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On a convergence of positive continuous additive functionals in terms of the smooth measures

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Abstract. A compactness of the Revuz map is established in the sense that the locally uniform convergence of a sequence of positive continuous additive functionals (PCAFs) is derived in terms of their smooth measures. To this end, we first introduce a metric on the space of measures of finite energy integrals and show some structures of the metric. Then, we show the compactness and give some examples of PCAFs that the convergence holds in terms of the associated smooth measures.

1 Introduction

The positive continuous additive functionals (PCAFs) play an important role in the study of Markov processes and the Dirichlet spaces, in particular, in the study of transformations of the Markov processes including time changes, Feynman-Kac and Girsanov transformations (see, e.g., [2, 7, 13]). There is a one-to-one correspondence between the class of PCAFs ($\mathbf{A}_c^+$ in notation) and the class of nonnegative Borel measures charging no set of zero capacities, called the *smooth measures* ($\mathcal{S}$ in notation). This correspondence is called the *Revuz correspondence*.

In the present article, we will focus on a convergence of PCAFs. In particular, we will study the locally uniform convergence of PCAFs in terms of some convergences of their smooth measures. Recently, in the case that the planar Brownian motion and the one-dimensional symmetric α-stable process, this type of the convergence has been studied in [1, 8, 12] to construct the Liouville Brownian motion and the Liouville α-stable process. In [3], using smooth measures, L^2-convergences of Feynman-Kac semigroups defined through the PCAFs are considered.

To explain a more specific situation, let E be a locally compact separable metric space and m a positive Radon measure on E with full support. We are given an m-symmetric Hunt process $\mathbb{M} = (X_t, \mathbb{P}_x)$ on E and the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is assumed to be regular. Then the PCAFs and the smooth measures $\mathcal{S}$ generated by $\mathbb{M}$ or $(\mathcal{E}, \mathcal{F})$ are defined. For example, if f is a nonnegative

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bounded Borel function on E, then $texts f A_t = \int_0^t f(X_s)ds$ defines a PCAF and the corresponding smooth measure (called the *Revuz measure* of $\{\mathrm{A}_t\}$) is $f(x)m(dx)$. For the one-dimensional Brownian motion, the delta measure $\delta_a(dx)$ at a point $a \in \mathbb{R}$ is a smooth measure whose PCAF is nothing but the local time $\{\ell(t,a)\}_{t\geq 0}$.

In the following, we will regard the Revuz correspondence as a map from the space $\mathcal{S}$ to $\mathbf{A}_c^+$, call it *Revuz map*, and study the convergence of PCAFs via some convergences of their smooth measures from this point of view. To the best of our knowledge, there does not seem to be much research on such "topological properties of the mapping from $\mathcal{S}$ to $\mathbf{A}_c^+$". Since $\mathcal{S}$ is quite large (which is indeed larger than the class of positive Radon measures charging no set of zero capacities) and so is $\mathbf{A}_c^+$, it is harder to put a topological structure to this space $\mathbf{A}_c^+$ through $\mathcal{S}$ in general. We therefore turn our attention to some restricted classes of smooth measures first. One is the set of positive Radon measures which are of finite energy integrals ($\mathcal{S}_0$ in notation). For any $\mu \in \mathcal{S}_0$ and $\alpha > 0$, there exists a unique element $U_\alpha\mu \in \mathcal{F}$, called the α-potential of μ, such that

$$\int_E \varphi(x)\mu(dx) = \mathcal{E}_\alpha(U_\alpha\mu, \varphi) := \mathcal{E}(U_\alpha\mu, \varphi) + \alpha(U_\alpha\mu, \varphi), \quad \varphi \in \mathcal{F} \cap C_0(E),$$

where $C_0(E)$ is the set of continuous functions on E with compact support and $(\cdot,\cdot)$ means the L^2-inner product of $L^2(E;m)$. Using the 1-potentials of measures μ and ν in $\mathcal{S}_0$, we can define a metric on $\mathcal{S}_0$ as follows:

$$\rho(\mu,\nu) := \sqrt{\mathcal{E}_1\big(U_1\mu - U_1\nu, U_1\mu - U_1\nu\big)}, \quad \mu,\nu \in \mathcal{S}_0.$$

Then we show that the space $\mathcal{S}_0$ is a Polish space with the metric ρ (Proposition 3.6). This result itself is well-known in the classical potential theory for the cases of the Newton potential and the Riesz potential (see [10]). Note that our result holds true for potential free situations as far as in a regular Dirichlet form setting. We then consider the locally uniform convergence of PCAFs for the $\mathbb{P}_x$-almost surely (for quasi everywhere $x \in E$) in terms of their smooth measures in $\mathcal{S}_0$ through this metric ρ. After establishing a convergence result of PCAFs via that of their smooth measures in $\mathcal{S}_0$ (Theorem 4.1), we turn to consider the case when the measures are in $\mathcal{S}$. Since a measure in $\mathcal{S}$ may not have the 1-potential in general, we can not directly apply the method for proving the convergence in the case of the measures of finite energy integrals. To overcome this difficulty, we take a proper increasing sequence of closed sets such that smooth measures restricted on each closed sets are of finite energy integrals. Then under some assumptions, we can obtain the convergence of PCAFs via the convergence of their smooth measures (Theorem 5.1).

We next introduce a class of smooth measures that are absolutely continuous with respect to the basic measure m. Then imposing some conditions on the Radon–Nikodým densities, we obtain a convergence of the PCAFs associated with such measures in $\mathcal{S}$ (Theorem 6.4). In particular, the convergence of PCAFs holds when the sequence of the densities of smooth measures converges in L^p for $1 \leq p < \infty$ (Proposition 6.5). We also stress that we can allow the exponents in the Lebesgue spaces for which the densities of smooth measures belong to may vary (Theorem 6.4).

The structure of this article is the following. In Section 2, we collect some basic notions and notations of the Dirichlet spaces and the Markov processes, especially,

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notions of capacity, smooth measures, measures of finite energy integrals, and additive functionals. After this, we show in Section 3 that the space $\mathcal{S}_0$ is indeed a Polish space with the metric ρ and the convergence of measures in $\mathcal{S}_0$ implies that the measures converge *vaguely*. Conversely, if a sequence of measures in $\mathcal{S}_0$ having the uniformly bounded 1-potentials converges vaguely, the measures converges *"weakly in $\mathcal{S}_0$ with respect to ρ"* (Proposition 3.9). Some algebraic structures of $\mathcal{S}_0$ are revealed in Proposition 3.10. In Section 4, we prove the convergence of PCAFs through the convergence of their measures in $\mathcal{S}_0$ with respect to ρ by using an idea in [7]. We proceed to consider the convergence of PCAFs through the convergence of their smooth measures in the class $\mathcal{S}$ in Section 5. In Section 6, we assume that the measures have densities. We introduce an approximating method of the PCAF associated with fm for a given density f. Even though the density function may not have L^p-integrability for all $p \in [1, \infty)$, if the density function is locally integrable and it can be approximated by a sequence f_n in the meaning of Definition 6.2, then the PCAF of fm is also approximated the PCAFs of $f_n m$. This provides one perspective to observe the topological structure of the space of PCAFs from the corresponding measure space. In Section 7, we first show some examples of the density functions concerned in Section 6. Finally the cases of a skew product diffusion process and a one-dimensional diffusion process (ODDP) are discussed in which, according to the fact that the PCAFs are expressed as integrals of the local time with respect to the smooth measures, the convergences of PCAFs are directly derived from the vague convergence of the measures not necessarily using the results obtained the previous sections.

2 Preliminaries

Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $E_\partial = E \cup \{\partial\}$ be the one-point compactification of E. When E is already compact, ∂ is regarded as an isolated point. Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x, \zeta)$ be an m-symmetric Hunt process on E, where $\{\mathcal{F}_t\}$ is the minimum augmented filtration and ζ is the life time associated with $\mathbb{M}$, that is, $\zeta(\omega) := \inf\{t \geq 0 : X_t = \partial\}$.

Let $\{p_t\}_{t\geq 0}$ be the transition semigroup of $\mathbb{M}$, that is, $p_t f(x) = \mathbb{E}_x[f(X_t)]$. By [7, Lemma 1.4.3], we know that $\{p_t\}_{t\geq 0}$ uniquely determines a strongly continuous Markovian semigroup $\{T_t\}_{t\geq 0}$ on $L^2(E; m)$. Then the Dirichlet form on $L^2(E; m)$ of $\mathbb{M}$ is defined by

$$\begin{cases} \mathcal{E}(u, v) = \lim\limits_{t\downarrow 0} \dfrac{1}{t}(u - T_t u, v), \\ \mathcal{F} = \left\{ u \in L^2(E; m) : \lim\limits_{t\downarrow 0} \dfrac{1}{t}(u - T_t u, u) < \infty \right\}, \end{cases}$$

where (f, g) (resp. $\|f\|_2$) means the L^2-inner product of f, g (resp. the L^2-norm of f). For $\alpha > 0$ and $u, v \in \mathcal{F}$, denote by $\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)$. Note that $\mathcal{F}$ is a Hilbert space with inner product $\mathcal{E}_1$. Let $\|\cdot\|_{\mathcal{E}_1}$ be the norm of $\mathcal{F}$ with respect to $\mathcal{E}_1$, that is,

$$\|f\|_{\mathcal{E}_1} := \sqrt{\mathcal{E}_1(f, f)} = \sqrt{\mathcal{E}(f, f) + (f, f)}, \quad f \in \mathcal{F}.$$

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Throughout this article, we assume that $(\mathcal{E}, \mathcal{F})$ is *regular* in the sense that $\mathcal{F} \cap C_0(E)$ is dense in $\mathcal{F}$ with respect to $\sqrt{\mathcal{E}_1}$-norm and is also dense in $C_0(E)$ with respect to sup-norm $\|\cdot\|_\infty$.

Now we define the (1-)capacity associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$. For an open set $O \subset E$,

$$\mathrm{Cap}\,(O) := \begin{cases} \inf\left\{\mathcal{E}_1(u,u) : u \in \mathcal{L}_O\right\}, & \text{if } \mathcal{L}_O \neq \varnothing, \\ \infty, & \text{if } \mathcal{L}_O = \varnothing, \end{cases}$$

where $\mathcal{L}_O := \{u \in \mathcal{F} : u \geq 1\ m\text{-a.e. on } O\}$, and for any set $A \subset E$,

$$\mathrm{Cap}\,(A) := \inf\left\{\mathrm{Cap}\,(O) : A \subset O,\ O \text{ is open}\right\}.$$

Then it is known that Cap is a Choquet capacity (see [7]). We call a set $A \subset E$ *exceptional* if $\mathrm{Cap}\,(A) = 0$. A statement depending on $x \in A$ is said to hold q.e. on A if there exists an exceptional set $N \subset A$ such that the statement is true for every $x \in A \backslash N$. Let u be a numerical function defined q.e. on E. The function u is said to be *quasi-continuous* if for any $\varepsilon > 0$, there exists an open set $G \subset E$ such that $\mathrm{Cap}\,(G) < \varepsilon$ and $u|_{E\backslash G}$ is finite and continuous. Here $u|_{E\backslash G}$ means the restriction of u to the set $E\backslash G$. By [7, Theorem 2.1.3], we see that each $u \in \mathcal{F}$ admits a quasi continuous version. Denote by $\widetilde{u}$ the quasi continuous version of $u \in \mathcal{F}$, that is, $\tilde{u}$ is quasi continuous and $\tilde{u} = u,\ m$-a.e. We say an increasing sequence $\{F_n\}$ of closed sets is a *nest* if $\mathrm{Cap}\,(E\backslash F_n)$ decreases to 0 as $n \to \infty$.

We now give the definition of smooth measures which play an important role in this article. A positive Borel measure μ on E is called *smooth* if the following two conditions are satisfied:

(S.1) μ charges no set of zero capacity,
(S.2) there exists an increasing sequence $\{F_n\}$ of closed sets such that

$$\mu(F_n) < \infty\ (n = 1, 2, \cdots) \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Cap}\,(K\backslash F_n) = 0 \quad \text{for any compact set } K.$$

We denote the set of all smooth measures by $\mathcal{S}$. An increasing sequence $\{F_n\}$ of closed sets satisfying the condition of the last part in *(S.2)* is called a *generalized nest*. A positive Radon measure μ on E is said to be of *finite energy integrals* if there exists a positive constant $C > 0$ such that

$$\int_E |v(x)|\mu(dx) \leq C\sqrt{\mathcal{E}_1(v,v)}, \quad v \in \mathcal{F} \cap C_0(E).$$

The set of all measures on E which are of finite energy integrals is denoted by $\mathcal{S}_0$. Then, according to the Riesz representation theorem, we find that $\mu \in \mathcal{S}_0$ if and only if there exists a unique element $U_\alpha\mu \in \mathcal{F}$, called the *$\alpha$-potential* of μ, for each $\alpha > 0$ such that

$$\mathcal{E}_\alpha(U_\alpha\mu, v) = \int_E \widetilde{v}(x)\mu(dx), \quad \text{for any } v \in \mathcal{F}.$$

We now introduce two subclasses of $\mathcal{S}_0$, $\mathcal{S}^1_{c,00}$ in addition to $\mathcal{S}_{00}$ (see [7, eq. (2.2.10)]), which are not just valuable in themselves but also useful to characterize the exceptional sets as follows:

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$$\mathcal{S}_{00} := \left\{\mu \in S_0 : \mu(E) < \infty \text{ and } \|U_1\mu\|_\infty < \infty\right\},$$
$$\mathcal{S}^1_{c,00} := \left\{\mu \in \mathcal{S}_0 : \text{supp}[\mu] \text{ is compact, } \mu(E) \le 1 \text{ and } \|U_1\mu\|_\infty \le 1\right\},$$

where $\|\cdot\|_\infty$ is the L^∞-norm with respect to the measure m.

