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Supersonic wing-body interference

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Introduction. We consider here the supersonic inviscid flow past a wing-body combination consisting of a semi-infinite plane wing symmetrically placed about an infinite convex cylinder. When the cylinder is circular in cross-section a formal solution for the Laplace transform of the velocity potential exists (Neilson (1), and Stewartson, 1951, unpublished). This solution is given in the form of a series which is only slowly convergent near the boundary surface separating the disturbed and the undisturbed regions. However, this slow convergence has been overcome by Waechter (4) using the Poisson Summation formula on the series solution. This enables the solution to be expressed in terms of integrals which can be evaluated as a convergent residue series, taking on different forms in the different regions of the interaction.

The solution to the problem in the neighbourhood of this bounding surface is mathematically related to short wave diffraction by a cylinder, and this relationship is exploited in the solution given here. Using this analogy, combined with boundary-layer analysis, Stewartson (2) examined the solution again for the circular case. An alternative approach is given in Jones (3) who develops the problem for an arbitrary cylinder using electromagnetic diffraction theory based on the method of Green's functions.

In this paper we show that Stewartson's approach can be extended to provide a straightforward derivation of results applicable to any smooth convex cylinder, provided that the curvature does not vanish.

We have considered only the case in which the wings are of rectangular planform, with no detailed discussion of the case in which the leading edge of the wing is swept back. Using the formulation of this latter problem given by Waechter (4) it follows that the solution valid in the shadow region, in which the disturbance from the leading edge creeps around the cylinder, will retain the same general form given by (3·31) and (3·32). However, the arbitrary constants involved in these solutions are determined by the solution valid near the wing intersection, and this will need to be modified. Further, it will be necessary to introduce an edge diffraction region separating the shadow region from that region dominated by the disturbance from the wing alone. In this region the dominant term in the solution propagates directly from the junction of the wing and the fuselage.

The results obtained are in agreement with those of Jones (3) for the general cylinder, and with Stewartson (2) and Waechter (4) for the circular cylinder.

1. *Basic equations and boundary conditions.* The wing is taken to be at a small angle

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of incidence, and we assume it to occupy the plane $y = 0$ for which $z \geqslant 0$, $|x| > a$. We consider the Laplace transform of the velocity potential $\phi(x, y, z)$ defined by

$$\Phi(x, y; s) = \int_0^\infty \phi(x, y, z)\, e^{-sz}\, dz. \tag{1·1}$$

The equation now becomes

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = s^2 \Phi, \tag{1·2a}$$

with the boundary conditions

$$\frac{\partial \Phi}{\partial y} = \frac{-1}{s} \quad \text{on} \quad y = 0 \quad (|x| > a) \tag{1·2b}$$

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on the cylinder} \tag{1·2c}$$

and

$$\nabla \Phi \to 0 \quad \text{as } |y| \to \infty \text{ for fixed } x, \tag{1·2d}$$

where $\partial/\partial n$ is the derivative normal to the cylinder.

An asymptotic solution to this problem is to be found for $s/\kappa \gg 1$, where $\kappa(x, y)$ is the curvature of the cylinder at the point (x, y).

2. *Region near wing intersection.* At a large distance from the cylinder we would expect its effect on Φ to become weak in comparison with that of the wings. Hence

$$\Phi \to \frac{1}{s^2} e^{-sy} \quad \text{as } |x| \to \infty \text{ for fixed } y. \tag{2·1}$$

Thus we make the substitution

$$\Phi = \frac{1}{s^2} e^{-sy} \Psi(x, y). \tag{2·2}$$

We assume that the cylinder is symmetric about the plane $y = 0$, and so for convenience we shall consider only that part of the cylinder in the first quadrant.

Let the cylinder be represented parametrically by

$$x = p(t), \quad y = q(t),$$

where t is the arc length from the point to the wing intersection, which is taken as $(a, 0)$. From a general point (x, y) let N and T be the points of contact of the normal and tangent to the cylinder, and let them have parameters t_N and t respectively. Denote the values of the functions $p(t)$, $q(t)$, $\kappa(t)$ at t_N and t by p_N, q_N, and κ_N, and by p, q, κ respectively. Further let the lengths PN and PT be r_N and r_T.

