \mathfrak{r}$.
(2) *For every singular cardinal κ of countable cofinality and every subset A of $[\kappa]^\omega$ that consists of cofinal subsets of κ and has cardinality at most θ, there exists a subset E of κ such that for every $a \in A$, both $a \cap E$ and $a \backslash E$ are infinite.*
(3) *There is a singular cardinal κ of countable cofinality with the property that for every subset A of $[\kappa]^\omega$ that consists of cofinal subsets of κ and has cardinality at most θ, there exists a subset E of κ such that for every $a \in A$, both $a \cap E$ and $a \backslash E$ are infinite.*

Proof First, assume that (1) holds and (2) fails. Then, there is a singular cardinal κ of countable cofinality and a subset A of $[\kappa]^\omega$ such that A consists of cofinal subsets

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of κ, $|A| \leq \theta$, and for every subset E of κ, there is $a \in A$ such that either $a \cap E$ or $a \backslash E$ is finite. Let $\langle \kappa_n \mid n < \omega \rangle$ be a strictly increasing sequence that is cofinal in κ with $\kappa_0 = 0$. Given $a \in A$, define

$$b_a = \{n < \omega \mid a \cap [\kappa_n, \kappa_{n+1}) \neq \emptyset\} \in [\omega]^\omega.$$

Since $|A| < \mathfrak{r}$, we can now find $c \in [\omega]^\omega$ with the property that for all $a \in A$, both $b_a \backslash c$ and $b_a \cap c$ are infinite. Set

$$E = \bigcup_{n \in c} [\kappa_n, \kappa_{n+1}).$$

By our assumptions, there is $a \in A$ with the property that either $a \backslash E$ or $a \cap E$ is finite. But this implies that either $b_a \backslash c$ or $b_a \cap c$ is finite, a contradiction.

Now, assume that (3) holds and (1) fails. Then, there is a singular cardinal κ of countable cofinality with the property that for every subset A of $[\kappa]^\omega$ that consists of cofinal subsets of κ and has cardinality at most θ, there exists a subset E of κ such that for every $a \in A$, both $a \cap E$ and $a \backslash E$ are infinite. Fix $A \subseteq [\omega]^\omega$ of cardinality $\mathfrak{r}$ such that for every $b \in [\omega]^\omega$, there exists $a \in A$ with the property that either $a \backslash b$ or $a \cap b$ is finite. Pick a s strictly increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ that is cofinal in κ and, given $a \in A$, define $b_a = \{\kappa_n \mid n \in a\} \in [\kappa]^\omega$. Since b_a is cofinal in κ for all $a \in A$, we can now find a subset E of κ such that for every $a \in A$, both $b_a \cap E$ and $b_a \backslash E$ are infinite. Set $c = \{n < \omega \mid \kappa_n \in E\} \in [\omega]^\omega$. If $a \in A$, then both $a \backslash c$ and $a \cap c$ are infinite, contradicting our assumptions on A. ■

Proof of Proposition 2.8 Let κ be a singular cardinal of countable cofinality, let A be a set of cardinality less than $\mathfrak{r}$, and let $n > 0$ be a natural number. Then, there exists a subset A' of $[\kappa]^\omega$ of cardinality less than $\mathfrak{r}$ that consists of cofinal sequences and has the property that for every closed unbounded subset C of κ such that the set $\{C\}$ is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$, there exists $a \in A'$ with $a \subseteq C$. Using Lemma 2.9, we can now find a subset E of κ such that for every $a \in A$, both $a \backslash E$ and $a \cap E$ are infinite. Then, both E and $\kappa \backslash E$ are $\Sigma_n(A)$-stationary in κ. ■

We close this section by comparing another aspect of the behavior of stationary sets with its counterpart in the definable context. While the collection of all closed unbounded subsets of a cardinal of uncountable cofinality is closed under intersections, and therefore, all of these subsets are stationary, these implications can obviously fail at a singular cardinal of countable cofinality, where all unbounded sets of order type ω are closed unbounded and stationarity coincides with coboundedness. The following lemma completely characterizes the settings in which these implications also hold in the definable context:

Lemma 2.10 *Given a class A and a natural number $n > 0$, the following statements are equivalent for every singular cardinal κ of countable cofinality:*

(1) *There is a cofinal function $c : \omega \longrightarrow \kappa$ that is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$.*
(2) *There are disjoint closed unbounded subsets C_0 and C_1 of κ with the property that the sets $\{C_0\}$ and $\{C_1\}$ are both definable by Σ_n-formulas with parameters in $A \cup \{\kappa\}$.*

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(3) *There are closed unbounded subsets C_0 and C_1 of κ with $C_0 \cap C_1$ bounded in κ and the property that the sets $\{C_0\}$ and $\{C_1\}$ are both definable by Σ_n-formulas with parameters in $A \cup \{\kappa\}$.*

Proof First, we assume that there is a cofinal function $c : \omega \longrightarrow \kappa$ that is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$. Define $C_0 = \{c(n) + \omega \mid n < \omega\}$ and $C_1 = \{c(n) + \omega + 1 \mid n < \omega\}$. Then, C_0 and C_1 are disjoint closed unbounded in κ, and the sets $\{C_0\}$ and $\{C_1\}$ are definable by Σ_n-formulas with parameters in $A \cup \{\kappa\}$. In the other direction, assume that there are closed unbounded subsets C_0 and C_1 of κ with $C_0 \cap C_1$ bounded in κ and the property that the sets $\{C_0\}$ and $\{C_1\}$ are both definable by Σ_n-formulas with parameters in $A \cup \{\kappa\}$. We define $\mu = \max(C_0 \cap C_1) < \kappa$ and let $c : \omega \longrightarrow \kappa$ denote the unique function with $c(0) = \mu$ and

$$c(2k + 2 - i) \;=\; \min(C_i \backslash (c(2k + 1 - i) + 1))$$

for all $k < \omega$ and $i < 2$. The Σ_n-Recursion Theorem then implies that c is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$. Moreover, we know that c is cofinal in κ because otherwise, $\mu < \sup_{k<\omega} c(k) \in C_0 \cap C_1$. ■

Using an argument similar to the one of the previous Lemma, we obtain the following corollary, which will later allow us to show that definable closed unbounded sets behave nicely in various settings.

Corollary 2.11 *Let κ be a singular cardinal of countable cofinality, let A be a class, and let $n > 0$ be a natural number with the property that there exists a subset of κ that is $\Sigma_n(A)$-stationary in κ and consists of cardinals. Then, the collection of all closed unbounded subsets C of κ with the property that the set $\{C\}$ is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$ is closed under intersections.*

2.3 The Σ_n-undefinability property

We introduce a notion that will allow us to show that various nonstationary sets of cardinals $E \subseteq \kappa$ of an uncountable cardinal κ are $\mathbf{\Sigma}_n$-stationary.

Definition 2.2 Given uncountable cardinals $\mu < \kappa$, an ordinal $\gamma \geq \kappa$, and a natural number n, we say that the cardinal κ has the $\Sigma_n(\mu, \gamma)$-*undefinability property* if no ordinal α in the interval $[\mu, \kappa)$ has the property that the set $\{\alpha\}$ is definable by a Σ_n-formula with parameters in the set $\mathrm{H}_\mu \cup \{\kappa, \gamma\}$. Moreover, we say that κ has the $\Sigma_n(\mu)$-*undefinability property* if it has the $\Sigma_n(\mu, \kappa)$-undefinability property.

The next lemma shows how this undefinability property is connected to Σ_n-stationarity:

Lemma 2.12 *Given uncountable cardinals $\mu < \kappa$, an ordinal $\gamma \geq \kappa$, and a natural number $n > 0$, if κ has the $\Sigma_n(\mu, \gamma)$-undefinability property, then the set $\{\mu\}$ is $\Sigma_n(\mathrm{H}_\mu \cup \{\gamma\})$-stationary in κ.*

Proof Let C be a closed unbounded subset of κ with the property that the set $\{C\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}_\mu \cup \{\kappa, \gamma\}$. Assume, toward a contradiction, that μ is not an element of C. Set $\nu = \min(C \backslash \mu) > \mu$. Then,

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$C \cap \mu \neq \varnothing$ because otherwise, $\nu = \min(C)$, and this would imply that the set $\{\nu\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}_\mu \cup \{\kappa, \gamma\}$. This allows us to define $\rho = \max(C \cap \mu) < \mu$. But then, $\nu = \min(C \backslash (\rho + 1))$, and hence, $\{\nu\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}_\mu \cup \{\kappa, \gamma\}$, a contradiction. ■

The following direct corollary of the above lemma will later allow us to isolate various examples of $\mathbf{\Sigma}_1$-stationary subsets of cardinals of countable cofinality that are not stationary (i.e., not cobounded) in the given limit cardinal.

Corollary 2.13 *Let κ be a limit cardinal, let $\gamma \geq \kappa$ be an ordinal, let $n > 0$ be a natural number, and let E be the set of uncountable cardinals $\mu < \kappa$ with the property that κ has the $\Sigma_n(\mu, \gamma)$-undefinability property for all $\mu \in E$. If E is unbounded in κ, then E is $\mathbf{\Sigma}_n(\{\gamma\})$-stationary in κ.*

3 Undefinability in the Dodd-Jensen core model

In the following, we establish basic definability and undefinability results dealing with *Dodd-Jensen core model* K^{DJ}. These results make use of the presentation of this model in [5] and [11]. In this setting, a set M is a *premouse at an ordinal $\mu > \omega$* if M is of the form J^U_η and

$$\langle M, \in, U \rangle \vDash \text{“ } U \text{ is a normal ultrafilter on } \mu. \text{”}$$

Moreover, if M is a premouse at μ, then we define the *lower part of M* to be the set $lp(M) = M \cap V_\mu$. Given a premouse M and an ordinal δ, an iteration of M of length δ is given by a sequence $\langle M_\alpha \mid \alpha < \delta \rangle$ of premice and a commuting system $\langle j_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta < \delta \rangle$ of Σ_1-elementary embeddings such that the following statements hold:

- $M = M_0$ and $j_{\alpha,\alpha} = \mathrm{id}_{M_\alpha}$ for all $\alpha < \delta$.
- If $\alpha + 1 < \delta$, $M_\alpha = J^U_\eta$ is a premouse at μ and $M_{\alpha+1} = J^W_\zeta$ is a premouse at ν, then $M_{\alpha+1}$ is the transitive collapse of the ultrapower of M_α using U, $j_{\alpha,\alpha+1}$ as the corresponding ultrapower embedding, $j_{\alpha,\alpha+1}(\mu) = \nu$, and
$$W = \{[f]_U \mid f \in {}^\mu M_\alpha \cap M_\alpha, \{\xi < \mu \mid f(\xi) \in U\} \in U\}.$$
- If $\gamma \in \mathrm{Lim} \cap \delta$, then $\langle M_\gamma, \langle j_{\alpha,\gamma} : M_\alpha \longrightarrow M_\gamma \mid \alpha < \gamma \rangle \rangle$ is a direct limit of $\langle \langle M_\alpha \mid \alpha < \gamma \rangle, \langle j_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta < \gamma \rangle \rangle$.

If such an iteration exists, then it is uniquely determined, and it is called the δ-iteration of M. A premouse M is then called a *mouse* if δ-iterations of M exist for all $\delta \in \mathrm{Ord}$. Note that since the iterability of a premouse can be checked in every transitive structure of uncountable ordinal height that contains the mouse and satisfies a sufficiently strong fragment of ZFC (see [11, Theorem 2.7]), it follows that the class of all mice is Σ_1-definable from every uncountable ordinal. We can now use such iterations to compare a mouse $M = J^U_\eta$ at some ordinal μ with a mouse $N = J^W_\zeta$ at an ordinal ν, in the sense that they allow us to find mice $M' = J^C_{\eta'}$ and $N' = J^C_{\zeta'}$ at the same ordinal ρ and Σ_1-elementary embeddings $j : M \longrightarrow M'$ and $i : N \longrightarrow N'$ with $j(\mu) = i(\nu) = \rho$ (see [5, Lemma 1.13]). In our arguments below, we will frequently make use of results of Dodd and Jensen (see [5, pp. 238-241]) that show that the

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Dodd-Jensen core model K^{DJ} is equal to the union of the constructible universe L and the lower parts $lp(M)$ of all mice M.

We now prove three lemmata that show that various objects that witness the accessibility of cardinals in K^{DJ} are simply definable.

Lemma 3.1 *If κ is an infinite cardinal that is not inaccessible in K^{DJ}, then the set $\{H_\kappa^{K^{DJ}}\}$ is definable by a Σ_1-formula with parameter κ.*

Proof Since the GCH holds in K^{DJ} and $H_{\aleph_0}^{K^{DJ}} = L_\omega$, we may assume that either κ is a successor cardinal in K^{DJ} or κ is singular in K^{DJ}. Moreover, we may assume that there exists a mouse because otherwise, $K^{DJ} = L$, and $\{H_\kappa^{K^{DJ}}\} = \{L_\kappa\}$ is Σ_1-definable in the desired way.