Theorem 2.1 (cf. [7, Theorem 2.2.3]) *The following conditions are equivalent for a Borel set $B \subset E$:*

(i) $Cap\,(B) = 0$,
(ii) $\mu(B) = 0$, *for any* $\mu \in \mathcal{S}_0$,
(iii) $\mu(B) = 0$, *for any* $\mu \in \mathcal{S}_{00}$,
(iv) $\mu(B) = 0$, *for any* $\mu \in \mathcal{S}^1_{c,00}$.

Proof The equivalences among (i), (ii), and (iii) are given in [7, Theorem 2.2.3]. Now, the implication "(iii) $\Rightarrow$ (iv)" is obvious from the inclusion $\mathcal{S}^1_{c,00} \subset \mathcal{S}_{00}$. Suppose that (iv) holds, namely $\mu(B) = 0$ for any $\mu \in \mathcal{S}^1_{c,00}$. For any $\mu \in \mathcal{S}_0$, there exists an increasing sequence $\{F_n\}$ of compact sets such that

(a) $1_{F_n}\mu \in \mathcal{S}_{00}$, $n = 1, 2, \ldots$,
(b) $\text{Cap}\,(K\backslash F_n) \to 0$, $n \to \infty$ for any compact set K,

by [7]. Put $c_n := (\|U_1(1_{F_n}\mu)\|_\infty + \mu(F_n) + 1)^{-1} > 0$. Since F_n is compact for each n, we see from (a),

$$c_n 1_{F_n}\mu \in \mathcal{S}^1_{c,00} \quad \text{for } n = 1, 2, \ldots.$$

So it follows from the assumption, $\mu(B \cap F_n) = 0$, $n = 1, 2, \ldots$, hence it holds that

$$\mu\left(B \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) = 0.$$

Noting that μ vanishes on $E\backslash\bigcup_{n=1}^{\infty} F_n$, we find that

$$\mu(B) \le \mu\left(B \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu\left(E\backslash\bigcup_{n=1}^{\infty} F_n\right) = 0.$$

Hence $\mu(B) = 0$ for any $\mu \in \mathcal{S}_0$ and this implies (ii). ∎

The following proposition holds.

Lemma 2.2 *It holds that*

$$\|U_\alpha\mu\|_\infty \le 2, \quad \textit{for any } \mu \in \mathcal{S}^1_{c,00} \textit{ and } \alpha \ge 1.$$

Proof By [2, Exercise 2.3.8],

$$U_\alpha\mu = U_1\mu - (\alpha - 1)R_\alpha U_1\mu,$$

where R_α denotes the α-resolvent associated with $\mathbb{M}$:

$$R_\alpha f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t} f(X_t)dt\right], \quad f \in \mathcal{B}(E),\ f \ge 0.$$

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Hence using the Markovian property of αR_α,

$$\|U_\alpha \mu\|_\infty \le \|U_1 \mu\|_\infty + (\alpha - 1)\|R_\alpha U_1 \mu\|_\infty \le 1 + \frac{\alpha - 1}{\alpha} \le 2.$$

We obtain this lemma. ∎

We now introduce the definition of a continuous additive functional (CAF).

Definition 2.3 An extended real valued (or numerical) function $\mathrm{A}_t(\omega)$, $t \ge 0$, $\omega \in \Omega$, is a *CAF* if the following conditions must hold:

(A.1) for each $t \ge 0$, $\mathrm{A}_t(\cdot)$ is $\mathcal{F}_t$-measurable,
(A.2) there exist a set $\Lambda \in \mathcal{F}_\infty$ and an exceptional set $N \subset E$ such that $\mathbb{P}_x(\Lambda) = 1$ for any $x \in E \backslash N$ and $\theta_t \Lambda \subset \Lambda$ for any $t > 0$, and moreover, for each $\omega \in \Lambda$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for any $t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for any $t \ge \zeta(\omega)$ and

$$A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega), \text{ for any } t, s \ge 0,$$

(A.3) for each $\omega \in \Lambda$, the map $t \mapsto \mathrm{A}_t(\omega)$ is continuous on $[0, \infty)$, where Λ is the set defined in (A.2).

When a CAF $\mathrm{A} = \{\mathrm{A}_t\}$ only takes values in $[0, \infty]$, we call A a *PCAF*. The set of all CAFs (resp. PCAFs) is denoted by $\mathbf{A}_c$ (resp. $\mathbf{A}_c^+$).

Two CAFs $\mathrm{A}^{(1)}$ and $\mathrm{A}^{(2)}$ are said to be *equivalent* if for each $t > 0$, $\mathbb{P}_x(\mathrm{A}_t^{(1)} = \mathrm{A}_t^{(2)}) = 1$ q.e. $x \in E$. If $\mathrm{A}^{(1)}$ and $\mathrm{A}^{(2)}$ are equivalent, we write $\mathrm{A}^{(1)} \sim \mathrm{A}^{(2)}$. It is clear that this binomial relation "$\sim$" defines an equivalence relation on $\mathbf{A}_c$.

The next theorem characterizes a concrete relation between measures in $\mathcal{S}$ and PCAFs (in $\mathbf{A}_c^+$ under this equivalence).

Theorem 2.4 [7, Theorem 5.1.4] *The quotient space of $\mathbf{A}_c^+$ under the equivalence relation "$\sim$" and the family $\mathcal{S}$ are in one-to-one correspondence under the following relation: For $\mu \in \mathcal{S}$ and $A \in \mathbf{A}_c^+$,*

$$\lim_{t \downarrow 0} \frac{1}{t} \int_E \mathbb{E}_x \left[\int_0^t f(X_s) dA_s \right] h(x) m(dx) = \int_E f(x) h(x) \mu(dx), \tag{2.1}$$

where $f \in \mathcal{B}_+(E)$ and h is any γ-excessive function ($\gamma \ge 0$), that is, h is nonnegative universally measurable and it satisfies $e^{-\gamma t} p_t h(x) \uparrow h(x)$, $t \downarrow 0$, $x \in E$.

The equation (2.1) is called the *Revuz correspondence*. In this article, we call it the *Revuz map* by viewing this correspondence from $\mathcal{S}$ to $\mathbf{A}_c^+$.

3 Some properties of measures of finite energy integrals

In this section, we consider some topological and algebraic properties of the space of measures which are of finite energy integrals, $\mathcal{S}_0$. As is mentioned in the preceding section, any measure μ in $\mathcal{S}_0$ has the 1-potential $U_1 \mu \in \mathcal{F}$. Then we define a function $\rho : \mathcal{S}_0 \times \mathcal{S}_0 \to [0, \infty)$ by

$$\rho(\mu, \nu) := \|U_1 \mu - U_1 \nu\|_{\mathcal{E}_1}, \quad \mu, \nu \in \mathcal{S}_0. \tag{3.1}$$

We show that the function ρ becomes a metric on $\mathcal{S}_0$ and has nicer properties.

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Lemma 3.1 *The function ρ defines a metric on $\mathcal{S}_0$.*

Proof For $\mu, \nu \in \mathcal{S}_0$, it is clear that $\rho(\mu,\nu) \geq 0$ and it holds that

$$\rho(\mu,\nu) = \|U_1\mu - U_1\nu\|_{\mathcal{E}_1} = \|U_1\nu - U_1\mu\|_{\mathcal{E}_1} = \rho(\nu,\mu).$$

For any $\xi \in \mathcal{S}_0$,

$$\rho(\mu,\nu) = \|U_1\mu - U_1\nu\|_{\mathcal{E}_1} \leq \|U_1\mu - U_1\xi\|_{\mathcal{E}_1} + \|U_1\xi - U_1\nu\|_{\mathcal{E}_1} = \rho(\mu,\xi) + \rho(\xi,\nu).$$

Hence, we obtain the triangle inequality with respect to ρ. Finally, assume that $\rho(\mu,\nu) = 0$. Then $U_1\mu = U_1\nu$. So it follows that

$$\int_E \varphi(x)\mu(dx) = \mathcal{E}_1(U_1\mu,\varphi) = \mathcal{E}_1(U_1\nu,\varphi) = \int_E \varphi(x)\nu(dx), \quad \text{for any} \varphi \in \mathcal{F} \cap C_0(E).$$

This implies that $\mu = \nu$ because μ, ν are Radon measures and the Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular. Therefore ρ defines a metric on $\mathcal{S}_0$. ∎

Remark 3.2 For $\alpha > 0$, if we define a function ρ_α on $\mathcal{S}_0 \times \mathcal{S}_0$ by

$$\rho_\alpha(\mu,\nu) := \|U_\alpha\mu - U_\alpha\nu\|_{\mathcal{E}_\alpha} := \sqrt{\mathcal{E}_\alpha(U_\alpha\mu - U_\alpha\nu, U_\alpha\mu - U_\alpha\nu)}, \quad \mu,\nu \in \mathcal{S}_0,$$

we find that

$$\sqrt{1/(\alpha \vee 1)}\,\rho(\mu,\nu) \leq \rho_\alpha(\mu,\nu) \leq \sqrt{(1/\alpha) \vee 1}\,\rho(\mu,\nu), \quad \mu,\nu \in \mathcal{S}_0$$

holds. Then it follows that ρ_α is also a metric on $\mathcal{S}_0$ and it is equivalent to $\rho = \rho_1$ for each $\alpha > 0$.

Put $f = \widetilde{U_1\mu}$ for $\mu \in \mathcal{S}_0$ and define for $n \in \mathbb{N}$,

$$g_n(x) = n(f(x) - nR_{n+1}f(x)). \tag{3.2}$$

Since f is 1-excessive, we know that $g_n \geq 0$ m-a.e. by [7, Theorem 2.2.1].

Lemma 3.3 *Let μ be a measure in $\mathcal{S}_0$ and put $f = \widetilde{U_1\mu}$. Then $R_1 g_n$ converges to f in $\mathcal{E}_1$-strongly as $n \to \infty$.*

This proof is the same as the first half of the proof of [2, Proposition 4.1.6], so we omit it.

Lemma 3.4 *The metric space $(\mathcal{S}_0,\rho)$ is separable.*

Proof Since E is a locally compact separable metric space and m is a Radon measure on E, we see that the space $L^2(E;m)$ is separable. Let $\mathfrak{D}$ be a countable and dense subset of $L^2(E;m)$. For a measure $\mu \in \mathcal{S}_0$, let f be a quasi continuous version of the 1-potential of μ, that is, $f = \widetilde{U_1\mu}$. Let g_n be the function defined in (3.2). Then we know that $g_n \in \mathcal{F} \subset L^2(E;m)$ with $g_n \geq 0$ for all $n \in \mathbb{N}$. By Lemma 3.3, the following holds:

$$R_1 g_n \to f, \quad \mathcal{E}_1\text{-strongly}.$$

Hence, we know that for any $\varepsilon > 0$, there exists a number $N \in \mathbb{N}$ such that $\rho(g_n m,\mu) < \varepsilon/2$ for $n \geq N$. Since g_n is in $L^2(E;m)$, there exists a function $h_n \in \mathfrak{D}$ with $h_n \geq 0$ such that $\|g_n - h_n\|_2 < \varepsilon/2$. Then noting that

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$$\begin{aligned}\rho(g_n m, h_n m)^2 &= \mathcal{E}_1(R_1 g_n - R_1 h_n, R_1 g_n - R_1 h_n) = \mathcal{E}_1(R_1(g_n - h_n), R_1(g_n - h_n)) \\ &= \int_E (g_n(x) - h_n(x)) R_1(g_n - h_n)(x) m(dx) \le \|g_n - h_n\|_2 \|R_1(g_n - h_n)\|_2 \\ &\le \|g_n - h_n\|_2^2 < \left(\frac{\varepsilon}{2}\right)^2,\end{aligned}$$

we obtain that $\rho(g_n m, h_n m) < \varepsilon/2$. If $n \ge N$, it holds that

$$\rho(\mu, h_n m) \le \rho(\mu, g_n m) + \rho(g_n m, h_n m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If we put $\mathcal{S}_a$ by the set $\{\mu \in \mathcal{S}_0 : \mu = f m,\ f \in \mathfrak{D}_+\}$, $\mathcal{S}_a$ is countable and dense in $\mathcal{S}_0$. Therefore the separability of $(\mathcal{S}_0, \rho)$ is proved. ∎

Lemma 3.5 *The metric space $(\mathcal{S}_0, \rho)$ is complete.*

Proof Let $\{\mu_n\} \subset \mathcal{S}_0$ be a ρ-Cauchy sequence. Noting that

$$\rho(\mu_m, \mu_n) = \|U_1 \mu_m - U_1 \mu_n\|_{\mathcal{E}_1}$$

and $(\mathcal{F}, \mathcal{E}_1)$ is a Hilbert space, we know that there exists a function $h \in \mathcal{F}$ such that

$$\lim_{n \to \infty} \|U_1 \mu_n - h\|_{\mathcal{E}_1} = 0.$$

Hence, we know that for any $v \in \mathcal{F} \cap C_0(E)$ with $v \ge 0$,

$$\lim_{n \to \infty} \mathcal{E}_1(U_1 \mu_n, v) = \mathcal{E}_1(h, v). \tag{3.3}$$

By [7, Theorem 2.2.1], the left-hand side of (3.3) is nonnegative, so is the right-hand side. Using [7, Theorem 2.2.1] again, we can see that h is a 1-potential, hence there exists $\nu \in \mathcal{S}_0$ such that $h = U_1 \nu$. This implies that

$$\lim_{n \to \infty} \rho(\mu_n, \nu) = \lim_{n \to \infty} \|U_1 \mu_n - U_1 \nu\|_{\mathcal{E}_1} = 0.$$

This proof is completed. ∎

By Lemmas 3.4 and 3.5, we obtain the following proposition.