Using (2·2), the equations (1·2) give

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - 2s \frac{\partial \Psi}{\partial y} = 0, \tag{2·3a}$$

together with

$$\frac{\partial \Psi}{\partial y} - s\Psi = -s \quad \text{on} \quad y = 0 \quad (|x| > a), \tag{2·3b}$$

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on the cylinder,} \tag{2·3c}$$

and

$$\Psi \to 1 \quad \text{as} \quad |x| \to \infty. \tag{2·3d}$$

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Using the following relations

$$(x-p_N)\,\dot{p}_N+(y-q_N)\,\dot{q}_N=0, \tag{2·4a}$$

$$r_N^2=(x-p_N)^2+(y-q_N)^2, \tag{2·4b}$$

$$\kappa_N=\dot{p}_N\ddot{q}_N-\ddot{p}_N\dot{q}_N, \tag{2·4c}$$

and

$$1=\dot{p}_N^2+\dot{q}_N^2, \tag{2·4d}$$

where

$$\dot{p}_N=\left.\frac{dp}{dt}\right|_{t=t_N}, \quad \text{etc.},$$

we can rewrite equations (2·3) in terms of independent variables (r_N, t_N), to give

$$\frac{\partial^2\Psi}{\partial r_N^2}+\frac{1}{(1+\kappa_N r_N)^2}\frac{\partial^2\Psi}{\partial t_N^2}+\left\{\frac{\kappa_N}{1+\kappa_N r_N}+2s\dot{p}_N\right\}\frac{\partial\Psi}{\partial r_N}-\left\{\frac{r_N\dot{\kappa}_N}{(1+\kappa_N r_N)^3}+\frac{2s\dot{q}_N}{1+\kappa_N r_N}\right\}\frac{\partial\Psi}{\partial t_N}=0, \tag{2·5a}$$

with the boundary conditions

$$\frac{1}{(1+\kappa_N r_N)}\frac{\partial\Psi}{\partial t_N}-s\Psi=-s \quad \text{on} \quad t_N=0+ \quad (r_N>0) \tag{2·5b}$$

$$\frac{\partial\Psi}{\partial r_N}+s\dot{p}_N\Psi=0 \quad \text{on} \quad r_N=0 \quad (t_N>0) \tag{2·5c}$$

and

$$\Psi\to 1 \quad \text{as} \quad r_N\to\infty. \tag{2·5d}$$

In (2·5*b*) we have used the fact that $p(t)$ is an even function of t and $q(t)$ is an odd function.

We now introduce stretched coordinates R, θ defined by

$$R=\kappa_N r_N\left(\frac{s}{\kappa_N}\right)^{\frac{2}{3}}, \quad \theta=\int_0^{t_N}\kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}}dt \tag{2·6}$$

and assume that R and θ are of order unity in the region of interest. On substituting (2·6) into (2·5) we obtain

$$\frac{\partial^2\Psi}{\partial R^2}+2\dot{p}_N\left(\frac{s}{\kappa_N}\right)^{\frac{1}{3}}\frac{\partial\Psi}{\partial R}-2\dot{q}_N\frac{\partial\Psi}{\partial\theta}=\tfrac{2}{3}\dot{q}_N\frac{\dot{\kappa}_N}{\kappa_N^2}\left(\frac{s}{\kappa_N}\right)^{-\frac{1}{3}}R\frac{\partial\Psi}{\partial R} \tag{2·7}$$

$$\left(\text{where} \quad \dot{\kappa}_N=\left.\frac{d\kappa}{dt}\right|_{t=t_N}\right)$$

having neglected terms of order $(s/\kappa)^{-\frac{2}{3}}$.

To this order of approximation it is easily seen that

$$\dot{p}_N\left(\frac{s}{\kappa_N}\right)^{\frac{1}{3}}=-\theta, \quad \dot{q}_N=1. \tag{2·8}$$

Also, using the symmetry of the body, we must have

$$\frac{d\kappa}{dt}=0 \quad \text{at} \quad t=0 \tag{2·9}$$

which implies that $\dot{\kappa}_N$ is of order $(s/\kappa_N)^{-\frac{1}{3}}$.