Claim *If M is a mouse at some $\nu > \kappa$ such that κ is either a successor cardinal or a singular cardinal in M, then $H_\kappa^{K^{DJ}} \subseteq M$.* ■

Proof of the Claim Fix $x \in H_\kappa^{K^{DJ}}$. Then, there exists a mouse N_0 at some cardinal $\mu < \kappa$ such that $|N_0| < \kappa$ and $x \in lp(N_0)$. Let N be the mouse obtained by iterating the top measure of N_0 κ-many times. Then clearly, we have $x \in lp(N)$, and κ is the critical point of the top measure of N and is therefore inaccessible in N. The coiteration of M and N, which can only involve the top measures of the two mice, results in Σ_1-elementary embeddings $\pi_M : M \longrightarrow M'$ and $\pi_N : N \longrightarrow N'$ with the following properties:

- κ is inaccessible in N', but not in M',
- $x \in lp(N')$,
- $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{M'}$, and
- one of M', N' is an initial segment of the other.

It is therefore clear that N' must be an initial segment of M', from which we conclude that $x \in \mathcal{P}^{M'}(\kappa)$ and therefore that $x \in M$. ■

As outlined earlier, the assumption that $K^{DJ} \neq L$ implies that K^{DJ} is equal to the union of all lower parts of mice. In particular, there exists a mouse M at an ordinal above κ with the property that κ is either a successor cardinal or a singular cardinal in M. By the above claim, we know that $H_\kappa^{K^{DJ}} = H_\kappa^M$ holds for every mouse M with these properties. By combining this implication with earlier remarks about the definability of the class of all mice, we can derive the statement of the lemma.

Lemma 3.2 *If κ is an infinite cardinal, then the set $\{\mathrm{cof}(\kappa)^{K^{DJ}}\}$ is definable by a Σ_1-formula with parameter κ.*

Proof This is trivial in the case κ is not singular in K^{DJ}. Assuming it is singular, we know that $\mathrm{cof}(\kappa)^{K^{DJ}}$ is the unique ordinal $\xi < \kappa$ with the property that ξ is regular in $H_\kappa^{K^{DJ}}$ and K^{DJ} contains a cofinal function $c : \xi \longrightarrow \kappa$. Since the class K^{DJ} is definable by a Σ_1-formula with parameter κ (see, for example, the proof of [14, Lemma 4.13]), we can apply Lemma 3.1 to conclude that the set $\{\mathrm{cof}(\kappa)^{K^{DJ}}\}$ is also definable by a Σ_1-formula with parameter κ. ■

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Lemma 3.3 *If κ is an infinite cardinal and c is the $<_{\mathrm{K}^{DJ}}$-least cofinal function from $\mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}$ to κ in K^{DJ}, then the set $\{c\}$ is definable by a Σ_1-formula with parameter κ.*

Proof This is an immediate consequence of the proof of [16, Lemma 2.3], which shows that the collection of initial segments of the restriction of $<_{\mathrm{K}^{DJ}}$ to $\mathcal{P}(\kappa)$ is definable by a Σ_1-formula with parameter κ, and Lemma 3.2 above. ■

We now use the above results to show that, in the case of singular cardinals κ of uncountable cardinality, the statement of Proposition 2.5 cannot be strengthened in ZFC. We will later improve this result to obtain $\mathrm{cof}(\kappa)$-many measurable cardinals from the existence of a singular cardinal κ of uncountable cofinality with the property that there exists a nonstationary $\mathbf{\Sigma}_1$-stationary subset of κ (see Theorem 5.1 and Corollary 5.2 below).

Corollary 3.4 *Assume that there is no inner model with a measurable cardinal. If κ is a singular cardinal of uncountable cofinality and S is $\Sigma_1(\mathcal{P}(\mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}))$-stationary in κ, then S is a stationary subset of κ.*

Proof By our assumption, the results of [4] ensure that κ is a singular cardinal in K^{DJ}. Using Lemma 3.3, we find a closed unbounded subset C_0 of κ of order-type $\mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}$ with the property that both the set $\{C_0\}$ and the monotone enumeration of C_0 are definable by a Σ_1-formula with parameter κ. Given an arbitrary closed unbounded subset C of κ, we then know that the intersection $C \cap C_0$ is a closed unbounded subset and the set $\{C \cap C_0\}$ is definable by a Σ_1-formula with parameters in $\mathcal{P}(\mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}) \cup \{\kappa\}$. This shows that every subset of κ that is $\Sigma_1(\mathcal{P}(\mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}))$-stationary in κ is stationary in κ. ■

We now derive further consequences of the above lemmata. The results of the subsequent sections will show that it is possible to use large cardinals to obtain singular cardinals κ where the negations of all of the listed statements hold.

Corollary 3.5 *Assume that there is no inner model with a measurable cardinal. If κ is a singular cardinal, then the following statements hold:*

(1) *If $\alpha < \kappa$, then the set $\{\alpha\}$ is not $\Sigma_1(\alpha)$-stationary in κ.*
(2) *There is an unbounded subset of κ that consists of cardinals and is not Σ_1-stationary.*
(3) *There exists a regressive function $r : \kappa \longrightarrow \kappa$ that is definable by a Σ_1-formula with parameter κ and is not constant on any unbounded subset of κ.*

Proof Set $\lambda = \mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}$ and let $c : \lambda \longrightarrow \kappa$ denote the the $<_{\mathrm{K}^{DJ}}$-least cofinal function from λ to κ in K^{DJ}. By our assumption, the results of [4] imply that $\lambda < \kappa$, and we can use Lemma 3.1 to show that the set $\mathrm{H}_\kappa^{\mathrm{K}^{DJ}}$ is definable by a Σ_1-formula with parameter κ. Moreover, Lemma 3.2 and Lemma 3.3 ensure that both the set $\{\lambda\}$ and the function c are definable by a Σ_1-formulas with parameter κ.

Now, fix $\alpha < \kappa$. If $\alpha \leq \lambda$, then $C = (\lambda, \kappa)$ is a closed unbounded subset of κ that is disjoint from $\{\alpha\}$ and has the property that the set $\{C\}$ is definable by a Σ_1-formula with parameter κ. In the other case, if $\lambda < \alpha$, then there is $\xi < \lambda$ with $c(\xi) > \alpha$, and $C = (c(\xi), \kappa)$ is a closed unbounded subset of κ disjoint from $\{\alpha\}$ with the property that the set $\{C\}$ is definable by a Σ_1-formula with parameters in $\alpha \cup \{\kappa\}$.

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Next, assume that κ is a limit of limit cardinals in K^{DJ}, and define C to be the closed unbounded set of all ordinals $\rho < \kappa$ with the property that, in K^{DJ}, the ordinal ρ is a limit cardinal. Since the set $\{\mathrm{H}_\kappa^{\mathrm{K}^{DJ}}\}$ is definable by a Σ_1-formula with parameter κ, it follows that set $\{C\}$ is definable in the same way. Now, let E denote the set of all successor cardinals of singular cardinals smaller than κ. Since our setup and the results of [4] ensure that all singular cardinals are singular in K^{DJ} and K^{DJ} computes the successors of these cardinals correctly, we know that each element of E is the successor of a singular cardinal in K^{DJ}, and this shows that $C \cap E = \emptyset$. In particular, we can conclude that E is an unbounded subset of κ that consists of cardinals and is not Σ_1-stationary.

We now assume that κ is not a limit of limit cardinals in K^{DJ}. Let η denote the least ordinal below κ with the property that the interval (η, κ) contains no ordinals that are limit cardinals in K^{DJ}, and we define C to be the set of ordinals in (η, κ) that are successor ordinals of ordinals that are cardinals in K^{DJ}. Our assumptions then imply that C is a cofinal subset of κ of order-type ω, and this implies that C is a closed unbounded subset of κ. Moreover, the fact that the set $\{\mathrm{H}_\kappa^{\mathrm{K}^{DJ}}\}$ is definable by a Σ_1-formula with parameter κ ensures that the set $\{C\}$ is definable in the same way. If we now define E to be the set of all cardinals in the interval (η, κ), then E is unbounded in κ, and since $C \cap E = \emptyset$ holds, we know that E is not Σ_1-stationary.

Finally, define $r : \kappa \longrightarrow \lambda$ to be the unique map with $c(\alpha) = 0$ for all $\alpha < \lambda$ and $c(\alpha) = \min\{\xi < \lambda \mid c(\xi) \geq \alpha\}$ for all $\lambda \leq \alpha < \kappa$. Then, r is regressive, and it is not constant on any unbounded subset of κ. Moreover, our earlier observations show that r is definable by a Σ_1-formula with parameter κ. ∎

Corollary 3.6 *Assume that there is no inner model with a measurable cardinal. If κ is a singular cardinal of countable cofinality, then the following statements hold:*

(1) *There are disjoint closed unbounded subsets C_0 and C_1 of κ with the property that the sets $\{C_0\}$ and $\{C_1\}$ are both definable by Σ_1-formulas with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$.*
(2) *There exists a subset A of H_κ of cardinality $\mathfrak{r}$ such that every subset of κ that is $\Sigma_1(A)$-stationary in κ contains a closed unbounded subset C of κ with the property that the set $\{C\}$ is definable by a Σ_1-formula with parameters in $A \cup \{\kappa\}$.*

Proof Set $\lambda = \mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}$. Our assumptions then imply that $\lambda < \kappa$. Let $c : \lambda \longrightarrow \kappa$ denote the $<_{\mathrm{K}^{DJ}}$-least cofinal function in K^{DJ}. Since $\mathrm{cof}(\lambda) = \omega$, we can also fix a cofinal function $d : \omega \longrightarrow \lambda$. Define $C_0 = \{\omega \cdot ((c \circ d)(i)) \mid i < \omega\}$ and $C_1 = \{\alpha + 1 \mid \alpha \in C_0\}$. Then, the sets C_0 and C_1 are disjoint closed unbounded subsets of κ. Moreover, Lemma 3.3 ensures that the sets $\{C_0\}$ and $\{C_1\}$ are both definable by Σ_1-formulas with parameters κ and d. Finally, pick a subset A of H_κ of cardinality $\mathfrak{r}$ such that $\omega \cup \{d\} \subseteq A$, $A \cap \kappa$ is cofinal in κ, and for every $b \in [\omega]^\omega$, there is $a \in A \cap [\omega]^\omega$ with the property that either $a \backslash b$ or $a \cap b$ is finite. Let S be a subset of κ that is $\Sigma_1(A)$-stationary in κ. The fact that $A \cap \kappa$ is unbounded in κ then implies that the set $b = \{i < \omega \mid (c \circ d)(i) \in S\}$ is infinite. Hence, there exists $a \in A$ with the property that either $a \backslash b$ or $a \cap b$ is finite. Set $C = \{(c \circ d)(i) \mid i \in a\}$. Then, C is closed unbounded in κ, and the set $\{C\}$ is definable by a Σ_1-formula with parameters in $A \cup \{\kappa\}$. Hence, we know that $C \cap S$ is unbounded in κ, and this shows that $a \cap b$ is infinite. We can now find $k < \omega$ with $a \backslash b \subseteq k$, and if we define

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$D = \{(c \circ d)(i) \mid k \leq i \in A\}$, then D is a closed unbounded subset of S and the set $\{D\}$ is definable by a Σ_1-formula with parameters in $A \cup \{\kappa\}$. ■

We end this section with another result about the definability of initial segments of the Dodd-Jensen core model that will be used in our characterizations of *stably measurable* cardinals below.

Lemma 3.7 *Let κ be an uncountable cardinal, and let $E \in \mathrm{K}^{DJ}$ be a subset of κ. If κ is regular in K^{DJ} and E is a bistationary subset of κ in K^{DJ}, then the set $\{\mathrm{H}_\kappa^{\mathrm{K}^{DJ}}\}$ is definable by a Σ_1-formula with parameter E.*

Proof Since the class K^{DJ} is definable by a Σ_1-formula with parameter κ in V, we may assume that $V = \mathrm{K}^{DJ}$ holds. Moreover, we may assume that there exists a mouse because otherwise, we have $\mathrm{K}^{DJ} = \mathrm{L}$, and $\{\mathrm{H}_\kappa^{\mathrm{K}^{DJ}}\} = \{\mathrm{L}_\kappa\}$ is definable by a Σ_1-formula with parameter E. 2 The desired statement is then a direct consequence of the following observation:

Claim *If M is a mouse with $E \in lp(M)$, then $\mathrm{H}_\kappa \subseteq M$.* ■

Proof of the Claim Assume, toward a contradiction, that there is an $x \in \mathrm{H}_\kappa \backslash M$. Then, there is a mouse N_0 with $|N_0| < \kappa$ and $x \in lp(N_0)$. As above, this allows us to find a mouse N at κ with $x \in lp(N)$. Let $\pi_0 : M \longrightarrow M'$ and $\pi_1 : N \longrightarrow N'$ be the Σ_1-elementary embeddings obtained by coiterating M and N (i.e., either M' is an initial segment of N' or N' is an initial segment of M'). Since $\mathrm{H}_\kappa^M = \mathrm{H}_\kappa^{M'}$ and $x \in lp(N) \backslash M$, we now know that M' is an initial segment of N'. This implies that $E \in \mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{M'} \subseteq \mathcal{P}(\kappa)^{N'} = \mathcal{P}(\kappa)^N$. Since the given N-ultrafilter on κ is equal to the restriction of the closed unbounded filter on κ to $\mathcal{P}(\kappa)^N$, we can conclude that E either contains a closed unbounded subset of κ or is disjoint from such a subset. This contradicts the bistationarity of E. ■

This claim now shows that H_κ is the unique set B with the property that there exists a mouse M with $E \in lp(M)$ and $B = \mathrm{H}_\kappa^M$. This directly yields the desired Σ_1-definition of $\{\mathrm{H}_\kappa\}$.