Proposition 3.6 *The space $\mathcal{S}_0$ is a Polish space.*

Example 3.7 Let $\mu = f m$ and $\mu_n = f_n m$ for $n \in \mathbb{N}$, where $f,\ f_n \in L^2(E; m) \cap \mathcal{B}_+(E)$ and $\|f - f_n\|_2 \to 0$ as $n \to \infty$. Then $\mu,\ \mu_n \in \mathcal{S}_0$ and $\rho(\mu, \mu_n) = \|R_1 f - R_1 f_n\|_{\mathcal{E}_1} \le \|f - f_n\|_2$. Hence $\rho(\mu, \mu_n) \to 0$ as $n \to \infty$.

Next we collect some convergence results of the measures of $\mathcal{S}_0$ related to ρ.

Proposition 3.8 *Assume that $\{\mu_n\}$ are measures in $\mathcal{S}_0$ which converges to some $\mu \in \mathcal{S}_0$ in ρ. Then it follows that $\{\mu_n\}$ converges to μ vaguely, that is, for any $\varphi \in C_0(E)$,*

$$\lim_{n \to \infty} \int_E \varphi(x) \mu_n(dx) = \int_E \varphi(x) \mu(dx).$$

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Proof Suppose that $\lim_{n\to\infty}\rho(\mu_n,\mu)=\|U_1\mu_n-U_1\mu\|_{\varepsilon_1}=0$. Then the sequence $\{\|U_1\mu_n\|_{\varepsilon_1}\}_{n\geq 1}$ is bounded in $\mathbb{R}$. Moreover, for any $\varphi\in\mathcal{F}\cap C_0(E)$,

$$\int_E \varphi(x)\mu_n(dx)=\mathcal{E}_1(\varphi,U_1\mu_n)\xrightarrow{n\to\infty}\mathcal{E}_1(\varphi,U_1\mu)=\int_E\varphi(x)\mu(dx). \tag{3.4}$$

By the regularity of the Dirichlet form $(\mathcal{E},\mathcal{F})$, we find that for any $\varphi\in C_0(E)$, there exists a sequence $\{\varphi_\ell\}\subset\mathcal{F}\cap C_0(E)$ such that

$$\lim_{\ell\to\infty}\|\varphi-\varphi_\ell\|_\infty=0.$$

For any compact set K and a relatively compact open set G with $\operatorname{supp}[\varphi]\subset\overset{\circ}{K}\subset K\subset G$, we can take a function $\eta\in\mathcal{F}\cap C_0(E)$ such that

$$\eta=1 \text{ on } K,\ \eta=0 \text{ on } G^c \text{ and } 0\leq\eta\leq 1 \text{ on } E,$$

where $\overset{\circ}{K}$ means the interior of K. Note that $\varphi_\ell\eta\in\mathcal{F}\cap C_0(E)$ and $\varphi=\varphi\eta$. Then we find that

$$\begin{aligned}\left|\int_E\varphi(x)\mu(dx)-\int_E\varphi(x)\mu_n(dx)\right|&=\left|\int_E\varphi(x)\eta(x)\mu(dx)-\int_E\varphi(x)\eta(x)\mu_n(dx)\right|\\&\leq\left|\int_E\varphi(x)\eta(x)\mu(dx)-\int_E\varphi_\ell(x)\eta(x)\mu(dx)\right|\\&\quad+\left|\int_E\varphi_\ell(x)\eta(x)\mu(dx)-\int_E\varphi_\ell(x)\eta(x)\mu_n(dx)\right|\\&\quad+\left|\int_E\varphi_\ell(x)\eta(x)\mu_n(dx)-\int_E\varphi(x)\eta(x)\mu_n(dx)\right|\\&=:\text{(I)}+\text{(II)}+\text{(III)}.\end{aligned} \tag{3.5}$$

By the definition of $\{\varphi_\ell\}$,

$$\text{(I)}\leq\int_E|\varphi(x)-\varphi_\ell(x)|\eta(x)\mu(dx)\leq\|\varphi-\varphi_\ell\|_\infty\int_E\eta(x)\mu(dx)\ \to\ 0\quad(\ell\to\infty).$$

We see from (3.4) that for each ℓ,

$$\begin{aligned}\text{(II)}&=\left|\int_E\varphi_\ell(x)\eta(x)\mu(dx)-\int_E\varphi_\ell(x)\eta(x)\mu_n(dx)\right|\\&=|\mathcal{E}_1(\varphi_\ell\eta,U_1\mu)-\mathcal{E}_1(\varphi_\ell\eta,U_1\mu_n)|\ \to\ 0\quad(n\to\infty).\end{aligned}$$

Finally, let us consider (III). By the Schwarz inequality,

$$\begin{aligned}\text{(III)}&\leq\int_E|\varphi_\ell(x)-\varphi(x)|\eta(x)\mu_n(dx)\leq\ \|\varphi_\ell-\varphi\|_\infty\int_E\eta(x)\mu_n(dx)\\&=\|\varphi_\ell-\varphi\|_\infty\mathcal{E}_1(\eta,U_1\mu_n)\leq\ \|\varphi_\ell-\varphi\|_\infty\|\eta\|_{\varepsilon_1}\|U_1\mu_n\|_{\varepsilon_1}.\end{aligned}$$

Since $\{\|U_1\mu_n\|_{\varepsilon_1}\}_{n=1}^\infty$ is bounded, we know that (III) tends to 0 as $n\to\infty$ and $\ell\to\infty$. Therefore, taking the limit as $n\to\infty$ and $\ell\to\infty$ in (3.5), we obtain this proposition. ∎

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A sequence of measures $\{\mu_n\}$ in $\mathcal{S}_0$ is said to converge to a measure $\mu \in \mathcal{S}_0$ *weakly with respect to* ρ if

$$\lim_{n\to\infty} \mathcal{E}_1(U_1\mu_n, \varphi) = \mathcal{E}_1(U_1\mu, \varphi), \quad \text{for any } \varphi \in \mathcal{F}.$$

The next proposition is a general result from functional analysis.

Proposition 3.9 *Let $\{\mu_n\}$ be a sequence in $\mathcal{S}_0$. Assume that $\{\mu_n\}$ is bounded with respect to ρ, that is, $\sup_n \mathcal{E}_1(U_1\mu_n, U_1\mu_n) < \infty$. If the measures $\{\mu_n\}$ converges to a measure μ vaguely, then $\mu \in \mathcal{S}_0$ and $\{\mu_n\}$ converges to μ weakly with respect to ρ.*

In the rest of this section, we show other simple but useful properties of $\mathcal{S}_0$.

Proposition 3.10 *The following properties hold for $\mathcal{S}_0$:*

(1) (monotonicity) *Let μ and ν be Borel measures on $(E, \mathcal{B}(E))$. Assume that $\mu \le \nu$, that is, $\mu(A) \le \nu(A)$ for $A \in \mathcal{B}(E)$. If $\nu \in \mathcal{S}_0$, then $\mu \in \mathcal{S}_0$ and $U_\alpha\mu \le U_\alpha\nu$ hold for $\alpha > 0$.*

(2) (convex cone) *If $\mu, \nu \in \mathcal{S}_0$, $a, b \ge 0$ and $\alpha > 0$, then $a\mu + b\nu \in \mathcal{S}_0$ and $U_\alpha(a\mu + b\nu) = aU_\alpha\mu + bU_\alpha\nu$.*

(3) (ideal) *Let $\mathcal{B}_{b,+}(E)$ be the set of all nonnegative bounded measurable functions on E. Then $\mathcal{B}_{b,+}(E)$ is an ideal in $\mathcal{S}_0$ (i.e., $\mathcal{B}_{b,+}(E)\mathcal{S}_0 \subset \mathcal{S}_0$) in the sense that $f\mu$ belongs to $\mathcal{S}_0$ whenever $\mu \in \mathcal{S}_0$ and $f \in \mathcal{B}_{b,+}(E)$.*

Proof (1). Since $\nu \in \mathcal{S}_0$, there exists a $C > 0$ such that

$$\int_E |v| d\mu \le \int_E |v| d\nu \le C\sqrt{\mathcal{E}_1(v, v)}, \quad v \in \mathcal{F} \cap C_0(E).$$

This implies that $\mu \in \mathcal{S}_0$ and then

$$\begin{aligned}\mathcal{E}_\alpha(\varphi, U_\alpha\nu - U_\alpha\mu) &= \mathcal{E}_\alpha(\varphi, U_\alpha\nu) - \mathcal{E}_\alpha(\varphi, U_\alpha\mu) \\ &= \int_E \varphi d\nu - \int_E \varphi d\mu = \int_E \varphi d(\nu - \mu) \ge 0\end{aligned}$$

holds for any $\varphi \in \mathcal{F} \cap C_0(E)$ with $\varphi \ge 0$. This means that $U_\alpha\nu - U_\alpha\mu$ is an α-potential whose measure is nothing but $\nu - \mu$ and $0 \le U_\alpha(\nu - \mu) = U_\alpha\nu - U_\alpha\mu$.

(2). As in the proof of (1), we see that for any $\mu, \nu \in \mathcal{S}_0$, $a, b \ge 0$, $\alpha > 0$ and $\varphi \in \mathcal{F} \cap C_0(E)$ with $\varphi \ge 0$,

$$\begin{aligned}\mathcal{E}_\alpha(\varphi, aU_\alpha\mu + bU_\alpha\nu) &= a\mathcal{E}_\alpha(\varphi, U_\alpha\mu) + b\mathcal{E}_\alpha(\varphi, U_\alpha\nu) \\ &= a\int_E \varphi d\mu + b\int_E \varphi d\nu = \int_E \varphi d(a\mu + b\nu) \ge 0.\end{aligned}$$

Then it follows that $a\mu + b\nu \in \mathcal{S}_0$ and $U_\alpha(a\mu + b\nu) = aU_\alpha\mu + bU_\alpha\nu$.

(3). Let $f \in \mathcal{B}_{b,+}(E)$ and $\mu \in \mathcal{S}_0$. Since $f\mu \le \|f\|_\infty\mu$, we can conclude that $f\mu \in \mathcal{S}_0$ by (1). ∎

Remark 3.11 In the introduction, we noted that Proposition 3.6 holds for the Newton potential or the Riesz potential in the classical potential theory. The results stated in this section moreover hold true to any potential kernels associated with regular Dirichlet forms, including, of course, the cases of the Newton potential (corresponding to the Brownian motion) or the Riesz potential (corresponding to the symmetric

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α-stable process). But, for the space of signed measures $\mathcal{S}_0 - \mathcal{S}_0$, it is not complete in general (see [10, Theorem 1.19]).

4 Compactness of Revuz maps in $\mathcal{S}_0$

In this section, we show that the Revuz map is compact in the sense that, for any bounded sequence in $\mathcal{S}_0$ with the metric ρ, the corresponding sequence of $\mathbf{A}_c^+$ by the Revuz map has a subsequence so that the subsequence converges locally uniformly (on $[0,\infty)$). Let ρ be the metric on $\mathcal{S}_0$ defined by (3.1).