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We can now write the equations for Ψ in the following form

$$\frac{\partial^2\Psi}{\partial R^2} - 2\theta\frac{\partial\Psi}{\partial R} - 2\frac{\partial\Psi}{\partial\theta} = 0, \tag{2·10a}$$

$$\Psi(R, 0) = 1, \tag{2·10b}$$

$$\frac{\partial\Psi}{\partial R} - \theta\Psi = 0 \quad \text{on} \quad R = 0, \tag{2·10c}$$

and

$$\Psi \to 1 \quad \text{as} \quad R \to \infty. \tag{2·10d}$$

Making use of equations (3·19) and (3·32) of Stewartson (2) we can write down the solution as

$$\Psi(R, \theta) = \tfrac{1}{2}\pi e^{R\theta - \frac{1}{6}\theta^3} \sum_{n=1}^{\infty} \frac{\mathrm{Ai}(2^{\frac{1}{3}}R - \beta_n)}{\beta_n[\mathrm{Ai}(-\beta_n)]^2} e^{-2^{-\frac{1}{3}}\beta_n\theta} \int_{-\beta_n}^{\infty} \mathrm{Ai}(u)\, du \tag{2·11}$$

where $-\beta_n$ represents the nth zero of $\mathrm{Ai}'(\rho)$ arranged in increasing order of magnitude, $\mathrm{Ai}(\rho)$ being the Airy function defined by

$$\mathrm{Ai}(\rho) = \frac{\rho^{\frac{1}{2}}}{\pi\sqrt{3}} K_{\frac{1}{3}}(\tfrac{2}{3}\rho^{\frac{3}{2}}) \tag{2·12}$$

in terms of the Bessel function with imaginary argument. Now $y = q_N - \dot{p}_N r_N$, and so expanding q and $\dot{p}$ about $t = 0$ we obtain

$$-sy = -st_N + \tfrac{1}{6}\theta^3 - R\theta + O\left\{\left(\frac{s}{\kappa_N}\right)^{-\frac{2}{3}}\right\}. \tag{2·13}$$

Hence the transformed velocity potential is given by

$$\Phi = \frac{\pi}{2s^2} e^{-st_N} \sum_{n=1}^{\infty} \frac{\mathrm{Ai}(2^{\frac{1}{3}}R - \beta_n)}{\beta_n[\mathrm{Ai}(-\beta_n)]^2} \exp\left\{-2^{-\frac{1}{3}}\beta_n \int_0^{t_N} \kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}} dt\right\} \int_{-\beta_n}^{\infty} \mathrm{Ai}(u)\, du. \tag{2·14}$$

The symmetry assumed for the validity of this result implies that the wings meet the fuselage at right angles. If the wings are offset from this plane of symmetry then this solution will need to be reformulated, and, as with the delta wing case, an edge diffraction region will have to be introduced.

3. *The structure of* Φ *for* $t > 0$. We begin by considering the way in which ϕ, the disturbance to the velocity potential, is propagated from the leading edge of the wing. For $|x| > a$, it spreads out along the surface of the Mach Wedge from the leading edge. Hence, the bounding surface for non-zero ϕ is given by

$$z = |y| \quad \text{for} \quad |x| > a.$$

Suppose that, on the cylinder, the disturbance is propagated along a curve given by

$$z = z'(t),$$

where t is the arc length from the wing intersection to the point under consideration (measured in the cross-section at $z = z'$). Down stream from $P'(p, q, z')$, the disturbance can travel directly to all points on and within the Mach cone

$$(x-p)^2 + (y-q)^2 = (z-z')^2.$$

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The condition that an adjacent point on the curve $[p(t+\delta t), q(t+\delta t), z'(t+\delta t)]$, should also be on this Mach cone is easily seen to reduce to