4 Large cardinals and Σ_1-stationary sets

In Section 2, we already gave examples of two important features of $\Sigma_1(A)$-stationary sets. First, the collection of these sets can be substantially larger than the collection of ordinary stationary sets. Second, this collection can possess structural features that resemble the behavior of stationary sets. We start by proving results for large cardinals and then extend these results to limits of large cardinals (not necessarily regular). Finally, we show that it is possible to derive similar consequences from Ramsey-theoretic properties that may hold on smaller cardinals.

4.1 Stably measurable cardinals

The following large cardinal property, introduced by Welch in [19], turns out to be closely connected to Σ_1-undefinability considerations.

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Definition 4.1 (Welch) An uncountable regular cardinal κ is *stably measurable* if there exists

- a transitive set M with $\mathrm{H}_\kappa \cup \{\kappa\} \subseteq M \prec_{\Sigma_1} \mathrm{H}_{\kappa^+}$,
- a transitive set N with $M \cup {}^{<\kappa}N \subseteq N$, and
- a normal, weakly amenable N-ultrafilter F on κ with the property that $\langle N, \in, F\rangle$ is iterable.

In [17], Sharpe and Welch defined an uncountable cardinal κ to be *iterable* if for every subset A of κ, there is a transitive model M of ZFC^- of cardinality κ with $A, \kappa \in M$ and a weakly amenable M-ultrafilter U on κ such that $\langle M, \in, U\rangle$ is ω_1-iterable. It is easy to see that all iterable cardinals are stably measurable. Moreover, all Ramsey cardinals are iterable (see [17, Lemma 5.2]), and this shows that all measurable cardinals are stably measurable. In the other direction, [19, Corollary 1.18] shows that if κ is a stably measurable cardinal, then $a^\#$ exists for every set of ordinals a in H_κ. Motivated by Corollary 3.5, we proceed toward the following result:

Theorem 4.1 *Let κ be a stably measurable cardinal.*

(1) *If $\mu < \kappa$ is an uncountable cardinal, then the singleton $\{\mu\}$ is $\Sigma_1(\mathrm{H}_\mu)$-stationary in κ.*
(2) *If E is an unbounded subset of κ that consists of cardinals, then E is $\mathbf{\Sigma}_1$-stationary in κ.*
(3) *If S is a $\mathbf{\Sigma}_1$-stationary subset of κ and $r : \kappa \longrightarrow \kappa$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$, then r is constant on a $\mathbf{\Sigma}_1$-stationary subset of S.*

Using Lemma 2.12 and Corollary 2.13, the first two statements of Theorem 4.1 directly follow from the next lemma:

Lemma 4.2 *If κ is a stably measurable cardinal, then κ has the $\Sigma_1(\mu)$-undefinability property for every uncountable cardinal $\mu < \kappa$.*

Proof Assume, toward a contradiction, that there is a Σ_1-formula $\varphi(v_0, v_1, v_2)$, an uncountable cardinal $\mu < \kappa$, an ordinal α in the interval $[\mu, \kappa)$, and $z \in \mathrm{H}_\mu$ such that α is the unique ordinal ξ with the property that $\varphi(\xi, \kappa, z)$ holds. Pick a transitive set M with $\mathrm{H}_\kappa \cup \{\kappa\} \subseteq M \prec_{\Sigma_1} \mathrm{H}_{\kappa^+}$, a transitive set N with $M \cup {}^{<\kappa}N \subseteq N$, and a weakly amenable N-ultrafilter F on κ with the property that $\langle N, \in, F\rangle$ is iterable. Now, pick an elementary submodel $\langle X, \in, F_0\rangle$ of $\langle N, \in, F\rangle$ of cardinality less than μ with $\mathrm{tc}(\{z\}) \cup \{\kappa, \alpha\} \subseteq X$, and let $\pi : X \longrightarrow N_0$ denote the corresponding transitive collapse. Then, $\pi(z) = z$ and $\pi(\alpha) < \pi(\kappa) < \mu \leq \alpha$. Moreover, if we set $U = \pi[F_0]$, then U is a weakly amenable N_0-ultrafilter on $\pi(\kappa)$ and [10, Theorem 19.15] implies that $\langle N_0, \in, U\rangle$ is iterable. This yields a transitive set N_1 and an elementary embedding $j : N_0 \longrightarrow N_1$ with $j(\pi(\kappa)) = \kappa$ and $j \restriction \mathrm{H}^{N_0}_{\pi(\kappa)} = \mathrm{id}_{\mathrm{H}^{N_0}_{\pi(\kappa)}}$. Since our setup ensures that $\varphi(\alpha, \kappa, z)$ holds in N, we now know that $\varphi(\pi(\alpha), \kappa, z)$ holds in N_1. By Σ_1-upwards absoluteness, this shows that $\varphi(\pi(\alpha), \kappa, z)$ holds in V, contradicting the uniqueness of α. ∎

We now work toward a proof of the third part of Theorem 4.1. The starting point for this is a result of Welch (see [19, Theorem 1.26]) proving that stably measurable

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cardinals have the $\mathbf{\Sigma}_1$-*club property* introduced in [14] (i.e., if κ is a stably measurable cardinal and E is a subset of κ with the property that the set $\{E\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$, then either E or $\kappa \setminus E$ contains a closed unbounded subset of κ). This result directly implies that if κ is a stably measurable cardinal and $r : \kappa \longrightarrow \kappa$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$, then for every ordinal $\alpha < \kappa$, there is a closed unbounded subset C_α of κ with the property that C_α is either contained in $r^{-1}\{\alpha\}$ or disjoint from $r^{-1}\{\alpha\}$. By forming the diagonal intersection $\triangle_{\alpha<\kappa} C_\alpha$, it is now easy to see that there is a unique ordinal $\alpha_* < \kappa$ with the property that $C_{\alpha_*} \subseteq r^{-1}\{\alpha_*\}$, and this directly implies that α_* is the unique ordinal smaller than κ with the property that $r^{-1}\{\alpha_*\}$ contains a closed unbounded subset of κ. The following lemma further strengthens this conclusion:

Lemma 4.3 *Let κ be a stably measurable cardinal, and let E be a subset of κ that contains a closed unbounded subset of κ and has the property that the set $\{E\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$. Then, there exists a closed unbounded subset C of κ with $C \subseteq E$ and the property that the set $\{C\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$.*

Proof Fix a Σ_1-formula $\varphi(v_0, v_1, v_2)$ and an element z of H_κ with the property that the set $\{E\}$ is definable by the formula $\varphi(v_0, v_1, v_2)$ and the parameters κ and z. By our assumptions on κ, there exists a transitive set M with $\mathrm{H}_\kappa \cup \{\kappa\} \subseteq M \prec_{\Sigma_1} \mathrm{H}_{\kappa^+}$, a transitive set N with $M \cup {}^{<\kappa}N \subseteq N$, and a normal, weakly amenable N-ultrafilter F on κ with the property that $\langle N, \in, F\rangle$ is iterable. This setup ensures that $E \in M$ and $\varphi(E, \kappa, z)$ holds in N. Moreover, the fact that E contains a closed unbounded subset of κ implies that M contains such a subset of E and since F is a normal N-ultrafilter, we can conclude that E is an element of F. Pick an elementary submodel $\langle X, \in, F_0\rangle$ of $\langle N, \in, F\rangle$ of cardinality less than κ with $\mathrm{tc}(\{z\}) \cup \{\kappa\} \subseteq X$, and let $\pi : X \longrightarrow N_0$ denote the corresponding transitive collapse. Then, $\pi(z) = z$, and if we set $U = \pi[F_0]$, then U is a weakly amenable N_0-ultrafilter on $\pi(\kappa)$ and [10, Theorem 19.15] ensures that $\langle N_0, \in, U\rangle$ is iterable. Let

$$\langle \langle N_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle j_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta \in \mathrm{Ord}\rangle\rangle$$

denote the linear iteration of $\langle N_0, \in, U\rangle$. Then, $(j_{0,\kappa} \circ \pi)(\kappa) = \kappa$, $(j_{0,\kappa} \circ \pi)(z) = z$, Σ_1-upwards absoluteness implies that $\varphi((j_{0,\kappa} \circ \pi)(E), \kappa, z)$ holds in V, and hence, we know that $(j_{0,\kappa} \circ \pi)(E) = E$. Moreover, the fact that $E \in F$ ensures that $\pi(E) \in U$ and therefore $(j_{0,\alpha} \circ \pi)(\kappa) \in (j_{0,\alpha+1} \circ \pi)(E) \subseteq E$ for all $\alpha < \kappa$. This shows that the closed unbounded subset $C = \{(j_{0,\alpha} \circ \pi)(\kappa) \mid \alpha < \kappa\}$ of κ is a subset of E. Finally, the set $\{C\}$ is definable by a Σ_1-formula with parameters κ, N_0, and U. ∎

Proof of Theorem 4.1 The last two lemmata prove the first two assertions of the theorem. To prove the third assertion, let κ be a stably measurable cardinal, let S be $\mathbf{\Sigma}_1$-stationary in κ, and let $r : \kappa \longrightarrow \kappa$ be a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$. Our earlier observations now show that there is a unique ordinal $\alpha < \kappa$ with the property that the set $r^{-1}\{\alpha\}$ contains a closed unbounded subset of κ. Since the set $\{r^{-1}\{\alpha\}\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$, Lemma 4.3 yields a closed unbounded subset C of κ

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with $C \subseteq r^{-1}\{\alpha\}$ and the property that the set $\{C\}$ is definable by a Σ_1-formula with parameters in $H_\kappa \cup \{\kappa\}$. Then, r is constant on $C \cap S$, and since κ is an uncountable regular cardinal, we know that this set is $\mathbf{\Sigma}_1$-stationary in κ. ∎

We can now show that in the Dodd–Jensen core model K^{DJ}, stably measurable cardinals are characterized by the $\mathbf{\Sigma}_1$-stationarity of unbounded sets of cardinals:

Theorem 4.4 *If $V = K^{DJ}$, then the following statements are equivalent for every cardinal $\kappa > \omega_1$:*

(1) *The cardinal κ is stably measurable.*
(2) *The set $\{H_\kappa\}$ is not definable by a Σ_1-formula with parameters in the set $H_\kappa \cup \{\kappa\}$.*
(3) *The cardinal κ is a limit cardinal, and every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1$-stationary in κ.*

Proof First, assume that κ is not stably measurable. Then, we can apply [19, Theorem 2.6] to find a bistationary subset E of κ with the property that the set $\{E\}$ is definable by a Σ_1-formula with parameters in $H_\kappa \cup \{\kappa\}$. Lemma 3.7 then shows that the set $\{H_\kappa\}$ is definable by a Σ_1-formula with parameters in $H_\kappa \cup \{\kappa\}$.

Now, assume that κ is a limit cardinal and the set $\{H_\kappa\}$ is definable by a Σ_1-formula with parameters in $H_\kappa \cup \{\kappa\}$. Let C denote the set of all limit cardinals smaller than κ. Our assumption then implies that the set $\{C\}$ is definable by a Σ_1-formula with parameters in $H_\kappa \cup \{\kappa\}$. If C is unbounded in κ, then the set of all successor cardinals smaller than κ is an unbounded subset of κ that consists of cardinals and is not $\mathbf{\Sigma}_1$-stationary in κ. In the other case, if C is bounded in κ, then $\mathrm{cof}(\kappa) = \omega$, Lemma 3.3 yields a cofinal function $c : \omega \longrightarrow \kappa$ that is definable by a Σ_1-formula with parameter κ, and the set $D = \{c(n) + 1 \mid n < \omega\}$ is closed unbounded in κ that contains no cardinals and has the property that the set $\{D\}$ is definable by a Σ_1-formula with parameter κ.

Together with Theorem 4.1, these computations establish all equivalences claimed in the theorem. ∎

Next, we show that stable measurability provides the exact consistency strength for the existence of limit cardinals with given undefinability property:

Theorem 4.5 *The following statements are equiconsistent over* ZFC*:*

(1) *There exists a stably measurable cardinal.*
(2) *There exists a limit cardinal κ with the property that every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1$-stationary in κ.*

Proof Assume, toward a contradiction, that there is no inner model with a stably measurable cardinal and κ is a limit cardinal with the property that every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1$-stationary in κ. Then, Corollary 3.5 shows that κ is regular. Since κ is not stably measurable in K^{DJ}, Theorem 4.4 now shows that in K^{DJ}, the set $\{H_\kappa\}$ is definable by a Σ_1-formula with parameters in $H_\kappa \cup \{\kappa\}$. Let C denote the set of all ordinals less than κ that are limit cardinals in K^{DJ}. Then, C is a closed unbounded subset of κ, and since the class K^{DJ} is definable by a Σ_1-formula with parameter κ, the set $\{C\}$ is definable by a Σ_1-formula with parameters in $H_\kappa \cup \{\kappa\}$. Now, let E denote the set of all successor cardinals of singular cardinals less than κ. The results of [4] then show that the elements of E are successor cardinals of singular cardinals in K^{DJ}, and hence, $C \cap E = \varnothing$. This shows that E is an unbounded subset of κ

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that consists of cardinals and is not $\mathbf{\Sigma}_1$-stationary. In combination with Theorem 4.1, these arguments prove the desired equivalence. ■

In Section 5, we show that the assertion in item (2) of the last theorem can consistently hold at $\kappa = \aleph_\omega$, starting from the consistency assumption of a measurable cardinal.