Theorem 4.1 *Let $A^n, A \in \mathbf{A}_c^+$. Denote by μ_n, μ their Revuz measures in $\mathcal{S}$ for $n \geq 1$. Assume that all μ_n and μ belong to $\mathcal{S}_0$. If $\lim_{n\to\infty} \rho(\mu_n, \mu) = 0$, then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that*

$$\mathbb{P}_x\left(\lim_{n_k\to\infty} A_t^{n_k} = A_t \text{ locally uniformly in } t \text{ on } [0,\infty)\right) = 1, \quad \text{q.e. } x \in E.$$

The proof of Theorem 4.1 is long. We divide the proof into Lemmas 4.2–4.6. The next lemma reveals a duality relation between the resolvent and the potential.

Proposition 4.2 *Let $\nu \in \mathcal{S}_0$ and $\alpha > 0$. If $f \in L^p(E;m) \cap \mathcal{B}_+(E)$ for some $1 \leq p < \infty$, then*

$$\mathbb{E}_\nu\left[\int_0^\infty e^{-\alpha s} f(X_s)^p ds\right] = \int_E f(x)^p U_\alpha \nu(x) m(dx) = \int_E R_\alpha(f^p)(x)\nu(dx). \tag{4.1}$$

Proof Let $\{K_n\}$ be an increasing sequence of compact sets satisfying $\cup_{n=1}^\infty K_n = E$ and put $f_n(x) := (f(x)^p \wedge n)\cdot 1_{K_n}(x)$ for each $n \in \mathbb{N}$ and $x \in E$. Then it is easy to see that for any $x \in E$,

$$0 \leq f_n(x) \leq f_{n+1}(x) \leq f(x)^p, \quad \lim_{n\to\infty} f_n(x) = f(x)^p, \quad f_n \in L^2(E;m) \cap \mathcal{B}_+(E).$$

By the correspondence between the Dirichlet form and the α-potential function, we see that

$$\begin{aligned}\mathbb{E}_\nu\left[\int_0^\infty e^{-\alpha s} f_n(X_s) ds\right] &= \int_E R_\alpha f_n(x)\nu(dx) \\ &= \mathcal{E}_\alpha(R_\alpha f_n, U_\alpha \nu) = \int_E f_n(x) U_\alpha \nu(x) m(dx),\end{aligned}$$

hence,

$$\mathbb{E}_\nu\left[\int_0^\infty e^{-\alpha s} f_n(X_s) ds\right] = \int_E R_\alpha f_n(x)\nu(dx) = \int_E f_n(x) U_\alpha \nu(x) m(dx).$$

Then, by making use of the monotone convergence theorem on the respective sides, we have (4.1) since $R_\alpha f(x)$ is written as an integral form:

$$R_\alpha f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t} f(X_t) dt\right]. \qquad \blacksquare$$

Let $f \in L^2(E;m) \cap \mathcal{B}_+(E)$ and put

$$\mathrm{A}^f(t) := \int_0^t e^{-s} f(X_s) ds. \tag{4.2}$$

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Note that the measure fm belongs to $\mathcal{S}_0$ and the 1-potential $U_1(fm)$ is equal to $R_1 f$, the 1-resolvent of f.

Lemma 4.3 *It holds that*

$$\mathbb{E}_\nu\left[\left(A^f(\infty)-A^g(\infty)\right)^2\right] \le 16\max\{\|R_1 f\|_{\mathcal{E}_1},\ \|R_1 g\|_{\mathcal{E}_1}\}\|R_1 f - R_1 g\|_{\mathcal{E}_1},$$

for any $\nu \in \mathcal{S}^1_{c,00}$ *and* $f, g \in L^2(E;m)\cap \mathcal{B}_+(E)$.

Proof For any $f, g \in L^2(E;m)\cap \mathcal{B}_+(E)$, put $h := f - g$. By the Markov property,

$$\begin{aligned}
\mathbb{E}_\nu\left[\left(\mathrm{A}^f(\infty)-\mathrm{A}^g(\infty)\right)^2\right] &= \mathbb{E}_\nu\left[\left(\int_0^\infty e^{-s}h(X_s)ds\right)^2\right]\\
&= 2\mathbb{E}_\nu\left[\int_0^\infty e^{-s}h(X_s)\left(\int_s^\infty e^{-u}h(X_u)du\right)ds\right]\\
&= 2\mathbb{E}_\nu\left[\int_0^\infty e^{-s}h(X_s)\left(\int_0^\infty e^{-(u+s)}h(X_{u+s})du\right)ds\right]\\
&= 2\mathbb{E}_\nu\left[\int_0^\infty e^{-2s}h(X_s)R_1h(X_s)ds\right].
\end{aligned}$$

Replacing f and α in Lemma 4.2(1) by $h\cdot R_1 h \in L^1(E,m)$ and 2 respectively, then the above is continue to

$$\begin{aligned}
&= 2\int_E R_2(h\cdot R_1h)(x)v(dx) = 2\int_E h(x)\cdot R_1h(x)\cdot U_2v(x)m(dx) = 2\mathcal{E}_1(R_1h, R_1(hU_2v))\\
&\le 2\|R_1h\|_{\mathcal{E}_1}\|R_1(hU_2v)\|_{\mathcal{E}_1} \le 2\|R_1h\|_{\mathcal{E}_1}\left\{\|R_1(fU_2v)\|_{\mathcal{E}_1}+\|R_1(gU_2v)\|_{\mathcal{E}_1}\right\}.
\end{aligned}$$

Here, since $f\ge 0$ and $U_2v\ge 0$,

$$\|R_1(fU_2v)\|^2_{\mathcal{E}_1} = \left(fU_2v, R_1(fU_2v)\right) \le \|U_2v\|^2_\infty(f, R_1f) \le 4\mathcal{E}_1(R_1f, R_1f),$$

where in the last inequality we use Lemma 2.2. Hence, we obtain

$$\mathbb{E}_\nu\left[\left(\mathrm{A}^f(\infty)-\mathrm{A}^g(\infty)\right)^2\right] \le 16\|R_1h\|_{\mathcal{E}_1}\max\{\|R_1f\|_{\mathcal{E}_1}, \|R_1g\|_{\mathcal{E}_1}\}.$$

■

Lemma 4.4 *Let* $f\in L^2(E;m)\cap\mathcal{B}_+(E)$. *Denote by* $A^f(t)$ *the CAF defined by (4.2) and put*

$$M^f(t) := A^f(t) + e^{-t}R_1f(X_t).$$

Then $\{M^f(t)\}_{t\ge 0}$ *is an* $(\mathcal{F}_t, \mathbb{P}_\nu)$*-martingale, where* $\nu\in\mathcal{S}^1_{c,00}$ *with* $\nu(E)=1$.

Proof For $t>s\ge 0$,

$$\begin{aligned}
\mathbb{E}_\nu\left[\mathrm{M}^f(t)\mid\mathcal{F}_s\right] &= \mathbb{E}_\nu\left[\mathrm{A}^f(t)+e^{-t}R_1f(X_t)\,\middle|\,\mathcal{F}_s\right]\\
&= \mathbb{E}_\nu\left[\mathrm{A}^f(\infty)-\int_t^\infty e^{-u}f(X_u)du + e^{-t}R_1f(X_t)\,\middle|\,\mathcal{F}_s\right]\\
&= \mathbb{E}_\nu\left[\mathrm{A}^f(\infty)\mid\mathcal{F}_s\right] - \mathbb{E}_\nu\left[\int_0^\infty e^{-(u+t)}f(X_{u+t})du\,\middle|\,\mathcal{F}_s\right]\\
&\quad + \mathbb{E}_\nu\left[e^{-t}R_1f(X_t)\mid\mathcal{F}_s\right] =: \mathbb{E}_\nu\left[\mathrm{A}^f(\infty)\mid\mathcal{F}_s\right] - (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

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By the Markov property,

$$\begin{aligned}(\mathrm{II}) &= \mathbb{E}_\nu\left[e^{-t}\int_0^\infty e^{-u}\mathbb{E}_{X_t}[f(X_u)]\,du\,\middle|\,\mathcal{F}_s\right]\\ &= \mathbb{E}_\nu\left[e^{-t}\int_0^\infty e^{-u}\mathbb{E}_x[f(X_{u+t})|\mathcal{F}_t]du\,\middle|\,\mathcal{F}_s\right]\\ &= \mathbb{E}_\nu\left[\int_0^\infty e^{-(u+t)}f(X_{u+t})du\,\middle|\,\mathcal{F}_s\right] = (\mathrm{I}).\end{aligned}$$

Thus we know that $\mathbb{E}_\nu[\mathsf{M}^f(t)\mid\mathcal{F}_s] = \mathbb{E}_\nu[\mathsf{A}^f(\infty)\mid\mathcal{F}_s]$. Then,

$$\begin{aligned}\mathbb{E}_\nu[\mathsf{A}^f(\infty)\mid\mathcal{F}_s] &= \mathbb{E}_\nu\left[\int_0^s e^{-u}f(X_u)du + \int_s^\infty e^{-u}f(X_u)du\,\middle|\,\mathcal{F}_s\right]\\ &= \mathsf{A}^f(s) + \mathbb{E}_\nu\left[\int_s^\infty e^{-u}f(X_u)du\,\middle|\,\mathcal{F}_s\right],\end{aligned}$$

where by using the Markov property, the above second term is rewritten by

$$\mathbb{E}_\nu\left[\int_0^\infty e^{-(u+s)}f(X_{u+s})du\,\middle|\,\mathcal{F}_s\right] = e^{-s}\int_E \mathbb{E}_{X_s}\left[\int_0^\infty e^{-u}f(X_u)du\right]\nu(dx) = e^{-s}R_1f(X_s).$$

That is, $\mathbb{E}_\nu[\mathsf{A}^f(\infty)\mid\mathcal{F}_s] = \mathsf{M}^f(s)$. Therefore, we can find that $\{\mathsf{M}^f(t)\}_{t\geq 0}$ is an $(\mathcal{F}_t,\mathbb{P}_\nu)$-martingale. ∎

Lemma 4.5 *Let $\{f_k\}_{k\in\mathbb{N}}\subset L^2(E;m)\cap\mathcal{B}_+(E)$ and $M^{f_k}(t)$ be given by (4.4) with f_k in place of f. If $\{R_1f_k\}_{k\in\mathbb{N}}$ converges to a function in $\mathcal{E}_1$-strongly, then there exists a subsequence $\{k_j\}$ of $\{k\}$ such that*

$$\mathbb{P}_x\left(M^{f_{k_j}}(t)\ \textit{converges uniformly in } t \textit{ on } [0,\infty)\ \textit{as } j\to\infty\right) = 1,\qquad \textit{q.e. } x\in E.$$

Proof Let ν be in $\mathcal{S}^1_{c,00}$ with $\nu(E)=1$. Note here that $\mathsf{M}^{f_k}(\infty) = \mathsf{A}^{f_k}(\infty)$ in $L^2(\mathbb{P}_\nu)$ and $\{\mathsf{M}^{f_k}(t)\}_{t\geq 0}$ is an $(\mathcal{F}_t,\mathbb{P}_\nu)$-martingale. By Doob's inequality and Lemma 4.3,

$$\mathbb{P}_\nu\left(\sup_{0\leq t\leq\infty}|\mathsf{M}^{f_k}(t)-\mathsf{M}^{f_\ell}(t)|>\varepsilon\right)\leq\frac{1}{\varepsilon^2}\mathbb{E}_\nu\left[\left(\mathsf{A}^{f_k}(\infty)-\mathsf{A}^{f_\ell}(\infty)\right)^2\right]\leq\frac{M}{\varepsilon^2}\|R_1f_k-R_1f_\ell\|_{\mathcal{E}_1},$$

where $M = 16\sup_{k\in\mathbb{N}}\|R_1f_k\|_{\mathcal{E}_1}$ $(<\infty)$. Since $\{R_1f_k\}_{k\geq 1}$ is an $\mathcal{E}_1$-Cauchy sequence, there exists a subsequence $\{k_j\}$ of $\{k\}$ such that

$$\|R_1f_{k_{j+1}}-R_1f_{k_j}\|_{\mathcal{E}_1}<2^{-3j}.$$

Now we set

$$\Lambda_j = \left\{\omega\in\Omega:\sup_{0\leq t\leq\infty}\left|\mathsf{M}^{f_{k_{j+1}}}(t,\omega)-\mathsf{M}^{f_{k_j}}(t,\omega)\right|>2^{-j}\right\},$$

and we know that $\mathbb{P}_\nu(\Lambda_j)\leq M\cdot 2^{-j}$, hence, $\sum_{j=1}^\infty\mathbb{P}_\nu(\Lambda_j)\leq M<\infty$ holds. Then the first Borel–Cantelli lemma provides us that

$$\mathbb{P}_\nu\left(\limsup_{j\to\infty}\Lambda_j\right) = 0.$$

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For any $\mu \in \mathcal{S}^1_{c,00}$, by setting $\nu(\cdot) = \mu(\cdot)/\mu(E)$,

$$\mathbb{P}_\mu\left(\limsup_{j\to\infty} \Lambda_j\right) = 0. \tag{4.3}$$

Combining (4.3) with Theorem 2.1, we get this lemma. ∎

In the following, we need to take subsequences repeatedly, so, in order to avoid confusion with notations, we introduce a space of functions which define subsequences. We denote by Φ all functions from $\mathbb{N}$ to itself which are strictly increasing. Note that $\{\phi(n)\}$ is nothing more than a subsequence of the sequence $\{n\}$ for a $\phi \in \Phi$.