$$\frac{dz}{dt} = 1.$$

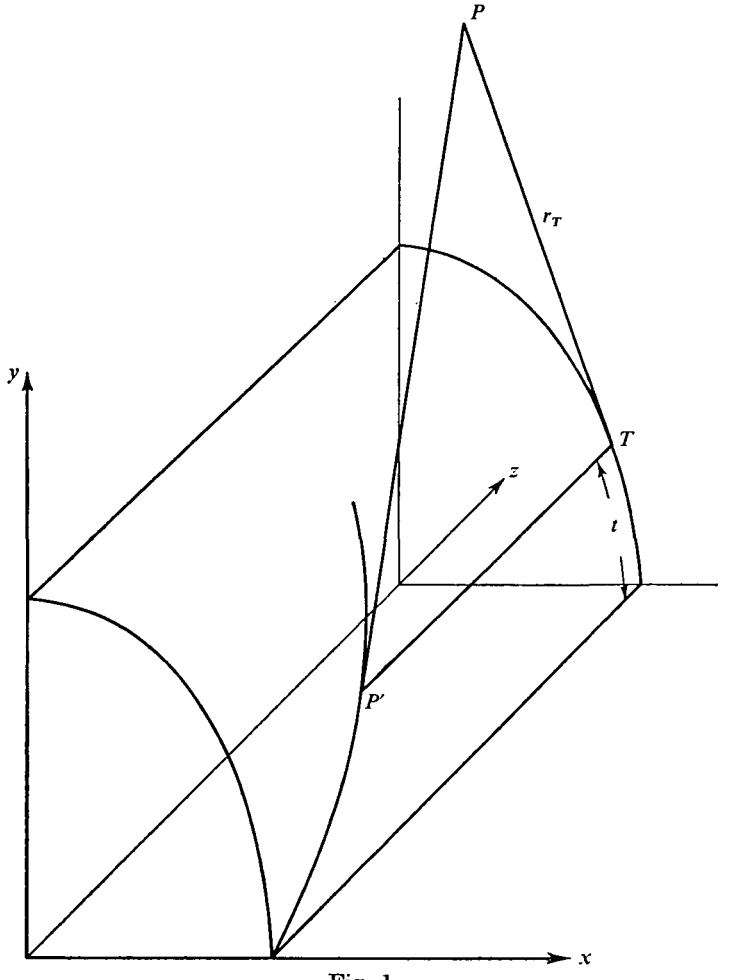


Fig. 1

Hence on the cylinder the boundary for non-zero values of ϕ will be given by the curve

$$z = t.$$

Suppose now that a general point away from the cylinder, $P(x, y, z)$ lies on the boundary surface for non-zero values of ϕ. It must clearly lie on the tangent to this curve from some point $P'(p, q, t)$; let T be the point (p, q, z), and $PT = r_T$. PT is therefore the tangent to the cylinder from P in the cross-sectional plane of P, as shown in Fig. 1.

Since P lies on the Mach cone from P' we see that

$$P'T = PT = r_T.$$

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Hence for a given x and y the disturbance is first seen where

$$z = t + r_T. \tag{3·1}$$

Further, t can be determined implicitly from the relationship

$$\frac{x-p}{y-q} = \frac{\dot{p}}{\dot{q}}, \tag{3·2}$$

p and q being known functions of t.

We now know that the boundary surface for non-zero ϕ is given by

$$z = y \qquad (|x| > a), \tag{3·3a}$$

$$z = t + r_T \quad (|x| < a). \tag{3·3b}$$

As we are only considering the first quadrant, we will now define a function $f(x, y)$ by

$$f = y \qquad (x > a) \tag{3·4a}$$

$$f = t + r_T \quad (0 < x < a). \tag{3·4b}$$

It can be shown that

$$\frac{\partial t}{\partial x} = -\frac{\dot{q}}{r_T \kappa}, \quad \frac{\partial t}{\partial y} = \frac{\dot{p}}{r_T \kappa}$$

$$\frac{\partial r_T}{\partial x} = \dot{p} + \frac{\dot{q}}{r_T \kappa}, \quad \frac{\partial r}{\partial y} = \dot{q} - \frac{\dot{p}}{r_T \kappa} \tag{3·5}$$

from which it follows that f satisfies the eikonal equation

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1. \tag{3·6}$$

Near the onset of the disturbance we assume that

$$\Phi(x, y) = F(x, y)\, e^{-sf}, \tag{3·7}$$

where F varies more slowly with s than does the exponential factor.

Using (3·6) and (3·7) the governing equation becomes

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - 2s\left\{\frac{\partial f}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial F}{\partial y}\right\} - s\left\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right\} F = 0. \tag{3·8}$$

In the region $x > a$ this reduces to

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - 2s\frac{\partial F}{\partial y} = 0 \tag{3·9a}$$

along with the boundary conditions

$$\frac{\partial F}{\partial y} - sF = -\frac{1}{s} \quad \text{on} \quad y = 0 \tag{3·9b}$$

and

$$\nabla\Phi \to 0 \quad \text{as} \quad y \to \infty \quad \text{for fixed } x. \tag{3·9c}$$

This latter condition will be satisfied due to the factor e^{-sy} in Φ provided that F has no exponential dependence on y.