4.2 Limits of measurable cardinals

We now continue by showing that many of the above results can be extended to larger classes of definable sets if the given cardinal is a (not necessarily regular) limit of measurable cardinals:

Theorem 4.6 *Let κ be a cardinal that is a limit of measurable cardinals.*

(1) *Every unbounded subset S of κ consisting of cardinals is $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary.*
(2) *If S is a $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary subset of κ and $r : \kappa \longrightarrow \kappa$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \mathrm{Ord}$, then r is constant on a $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary subset of S.*

The starting point of the proof of this theorem is the following technical lemma that generalizes [13, Lemma 1.1.25] to embeddings given by linear iterations.

Lemma 4.7 *Suppose that $\kappa_0 < \kappa_1$ are measurable cardinals and $\alpha_0 < \kappa_1$ is an ordinal. Given $i < 2$, let U_i be a normal ultrafilter on κ_i, and let*

$$\langle \langle M^i_\alpha \mid \alpha \in \mathrm{Ord} \rangle, \langle j^i_{\alpha,\beta} : M^i_\alpha \longrightarrow M^i_\beta \mid \alpha \leq \beta \in \mathrm{Ord} \rangle \rangle$$

denote the linear iteration of $\langle \mathrm{V}, \in, U_i \rangle$. In addition, let

$$\langle \langle M^*_\alpha \mid \alpha \in \mathrm{Ord} \rangle, \langle j^*_{\alpha,\beta} : M^*_\alpha \longrightarrow M^*_\beta \mid \alpha \leq \beta \in \mathrm{Ord} \rangle \rangle$$

denote the linear iteration of $\langle M^0_{\alpha_0}, \in, j^0_{0,\alpha_0}(U_1) \rangle$. Then, $j^*_{0,\alpha} \restriction \mathrm{Ord} = j^1_{0,\alpha} \restriction \mathrm{Ord}$ for all $\alpha \in \mathrm{Ord}$.

Proof Fix an ordinal $\alpha_1 < \kappa$ with the property that $j^*_{0,\alpha} \restriction \mathrm{Ord} = j^1_{0,\alpha} \restriction \mathrm{Ord}$ holds for all $\alpha < \alpha_1$. Set $M^0 = M^0_{\alpha_0}$, $M^1 = M^1_{\alpha_1}$, $M^* = M^*_{\alpha_1}$, $j^0 = j^0_{0,\alpha_0}$, $j^1 = j^1_{0,\alpha_1}$, $j^* = j^*_{0,\alpha_1}$, and $U_* = j^0(U_1)$. Then, $j^0(\kappa_1) = \kappa_1$. In addition, if $i < 2$, then we define $\kappa^i_\alpha = j^i_{0,\alpha}(\kappa_i)$ for all $\alpha \in \mathrm{Ord}$ and $C^i = \{\kappa^i_\alpha \mid \alpha < \alpha_i\}$. An application of [10, Lemma 19.6] then shows that every element of M^* is of the form $j^0(f)(c)$ with $n < \omega$, $f : [\kappa_0]^n \longrightarrow V$ and $c \in [C^0]^n$. In addition, given $i < 2$ and $n < \omega$, we define

$$U^n_i \;=\; \{X \subseteq [\kappa_i]^n \mid \exists E \in U_i \; [E]^n \subseteq X\}.$$

The results of [10, Chapter 19] then show that U^n_i is a $<\kappa_i$-complete ultrafilter on $[\kappa_i]^n$. Set $U^n_* = j^0(U^n_1)$ for all $n < \omega$. Our assumption now implies that $j^*_{0,\alpha}(\kappa_1) = \kappa^1_\alpha$ holds for all $\alpha < \alpha_1$. In particular, we know that every ordinal is of the form $j^*(g)(d)$ with $n < \omega$, $g : [\kappa_1]^n \longrightarrow \mathrm{Ord}$ in M_0 and $d \in [C^1]^n$.

Claim *If $n < \omega$, $B \in U^n_1$, and $B_* \in U^n_*$, then $j^0[B] \cap B_* \neq \emptyset$.* ■

Proof of the Claim Pick $n < \omega$, $f : [\kappa_0]^n \longrightarrow V$ and $c \in [C^0]^n$ with $B_* = j^0(f)(c)$. By [10, Lemma 19.9], we then have

$$D \;=\; \{a \in [\kappa_0]^n \mid f(a) \in U^n_1\} \;\in\; U^n_0.$$

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Since U_0^n has cardinality less than κ_1, the $<\kappa_1$-completeness of U_1^n implies that $E = \bigcap\{f(a) \mid a \in D\}$ is an element of U_1^n. Pick $b \in B \cap E$. Then,

$$D \subseteq \{a \in [\kappa_0]^n \mid b \in f(a)\} \in U_0^n,$$

and [10, Lemma 19.9] shows that $j^0(b) \in j^0[B] \cap j^0(f)(c) = j^0[B] \cap B_* \neq \emptyset$. ■

Claim *If $n < \omega$ and $g_0, g_1 : [\kappa_1]^n \longrightarrow$ Ord are functions in M^0 with*

$$\{b \in [\kappa_1]^n \mid g_0(b) = g_1(b)\} \in U_*^n,$$

then

$$\{b \in [\kappa_1]^n \mid g_0(j^0(b)) = g_1(j^0(b))\} \in U_1^n.$$

Proof of the Claim Assume, toward a contradiction, that the above conclusion fails. Then, we know that the set $B = \{b \in [\kappa_1]^n \mid g_0(j^0(b)) \neq g_1(j^0(b))\}$ is an element of U_1^n. In this situation, we can use our first claim to find $b \in B$ with $g_0(j^0(b)) = g_1(j^0(b))$, a contradiction.

The same proof also yields the following implication:

Claim *If $n < \omega$ and $g_0, g_1 : [\kappa_1]^n \longrightarrow$ Ord are functions in M^0 such that the set $\{b \in [\kappa_1]^n \mid g_0(b) < g_1(b)\}$ is an element of U_*^n, then the set*

$$\{b \in [\kappa_1]^n \mid g_0(j^0(b)) < g_1(j^0(b))\}$$

is an element of U_1^n. ■

Claim *For every function $f : [\kappa_1]^n \longrightarrow$ Ord in V, there is a function $g : [\kappa_1]^n \longrightarrow$ Ord in M^0 with*

$$\{b \in [\kappa_1]^n \mid f(b) = g(j^0(b))\} \in U_1^n.$$

Proof of the Claim Given $b \in [\kappa_1]^n$, pick $m_b < \omega$, $f_b : [\kappa_0]^{m_b} \longrightarrow$ Ord in V and $c_b \in [C^0]^{m_b}$ with $f(b) = j^0(f_b)(c_b)$. Then, there is $m < \omega$ and $c \in [C^0]^m$ with

$$B = \{b \in [\kappa_1]^n \mid m_b = m,\ c_b = c\} \in U_1^n.$$

Define

$$G : [\kappa_0]^m \times B \longrightarrow \text{Ord};\ (a, b) \longmapsto f_b(a)$$

and

$$g : [\kappa_1]^n \longrightarrow \text{Ord};\ b \longmapsto j^0(G)(c, b).$$

Then, g is an element of M^0, and

$$g(j^0(b)) = j^0(G)(c, j^0(b)) = j^0(f_b)(c) = f(b)$$

holds for all $b \in B \in U_1^n$. ■

Claim *If $g : [\kappa_1]^1 \longrightarrow$ Ord is an element of M^0, then*

$$j^1(g \circ j^0)(\{\kappa_1\}) = j^1(g)(\{\kappa_1\}).$$

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Proof of the Claim Since standard arguments show that $\{\{\gamma\} \mid \gamma < \kappa_1, j^0(\gamma) = \gamma\}$ is an element of U_1^1, we know that the set $\{\{\gamma\} \mid \gamma < \kappa_1, (g \circ j^0)(\gamma) = g(\gamma)\}$ is also contained in U_1^1, and hence, an application of [10, Lemma 19.9] shows that we have $j^1(g \circ j^0)(\{\kappa_1\}) = j^1(g)(\{\kappa_1\})$. ∎

By our second claim and [10, Lemma 19.9], there exists a well-defined class function $F : \mathrm{Ord} \longrightarrow \mathrm{Ord}$ with the property that

$$F(j^*(g)(c)) \;=\; j^1(g \circ j^0)(c)$$

holds for all $n < \omega$, $g : [\kappa_1]^n \longrightarrow \mathrm{Ord}$ in M^0 and $c \in [C^1]^n$. Moreover, our third claim shows that F is order-preserving. Finally, since every ordinal is of the form $j^1(f)(c)$ for some $n < \omega$, $f : [\kappa_1]^n \longrightarrow \mathrm{Ord}$ in V and $c \in [C^1]^n$, our fourth claim shows that F is also surjective. In combination, this shows that F is the identity on Ord. Now, fix an ordinal γ, and let g_γ denote the constant function on $[\kappa_1]^1$ with value γ. Then, we have

$$\begin{aligned} j^*(\gamma) \;&=\; j^*(g_\gamma)(\{\kappa_1\}) \;=\; F(j^*(g_\gamma)(\{\kappa_1\})) \\ &=\; j^1(g_\gamma \circ j^0)(\{\kappa_1\}) \;=\; j^1(g_\gamma)(\{\kappa_1\}) \;=\; j^1(\gamma), \end{aligned}$$

and this proves the statement of the lemma.

Using the above results, we can now drive a variation of [13, Lemma 1.1.27] for iterations. The following lemma will be the main tool used in the proofs of this section.

Lemma 4.8 *Let $\lambda \geq \aleph_0$ be a regular cardinal, let $\vec{\kappa} = \langle \kappa_\xi \mid \xi < \lambda \rangle$ be a strictly increasing sequence of measurable cardinals with supremum κ, and let $\vec{U} = \langle U_\xi \mid \xi < \lambda \rangle$ be a sequence with the property that U_ξ is a normal ultrafilter on κ_ξ for all $\xi < \lambda$. Assume that either $\kappa = \lambda$ or $\lambda < \kappa_0$. Given $\xi < \lambda$, let*

$$\langle \langle M_\alpha^\xi \mid \alpha \in \mathrm{Ord} \rangle, \langle j_{\alpha,\beta}^\xi : M_\alpha^\xi \longrightarrow M_\beta^\xi \mid \alpha \leq \beta \in \mathrm{Ord} \rangle \rangle$$

denote the linear iteration of $\langle \mathrm{V}, \in, U_\xi \rangle$. Then, for all ordinals γ and eventually all $\xi < \lambda$, we have $j_{0,\alpha}^\xi(\gamma) = \gamma$ for all $\alpha < \kappa$.

Proof Assume, toward a contradiction, that there is an ordinal γ such that for unboundedly many $\xi < \lambda$, there is an $\alpha < \kappa$ with $j_{0,\alpha}^\xi(\gamma) > \gamma$. Let γ be minimal with this property. Pick $\zeta < \lambda$ and $\alpha_0 < \kappa$ with $j_{0,\alpha_0}^\zeta(\gamma) > \gamma$. Set $j = j_{0,\alpha_0}^\zeta$ and $M = M_{\alpha_0}^\zeta$. Then, $j(\lambda) = \lambda$ and $j(\kappa) = \kappa$. Moreover, if we set $j(\vec{\kappa}) = \langle \kappa'_\xi \mid \xi < \lambda \rangle$ and $j(\vec{U}) = \langle U'_\xi \mid \xi < \lambda \rangle$, then the fact that $j(\gamma) > \gamma$ yields an ordinal $\rho < \lambda$ with the property that $i_{0,\alpha}^\xi(\gamma) = \gamma$ holds for all $\rho \leq \xi < \lambda$ and all $\alpha < \kappa$, where

$$\langle \langle N_\alpha^\xi \mid \alpha \in \mathrm{Ord} \rangle, \langle i_{\alpha,\beta}^\xi : N_\alpha^\xi \longrightarrow N_\beta^\xi \mid \alpha \leq \beta \in \mathrm{Ord} \rangle \rangle$$

denotes the linear iteration of $\langle M, \in, U'_\xi \rangle$. By our assumption, we are now able to pick $\rho < \xi < \lambda$ and $\alpha_1 < \kappa$ such that $\kappa_\xi > \alpha_0$ and $j_{0,\alpha_1}^\xi(\gamma) > \gamma$. In this situation, Lemma 4.7 implies that $i_{0,\alpha_1}^{j(\xi)}(\gamma) = j_{0,\alpha_1}^\xi(\gamma) > \gamma$, and since $\rho \leq \xi \leq j(\xi) < \lambda$, this yields a contradiction. ∎

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With the help of Lemma 4.8, we can now work toward a proof of the first part of Theorem 4.6.