Lemma 4.6 *Let A^n, $A \in \mathbf{A}_c^+$. Denote by μ_n and μ, their Revuz measures in $\mathcal{S}$ for $n \geq 1$. Assume that $\mu_n = f_n m$ and $\mu = fm$ for some f_n, $f \in L^2(E;m) \cap \mathcal{B}_+(E)$. If $\lim_{n\to\infty} \rho(\mu_n, \mu) = 0$, then there exists a $\phi \in \Phi$ such that*

$$\mathbb{P}_x\left(\lim_{n\to\infty} A_t^{\phi(n)} = A_t \text{ locally uniformly in } t \text{ on } [0,\infty)\right) = 1, \quad \text{q.e. } x \in E.$$

Proof We use an idea in the proof of [7, Theorem 5.1.1]. For each $n \in \mathbb{N}$, we put

$$\mathrm{B}_n(t) := \int_0^t e^{-s} f_n(X_s)ds \quad \text{and} \quad \mathrm{M}_n(t) := \mathrm{B}_n(t) + e^{-t}R_1 f_n(X_t).$$

Since $\lim_{n\to\infty} \|R_1 f_n - U_1\mu\|_{\mathcal{E}_1} = 0$, by Lemma 4.5, there is a $\phi_1 \in \Phi$ such that

$$\mathbb{P}_x\left(\mathrm{M}_{\phi_1(n)}(t) \text{ converges uniformly in } t \text{ on } [0,\infty) \text{ as } n \to \infty\right) = 1, \quad \text{q.e. } x \in E.$$

Further, by means of [7, Lemma 5.1.2(i)], there is a $\phi_2 \in \Phi$ such that

$$\mathbb{P}_x(R_1 f_{\psi_2(n)}(X_t) \text{ converges locally uniformly in } t \text{ on } [0,\infty) \text{ as } n \to \infty) = 1, \quad \text{q.e. } x \in E,$$

where $\psi_2 = \phi_2 \circ \phi_1$. Since $\mathrm{M}_n(\infty) = \mathrm{B}_n(\infty)$ in $L^2(\mathbb{P}_\nu)$ ($n \in \mathbb{N}$, $\nu \in \mathcal{S}^1_{c,00}$), there exists an exceptional set $N \subset E$ such that $\mathbb{P}_x(\Lambda) = 1$ for any $x \in E\backslash N$, where

$$\tilde{\Omega} = \left\{\omega \in \Omega \mid t \geq 0,\ X_t(\omega) \in E\backslash N, X_{t-}(\omega) \in E\backslash N\right\},$$

$$\Lambda = \left\{\omega \in \tilde{\Omega} \;\middle|\; \begin{array}{l} \mathrm{B}_{\psi_2(n)}(\infty,\omega) < \infty \text{ and} \\ \mathrm{B}_{\psi_2(n)}(t,\omega) \text{ converges locally uniformly in } t \text{ on } [0,\infty) \text{ as } n \to \infty \end{array}\right\}.$$

Now we define $\mathrm{B}(t,\omega)$ and $\tilde{\mathrm{A}}(t,\omega)$ by

$$\mathrm{B}(t,\omega) = \begin{cases} \lim_{n\to\infty} \mathrm{B}_{\psi_2(n)}(t,\omega), & \omega \in \Lambda, \\ 0, & \omega \notin \Lambda, \end{cases}$$

$$\tilde{\mathrm{A}}(t,\omega) = \int_0^t e^s \mathrm{B}(ds,\omega), \quad \omega \in \tilde{\Omega}.$$

Noting

$$\mathrm{A}_t^n = \int_0^t f_n(X_s)ds = \int_0^t e^s \mathrm{B}_n(ds) = e^t \mathrm{B}_n(t) - \int_0^t e^s \mathrm{B}_n(s)ds$$

and $\tilde{\mathrm{A}}(t) = e^t \mathrm{B}(t) - \int_0^t e^s \mathrm{B}(s)ds$, we find

$$\mathbb{P}_x\left(\mathrm{A}_t^{\psi_2(n)} \text{ converges to } \tilde{\mathrm{A}}_t \text{ locally uniformly in } t \text{ on } [0,\infty) \text{ as } n \to \infty\right) = 1, \quad \text{q.e. } x \in E.$$

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We claim the following:

$$\mathbb{E}_\nu\left[\int_0^\infty e^{-t}d\widetilde{\mathrm{A}}(s)\right] = \int_E \widetilde{U_1\mu}(x)\nu(dx), \quad \nu \in \mathcal{S}^1_{c,00}. \tag{4.4}$$

Note that

$$\lim_{t\to\infty} e^{-t}\limsup_{n\to\infty}\mathbb{E}_\nu\left[R_1 f_n(X_t)\right] = 0, \tag{4.5}$$

which follows from

$$\begin{aligned}\mathbb{E}_\nu\left[R_1 f_n(X_t)\right] &= \int_E p_t(R_1 f_n)(x)\nu(dx) = \mathcal{E}_1(p_t(R_1 f_n), U_1\nu) \le \|p_t(R_1 f_n)\|_{\mathcal{E}_1}\\ &\le \|R_1 f_n\|_{\mathcal{E}_1} \le \sup_n \|R_1 f_n\|_{\mathcal{E}_1} < \infty.\end{aligned}$$

We next note that

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}_\nu[\mathrm{M}_{\psi_2(n)}(t)] &= \lim_{n\to\infty}\mathbb{E}_\nu[\mathrm{M}_{\psi_2(n)}(\infty)] = \lim_{n\to\infty}\mathbb{E}_\nu[\mathrm{B}_{\psi_2(n)}(\infty)]\\ &= \lim_{n\to\infty}\mathbb{E}_\nu\left[\int_0^\infty e^{-s}f_{\psi_2(n)}(X_s)ds\right] = \lim_{n\to\infty}\int_E R_1 f_{\psi_2(n)}(x)\nu(dx)\\ &= \lim_{n\to\infty}\mathcal{E}_1(R_1 f_{\psi_2(n)}, U_1\nu) = \int_E \widetilde{U_1\mu}(x)\nu(dx),\end{aligned}$$

and hence

$$\begin{aligned}\mathbb{E}_\nu[B(t)] &= \lim_{n\to\infty}\mathbb{E}_\nu[\mathrm{B}_{\psi_2(n)}(t)] = \lim_{n\to\infty}\left(\mathbb{E}_\nu[\mathrm{M}_{\psi_2(n)}(t)] - e^{-t}\mathbb{E}_\nu\left[R_1 f_{\psi_2(n)}(X_t)\right]\right)\\ &= \int_E \widetilde{U_1\mu}(x)\nu(dx) - e^{-t}\lim_{n\to\infty}\mathbb{E}_\nu\left[R_1 f_{\psi_2(n)}(X_t)\right].\end{aligned}$$

Combining this with (4.5), we have

$$\mathbb{E}_\nu[\mathrm{B}(\infty)] = \int_E \widetilde{U_1\mu}(x)\nu(dx),$$

which implies (4.4). Since (4.4) implies $\mathrm{A} \sim \widetilde{\mathrm{A}}$, we obtain the lemma. ∎

Proof of Theorem 4.1 Since the 1-potential $U_1\mu_n$ of μ_n is 1-excessive, we can find a Borel measurable and quasi continuous version u_n of $U_1\mu_n$ and properly exceptional set N_n for each $n \in \mathbb{N}$ such that

$$\begin{cases} kR_{k+1}u_n(x) \uparrow u_n(x) \quad (k\to\infty), & x \in E\backslash N_n\\ u_n(x) = 0, & x \in N_n.\end{cases}$$

Now we put

$$g_n^k(x) := \begin{cases} k(u_n(x) - kR_{k+1}u_n(x)), & x \in E\backslash N_n\\ 0, & x \in N_n.\end{cases}$$

By Lemma 3.3, we know that $R_1 g_n^k \to u_n$, $\mathcal{E}_1$-strongly as $k\to\infty$. Now let

$$\mathrm{A}_n^k(t) := \int_0^t g_n^k(X_s)ds.$$

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By means of Lemma 4.6, for each $n \in \mathbb{N}$, there exist $\psi^n \in \Phi$ such that

$$\mathbb{P}_x\left(\mathrm{A}_n^{\psi^n(k)}(t) \text{ converges to } \mathrm{A}_t^n \text{ locally uniformly in } t \text{ on } [0,\infty) \text{ as } k \to \infty\right) = 1, \quad \text{q.e. } x \in E.$$

Let $T > 0$, $\varepsilon > 0$ and $\nu \in \mathcal{S}^1_{c,00}$. Then there are numbers $k_o^n \in \mathbb{N}$ such that

$$\mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|\mathrm{A}_n^{\psi^n(k)}(t) - \mathrm{A}_t^n\right| > \frac{\varepsilon}{2}\right) < 1/n, \quad \text{for any } k \ge k_o^n,\ n \in \mathbb{N}.$$

Now, since $R_1 g_n^{\psi^n(k)}$ converges to u_n in $\mathcal{E}_1$-strongly as $k \to \infty$ for each $n \in \mathbb{N}$, there is a $k_*^n \ge k_o^n$ such that

$$\left\|R_1 g_n^{\psi^n(k)} - u_n\right\|_{\mathcal{E}_1} < 1/n \quad \text{for any } k \ge k_*^n.$$

Let u be a quasi continuous version of $U_1\mu$. Putting $h_n := g_n^{\psi^n(k_*^n)}$, we get

$$\|R_1 h_n - u\|_{\mathcal{E}_1} \le \|R_1 h_n - u_n\|_{\mathcal{E}_1} + \|u_n - u\|_{\mathcal{E}_1} < \frac{1}{n} + \|u_n - u\|_{\mathcal{E}_1} \to 0 \quad (n \to \infty).$$

Hence we know that $R_1 h_n$ converges to u in $\mathcal{E}_1$-strongly as $n \to \infty$, that is, $\rho(h_n m, \mu) \to 0$. By means of Lemma 4.6, there is a $\phi_* \in \Phi$ such that

$$\mathbb{P}_x\left(\mathrm{A}_{n_*}^{\psi^{n_*}(k_*^{n_*})}(t) \text{ converges to } \mathrm{A}_t \text{ locally uniformly in } t \text{ on } [0,\infty) \text{ as } n \to \infty\right) = 1, \quad \text{q.e. } x \in E,$$

where $n_* = \phi_*(n)$. Therefore we obtain

$$\begin{aligned}
&\mathbb{P}_\nu\left(\sup_{0\le t\le T}|\mathrm{A}_t^{n_*} - \mathrm{A}_t| > \varepsilon\right) \\
&\le \mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|\mathrm{A}_t^{n_*} - \mathrm{A}_{n_*}^{\psi^{n_*}(k_*^{n_*})}(t)\right| > \frac{\varepsilon}{2}\right) + \mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|\mathrm{A}_{n_*}^{\psi^{n_*}(k_*^{n_*})}(t) - \mathrm{A}_t\right| > \frac{\varepsilon}{2}\right) \to 0 \ (n \to \infty).
\end{aligned}$$

Finally applying the first Borel–Cantelli lemma and Theorem 2.1, we arrive at the theorem. ∎

Example 4.7 Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form $(\frac{1}{2}\mathbb{D}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$, $d \ge 2$. Let μ be the Lebesgue measure on a $(d-1)$-dimensional hyperplane $F \subset \mathbb{R}^d$. Let f be a nonnegative bounded measurable function on $\mathbb{R}^d$ whose support, denoted by K, is a compact subset of F. By means of [7, Exercise 2.2.1], $1_K\mu \in \mathcal{S}_0$.

Take a sequence of increasing sequence of measurable functions $\{f_n\}$ defined on F so that

$$0 \le f_n(x) \le f_{n+1}(x) \le f(x), \quad \lim_{n\to\infty} f_n(x) = f(x), \quad \mu\text{-a.e. } x \in F.$$

Set $\nu_n := f_n\mu$ for $n \in \mathbb{N}$ and $\nu := f\mu$. Then ν_n and ν are not absolutely continuous with respect to the Lebesgue measure dx on $\mathbb{R}^d$. By noting the estimates $f_n\mu \le f\mu \le \|f\|_\infty 1_K\mu$, it follows that $\nu_n,\ \nu \in \mathcal{S}_0$ for $n \in \mathbb{N}$ from Proposition 3.10. Moreover, we find that

$$\lim_{n\to\infty} \rho(\nu, \nu_n) = 0.$$

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In fact, since $U_1(v_n) \le U_1(fv) \le \|f\|_\infty U_1(1_K\mu)$ from Proposition 3.10 (1) for any n, we see that for any n,

$$\rho(v,v_n)^2 = \mathcal{E}_1(U_1v - U_1v_n, U_1v - U_1v_n) = \int_F \left(\widetilde{U_1v} - \widetilde{U_1v_n}\right)(f-f_n)d\mu$$
$$\le \|f\|_\infty \int_K \widetilde{U_1(1_K\mu)}(f-f_n)d\mu.$$

Then the right-hand side goes to 0 as n tending to ∞ by the dominated convergence theorem.