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The solution to (3·9) is

$$F = \frac{1}{s^2} \quad \text{for} \quad x > a. \tag{3·10}$$

Now $$\frac{\partial}{\partial n} = \dot{q}\frac{\partial}{\partial x} - \dot{p}\frac{\partial}{\partial y}, \quad \text{and so} \quad \frac{\partial f}{\partial n} = 0.$$

Hence in the region $0 < x < a$ equation (3·8) becomes

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - 2s\left(\dot{p}\frac{\partial F}{\partial x} + \dot{q}\frac{\partial F}{\partial y}\right) - \frac{sF}{r_T} = 0, \tag{3·11a}$$

along with boundary conditions

$$\dot{q}\frac{\partial F}{\partial x} - \dot{p}\frac{\partial F}{\partial y} = 0 \quad \text{on} \quad r_T = 0 \tag{3·11b}$$

and $$\nabla\phi \to 0 \quad \text{as} \quad |y| \to \infty \quad \text{for fixed } x. \tag{3·11c}$$

As a first approximation to (3·11*a*) we neglect the terms independent of s; writing the resulting equation in terms of independent variables (r_T, t) we get

$$2\frac{\partial F}{\partial r_T} + \frac{F}{r_T} = 0 \tag{3·12}$$

which has the solution

$$F = \frac{1}{r_T^{\frac{1}{2}}} H(t), \tag{3·13}$$

where H is an arbitrary function of the arc length t. Clearly this solution must break down as r_T approaches zero, and so our assumption that

$$\frac{1}{s}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)$$

is negligible in comparison to the retained terms in (3·12) must be invalid in a region near $r_T = 0$. This is overcome by using a boundary layer type of approximation to the solution, and then carrying out a matching procedure for the two solutions.

In examining this boundary layer region the coordinates (r_N, t_N) will be used as independent variables, and so it is first of all necessary to examine the relation between r_N and r_T close to the surface.

By expressing (x, y) in terms of (r_N, t_N) and (r_T, t)

$$\begin{aligned}(x, y) &= (p_N + \dot{q}_N r_N,\ q_N - \dot{p}_N r_N) \\ &= (p + r_T\dot{p},\ q + r_T\dot{q})\end{aligned} \tag{3·14}$$

and expanding p, q as Taylor series in $(t_N - t)$ about p_N, q_N, the following formulae can be derived

$$t_N = t + r_T - \tfrac{1}{3}\kappa_N^2 r_T^3 + O(r_T^4), \tag{3·15a}$$

$$r_N = \tfrac{1}{2}\kappa_N r_T^2 - \tfrac{1}{3}\dot{\kappa}_N r_T^3 + O(r_T^4), \tag{3·15b}$$

from which we see that

$$r_T = \frac{1}{\kappa_N}(2\kappa_N r_N)^{\frac{1}{2}}\left\{1 + \frac{1}{3}\frac{\dot{\kappa}_N}{\kappa_N^2}(2\kappa_N r_N)^{\frac{1}{2}}\right\} + O\{(\kappa_N r_N)^{\frac{3}{2}}\}. \tag{3·15c}$$

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Also
$$\frac{\partial \kappa_N}{\partial r_T} = \dot{\kappa}_N + O(r_T^2) \tag{3·15d}$$
and so it follows that
$$\frac{\partial r_N}{\partial r_T} = (2\kappa_N r_N)^{\frac{1}{2}}\left\{1 - \frac{1}{6}\frac{\dot{\kappa}_N}{\kappa_N^2}(2\kappa_N r_N)^{\frac{1}{2}}\right\} + O\{(\kappa_N r_N)^{\frac{3}{2}}\} \tag{3·16a}$$
and
$$\frac{\partial t_N}{\partial r_T} = 1 - 2\kappa_N r_N + O\{(\kappa_N r_N)^{\frac{3}{2}}\}. \tag{3·16b}$$

We shall use the same scaling of the variables here as was used in section 2:
$$R = \kappa_N r_N \left(\frac{s}{\kappa_N}\right)^{\frac{2}{3}}, \quad \theta = \int_0^{t_N} \kappa \left(\frac{s}{\kappa}\right)^{\frac{1}{3}} dt, \tag{3·17}$$
the justification for the 'stretching' of the t coordinate becoming apparent later.