Lemma 4.9 *In the setting of Lemma 4.8, for every ordinal $\gamma \geq \kappa$, there exists an ordinal $\lambda_0 < \lambda$ with the property that the cardinal κ has the $\Sigma_1(\kappa_\xi, \gamma)$-undefinability property for all $\lambda_0 \leq \xi < \lambda$.*

Proof Lemma 4.8 yields $\lambda_0 < \lambda$ with the property that $j^\xi_{0,\alpha}(\kappa) = \kappa$ and $j^\xi_{0,\alpha}(\gamma) = \gamma$ holds for all $\lambda_0 \leq \xi < \lambda$ and $\alpha < \kappa$. Assume, toward a contradiction, that there is a Σ_1-formula $\varphi(v_0, \ldots, v_3)$, and ordinal $\lambda_0 \leq \xi < \lambda$, an element z of H_{κ_ξ}, and an ordinal α in the interval $[\kappa_\xi, \kappa)$ such that the set $\{\alpha\}$ is definable by the formula $\varphi(v_0, \ldots, v_3)$ and the parameters κ, γ, and z. Pick an ordinal $\beta < \kappa$ with $j^\xi_{0,\beta}(\kappa_\xi) > \alpha$. Then, we know that $j^\xi_{0,\beta}(\alpha) > \alpha$, and elementarity implies that $\varphi(j^\xi_{0,\beta}(\alpha), \kappa, \gamma, z)$ holds in M^ξ_β. But then, Σ_1-upwards absoluteness implies that $\varphi(j^\xi_{0,\beta}(\alpha), \kappa, \gamma, z)$ holds in V, contradicting our assumptions. ∎

Lemma 4.10 *Let κ be a cardinal that is a limit of measurable cardinals. Then, every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary in κ.*

Proof Work in the setting of Lemma 4.8. Let E be an unbounded subset of κ that consists of cardinals, and fix a Σ_1-formula $\varphi(v_0, v_1, v_2)$, an ordinal γ, and $z \in \mathrm{H}_\kappa$ with the property that there exists a closed unbounded subset C of κ that is the unique set x with the property that $\varphi(x, \gamma, z)$ holds. By combining Lemma 4.8 and Lemma 4.9 with Corollary 2.13, we can find $\xi < \lambda$ such that κ_ξ is an element of C, $z \in \mathrm{H}_{\kappa_\xi}$, and for all $\alpha < \kappa$, we have $j^\xi_{0,\alpha}(\kappa) = \kappa$ and $j^\xi_{0,\alpha}(\gamma) = \gamma$. In this situation, we can find a cardinal $\kappa_\xi < \mu \in E$ with $j^\xi_{0,\mu}(\kappa_\xi) = \mu$. Then, $\varphi(j^\xi_{0,\mu}(C), \gamma, z)$ holds in M^ξ_μ, and since Σ_1-upwards absoluteness implies that this statement also holds in V, it follows that $j^\xi_{0,\mu}(C) = C$. Moreover, since $\kappa_\xi \in C$, we can now use elementarity to conclude that $\mu = j^\xi_{0,\mu}(\kappa_\xi) \in C \cap E \neq \emptyset$. ∎

The next result generalizes [19, Theorem 1.26] and Lemma 4.3 to (possibly singular) limits of singular cardinals:

Lemma 4.11 *Let κ be a cardinal that is a limit of measurable cardinals, and let E be a subset of κ with the property that the set $\{E\}$ is definable by a Σ_1-formula with parameters in $\mathrm{Ord} \cup \mathrm{H}_\kappa$. Then, there exists a closed unbounded subset C of κ of order type κ with the property that the set $\{C\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$, and either $C \subseteq E$ or $C \cap E = \emptyset$ holds.*

Proof We may assume that there is a Σ_1-formula $\varphi(v_0, v_1, v_2)$, an ordinal γ, and $z \in \mathrm{H}_\kappa$ with the property that E is the unique set x satisfying $\varphi(x, \gamma, z)$. By Lemma 4.8, there exists a measurable cardinal $\kappa_0 < \kappa$ with $z \in \mathrm{H}_{\kappa_0}$ and a normal ultrafilter U on κ_0 satisfying $j_{0,\alpha}(\kappa) = \kappa$ and $j_{0,\alpha}(\gamma) = \gamma$ for all $\alpha < \kappa$, where

$$\langle \langle M_\alpha \mid \alpha \in \mathrm{Ord} \rangle, \langle j_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta \in \mathrm{Ord} \rangle \rangle$$

is the linear iteration of $\langle V, \in, U \rangle$. Given an ordinal $\alpha < \kappa$, elementarity ensures that $\varphi(j_{0,\alpha}(E), \gamma, z)$ holds in M_α. In this situation, Σ_1-upwards absoluteness implies that $\varphi(j_{0,\alpha}(E), \gamma, z)$ holds in V for all $\alpha < \kappa$, and this allows us to conclude that

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$j_{0,\alpha}(E) = E$ holds for all $\alpha < \kappa$. If κ_0 is an element of E, then these observations show that $\{j_{0,\alpha}(\kappa_0) \mid \alpha < \kappa\}$ is a closed unbounded subset of E of order type κ. In the other case, if $\kappa_0 \notin E$, then $\{j_{0,\alpha}(\kappa_0) \mid \alpha < \kappa\}$ is a closed unbounded subset of E of order type κ that is disjoint from E.

Now, pick a transitive model N of ZFC^- such that N contains κ and U and the model $\langle N, \in, U\rangle$ is linearly iterable. Let

$$\langle\langle N_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle i_{\alpha,\beta} : N_\alpha \longrightarrow N_\beta \mid \alpha \leq \beta \in \mathrm{Ord}\rangle\rangle$$

denote the linear iteration of $\langle N, \in, U\rangle$. In this situation, the fact that $\mathcal{P}(\kappa_0) \subseteq N$ implies that $j_{0,\alpha}(\kappa_0) = i_{0,\alpha}(\kappa_0)$ holds for all $\alpha < \kappa$. Since the class of all sets N with the above properties is definable by a Σ_1-formula with parameter U, we can conclude that the set $\{\{j_{0,\alpha}(\kappa_0) \mid \alpha < \kappa\}\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$. ∎

The techniques used in the above proofs also allow us to prove the second part of Theorem 4.6:

Proof of Theorem 4.6 Let κ be a limit of measurable cardinals. Then, Lemma 4.10 shows that every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary. Now, let S be a $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary subset of κ, and let $r : \kappa \longrightarrow \kappa$ be a regressive function that is definable by a Σ_1-formula $\varphi(v_0, \ldots, v_3)$, an ordinal γ, and an element z of H_κ. As in the proof of Lemma 4.11, we can now find a measurable cardinal $\kappa_0 < \kappa$ with $z \in \mathrm{H}_{\kappa_0}$ and a normal ultrafilter U on κ_0 with the property that $j_{0,\alpha}(\gamma) = \gamma$ holds for all $\alpha < \kappa$, where

$$\langle\langle M_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle j_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta \in \mathrm{Ord}\rangle\rangle$$

denotes the linear iteration of $\langle V, \in, U\rangle$. Let $r(\kappa_0) = \xi < \kappa_0$. Then, $\varphi(\kappa_0, \xi, \gamma, z)$ holds in V, and for all $\alpha < \kappa$, elementarity implies that $\varphi(j_{0,\alpha}(\kappa_0), \xi, \gamma, z)$ holds in M_α. Given $\alpha < \kappa$, Σ_1-upwards absoluteness now implies that $\varphi(j_{0,\alpha}(\kappa_0), \xi, \gamma, z)$ holds in V, and hence, $r(j_{0,\alpha}(\kappa_0)) = \xi$. This shows that the restriction of r to the closed unbounded subset $C = \{j_{0,\alpha}(\kappa_0) \mid \alpha < \kappa\}$ of κ is constant with value ξ. Moreover, the proof of Lemma 4.11 shows that the set $\{C\}$ is definable by a Σ_1-formula with parameters κ and U. Since Corollary 2.11 and Lemma 4.10 show that the set $C \cap S$ is $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary in κ, these arguments show that r is constant on a $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary subset of S. ∎

4.3 Partition properties

Remember that, given uncountable cardinals $\mu < \kappa$, the cardinal κ is *μ-Rowbottom* if the square brackets partition relation $\kappa \longrightarrow [\kappa]^{<\omega}_{\lambda,<\mu}$ holds true for all $\lambda < \kappa$ (i.e., for every $\lambda < \kappa$ and every function $c : [\kappa]^{<\omega} \longrightarrow \lambda$, there exists $H \in [\kappa]^\kappa$ with the property that $|c[[H]^{<\omega}]| < \mu$). Moreover, ω_1-Rowbottom cardinals are called *Rowbottom cardinals*. The following lemma connects this partition property to the $\mathbf{\Sigma}_1$-undefinability property:

Lemma 4.12 *Let κ be a μ-Rowbottom cardinal. If either $\kappa = \omega_\omega$ or $\rho^{<\mu} < \kappa$ holds for all $\rho < \kappa$, then κ has the $\Sigma_1(\mu)$-undefinability property.*

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Proof Assume, toward a contradiction, that there is a Σ_1-formula $\varphi(v_0, v_1, v_2)$, an element z of H_μ, and an ordinal α in $[\mu, \kappa)$ that is the unique set x such that $\varphi(x, \kappa, z)$ holds.

Claim *There exists an elementary submodel X of H_{κ^+} satisfying $\mathrm{tc}(\{z\}) \cup \{\kappa, \alpha\} \subseteq X$, $|X \cap \alpha| < |\alpha|$ and $|X \cap \kappa| = \kappa$.*

Proof of the Claim Define $\mathcal{L}$ to be the countable first-order language that extends $\mathcal{L}_\in$ by a unary predicate symbol $\dot{P}$, a unary function symbol $\dot{s}$, a constant symbol $\dot{\kappa}$, a constant symbol $\dot{\alpha}$, and a constant symbol $\dot{c}_y$ for every $y \in \mathrm{tc}(\{z\})$. Pick an elementary submodel M of H_{κ^+} of cardinality κ with $\mathrm{tc}(\{z\}) \cup (\kappa + 1) \subseteq M$ and a surjection $s : \kappa \longrightarrow M$. Now, let A denote an $\mathcal{L}$-expansion of $\langle M, \in\rangle$ with $\dot{P}^A = \alpha$, $\dot{s}^A \restriction \kappa = s$, $\dot{\kappa}^A = \kappa$, $\dot{\alpha}^A = \alpha$, and $\dot{c}^A_y = y$ for all $y \in \mathrm{tc}(\{z\})$. In addition, we define $\rho = |\alpha| \geq \mu$.

First, assume that $\rho^{<\mu} < \kappa$ holds. Then, [10, Theorem 8.5] yields an elementary substructure X of A of cardinality κ with $|X \cap \alpha| < \mu \leq \rho$. This setup then ensures that $\mathrm{tc}(\{z\}) \cup \{\kappa, \alpha\} \subseteq X$, and since $\mathrm{ran}(s \restriction X) = X$, we also know that $|X \cap \kappa| = \kappa$.

In the other case, assume that $\kappa = \omega_\omega$. Then, ρ is regular and κ is also ρ-Rowbottom. Another application of [10, Theorem 8.5] now produces an elementary substructure X of A of cardinality κ with $|X \cap \alpha| < \rho$. As above, we can conclude that $\mathrm{tc}(\{z\}) \cup \{\kappa, \alpha\} \subseteq X$ and $|X \cap \kappa| = \kappa$. ■

Let $\pi : X \longrightarrow N$ denote the corresponding transitive collapse. Then, our setup ensures that $\pi(\kappa) = \kappa$, $\pi(z) = z$, and $\pi(\alpha) < \alpha$. Moreover, since Σ_1-absoluteness causes $\varphi(\alpha, \kappa, z)$ to hold in H_{κ^+} and X, we know that $\varphi(\pi(\alpha), \kappa, z)$ holds in N. But then, Σ_1-upwards absoluteness implies that this statement holds in V, contradicting our assumptions. ■

Recall that a cardinal κ is *Jónsson* if for every function $f : [\kappa]^{<\omega} \longrightarrow \kappa$, there is a proper subset H of κ of cardinality κ with $f[[H]^{<\omega}] \subseteq H$. Motivated by the notoriously open question whether the first limit cardinal ω_ω can be Jónsson, we show that this assumption causes analogs of central results from the previous two sections to hold at ω_ω.

Theorem 4.13 *If ω_ω is a Jónsson cardinal, then the following statements hold:*

(1) *Every infinite subset of $\{\omega_n \mid n < \omega\}$ is $\mathbf{\Sigma}_1$-stationary in ω_ω.*
(2) *If $r : \omega_\omega \longrightarrow \omega_\omega$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}_{\aleph_\omega} \cup \{\omega_\omega\}$, then r is constant on an infinite subset of $\{\omega_n \mid n < \omega\}$.*

Proof Since ω_ω is the least Jónsson cardinal, we know that ω_ω is ω_n-Rowbottom for some $0 < n < \omega$ (see [10, Proposition 8.15]). Then, Lemma 4.12 implies that ω_ω has the $\Sigma_1(\omega_k)$-undefinability property for all $n \leq k < \omega$. An application of Corollary 2.13 then shows that every infinite subset of $\{\omega_m \mid m < \omega\}$ is $\mathbf{\Sigma}_1$-stationary in ω_ω.