5 Compactness of Revuz maps in $\mathcal{S}$

In this section, we discuss the convergence of PCAFs associated with general smooth measures.

Theorem 5.1 *Let A^n, $A \in \mathbf{A}_c^+$. Denote by μ_n and μ their Revuz measures in $\mathcal{S}$ for $n \ge 1$. Assume there exists an increasing sequence of closed sets $\{F_k\}$ such that*

$$Cap\left(E\backslash\bigcup_{k=1}^{\infty} F_k\right) = 0; \tag{5.1}$$

(Sa) *for each $n, k \in \mathbb{N}$, $1_{F_k}\mu_n, 1_{F_k}\mu \in \mathcal{S}_0$ and, for each $k \in \mathbb{N}$, $\lim_{n\to\infty} \rho(1_{F_k}\mu_n, 1_{F_k}\mu) = 0$;*

(Sa) $\lim_{k\to\infty} \sup_{v\in\mathcal{S}^1_{c,00}} \sup_{n\in\mathbb{N}} \int_{F_k^c} \widetilde{U_1v}\, d\mu_n = 0$;

(Sa) $\lim_{k\to\infty} \sup_{v\in\mathcal{S}^1_{c,00}} \int_{F_k^c} \widetilde{U_1v}\, d\mu = 0$.

Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\mathbb{P}_x\left(\lim_{n_k\to\infty} A_t^{n_k} = A_t \ \text{ locally uniformly in } t \text{ on } [0,\infty)\right) = 1, \quad q.e.\ x \in E.$$

Proof Put $\Gamma_k = F_{k+1}\backslash F_k$, $\mu_n^{(k)} = 1_{\Gamma_k}\mu_n$ and $\mu^{(k)} = 1_{\Gamma_k}\mu (k, n \in \mathbb{N})$. By means of (Sa), $\mu_n^{(k)}, \mu^{(k)} \in \mathcal{S}_0$. Let $\mathrm{A}_n^{(k)}$, $\mathrm{A}^{(k)}$ be the PCAFs corresponding to $\mu_n^{(k)}$, $\mu^{(k)}$, respectively. By virtue of [7, Theorem 5.1.3], $\mathrm{A}_n^{(k)} \sim 1_{\Gamma_k}\mathrm{A}_n$ and $\mathrm{A}^{(k)} \sim 1_{\Gamma_k}\mathrm{A}$.

Let $\varepsilon > 0$, $T > 0$ and $v \in \mathcal{S}_{00}$. We show that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{n_k\to\infty} \mathbb{P}_v\left(\sup_{0\le t\le T} |\mathrm{A}_t^{n_k} - \mathrm{A}_t| > \varepsilon\right) = 0. \tag{5.2}$$

To prove this, divide the above into three terms for j, n as follows:

$$\mathbb{P}_v\left(\sup_{0\le t\le T} \left|\mathrm{A}_t^n - \mathrm{A}_t\right| > \varepsilon\right) \le \mathbb{P}_v\left(\sup_{0\le t\le T} \left|\mathrm{A}_t^n - 1_{F_j}\mathrm{A}_t^n\right| > \frac{\varepsilon}{3}\right) + \mathbb{P}_v\left(\sup_{0\le t\le T} \left|1_{F_j}\mathrm{A}_t^n - 1_{F_j}\mathrm{A}_t\right| > \frac{\varepsilon}{3}\right)$$
$$+ \mathbb{P}_v\left(\sup_{0\le t\le T} \left|1_{F_j}\mathrm{A}_t - \mathrm{A}_t\right| > \frac{\varepsilon}{3}\right)$$
$$=: (\mathrm{I})_{j,n} + (\mathrm{II})_{j,n} + (\mathrm{III})_j.$$

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By the Chebyshev inequality and the inequality, $1 \le e^{T-s}$ for $0 \le s \le T$, the term $(\mathrm{III})_j$ is estimated as

$$(\mathrm{III})_j \le \frac{3e^T}{\varepsilon}\mathbb{E}_\nu\left[\int_0^T e^{-s}1_{F_j^c}(X_s)dA_s\right] = \frac{3e^T}{\varepsilon}\sum_{k=j}^{\infty}\mathbb{E}_\nu\left[\int_0^\infty e^{-s}1_{\Gamma_k}(X_s)dA_s\right].$$

Then the expectation part is rewritten as follows:

$$\begin{aligned}\mathbb{E}_\nu\left[\int_0^\infty e^{-s}1_{\Gamma_k}(X_s)dA_s\right] &= \int_E \mathbb{E}_x\left[\int_0^\infty e^{-s}dA_s^{(k)}\right]\nu(dx) = \int_E \widetilde{U_1\mu^{(k)}}\,d\nu \\ &= \mathcal{E}_1\left(U_1\mu^{(k)}, U_1\nu\right) = \int_E \widetilde{U_1\nu}\,d\mu^{(k)} = \int_{\Gamma_k}\widetilde{U_1\nu}\,d\mu.\end{aligned}$$

Thus, we have that

$$(\mathrm{III})_j \le \frac{3e^T}{\varepsilon}\sum_{k=j}^{\infty}\int_{\Gamma_k}\widetilde{U_1\nu}\,d\mu = \frac{3e^T}{\varepsilon}\int_{F_j^c}\widetilde{U_1\nu}\,d\mu \le \frac{3e^T}{\varepsilon}\sup_{\nu\in\mathcal{S}^1_{c,00}}\int_{F_j^c}\widetilde{U_1\nu}d\mu.$$

Let us fix an $\varepsilon' > 0$ arbitrarily. Then, by means of (Sc), there is an j_1 such that $(\mathrm{III})_j < \varepsilon'$ for any $j \ge j_1$. In the same way as above, we get the following:

$$(\mathrm{I})_{j,n} \le \frac{3e^T}{\varepsilon}\sup_{\nu\in\mathcal{S}^1_{c,00}}\sup_{n\in\mathbb{N}}\int_{F_j^c}\widetilde{U_1\nu}\,d\mu_n.$$

By means of (Sb), there is an j_2 $(\ge j_1)$ such that $(\mathrm{I})_{j,n} < \varepsilon'$ for any $j \ge j_2, n \in \mathbb{N}$. Finally we consider $(\mathrm{II})_{j,n}$ with $j = j_2$. Noting (Sa) and using Theorem 4.1, we find a subsequence $\{n_k\}$ such that $\lim_{n_k\to\infty}(\mathrm{II})_{j_2,n_k} = 0$. Thus we obtain

$$\limsup_{n_k\to\infty}\mathbb{P}_\nu\left(\sup_{0\le t\le T}|A_t^{n_k} - A_t| > \varepsilon\right) \le 2\varepsilon'.$$

Letting $\varepsilon' \downarrow 0$ leads us to (5.2). Applying the first Borel–Cantelli lemma and Theorem 2.1, we arrive at the theorem. ∎

Remark 5.2

(i) If μ_n and μ are absolutely continuous with respect to m, then the assumption (5.1) can be replaced by

$$m\left(E\setminus\bigcup_{k=1}^{\infty}F_k\right) = 0.$$

(ii) If $\mu_n, \mu \in \mathcal{S}_0$ satisfy the assumption of Theorem 4.1, that is, $\lim_{n\to\infty}\rho(\mu_n,\mu) = 0$, then the assumption of Theorem 5.1 is satisfied with $F_k = E$ $(k \in \mathbb{N})$.

6 A class of absolutely continuous measures that are smooth

In this section, we define a class of smooth measures that are absolutely continuous with respect to the basic measure m. We say that a measurable function f on E is *locally in* $L^1(E;m)$ *in the broad sense* ($u \in \dot{L}^1_{\mathrm{loc}}(E;m)$ in notation) if there exists an nest $\{F_n\}$ consisting of compact sets such that $f \in L^1(F_n;m)$ for any n. It is clear that $L^p(E;m) \subset L^1_{\mathrm{loc}}(E;m) \subset \dot{L}^1_{\mathrm{loc}}(E;m)$ for any $p \in [1,\infty]$.

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Lemma 6.1 *Assume that $f \in \dot{L}^1_{loc}(E;m) \cap \mathcal{B}_+(E)$ and put $\mu := fm$. Then μ belongs to $\mathcal{S}$.*

Proof Let $f \in \dot{L}^1_{\mathrm{loc}}(E;m) \cap \mathcal{B}_+(E)$ and $\{F_n\}$ be an increasing sequence of compact sets satisfying the above condition. By the assumption of $\{F_n\}$, it is obvious that $\mu(F_n) < \infty$ for any $n \in \mathbb{N}$. Hence, $\mu = fm$ is a smooth measure. ■

Definition 6.2 Write $\mathfrak{A}$ for the set of functions f satisfying that $f \in \dot{L}^1_{\mathrm{loc}}(E;m) \cap \mathcal{B}_+(E)$ and there exist a sequence of functions $\{f_n\} \subset \dot{L}^1_{\mathrm{loc}}(E;m) \cap \mathcal{B}_+(E)$, an increasing sequence of closed sets $\{F_n\}$ satisfying $m\left(E \setminus \bigcup_{n=1}^{\infty} F_n\right) = 0$, and sequences $\{p_n\}, \{q_n\}, \{r_n\}$ with $1 \le p_n, q_n, r_n < \infty$ such that

(Aa) $\lim_{n\to\infty} \|f_n - f\|_{L^{p_n}(F_n)} = 0$;

(Ab) one of the following conditions is satisfied;

(Ab1) $\lim_{n\to\infty} \sum_{k=n}^{\infty} \|f_n\|_{L^{q_k}(F_{k+1}\setminus F_k)} \cdot (1 \wedge m(F_{k+1}\setminus F_k)^{1/q_k'}) = 0$,

(Ab2) $\lim_{n\to\infty} C^{q_n} \|f_n\|^{q_n}_{L^{q_n}(E\setminus F_n)} = 0$ for any $C > 1$,

(Ac) one of the following conditions is satisfied;

(Ac1) $\lim_{n\to\infty} \sum_{k=n}^{\infty} \|f\|_{L^{r_k}(F_{k+1}\setminus F_k)} \cdot (1 \wedge m(F_{k+1}\setminus F_k)^{1/r_k'}) = 0$,

(Ac2) $\lim_{n\to\infty} C^{r_n} \|f\|^{r_n}_{L^{r_n}(E\setminus F_n)} = 0$ for any $C > 1$,

where q_k' (resp. r_k') satisfies that $1/q_k + 1/q_k' = 1$ (resp. $1/r_k + 1/r_k' = 1$). We call the sequence $\{f_n\}$ an $\mathfrak{A}$*-approximating sequence of* f.

Remark 6.3

(i) There exists an example such that the condition (Ab1) (resp. (Ab2)) is satisfied but (Ab2) (resp. (Ab1)) fails for some $\{q_n\}$ and $\{F_n\}$. Similarly, there exists an example so that the same conclusions go for (Ac1) and (Ac2) to some $\{r_n\}$ and $\{F_n\}$ (see Section 7.1).

(ii) Let $f_n, f \in L^p(E;m) \cap \mathcal{B}_+(E)$ $(n \in \mathbb{N})$ such that $\|f_n - f\|_{L^p} \to 0$ as $n \to \infty$, where $1 \le p < \infty$. Then $\{f_n\}$ is an $\mathfrak{A}$-approximating sequence of f. Indeed, taking an increasing sequence $\{F_n\}$ of compact sets satisfying $m\left(E \setminus \bigcup_{n=1}^{\infty} F_n\right) = 0$, it is easy to see that $f_n, f \in \dot{L}^1_{\mathrm{loc}}(E;m) \cap \mathcal{B}_+(E)$ and the conditions (Aa), (Ab2), and (Ac2) are satisfied with $p_n = q_n = r_n = p$ $(n \in \mathbb{N})$. So, $L^p(E;m) \cap \mathcal{B}_+(E) \subset \mathfrak{A}$ holds for $1 \le p < \infty$.

(iii) We note that there is an $f \in \mathfrak{A}$ such that $f \notin L^p(E;m)$ for any $p \in [1,\infty]$. We emphasize that the integrability exponents p_n, q_n, r_n are allowed to vary for functions of $\mathfrak{A}$ (see also Section 7.1).