Using the above results, and substituting into equations (3·11) we obtain
$$\frac{\partial^2 F}{\partial R^2} - 2(2R)^{\frac{1}{2}}\frac{\partial F}{\partial R} - \frac{F}{(2R)^{\frac{1}{2}}} - 2\frac{\partial F}{\partial \theta} + \frac{1}{3}\frac{\dot{\kappa}_N}{\kappa_N^2}\left(\frac{s}{\kappa_N}\right)^{-\frac{1}{3}} F = O\left\{\left(\frac{s}{\kappa_N}\right)^{-\frac{2}{3}}\right\} \tag{3·18a}$$
along with the boundary condition
$$\frac{\partial F}{\partial R} = 0 \quad \text{on} \quad R = 0 \tag{3·18b}$$
plus a condition obtained in matching this solution to that given by (3·13) away from the cylinder. The condition (3·18b) follows from (3·11b) on noting that when $R = 0$, then $t = t_N$.

If we write
$$F_n(R, \theta) = \kappa_N^{\frac{1}{6}} L_n(R) \exp\{\tfrac{1}{3}(2R)^{\frac{3}{2}} - (B_n/2^{\frac{1}{3}})\theta\} \tag{3·19}$$
and substitute into equations (3·15) we obtain the following equations in $L_n(R)$
$$L_n'' - (2R - 2^{\frac{2}{3}}B_n) L_n = 0, \tag{3·20a}$$
together with
$$L_n'(0) = 0 \quad \text{and} \quad L_n(\infty) = 0, \tag{3·20b}$$
the latter condition being necessary to make a match with (3·13) possible. These equations imply that $B_n = \beta_n$, where β_n was defined in the second section, and that
$$L_n(R) = \mathrm{Ai}\,(2^{\frac{1}{3}}R - \beta_n). \tag{3·21}$$

Hence the general solution to equations (3·18) is given by
$$F(R, \theta) = \sum_{n=1}^{\infty} \kappa_N^{\frac{1}{6}} \alpha_n \,\mathrm{Ai}\,(2^{\frac{1}{3}}R - \beta_n) \exp\left\{\tfrac{1}{3}(2R)^{\frac{3}{2}} - \frac{\beta_n}{2^{\frac{1}{3}}}\theta\right\}, \tag{3·22}$$
the error being of $O\{(s/\kappa_N)^{-\frac{2}{3}}\}$, where α_n is constant.

In order to match the solutions (3·13) and (3·22) we must expand the latter for large values of R, and compare it with (3·13) for small values of r_T. As we are considering the solution in the limit as $s/\kappa_N \to \infty$ it is possible to have R large at the same time as r_N (and hence r_T) is small.

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We know that, for large ρ

$$\mathrm{Ai}\,(\rho) \sim [2\pi^{\frac{1}{2}}\rho^{\frac{1}{4}}]^{-1}\exp\{-\tfrac{2}{3}\rho^{\frac{3}{2}}\}, \tag{3·23}$$

and so when R is large

$$F \sim \frac{1}{2\pi^{\frac{1}{2}}}\sum_{n=1}^{\infty}\frac{\kappa_N^{\frac{1}{6}}\alpha_n}{2^{\frac{1}{12}}(\kappa_N r_N)^{\frac{1}{4}}}\left(\frac{\kappa_N}{s}\right)^{\frac{1}{6}}\exp\{\tfrac{1}{3}(2R)^{\frac{3}{2}}-\tfrac{2}{3}(2^{\frac{1}{3}}R-\beta_n)^{\frac{3}{2}}-\frac{\beta_n}{2^{\frac{1}{3}}}\int_0^{t_N}\kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}}dt\Big\}. \tag{3·24}$$

However, within the boundary layer region,

$$\int_t^{t_N}\kappa^{\frac{2}{3}}dt = \kappa_N^{-\frac{1}{3}}(2\kappa_N r_N)^{\frac{1}{2}}+O\{(\kappa_N r_N)^{\frac{3}{2}}\}, \tag{3·25}$$

using equations (3·15).