Now, assume that $r : \omega_\omega \longrightarrow \omega_\omega$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}_{\aleph_\omega} \cup \{\omega_\omega\}$. Then, we can find $0 < n < \omega$ and $z \in \mathrm{H}_{\aleph_n}$ such that ω_ω is ω_n-Rowbottom and r is definable by a Σ_1-formula $\varphi(v_0, \ldots, v_3)$ and the parameters ω_ω and z. By repeating the proof of Lemma 4.12, we can find an elementary submodel X of $\mathrm{H}_{\aleph_{\omega+1}}$ with $\mathrm{tc}(\{z\}) \cup \{\kappa\} \subseteq X$, $|X \cap \omega_\omega| = \aleph_\omega$, and $|X \cap \omega_n| = \aleph_{n-1}$. Let $\pi : X \longrightarrow N$ denote the corresponding transitive collapse. Then, $\omega_\omega \in N$, $\pi(\omega_\omega) = \omega_\omega$ and $\omega_n \in \mathrm{ran}(\pi)$. Set $\lambda = \pi^{-1}(\omega_n) < \omega_n$. Then, $\pi \restriction \lambda = \mathrm{id}_\lambda$,

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and an easy induction shows that $\pi(\omega_k) < \omega_k$ holds for all $n \leq k < \omega$. In particular, we know that $\pi^{-1}(\alpha) > \alpha$ holds for all $\omega_n \leq \alpha < \omega_\omega$. By elementarity, the formula $\varphi(\nu_0, \dots, \nu_3)$ and the parameters ω_ω and z define a regressive function in N. In particular, Σ_1-upwards absoluteness implies that $\varphi(\alpha, r(\alpha), \omega_\omega, z)$ holds in N for all $\alpha < \omega_\omega$. We now define a strictly increasing sequence $\langle \mu_i \mid i < \omega \rangle$ of ordinals in the interval $[\omega_n, \omega_\omega)$ by setting $\mu_0 = \omega_n$ and $\mu_{i+1} = \pi^{-1}(\mu_i)$ for all $i < \omega$. We can then inductively show that each μ_i is a cardinal with $r(\mu_i) = r(\lambda)$, and hence, r is constant on an infinite subset of $\{\omega_m \mid m < \omega\}$. ■

We end this section by presenting an example of an application of the concepts isolated in this paper to reduce the class of models of set theory in which ω_ω possesses strong partition properties. In particular, we will show that ω_ω is not ω_2-Rowbottom in the standard models of strong forcing axioms, where the given axiom was forced over a model of the GCH by turning some large cardinal into ω_2. This implication will be a direct consequence of the following observation:

Lemma 4.14 *Assume that there is a natural number $n_* > 1$ such that there are no special ω_{n_*}-Aronszajn trees, and for all $n_* < n < \omega$, there are special ω_n-Aronszajn trees. Then, the set $\{\omega_{n_*}\}$ is definable by a Σ_1-formula with parameter ω_ω, and the cardinal ω_ω is not ω_{n_*}-Rowbottom.*

Proof Consider the collection of all transitive models M of ZFC^- with the following properties:

- $\omega_\omega + 1 \subseteq M$.
- $\omega_\omega = \omega_\omega^M$.
- In M, for every $n_* < n < \omega$, there is a special ω_n^M-Aronszajn tree.

The collection of such models M is not empty as it includes $\mathrm{H}_{\aleph_{\omega+1}}$. It is clear that for each model M in this collection, we have $\omega_{n_*}^M = \omega_{n_*}$. Hence, we can conclude that the set $\{\omega_{n_*}\}$ is definable by the Σ_1-formula with parameter ω_ω that says there is a model M with the above properties and x is equal to $\omega_{n_*}^M$. In particular, this shows that the cardinal ω_ω does not have the $\Sigma_1(\omega_{n_*})$-undefinability property, and hence, Lemma 4.12 shows that ω_ω is not ω_{n_*}-Rowbottom. ■

If we start in a model of the GCH containing a supercompact cardinal and use the canonical forcing to force the validity of some strong forcing axiom, like PFA or MM^{++}, then a result of Baumgartner shows that the tree property holds at ω_2, and since the GCH holds above $\aleph_0$, a result of Specker ensures that there are special ω_n-Aronszajn trees for all $2 < n < \omega$. Therefore, the above lemma shows that ω_ω is not ω_2-Rowbottom in these models. This observation should be compared with the consistency results of König in [12].

5 Equiconsistency results

We build on the results of the previous sections to obtain more equiconsistency results that witness various ways by which different types of Σ_1-definable closed unbounded sets of cardinals κ fail to approximate the closed unbounded filter on κ.

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5.1 Limits of uncountably many measurable cardinals

We start by showing that, in the case of singular cardinals of uncountable cofinality, Theorem 4.6 provides the right consistency strength for both of the listed conclusions. In contrast, we will later show that the consistency strength for singular cardinals of countable cofinality is merely a single measurable cardinal.

Theorem 5.1 *Let κ be a singular cardinal of uncountable cofinality. If there is no inner model with $\mathrm{cof}(\kappa)$-many measurable cardinals, then the following statements hold:*

(1) *Every subset of κ that is $\mathbf{\Sigma}_1$-stationary in κ is stationary in κ. In particular, there is an unbounded subset of κ that consists of cardinals and is not $\mathbf{\Sigma}_1$-stationary in κ.*
(2) *There exists a regressive function $r : \kappa \longrightarrow \kappa$ that is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$ and is not constant on any unbounded subset of κ.*

Proof By [11, Theorem 2.14], our assumptions imply that 0^{long} (as defined in [11, Definition 2.13]) does not exist. Let U_{can} denote the canonical sequence of measures, and let $\mathrm{K}[U_{can}]$ denote the canonical core model (as defined in [11, Definition 3.15]). Then, our assumption ensures that $\mathrm{dom}(U_{can})$ has order-type less than $\mathrm{cof}(\kappa)$. Moreover, since $\mathrm{cof}(\kappa)$ is uncountable, we can apply [11, Theorem 3.23] to show that κ is not measurable in $\mathrm{K}[U_{can}]$. But this allows us to use [11, Theorem 3.20] to conclude that κ is singular in $\mathrm{K}[U_{can}]$. Set $U = U_{can} \restriction \kappa$ and $\mathrm{K} = \mathrm{K}[U]$ (see [11, Definition 3.1]). Then, $U \in \mathrm{H}_\kappa$ and [11, Theorem 3.9] shows that $\mathcal{P}(\kappa)^{\mathrm{K}[U_{can}]} \subseteq \mathrm{K}$. In particular, we know that κ is singular in K, and we can define $c : \mathrm{cof}(\kappa)^{\mathrm{K}} \longrightarrow \kappa$ to be the $<_{\mathrm{K}[U]}$-least cofinal function in K (see [11, Theorem 3.4]). Then, [16, Lemma 2.3] shows that c is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$. But this shows that there is a closed unbounded subset C of κ of order-type $\mathrm{cof}(\kappa)^{\mathrm{K}}$ such that $\min(C) > \mathrm{cof}(\kappa)^{\mathrm{K}}$, and the set $\{C\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$. As in the proof of Corollary 3.4, we now know that every subset of κ that is $\mathbf{\Sigma}_1$-stationary in κ is stationary in κ. Finally, the function

$$r : \kappa \longrightarrow \mathrm{cof}(\kappa)^{\mathrm{K}};\ \alpha \longmapsto \mathrm{otp}\,(C \cap \alpha)$$

is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$ and is not constant on any unbounded subset of κ. ■

By combining this result with Theorem 4.6, we obtain the following equiconsistency:

Corollary 5.2 *The following statements are equiconsistent over* ZFC*:*

(1) *There exist uncountably many measurable cardinals.*
(2) *There exists a singular cardinal κ of uncountable cofinality with the property that some nonstationary subset of κ is $\mathbf{\Sigma}_1$-stationary in κ.*
(3) *There exists a singular cardinal κ of uncountable cofinality with the property that some nonstationary subset of κ is $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary in κ.*

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5.2 Countable cofinalities

The aim of this section is to prove the following equiconsistency result:

Theorem 5.3 *The following statements are equiconsistent over* ZFC*:*

(1) *There is a measurable cardinal.*
(2) *Every unbounded subset of* $\{\omega_n \mid n < \omega\}$ *is* $\mathbf{\Sigma}_1(\mathrm{Ord})$*-stationary in* ω_ω*.*
(3) *There is a singular cardinal* κ *of countable cofinality and a subset of* κ *that consists of cardinals and is* $\mathbf{\Sigma}_1(\mathrm{Ord})$*-stationary in* κ*.*
(4) *There is a singular cardinal* κ *of countable cofinality and a subset of* κ *that consists of cardinals and is* $\mathbf{\Sigma}_1$*-stationary in* κ*.*

Remark 5.4 The restriction to $\mathbf{\Sigma}_1(\mathrm{Ord})$-definability in (2) of the theorem is optimal since the sequence $\langle \omega_n \mid n < \omega \rangle$ is $\Sigma_2(\mathrm{Ord})$-definable.

In the following arguments, we will make use of the Easton-support collapse version from [18] of a universal collapse forcing (see [6] for a comprehensive background).

Definition 5.1 Suppose that κ is a Mahlo cardinal, and let

$$I_\kappa = \{\omega\} \cup \{\gamma < \kappa \mid \gamma \text{ is inaccessible}\}.$$

For each $\gamma \in I_\kappa$, we define $\mathbb{Q}(\gamma, <\kappa)$ to be the product

$$\mathbb{Q}(\gamma, <\kappa) = \prod_{\delta \in I_\kappa \setminus \gamma} \mathrm{Col}(\delta, <\kappa)$$

with the Easton-support.

Remark 5.5 The partial order $\mathbb{Q}(\gamma, <\kappa)$ is clearly definable from the ordinals γ and κ, and forcing with $\mathbb{Q}(\gamma, <\kappa)$ collapses all cardinals in the interval (γ, κ). Since this partial order is $<\gamma$-closed and satisfies the κ-chain condition, the cardinal κ becomes γ^+ in $\mathbb{Q}(\gamma, <\kappa)$-generic extensions. In the following, we will rely on the following two useful features of this partial order:

(1) If $\delta < \kappa$ is a Mahlo cardinal and $\gamma \in I_\delta$, then $\mathbb{Q}(\gamma, <\delta) \times \mathbb{Q}(\delta, <\kappa)$ is a regular subforcing of $\mathbb{Q}(\gamma, <\kappa)$.
(2) $\mathbb{Q}(\gamma, <\kappa)$ is weakly homogeneous.

The proofs are left to the reader (see also [18]).

Before proving Theorem 5.3, we review some basic facts about Prikry forcing and products of collapse forcings. Suppose that κ is a measurable cardinal and U is a normal measure on κ. We let $\mathbb{P}_U$ denote the Prikry forcing given by U. Conditions $\mathbb{P}_U$ are then of the form $p = \langle s_p, A_p \rangle$, where $s_p \in [\kappa]^{<\omega}$ is a finite sequence of Mahlo cardinals and $A_p \in U$ consists of Mahlo cardinals and satisfies $\min(A_p) > \max(s_p)$. We then have $p \leq_{\mathbb{P}_U} q$ if s_p end-extends s_q, $A_p \subseteq A_q$, and $s_p \setminus s_q \subseteq A_q$. If, moreover, $s_p = s_q$ holds, then we say p is a direct extension of q, denoted by $p \leq^*_{\mathbb{P}_U} q$. Given a condition p in $\mathbb{P}_U$ and $t \in [A_p]^{<\omega}$, we let $p^\frown t$ denote the condition $\langle s_p \cup t, A_p \setminus (\max(t)+1) \rangle$. Finally, if G is $\mathbb{P}_U$-generic filter over V, then its associated Prikry sequence $\vec{\kappa}^G = \langle \kappa^G_n \mid n < \omega \rangle$ is defined by $\vec{\kappa}^G = \bigcup \{s \in [\kappa]^{<\omega} \mid \exists A \in U\ \langle s, A \rangle \in G\}$. Below, we record some of the basic properties of $\mathbb{P}_U$ that will be used in the following arguments (see [8] for details).