Theorem 6.4 *Let $f \in \mathfrak{A}$ and $\{f_n\}$ be an $\mathfrak{A}$-approximating sequence of f. Let $\{A^n_t\}$ and $\{A_t\}$ be PCAFs corresponding to $f_n m$ and fm, respectively. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that*

$$\mathbb{P}_x\left(A^{n_k}_t \text{ converges to } A_t \text{ locally uniformly in } t \text{ on } [0,\infty)\right) = 1, \quad \text{q.e. } x \in E.$$

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Proof Take $\varepsilon > 0$ and $T > 0$. For $\nu \in \mathcal{S}^1_{c,00}$, we have

$$\begin{aligned}\mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|\mathsf{A}^n_t - \mathsf{A}_t\right| > \varepsilon\right) &\le \mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|\mathsf{A}^n_t - 1_{F_n}\cdot\mathsf{A}^n_t\right| > \frac{\varepsilon}{3}\right) + \mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|1_{F_n}\cdot\mathsf{A}^n_t - 1_{F_n}\cdot\mathsf{A}_t\right| > \frac{\varepsilon}{3}\right)\\ &\quad + \mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|1_{F_n}\cdot\mathsf{A}_t - \mathsf{A}_t\right| > \frac{\varepsilon}{3}\right)\\ &=: \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)}.\end{aligned}$$

Here $1_{F_n}\cdot\mathsf{A}^n_t = \int_0^t 1_{F_n}(X_s)f_n(X_s)ds$. We start with (II). By Chebyshev's and Hölder's inequalities,

$$\begin{aligned}\mathrm{(II)} &= \mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|\int_0^t 1_{F_n}\cdot(f_n - f)(X_s)\,ds\right| > \frac{\varepsilon}{3}\right)\\ &\le \frac{3}{\varepsilon}\mathbb{E}_\nu\left[\int_0^T 1_{F_n}(X_s)|f_n - f|(X_s)\,ds\right]\\ &\le \frac{3e^T}{\varepsilon}\mathbb{E}_\nu\left[\left(\int_0^T e^{-s}1_{F_n}(X_s)|f_n - f|(X_s)\,ds\right)^{p_n}\right]^{1/p_n}\\ &= \frac{3e^T}{\varepsilon}\mathbb{E}_\nu\left[\left(\int_0^T e^{(-s)(1/p_n+1/p_n')}1_{F_n}(X_s)|f_n - f|(X_s)\,ds\right)^{p_n}\right]^{1/p_n}\\ &\le \frac{3e^T}{\varepsilon}\mathbb{E}_\nu\left[\left(\int_0^T e^{-s}1_{F_n}(X_s)|f_n - f|^{p_n}(X_s)\,ds\right)\left(\int_0^T e^{-s}ds\right)^{p_n/p_n'}\right]^{1/p_n}\\ &\le \frac{3e^T}{\varepsilon}\mathbb{E}_\nu\left[\int_0^\infty e^{-s}1_{F_n}(X_s)|f_n - f|^{p_n}(X_s)\,ds\right]^{1/p_n}.\end{aligned}$$

By Lemma 4.2,

$$\begin{aligned}\mathrm{(II)} &\le \frac{3e^T}{\varepsilon}\left(\int_E R_1(1_{F_n}|f_n - f|^{p_n})(x)\,\nu(dx)\right)^{1/p_n}\\ &= \frac{3e^T}{\varepsilon}\left(\int_E 1_{F_n}|f_n - f|^{p_n}(x)\cdot U_1\nu(x)\,m(dx)\right)^{1/p_n}\\ &\le \frac{3e^T}{\varepsilon}\|f_n - f\|_{L^{p_n}(F_n)}.\end{aligned}$$

Hence (II) tends to 0 as $n\to\infty$ by the assumption (Aa) in Definition 6.2.

Next we consider (I). First we assume (Ab1). Put $\Gamma_k = F_{k+1}\backslash F_k$. Then, by the Chebyshev inequality,

$$\mathrm{(I)} \le \frac{3}{\varepsilon}\mathbb{E}_\nu\left[\sup_{0\le t\le T}\int_0^t 1_{F_n^c}(X_s)f_n(X_s)ds\right] \le \frac{3}{\varepsilon}\sum_{k=n}^\infty \mathbb{E}_\nu\left[\int_0^T 1_{\Gamma_k}(X_s)f_n(X_s)ds\right].$$

We obtain by the Hölder inequality and Lemma 4.2,

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$$\begin{aligned}\mathbb{E}_\nu\left[\int_0^T 1_{\Gamma_k}(X_s)f_n(X_s)ds\right] &\le \mathbb{E}_\nu\left[\int_0^T 1_{\Gamma_k}(X_s)f_n(X_s)^{q_k}ds\right]^{1/q_k}\cdot \mathbb{E}_\nu\left[\int_0^T 1^{q_k'}ds\right]^{1/q_k'}\\ &\le e^{T/q_k}(T\cdot\nu(E))^{1/q_k'}\cdot\mathbb{E}_\nu\left[\int_0^\infty e^{-s}1_{\Gamma_k}(X_s)f_n(X_s)^{q_k}ds\right]^{1/q_k}\\ &\le e^{T/q_k}(T\cdot\nu(E))^{1/q_k'}\left(\int_E 1_{\Gamma_k}(x)f_n(x)^{q_k}U_1\nu(x)\,m(dx)\right)^{1/q_k}\\ &\le C_1\|f_n\|_{L^{q_k}(\Gamma_k)},\end{aligned}$$

where $C_1 := \sup_{k\in\mathbb{N}} e^{T/q_k}T^{1/q_k'} < \infty$ by virtue of $1/q_k,\ 1/q_k' \in (0,1]$. On the other hand, we can also estimate as follows:

$$\begin{aligned}\mathbb{E}_\nu\left[\int_0^T 1_{\Gamma_k}(X_s)f_n(X_s)ds\right] &\le \mathbb{E}_\nu\left[\int_0^T 1_{\Gamma_k}(X_s)f_n(X_s)^{q_k}ds\right]^{1/q_k}\cdot\mathbb{E}_\nu\left[\int_0^T 1_{\Gamma_k}(X_s)\,ds\right]^{1/q_k'}\\ &\le e^{T(1/q_k+1/q_k')}\left(\int_E R_1(1_{\Gamma_k}f_n^{q_k})(x)\,\nu(dx)\right)^{1/q_k}\cdot\left(\int_E R_1(1_{\Gamma_k})(x)\,\nu(dx)\right)^{1/q_k'}\\ &= e^T\left(\int_E 1_{\Gamma_k}(x)f_n^{q_k}(x)U_1\nu(x)\,m(dx)\right)^{1/q_k}\cdot\left(\int_E 1_{\Gamma_k}(x)U_1\nu(x)\,m(dx)\right)^{1/q_k'}\\ &\le e^T\|f_n\|_{L^{q_k}(\Gamma_k)}m(\Gamma_k)^{1/q_k'},\end{aligned}$$

Hence, by (Ab1) we can see the following:

$$\text{(I)} \le \frac{3}{\varepsilon}\left(C_1+e^T\right)\sum_{k=n}^\infty \|f_n\|_{L^{q_k}(\Gamma_k)}\left(1\wedge m(\Gamma_k)^{1/q_k'}\right)\to 0 \quad \text{as } n\to\infty.$$

We next assume (Ab2). By the Chebyshev inequality,

$$\begin{aligned}\text{(I)} &\le \left(\frac{3}{\varepsilon}\right)^{q_n}\mathbb{E}_\nu\left[\left(\int_0^T e^{T-s}1_{F_n^c}(X_s)f_n(X_s)\,ds\right)^{q_n}\right]\\ &\le \left(\frac{3e^T}{\varepsilon}\right)^{q_n}\mathbb{E}_\nu\left[\left(\int_0^\infty e^{-s}1_{F_n^c}(X_s)f_n(X_s)\,ds\right)^{q_n}\right]\\ &\le \left(\frac{3e^T}{\varepsilon}\right)^{q_n}\mathbb{E}_\nu\left[\int_0^\infty e^{-s}1_{F_n^c}(X_s)f_n(X_s)^{q_n}\,ds\right],\end{aligned}$$

where we use the Jensen inequality for the last inequality on account that $e^{-s}ds$ is the probability measure on $(0,\infty)$. Thus we can estimate (I) as follows:

$$\begin{aligned}\text{(I)} &\le \left(\frac{3e^T}{\varepsilon}\right)^{q_n}\int_E R_1(1_{F_n^c}f_n^{q_n})(x)\,\nu(dx) \le \left(\frac{3e^T}{\varepsilon}\right)^{q_n}\int_E 1_{F_n^c}(x)f_n^{q_n}(x)U_1\nu(x)\,m(dx)\\ &\le \left(\frac{3e^T}{\varepsilon}\right)^{q_n}\|f_n\|^{q_n}_{L^{q_n}(F_n^c)} \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

For (III), using the similar arguments to (II) with (Ac2), we can obtain that (III) tends to 0 as $n\to\infty$.

Thus, we know that

$$\lim_{n\to\infty}\mathbb{P}_\nu\left(\sup_{0\le t\le T}|\mathsf{A}_t^n-\mathsf{A}_t|>\varepsilon\right)=0.$$

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Therefore, applying the first Borel–Cantelli lemma and Theorem 2.1, we obtain the theorem. ■

Note that, for a sequence of functions $\{f_n\}$ and f satisfying the conditions of Theorem 6.4, it is not clear in general whether the measures $\{\mu_n\} := \{f_n m\}$ and $\mu := fm$ satisfy the conditions of Theorem 5.1 or not.

We now consider the special case that, for a sequence of densities $\{f_n\}$ and f, the functions f_n converges to f in $L^p(E;m)$ *strongly* for some p $(1 \le p < \infty)$. In this case, we can directly prove the convergence of PCAFs, not using the class $\mathfrak{A}$ and an increasing sequence of closed sets $\{F_n\}$. Also we would like to emphasize that the convergence of PCAFs holds by Theorem 4.1.

Proposition 6.5 *Let $1 \le p < \infty$ and $f_n, f \in L^p(E;m) \cap \mathcal{B}_+(E)$ for $n \in \mathbb{N}$. Put $\mu_n = f_n m$ and $\mu = fm$. Assume that $f_n \to f$ in L^p. Then there exists a subsequence $\{n_k\}$ such that*

$$\mathbb{P}_x\left(\int_0^t f_{n_k}(X_s)ds \text{ converges to } \int_0^t f(X_s)ds \text{ locally uniformly in } t \text{ on } [0,\infty)\right) = 1, \quad \text{q.e. } x \in E.$$

Proof Assume that $1 \le p < \infty$ and take any $\nu \in \mathcal{S}^1_{c,00}$.

Case I. $2 < p < \infty$: For any $\varepsilon > 0$ and $T > 0$, by making use of the Chebyshev inequality and then the Jensen inequality (since $p/2 > 1$), we have

$$\begin{aligned}
\mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|A^n_t - A_t\right| > \varepsilon\right) &\le \mathbb{P}_\nu\left(\int_0^T \left|f_n(X_s) - f(X_s)\right|ds > \varepsilon\right) \\
&\le \frac{1}{\varepsilon^{p/2}}\mathbb{E}_\nu\left[\left(\int_0^\infty \left|f_n(X_s) - f(X_s)\right|\right)^{p/2}\right] \\
&\le \frac{T^{p/2-1}}{\varepsilon^{p/2}}\mathbb{E}_\nu\left[\int_0^T \left|f_n(X_s) - f(X_s)\right|^{p/2}ds\right] \\
&\le \frac{e^T T^{p/2-1}}{\varepsilon^{p/2}}\mathbb{E}_\nu\left[\int_0^\infty e^{-s}\left|f_n(X_s) - f(X_s)\right|^{p/2}ds\right] \\
&= C_2\mathcal{E}_1(U_1\nu, R_1(|f_n - f|^{p/2})),
\end{aligned}$$

where $C_2 := e^T T^{p/2-1}/\varepsilon^{p/2}$. Note here that $|f_n - f|^{p/2} \in L^2(E;m)$ since f_n and f belong to $L^p(E;m)$. So, by the Schwarz inequality,

$$\begin{aligned}
\mathbb{P}_\nu\left(\sup_{0\le t\le T}\left|A^n_t - A_t\right| > \varepsilon\right) &\le C_2\mathcal{E}_1(U_1\nu, R_1(|f_n - f|^{p/2})) = C_2\int_E \left|f_n - f\right|^{p/2}(x)\widetilde{U_1\nu}(x)m(dx) \\
&\le C_2\sqrt{\int_E \left|f_n - f\right|^p(x)m(dx)}\sqrt{\int_E \widetilde{U_1\nu}(x)^2 m(dx)} \\
&= C_2\|f_n - f\|_p^{p/2}\sqrt{\mathcal{E}_1(R_1(U_1\nu), U_1\nu)} \\
&= C_2\|f_n - f\|_p^{p/2}\sqrt{\int_E \widetilde{R_1(U_1\nu)}(x)\nu(dx)} \\
&\le C_2\|f_n - f\|_p^{p/2}\sqrt{\nu(E)} \le C_2\|f_n - f\|_p^{p/2}
\end{aligned}$$

and the last term goes to 0 as $n \to \infty$ (uniformly in ν).