Hence we can write

$$\tfrac{2}{3}(2^{\frac{1}{3}}R-\beta_n)^{\frac{3}{2}}+\frac{\beta_n}{2^{\frac{1}{3}}}\int_0^{t_N}\kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}}dt \sim \tfrac{1}{3}(2R)^{\frac{3}{2}}+\frac{\beta_n}{2^{\frac{1}{3}}}\int_0^{t}\kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}}dt. \tag{3·26}$$

Substituting this in (3·24) gives

$$F \sim \frac{1}{2\pi^{\frac{1}{2}}}\left(\frac{\kappa_N}{2r_N}\right)^{\frac{1}{4}}\frac{1}{(\kappa\kappa_N)^{\frac{1}{6}}}\sum_{n=1}^{\infty}\alpha_n\left(\frac{2\kappa}{s}\right)^{\frac{1}{6}}\exp\Big\{-2^{-\frac{1}{3}}\beta_n\int_0^{t}\kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}}dt\Big\}. \tag{3·27}$$

Also, by expanding $\kappa_N^{-\frac{1}{6}}$ about $t = t_N$, and making use of equations (3·15) it follows that

$$\left(\frac{\kappa_N}{2r_N}\right)^{\frac{1}{4}}(\kappa\kappa_N)^{-\frac{1}{6}} \sim r_T^{-\frac{1}{2}}\kappa^{-\frac{1}{3}}. \tag{3·28}$$

Putting this into (3·27) we see that

$$F \sim \frac{1}{2\pi^{\frac{1}{2}}r_T^{\frac{1}{2}}}\sum_{n=1}^{\infty}\frac{\alpha_n}{\kappa^{\frac{1}{3}}}\left(\frac{2\kappa}{s}\right)^{\frac{1}{6}}\exp\Big\{-2^{-\frac{1}{3}}\beta_n\int_0^{t}\kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}}dt\Big\} \tag{3·29}$$

and by comparison with (3·13) we see that the arbitrary function H is now determined:

$$H(t) = \frac{1}{2\pi^{\frac{1}{2}}}\sum_{n=1}^{\infty}\frac{\alpha_n}{\kappa^{\frac{1}{3}}}\left(\frac{2\kappa}{s}\right)^{\frac{1}{6}}\exp\Big\{-2^{-\frac{1}{3}}\beta_n\int_0^{t}\kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}}dt\Big\}, \tag{3·30}$$

where the constants, α_n, have yet to be determined.

We can therefore write the solution for the transformed potential Φ as

$$\Phi = \exp\{-s(t+r_T)\}\sum_{n=1}^{\infty}\kappa_N^{\frac{1}{6}}\alpha_n\,\mathrm{Ai}\,(2^{\frac{1}{3}}R-\beta_n)\exp\left(\tfrac{1}{3}(2R)^{\frac{3}{2}}-2^{-\frac{1}{3}}\beta_n\int_0^{t_N}\kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}}dt\right) \tag{3·31}$$

near the cylinder, and

$$\Phi = \frac{1}{2\pi^{\frac{1}{2}}r_T^{\frac{1}{2}}}\exp\{-s(t+r_T)\}\sum_{n=1}^{\infty}\alpha_n\left(\frac{2}{s\kappa}\right)^{\frac{1}{6}}\exp\Big\{-2^{-\frac{1}{3}}\beta_n\int_0^{t}\kappa\left(\frac{s}{\kappa}\right)^{\frac{1}{3}}dt\Big\} \tag{3·32}$$

away from the cylinder. Furthermore, the α_n can be determined by requiring that the solution given by (3·31), as $t \to 0$ matches the solution obtained in the region of the wing intersection, given by equation (2·14).

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Again using equations (3·15) we can show that for small $\kappa_N r_N$

$$\tfrac{1}{3}(2R)^{\frac{3}{2}} - s(t+r_T) \sim -st_N. \tag{3·33}$$

Thus if κ_0 denotes the curvature at $t = 0$, then for small t, (3·31) becomes

$$\Phi \sim \frac{1}{s^2} e^{-st_N} \sum_{n=1}^{\infty} s^2 \kappa_0^{\frac{1}{3}} \alpha_n \operatorname{Ai}(2^{\frac{1}{3}}R - \beta_n) \exp\left\{-2^{-\frac{1}{3}}\beta_n \int_0^{t_N} \kappa \left(\frac{s}{\kappa}\right)^{\frac{1}{3}} dt\right\}, \tag{3·34}$$

and so we see that

$$\alpha_n = \frac{\pi}{2s^2\kappa_0^{\frac{1}{3}}} \int_{-\beta_n}^{\infty} \operatorname{Ai}(u)\, du \{\beta_n [\operatorname{Ai}(-\beta_n)]^2\}^{-1}. \tag{3·35}$$