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Fact 5.6

(1) *The direct extension order $\leq^*_{\mathbb{P}_U}$ is $<\kappa$-closed.*
(2) *Forcing with $\mathbb{P}_U$ does not introduce new bounded subsets of κ.*
(3) *The partial order $\mathbb{P}_U$ satisfies the κ^+-chain condition.*
(4) *(Prikry Property) For every condition q in $\mathbb{P}_U$ and every statement σ in the forcing language of $\mathbb{P}_U$, there is $p \leq^*_{\mathbb{P}_U} q$ which decides σ.*
(5) *(Strong Prikry Property) For every condition q in $\mathbb{P}_U$ and every dense open set D of $\mathbb{P}_U$, there is $p \leq^*_{\mathbb{P}_U} q$ and $n < \omega$ such that D contains all conditions of the form $p^\frown t$ with $t \in [A_p]^n$.*
(6) *(Name-Capturing Property) For every condition q in $\mathbb{P}_U$ and every $\mathbb{P}_U$-name $\dot{f}$ for a function with domain ω with the property that $\dot{f}^G(n) \in \mathrm{H}_{\kappa^G_n}$ holds whenever $n < \omega$ and G is $\mathbb{P}_U$-generic over V with $q \in G$, there is a condition $p \leq^*_{\mathbb{P}_U} q$ and a function $F : [A_p]^{<\omega} \longrightarrow \mathrm{H}_\kappa$ so that*

$$p^\frown t \Vdash_{\mathbb{P}_U} \text{“}\dot{f}(|\check{s}_p \cup \check{t}|) = \check{F}(\check{t})\text{”}$$

holds for every $t \in [A^*]^{<\omega}$.
(7) *A $\mathbb{P}_U$-generic filter G is generated by its induced Prikry sequence $\vec{\kappa}^G$ in the sense that $G = \{\langle \vec{\kappa}^G \restriction n, A\rangle \mid n < \omega,\ \vec{\kappa}^G \backslash n \subseteq A \in U\}$.*
(8) *(Mathias Criterion) A sequence $\vec{\kappa} = \langle \kappa_n \mid n < \omega\rangle$ generates a $\mathbb{P}_U$-generic filter if and only if $\vec{\kappa}\backslash A$ is finite for every $A \in U$.*

To push the construction down to ω_ω, we force with a product of collapse posets after adding a Prikry forcing. Let $\vec{\rho} = \langle \rho_n \mid n < \ell_{\vec{\rho}}\rangle$ be a strictly increasing sequence of Mahlo cardinals of length $0 < \ell_{\vec{\rho}} \leq \omega$ and define $\mathbb{C}_{\vec{\rho}}$ to be the product

$$\mathbb{Q}(\omega, <\rho_0) \times \prod_{0<n<\ell_{\vec{\rho}}} \mathbb{Q}(\rho_{n-1}, <\rho_n)$$

with full support. Therefore, conditions in $\mathbb{C}_{\vec{\rho}}$ are sequences $q = \langle q_n \mid n < \ell_{\vec{\rho}}\rangle$ with $q_0 \in \mathbb{Q}(\omega, <\rho_0)$ and $q_n \in \mathbb{Q}(\rho_{n-1}, <\rho_n)$ for all $0 < n < \ell_{\vec{\rho}}$.

Remark 5.7

(1) Standard arguments about product forcings show that if G is $\mathbb{C}_{\vec{\rho}}$-generic over V, then $\rho_n = \omega_{n+1}^{V[G]}$ holds for every $n < \ell_{\vec{\rho}}$.
(2) By the absorption argument for the Easton-support collapse product (Remark 5.5), if s is a finite strictly increasing sequences of Mahlo cardinals and t is a subsequence of s with $\max(s) = \max(t)$, then $\mathbb{C}_s$ is a regular subforcing of $\mathbb{C}_t$. This directly implies that if $\vec{\rho}$ is a strictly increasing sequence of Mahlo cardinals of length ω and s is a finite subset of $\vec{\rho}$, then $\mathbb{C}_{\vec{\rho}}$ is a regular subforcing of $\mathbb{C}_{\vec{\rho}\backslash s}$. Moreover, the associated forcing projection from $\mathbb{C}_{\vec{\rho}\backslash s}$ to $\mathbb{C}_{\vec{\rho}}$ is the identity on the components below $\min(s)$ and above $\min(\vec{\rho}\backslash(\max(s)+1))$.
(3) The forcing $\mathbb{C}_{\vec{\rho}}$ is weakly homogeneous. Therefore, if some condition in $\mathbb{C}_{\vec{\rho}}$ forces a statement with ground model parameters to hold, then every condition forces this statement to hold. Similarly, for every sequence $\vec{\rho}$ and an initial segment s, the quotient forcing $\mathbb{C}_{\vec{\rho}}/\mathbb{C}_s$ is also weakly homogeneous.

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In the following, let $\dot{\mathbb{C}}$ denote the canonical $\mathbb{P}_U$-name for a partial order with the property that $\dot{\mathbb{C}}^G = \mathbb{C}_{\vec{\kappa}^G}$ holds whenever G is $\mathbb{P}_U$-generic over V. The argument of the following Lemma will give the forcing direction of Theorem 5.3.

Lemma 5.8 *If $G * H$ is $(\mathbb{P}_U * \dot{\mathbb{C}})$-generic over V and $\gamma \geq \kappa$ is an ordinal, then in $V[G,H]$, the cardinal ω_ω has the $\Sigma_1(\omega_n, \gamma)$-undefinability property for all sufficiently large natural numbers n.*

Proof Fix a condition p_* in $\mathbb{P}_U$. For each natural number $n \geq |s_{p_*}|$, let σ_n be the statement in the forcing language of $\mathbb{P}_U$ that says that there is a condition in $\dot{\mathbb{C}}$ which forces that κ does not have the $\Sigma_1(\dot{\kappa}_n, \gamma)$-undefinability property, where $\dot{\kappa}_n$ denotes the canonical $\mathbb{P}_U$-name for κ_n^G. By the Prikry property, for each $|s_{p_*}| \leq n < \omega$, there is a condition $p_n \leq^*_{\mathbb{P}_U} p_*$ that decides σ_n. We complete the proof by showing that for all $|s_{p_*}| \leq n < \omega$, the condition p_n forces $\neg\sigma_n$ to hold. Suppose otherwise that $p_n \Vdash_{\mathbb{P}_U} \sigma_n$ for some $|s_{p_*}| \leq n < \omega$. Pick a condition $p \leq_{\mathbb{P}_U} p_n$ with $|s_p| = n$. Since $p \Vdash_{\mathbb{P}_U} \sigma_n$, there is a $\mathbb{P}_U$-name $\dot{q}$ for a condition in $\dot{\mathbb{C}}$ so that

$$\langle p, \dot{q} \rangle \Vdash_{\mathbb{P}_U * \dot{\mathbb{C}}} \text{“}\kappa \textit{ does not have the } \Sigma_1(\dot{\kappa}_n, \check{\gamma})\textit{-undefinability property.}\text{”}$$

Since $\dot{\kappa}_n$ is a $\mathbb{P}_U$-name and $\dot{\mathbb{C}}$ is forced to be weakly homogeneous (see Remark 5.7), we can assume that $\dot{q}$ is the name for the trivial condition $\mathbb{1}_{\dot{\mathbb{C}}}$ in $\dot{\mathbb{C}}$. This allows us to find $(\mathbb{P}_U * \dot{\mathbb{C}})$-names $\dot{x}$, $\dot{\alpha}$, and $\dot{\tau}$ so that $\langle p, \mathbb{1}_{\dot{\mathbb{C}}} \rangle$ forces the following statements to hold:

- $\dot{x}$ is an element of $\mathrm{H}_{\dot{\kappa}_n}$.
- $\dot{\alpha}$ is an ordinal in the interval $[\dot{\kappa}_n, \check{\kappa})$.
- $\dot{\tau}$ is the Gödel number of a Σ_1-formula $\varphi(v_0, \dots, v_3)$ that defines the set $\{\dot{\alpha}\}$ using the parameters $\check{\kappa}$, $\check{\gamma}$, and $\dot{x}$.

Making another use of the fact that $\dot{\mathbb{C}}$ is forced to be a weakly homogeneous partial order, we may assume $\dot{\alpha}$ and $\dot{\tau}$ are $\mathbb{P}_U$-names. Moreover, using the Prikry property of $\mathbb{P}_U$, we may also assume that p decides that $\dot{\tau}$ codes a given Σ_1-formula $\varphi(v_0, \dots, v_3)$. Since the quotient forcing $\mathbb{C}_{\vec{\kappa}^G}/\mathbb{C}_{\vec{\kappa}^G \restriction (n+1)}$ is weakly homogeneous whenever G is $\mathbb{P}_U$-generic over V, we may assume that $\dot{x}$ is a $(\mathbb{P}_U * \dot{\mathbb{C}}_n)$-name, where $\dot{\mathbb{C}}_n$ is the canonical $\mathbb{P}_U$-name for $\mathbb{C}_{\vec{\kappa}^G \restriction (n+1)}$. In addition, since forcing with $\mathbb{P}_U$ does not add new bounded sets to κ and $\dot{\mathbb{C}}_n$ is forced to satisfy the $\dot{\kappa}_n$-chain condition, there is a $\mathbb{P}_U$-name $\dot{y}$ with the property that whenever G is $\mathbb{P}_U$-generic over V with $p \in G$, then $\dot{y}^G$ is a $\mathbb{C}_{\vec{\kappa}^G \restriction (n+1)}$-name in $\mathrm{H}^V_{\kappa_n^G}$ that $\mathbb{C}_{\vec{\kappa}^G \restriction (n+1)}$ forces to be equal to $\dot{x}$. The fact that $p^\frown\langle \rho \rangle \Vdash_{\mathbb{P}_U} \text{“}\dot{\kappa}_n = \check{\rho}\text{”}$ holds for every $\rho \in A_p$ allows us to use the name-capturing property of $\mathbb{P}_U$ (see Fact 5.6) to find a function $Y : A_p \longrightarrow \mathrm{H}_\kappa$ with the property that for every $\rho \in A_p$, the set $Y(\rho)$ is a $\mathbb{C}_{s_p \cup \{\rho\}}$-name in H_ρ with $p^\frown\langle \rho \rangle \Vdash_{\mathbb{P}_U} \text{“}\dot{y} = \check{Y}(\check{\rho})\text{.”}$ Using the normality of U, we can find $A \subseteq A_p$ in U and $\dot{y}_0$ in H_κ with the property that $Y(\rho) = \dot{y}_0$ holds for all $\rho \in A$. Let $\rho_0 = \min(A)$. Then, we can find an ordinal α in the interval $[\rho_0, \kappa)$ and a condition $r_0 \leq_{\mathbb{P}_U} p^\frown\langle \rho_0 \rangle$ with $\alpha < \max(s_{r_0})$ and $r_0 \Vdash_{\mathbb{P}_U} \text{“}\dot{\alpha} = \check{\alpha}\text{.”}$ Set $\rho_1 = \min(A \cap A_{r_0})$ and $r_1 = \langle s_p \cup \{\rho_1\}, A_{r_0} \backslash (\rho_1 + 1) \rangle$. Then, r_1 is a condition in $\mathbb{P}_U$ that strengthens $p^\frown\langle \rho_1 \rangle$. Let $G_1 * H_1$ be $(\mathbb{P}_U * \dot{\mathbb{C}})$-generic over V with $r_1 \in G_1$. We then have $\kappa_n^{G_1} = \rho_1 > \max(s_{r_0}) > \alpha$. By the Mathias criterion for $\mathbb{P}_U$ (see Fact 5.6), the sequence $s_{r_0}{}^\frown(\vec{\kappa}^{G_1} \restriction [n, \omega))$ generates a $\mathbb{P}_U$-generic filter

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G_0 over V that is an element of $V[G_1]$. It is then clear that r_0 is an element of G_0 and $V[G_0] = V[G_1]$. Moreover, since $\vec{\kappa}^{G_1} \restriction n = s_p = s_{r_0} \restriction n$, it follows that $\vec{\kappa}^{G_1}$ is a subsequence of $\vec{\kappa}^{G_0}$ with finite difference, and hence, the absorption property of Easton collapse posets (see Remark 5.5) ensures that the partial order $\mathbb{C}_{\vec{\kappa}^{G_0}}$ is a regular subforcing of the partial order $\mathbb{C}_{\vec{\kappa}^{G_1}}$. Let $H_0 \in V[G_0, H_1]$ denote the filter on $\mathbb{C}_{\vec{\kappa}^{G_0}}$ induced by H_1. Since H_0 and H_1 induce the same filter on $\mathbb{C}_{s_p \cup \{\rho_0\}}$, we know that $\dot{y}_0^{H_0} = \dot{y}_0^{H_1}$. This implies that $\varphi(\alpha, \kappa, \gamma, \dot{y}_0^{H_1})$ holds in $V[G_0, H_0]$, and by Σ_1-upwards absoluteness, this statement also holds in $V[G_0, H_1]$. But this yields a contradiction because the fact that r_1 is an element of G_1 implies that $\dot{\alpha}^{G_1} \geq \kappa_n^{G_1} = \rho_1 > \alpha$ and $\dot{\alpha}^{G_1}$ is the unique element a of $V[G_0, H_1]$ with the property that $\varphi(a, \kappa, \gamma, \dot{y}_0^{H_1})$ holds. ■

A combination of the above lemma with Corollary 2.13 now directly yields the following result:

Corollary 5.9 *If $G * H$ is $(\mathbb{P}_U * \dot{\mathbb{C}})$-generic over V, then in $V[G, H]$, every unbounded subset of $\{\omega_n \mid n < \omega\}$ is $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary in ω_ω.*

We are now ready to prove the main result of this section:

Proof of Theorem 5.3 First, let κ be a singular cardinal of countable cofinality, and let E be a subset of κ that consists of cardinals and is $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary in κ. Assume, toward contradiction, that there is no inner model with a measurable cardinal. Then, κ is singular in the Dodd-Jensen core model K^{DJ}. By Lemma 3.3, there is a cofinal function $c : \mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}} \longrightarrow \kappa$ which is definable by a Σ_1-formula with parameter κ. In addition, pick a cofinal function $g : \omega \longrightarrow \mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}$. Then the composition $f = c \circ g : \omega \longrightarrow \kappa$ is cofinal in κ and Σ_1-definable from the parameters κ and $g \in \mathrm{H}_\kappa$. Then, the $C = \{f(n) + 1 \mid n < \omega\}$ is closed unbounded in κ, and it is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa\}$. Since it consists of successor ordinals, it is disjoint from E, contradicting our assumptions.