When the curvature is constant, equations (3·31) and (3·32) agree with the corresponding results by Stewartson ((2), equations (5·7) and (5·15)) and Waechter ((4), equations (3·28) and (3·32)). They are also in agreement in the general case with Jones ((3), equations (74) and (78)). The additional terms given by Jones and Waechter correspond to the disturbance propagating around the cylinder from the opposite wing intersection. In general, due to the exponential decay, only one of these terms will be important, but the nature of this second solution follows immediately from the ambiguity in the definition of the coordinates (r_T, t).

4. *Solution near $x = a$.* Near the onset of the disturbance, the solution for Φ when $x > a$ is given by

$$\Phi = \frac{1}{s^2} e^{-sy}. \tag{4·1}$$

Also we have seen that for $x < a$

$$\Phi = \frac{1}{2\pi^{\frac{1}{2}} r_T^{\frac{1}{2}}} \exp[-s(t+r_T)] \sum_{n=1}^{\infty} \frac{\alpha_n}{\kappa^{\frac{1}{3}}} \left(\frac{2\kappa}{s}\right)^{\frac{1}{6}} \exp\left\{-2^{-\frac{1}{3}}\beta_n \int_0^t \kappa \left(\frac{s}{\kappa}\right)^{\frac{1}{3}} dt\right\}. \tag{4·2}$$

We now turn our attention to the behaviour of this solution as $x \to a-$.

Using the relations

$$(x-p)\dot{q} - (y-q)\dot{p} = 0, \tag{4·3a}$$

$$r_T = \frac{y-q}{\dot{q}}, \tag{4·3b}$$

and expanding p and q for small t we can show that

$$t = \frac{a-x}{\kappa_0 y} + \frac{(a-x)^2}{2\kappa_0^2 y^3} + O(a-x)^3, \tag{4·4a}$$

$$t + r_T = y + \frac{(a-x)^2}{2y} + O(a-x)^3, \tag{4·4b}$$

$$r_T^{-\frac{1}{2}} = \frac{1}{y^{\frac{1}{2}}} + \frac{a-x}{2\kappa_0 y^{\frac{5}{2}}} + O(a-x)^2. \tag{4·4c}$$

Hence as $x \to a-$

$$\Phi \sim \frac{1}{2\pi^{\frac{1}{2}} y^{\frac{1}{2}}} \left(1 + \frac{a-x}{2\kappa_0 y^2}\right) \sum_{n=1}^{\infty} \alpha_n \left(\frac{2}{\kappa_0 s}\right)^{\frac{1}{6}} \exp\left\{-s\left(y + \frac{(a-x)^2}{2y}\right) - \beta_n \left(\frac{s}{2\kappa_0}\right)^{\frac{1}{3}} \kappa_0 t\right\}. \tag{4·5}$$

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Making use of the substitution

$$\sigma = \left(\frac{s}{2\kappa_0}\right)^{\frac{1}{3}} \kappa_0 t \tag{4·6}$$

we are now able to write

$$\Phi \sim \left(\frac{y}{2\pi s}\right)^{\frac{1}{2}} \frac{1}{a-x} \exp\left\{-s\left(y + \frac{(a-x)^2}{2y}\right)\right\} \sum_{n=1}^{\infty} \alpha_n \sigma e^{-\beta_n \sigma}. \tag{4·7}$$

It follows from Stewartson's paper that near $x = a$ the solution is

$$\Phi = \frac{1}{s^2} e^{-sy} G(\eta), \tag{4·8a}$$

where

$$\eta = (x-a)\left(\frac{s}{2y}\right)^{\frac{1}{2}} \tag{4·8b}$$

and

$$G(\eta) = \frac{1}{\pi} \int_{-\infty}^{\eta} e^{-u^2} du. \tag{4·8c}$$

Thus we see that there is a transition region of thickness of $O\{(y/s)^{\frac{1}{2}}\}$ through which Φ changes rapidly from the form given by (4·1) for $x > a$ to that of (4·7) for $x < a$.

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