The above computations show that (4) implies (1) in the statement of the theorem. This completes the proof of the theorem because the implications from (2) to (3) and from (3) to (4) are trivial, and the implication from (1) to (2) is given by Corollary 5.9. ■

By combining Theorem 5.3 with the second part of Corollary 3.6, we derive the following equiconsistency result that shows that the statement of Proposition 2.8 is optimal in ZFC.

Corollary 5.10 *The following statements are equiconsistent over* ZFC*:*

(1) *There is a measurable cardinal.*
(2) *There is a singular cardinal κ of countable cofinality with the property that for every subset A of H_κ of cardinality $\mathfrak{r}$, there exists a subset E of κ such that both E and $\kappa \backslash E$ are $\Sigma_1(A)$-stationary.*
(3) *There is a singular cardinal κ of countable cofinality with the property that there exists a subset E of κ such that both E and $\kappa \backslash E$ are $\mathbf{\Sigma}_1(\mathrm{Ord})$-stationary.*

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5.3 Successors of regular cardinals

We continue by studying $\Sigma_1(A)$-stationary subsets of successor cardinals. In this section, we prove the following equiconsistency result for successors of regular cardinals:

Theorem 5.11 *The following statements are equiconsistent over* ZFC*:*

(1) *There is a Mahlo cardinal.*
(2) *There is a regular cardinal μ with the property that the set $\{\mu\}$ is $\Sigma_1(\mathrm{H}_\mu)$-stationary in μ^+.*
(3) *There is a regular cardinal μ with the property that the set $\{\mu\}$ is Σ_1-stationary in μ^+.*

Lemma 5.12 *If κ is a Mahlo cardinal, then there exists an inaccessible cardinal $\delta < \kappa$ with the property that $\mathbb{Q}(\delta, <\kappa)$ forces κ to have the $\Sigma_1(\delta)$-undefinability property.*

Proof Assume, toward a contradiction, that no such δ exists. Then, for every inaccessible $\gamma \in I_\kappa$, the fact that $\mathbb{Q}(\gamma, <\kappa)$ is weakly homogeneous (see Remarks 5.5 and 5.7) allows us to find x_γ, α_γ, and τ_γ such that the following statements hold:

- x_γ is an element of H_γ.
- α_γ is an ordinal in the interval $[\gamma, \kappa)$.
- τ_γ is a Gödel number of a Σ_1-formula $\varphi(v_0, v_1, v_2)$ with the property that the trivial condition of $\mathbb{Q}(\gamma, <\kappa)$ forces that α_γ is the unique set a such that $\varphi(a, \kappa, x_\gamma)$ holds.

Since κ is a Mahlo cardinal, we can find a stationary subset S of κ consisting of inaccessible cardinals, an element x of H_κ, and a Σ_1-formula $\varphi(v_0, v_1, v_2)$ such that for all $\gamma \in S$, we have $x_\gamma = x$ and τ_γ is a Gödel number of $\varphi(v_0, v_1, v_2)$. Let $\gamma_0 = \min(S)$ and $\gamma_1 = \min(S \backslash (\alpha_{\gamma_0} + 1))$. Clearly, we then have $\gamma_0 < \gamma_1$ and $\alpha_{\gamma_1} > \alpha_{\gamma_0}$. Let G_0 be $\mathbb{Q}(\gamma_0, <\kappa)$-generic over V. As $\mathbb{Q}(\gamma_1, <\kappa)$ is a regular subforcing of $\mathbb{Q}(\gamma_0, <\kappa)$, we can now find $G_1 \in V[G_0]$ that is $\mathbb{Q}(\gamma_1, <\kappa)$-generic over V. In this situation, we know that $\varphi(\alpha_{\gamma_1}, \kappa, x)$ holds in $V[G_1]$, and by Σ_1-upwards absoluteness, this statement also holds in $V[G_0]$. But this yields a contradiction because $\alpha_{\gamma_1} > \alpha_{\gamma_0}$, and α_{γ_0} is the unique element a of $V[G_0]$ with the property that $\varphi(a, \kappa, x)$ holds in $V[G_0]$. ■

Proof of Theorem 5.11 Suppose that μ is a regular cardinal so that the set $\{\mu\}$ is Σ_1-stationary in μ^+. Assume, toward a contradiction, that $\kappa = \mu^+$ is not a Mahlo cardinal in L. Then, there is a constructible closed unbounded subset of κ whose elements are singular in L. Let C denote the $<_{\mathrm{L}}$-least such subset of κ. Then, the set $\{C\}$ is definable by a Σ_1-formula with parameter κ, and hence, we know that μ is an element of C. But this is a contradiction because all elements of C are singular.

The above computations yield the implication from (3) to (1) in the statement of the theorem. The implication from (1) to (2) is given by a combination of Lemma 2.12 and Lemma 5.12. Finally, the implication from (2) to (3) is trivial. ■

5.4 Successors of singular cardinals

The last arguments used in the above proof of Theorem 5.11 do not apply if we consider successors of singular cardinals. Indeed, it turns out that the corresponding

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assumption has much higher consistency strength. In one direction, we show that an analogous statement holds for successors of limits of measurable cardinals:

Theorem 5.13 *If κ is a limit of measurable cardinals, then the cardinal κ^+ has the $\Sigma_1(\kappa)$-undefinability property.*

Proof Assume, toward a contradiction, that there is a Σ_1-formula $\varphi(v_0, v_1, v_2)$, an ordinal α in the interval $[\kappa, \kappa^+)$, and an element z of H_κ such that α is the unique set a with the property that $\varphi(a, \kappa^+, z)$ holds. Pick a measurable cardinal $\delta < \kappa$ such that $z \in \mathrm{H}_\delta$ and $\mathrm{cof}(\kappa) \in \delta \cup \{\kappa\}$. Pick a normal ultrafilter U on δ, and let

$$\langle \langle M_\alpha \mid \alpha \in \mathrm{Ord} \rangle, \langle j_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta \in \mathrm{Ord} \rangle \rangle$$

denote the linear iteration of $\langle V, \in, U \rangle$. Standard arguments now allow us to conclude that $j_{0,\gamma}(\kappa^+) = \kappa^+$ and $j_{0,\gamma}(z) = z$ holds for all $\gamma < \kappa^+$ (see, for example, [15, Lemma 7.3]). We can now pick $\gamma < \kappa^+$ with $j_{0,\gamma}(\delta) > \alpha$. Elementarity then implies that $\varphi(j_{0,\gamma}(\alpha), \kappa^+, z)$ holds in M_γ, and by Σ_1-upwards absoluteness, this shows that $\varphi(j_{0,\gamma}(\alpha), \kappa^+, z)$ also holds in V. Since $j_{0,\gamma}(\alpha) > \alpha$, this yields a contradiction. ■

In combination with Lemma 2.12, the above result shows that if κ is a limit of measurable cardinals, then the set $\{\kappa\}$ is $\Sigma_1(\mathrm{H}_\kappa)$-stationary in κ^+. We end this section by showing that in the case of successors of singular cardinals, the used large cardinal assumption is optimal. In the proof of this result, we again rely on the results of [11].

Theorem 5.14 *Let κ be a singular cardinal with the property that the set $\{\kappa\}$ is $\Sigma_1(\mathrm{H}_\kappa)$-stationary in κ^+. Then, there is an inner model with $\mathrm{cof}(\kappa)$-many measurable cardinals.*

Proof Assume, toward a contradiction, that the above conclusion fails. An application of [11, Theorem 2.14] then shows that 0^{long} does not exist. Let U_{can} denote the canonical sequence of measures, and let $\mathrm{K}[U_{can}]$ denote the canonical core model. Our assumption then implies that $\mathrm{dom}(U_{can} \restriction \kappa)$ is a bounded subset of κ.

Claim $\kappa^+ = (\kappa^+)^{\mathrm{K}[U_{can}]}$. ■

Proof of the Claim First, if $\kappa \notin \mathrm{dom}(U_{can})$, then the fact that $\mathrm{dom}(U_{can} \restriction \kappa)$ is bounded in κ allows us to use [11, Theorem 3.20] to derive the desired conclusion. Hence, we may assume that κ is an element of $\mathrm{dom}(U_{can})$. An application of [11, Theorem 3.23] then shows that $\mathrm{cof}(\kappa) = \omega$, and there is a generic extension of $\mathrm{K}[U_{can}]$ by finitely-many Prikry forcings that computes κ^+ correctly. But this also means that $\mathrm{K}[U_{can}]$ computes κ^+ correctly. ■

Set $U = U_{can} \restriction \kappa \in \mathrm{H}_\kappa$. By [11, Theorem 3.9], we then have $\mathcal{P}(\kappa)^{\mathrm{K}[U_{can}]} \subseteq \mathrm{K}[U]$, and hence, the above claim shows that $\kappa^+ = (\kappa^+)^{\mathrm{K}[U]}$.

Claim *The set $\{\kappa\}$ is definable by a Σ_1-formula with parameters κ^+ and U.*

Proof of the Claim If $\zeta < \kappa^+$ is an ordinal and M is a U-mouse (i.e., an iterable premouse over U; see [11, Definition 2.5]) with $\kappa^+ \in lp(M)$ and $\kappa^+ = (\zeta^+)^M$, then a direct adaptation of the proof of Lemma 3.1 shows that $\mathcal{P}(\kappa) \subseteq M$, and hence, $\kappa = \zeta$. Since there exists a U-mouse that satisfies the listed properties with respect to κ, this observation yields the desired definition of the set $\{\kappa\}$. ■

The above claim directly yields a contradiction because it implies that the set $\{(\kappa, \kappa^+)\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}_\kappa \cup \{\kappa^+\}$.

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6 Open Problems

There are many natural ways to vary Definition 2.2. For example, given a cardinal κ greater than ω_1, we may ask whether for some uncountable ordinal $\alpha < \kappa$, the set $\{\alpha\}$ is definable by a Σ_1-formula with parameter κ. Since our arguments to derive consistency strength from the undefinability property (as in the proof of Theorem 5.3) cannot be directly adjusted to this variation, we arrive at the following question:

Question 6.1 *Assume that for every uncountable ordinal $\alpha < \omega_\omega$, the set $\{\alpha\}$ is not definable by a Σ_1-formula with parameter ω_ω. Does $0^\#$ exist?*

In Section 4.3, we show that many of the implications of large cardinals on $\Sigma_1(A)$-stationary sets can also be derived for smaller cardinals in the case where these cardinals possess strong partition properties. For some of these implications, it is natural to ask whether they can be strengthened. First, since Lemma 4.12 relies on additional assumptions on the given Rowbottom cardinal, we ask whether these assumptions can be omitted:

Question 6.2 *Does every μ-Rowbottom cardinal have the $\Sigma_1(\mu)$-undefinability property?*

Second, when we consider Σ_1-definable regressive functions and compare the third part of Theorem 4.1 and the second part of Theorem 4.6 with the second part of Theorem 4.13, then we notice that our result for the case where ω_ω is Jónsson is restricted to $\mathbf{\Sigma}_1$-stationary sets consisting of cardinals. We therefore ask if the given conclusion can also be extended to arbitrary $\mathbf{\Sigma}_1$-stationary subsets in this setting.

Question 6.3 *Assume that ω_ω is a Jónsson cardinal, S is a $\mathbf{\Sigma}_1$-stationary subset of ω_ω, and $r : \omega_\omega \longrightarrow \omega_\omega$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}_{\aleph_\omega} \cup \{\omega_\omega\}$. Is r constant on a $\mathbf{\Sigma}_1$-stationary subset of S?*

Finally, let us say that a cardinal κ is *strongly measurable with respect to $\Sigma_1(\mathrm{Ord})$-clubs* if there is an inner model W in which κ is measurable such that the collection of $\Sigma_1(\mathrm{Ord})$-closed unbounded subsets of κ (in V) is generated by the intersection filter $F \in W$ of $\eta < \kappa$ many κ-complete ultrafilters on κ in W. The proof of Theorem 5.3 shows that ω_ω can be strongly measurable with respect to Σ_1-definable clubs. Can the same hold for other singular cardinals such as ω_{ω_1}?

Question 6.4 *Can ω_{ω_1} be strongly measurable with respect to $\Sigma_1(\mathrm{Ord})$-clubs?*

See [3] for an analog of Prikry forcing, that changes the cofinality of a cardinal κ to ω_1 by a homogeneous poset.

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