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Case II. $1 \le p \le 2$: For any $\varepsilon > 0$ and $T > 0$, similar to the previous case, we see from the Chebyshev and Jensen inequality and (4.1),

$$\begin{aligned}
\mathbb{P}_v\left(\sup_{0\le t\le T}\left|A_t^n - A_t\right| > \varepsilon\right) &\le \mathbb{P}_v\left(\int_0^T \left|f_n(X_s) - f(X_s)\right|ds > \varepsilon\right) \\
&\le \frac{1}{\varepsilon^p}\mathbb{E}_v\left[\left(\int_0^T \left|f_n(X_s) - f(X_s)\right|ds\right)^p\right] \\
&\le \frac{T^{p-1}e^T}{\varepsilon^p}\mathbb{E}_v\left[\int_0^\infty e^{-s}\left|f_n(X_s) - f(X_s)\right|^p ds\right] \\
&= \frac{T^{p-1}e^T}{\varepsilon^p}\int_E \left|f_n - f\right|^p(x)U_1v(x)m(dx) \\
&\le \frac{T^{p-1}e^T}{\varepsilon}\|f_n - f\|_p^p \to 0 \quad \text{as } n\to\infty \text{ (uniformly in } v).
\end{aligned}$$

■

Remark 6.6 If f_n converges to f in $L^2(E;m)$, the 1-resolvent R_1f_n also converges to R_1f in $L^2(E;m)$. This means that f_nm converges to fm with respect to ρ-metric. Hence by Lemma 4.6, we can obtain Proposition 6.5 for $p = 2$.

7 Some examples and remarks

7.1 A class of absolutely continuous smooth measures

As mentioned in Remark 6.3, we can construct approximating sequences $\{f_n\}$ and $\{F_n\}$ for $f \in L^p(E;m) \cap \mathcal{B}_+(E)$ $(1 \le p < \infty)$. This means that $L^p(E;m) \cap \mathcal{B}_+(E) \subset \mathfrak{A}$ for $1 \le p < \infty$. In this subsection, we introduce several examples of functions having some singularities so that they satisfy Definition 6.2. Hence for such examples we obtain the compactness of Revuz map by Theorem 6.4. To this end, let $\mathsf{M} = (\mathbb{P}_x, X_t)$ be the Brownian motion on $\mathbb{R}^d$ and $m(dx) = dx$ the Lebesgue measure on $\mathbb{R}^d$ $(d \ge 2)$. We start with a simple example.

Example 7.1 Assume $\beta \in \mathbb{R}$. Let

$$f(x) = |x|^{-\beta}, \quad x \in \mathbb{R}^d.$$

Put $K_n = \{x \in \mathbb{R}^d : 1/n \le |x| \le n\}$ for $n \in \mathbb{N}$. Then we see that $\{K_n\}$ is an increasing sequence of compact sets so that $\bigcup_{n=1}^\infty K_n = \mathbb{R}^d$ *q.e.* and $f \in L^1(K_n)$ for $n \ge 1$. So $f \in \dot{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, and $fm = |x|^{-\beta}m \in \mathcal{S}$ by Lemma 6.1 (see also [7, p. 239, Example 5.1.1]). Of course we know $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ if $\beta < d$.

On the other hand, assume $0 < \beta < d$. Then $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ but $f \notin L^p(\mathbb{R}^d)$ for any $p \in [1,\infty]$. We now put $F_n = \{x \in \mathbb{R}^d : |x| \le n\}$ and set $f_n = 1_{F_n}f$ for $n \ge 1$. Noting that f, $f_n \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{B}_+(\mathbb{R}^d)$ and $\mathbb{R}^d = \bigcup_{n=1}^\infty F_n$, we see that $f \in \mathfrak{A}$ and $\{f_n\}$ is an $\mathfrak{A}$-approximating sequence of f with $\{F_n\}$ as an increasing sequence of closed sets. Indeed, (Aa) (resp. (Ab1) and (Ab2)) is satisfied for any $p_n \in [1,\infty)$ (resp. for any $q_n \in [1,\infty)$ and $C > 1$) since $f - f_n = 0$ on F_n (resp. $f_n = 0$ on $\mathbb{R}^d \setminus F_n$) hold for all $n \ge 1$.

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As for (Ac), set $r_n = \frac{d+1}{\beta}$ (> 1) for $n \geq 1$. Then it follows that

$$C^{r_n} \|f\|^{r_n}_{L^{r_n}(\mathbb{R}^d \setminus F_n)} = C^{(d+1)/\beta} \int_{|x| \geq n} |x|^{-d-1} dx \to 0 \quad (n \to \infty)$$

for any $C > 1$.

We introduce another example for which the approximating sequence of parameters varies ($\{r_n\}$, in this case).

Example 7.2 We put

$$f(x) = |x|^{-d} (\log |x|)^{-2} 1_{B(0,1/e)}(x) + \sum_{n=1}^{\infty} |x|^{-d+1/\log(n+1)} 1_{K_n}(x), \quad x \in \mathbb{R}^d,$$

where $K_n = \{x \in \mathbb{R} : n \leq |x| < n+1\}$ for $n \geq 1$. Then $f \notin L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$ but $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathfrak{A}$. We first show that $f \notin L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$. When $p = 1$,

$$\begin{aligned} \|f\|_{L^1} &\geq \sum_{n=1}^{\infty} \int_{n \leq |x| < n+1} |x|^{-d+1/\log(n+1)} dx \\ &= c_d \sum_{n=1}^{\infty} \int_n^{n+1} u^{-1-1/\log(n+1)} du \\ &\geq c_d \sum_{n=1}^{\infty} (n+1)^{-1} \underbrace{(n+1)^{-1/\log(n+1)}}_{\text{bounded in } n} = \infty. \end{aligned}$$

When $1 < p < \infty$,

$$\begin{aligned} \|f\|^p_{L^p} &\geq \int_{|x|<1/e} |x|^{-pd} \big(\log |x|\big)^{-2p} dx = c_d \int_0^{1/e} u^{-d(p-1)-1} (\log u)^{-2p} du \quad (t = \log u) \\ &= c_d \int_1^{\infty} e^{d(p-1)t} t^{-2p} dt = \infty. \end{aligned}$$

As for the locally integrability of f, it is enough to see the integrability around the origin:

$$\int_{B(0,1/e)} |x|^{-d} \big(\log |x|\big)^{-2} dx = c_d \int_0^{1/e} u^{-1} (\log u)^{-2} du = c_d \int_1^{\infty} t^{-2} dt < \infty \quad (t = -\log u).$$

We now show $f \in \mathfrak{A}$. To this end, put $F_n = \{x \in \mathbb{R}^d : |x| \leq n\}$, $p_n = q_n = 1$, $r_n = \log(n+1)$ and set $f_n = f 1_{F_n}$ for $n \geq 1$. As in the previous examples, we see that $\|f - f_n\|_{L^{p_n}(F_n)} = \|f_n\|_{L^{q_n}(F_n^c)} = 0$ for any $n \geq 1$, hence (Aa) and (Ab2) hold. We finally show (Ac1). Let $C > 1$. Then for any $n \geq 2$,

$$\begin{aligned} \sum_{k=n}^{\infty} \|f\|_{L^{r_k}(F_{k+1} \setminus F_k)} &= \sum_{k=n}^{\infty} \Big(\int_{k \leq |x| < k+1} |x|^{-d \log(k+1)+1} dx \Big)^{1/\log(k+1)} \\ &= \sum_{k=n}^{\infty} \Big(c_d \int_k^{k+1} u^{-d(\log(k+1)-1)} du \Big)^{1/\log(k+1)} \\ &\leq \sum_{k=n}^{\infty} k^{-d} \underbrace{(c_d k^d)^{1/\log(k+1)}}_{\text{bounded in } k} \longrightarrow 0 \quad (n \to \infty) \end{aligned}$$

because $d \geq 2$.

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We noted in Remark 6.3 that the condition (Ab1) (resp. (Ab2)) is not necessarily stronger than (Ab2) (resp. (Ab1)) and so are the conditions (Ac1) and (Ac2). The following example shows such a situation indeed happens for the conditions (Ac1) and (Ac2).

Example 7.3

(i) Set $f(x) = |x|^{-1}1_{B(0,1)^c}(x),\ x \in \mathbb{R}^d$. Then we find that the condition (Ac2) holds but not (Ac1) for $r_n = d+1$ and $F_n = \{|x| \le n\}$. In fact, similar to (7.1), we see that, for any $C > 1$ and $n \ge 1$,

$$C^{r_n}\|f\|^{r_n}_{L^{r_n}(F_n^c)} = C^{d+1}\int_{|x|\ge n}|x|^{-d-1}dx = c_d C^{d+1}\int_n^\infty u^{-2}du \to 0 \quad (n\to\infty)$$

holds, and then (Ac2) is satisfied. As a result, it follows $f \in \mathfrak{A}$. But we see (Ac1) fails since, for any $n \ge 1$,

$$\begin{aligned}\sum_{k=n}^\infty \|f\|_{L^{r_k}(F_{k+1}\setminus F_k)} &= \sum_{k=n}^\infty \Big(\int_{k<|x|\le k+1}|x|^{-d-1}dx\Big)^{1/(d+1)}\\ &= \sum_{k=n}^\infty \Big(\frac{c_d}{k(k+1)}\Big)^{1/(d+1)} \ge c_d^{1/(d+1)}\sum_{k=n}^\infty (k+1)^{-2/(d+1)} = \infty\end{aligned}$$

and $m(F_{k+1}\setminus F_k) = \mathrm{vol}(\{k<|x|\le k+1\}) \ge c_d k^{d-1} > 1$ for $k \ge 1$.

(ii) Put $f(x) = \sum_{k=1}^\infty |x|^d 1_{K_{2k-1}}(x) + \sum_{k=1}^\infty |x|^{-d}1_{K_{2n}}(x),\ x\in\mathbb{R}^d$, where

$$\begin{cases} K_{2k-1} &:= \Big\{x\in\mathbb{R}^d :\ (2k-1)^{1/d} < |x| \le \big(2k-1+(2k-1)^{-3}\big)^{1/d}\Big\},\\ K_{2k} &:= \Big\{x\in\mathbb{R}^d :\ (2k)^{1/d} < |x| \le \big(2k+(2k)^{-\delta\log(2k+2)}\big)^{1/d}\Big\}\end{cases}$$

for $k \ge 1$ and for some $\delta > 1$. Then, for $F_n = \{|x| \le n^{1/d}\}$ and $r_n = \begin{cases}\log(n+1) & \text{if } n \text{ is even},\\ 1 & \text{if } n \text{ is odd},\end{cases}$ we see that the condition (Ac1) holds but (Ac2) fails. Take any $C > 1$ and any even number $n \ge 1$, namely $n = 2m$ for any $m \ge 1$, then we see that

$$\begin{aligned}C^{r_n}\|f\|^{r_n}_{L^{r_n}(F_n^c)} &= C^{\log(2m+1)}\int_{|x|>(2m)^{1/d}}|f(x)|^{\log(2m+2)}dx\\ &\ge C^{\log(2m+1)}\sum_{\substack{k\ge 2m\\ k=2\ell-1:\text{odd}}}\ \int_{k^{1/d}<|x|\le(k+1/k^3)^{1/d}}|x|^{d\log(2m+2)}dx\\ &\ge c_d C^{\log(2m+1)}\sum_{\ell=m+1}^\infty (2\ell-1)^{\log(2m+2)-3} = \infty.\end{aligned}$$

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Thus this means (Ac2) fails. On the other hand, for $n \geq 2$,

$$\begin{aligned}\sum_{k=n}^{\infty} \|f\|_{L^{r_k}(F_{k+1}\setminus F_k)} &= \sum_{k=n}^{\infty} \Big(\int_{k^{1/d}<|x|\leq(k+1)^{1/d}} |f(x)|^{r_k} dx \Big)^{1/r_k} \\ &= \sum_{\substack{k\geq n \\ k=2m}}^{\infty} \Big(\int_{(2m)^{1/d}<|x|\leq(2m+1)^{1/d}} |f(x)|^{r_{2m}} dx \Big)^{1/r_{2m}} \\ &\quad + \sum_{\substack{k\geq n \\ k=2m-1}}^{\infty} \Big(\int_{(2m-1)^{1/d}<|x|\leq(2m)^{1/d}} |f(x)|^{r_{2m-1}} dx \Big)^{1/r_{2m-1}} \\ &= \sum_{\substack{k\geq n \\ k=2m}}^{\infty} \Big(\int_{K_{2m}} |x|^{-d\log(2m+2)} dx \Big)^{1/\log(2m+2)} + \sum_{\substack{k\geq n \\ k=2m-1}}^{\infty} \Big(\int_{K_{2m-1}} |x|^{d} dx \Big) \\ &\leq c_d \sum_{m=[n/2]}^{\infty} (2m)^{-\delta-1} + c_d \sum_{m=[(n+1)/2]}^{\infty} (2m+1)\cdot(2m-1)^{-3}\end{aligned}$$

goes to 0 when $n \to \infty$, and hence, (Ac1) holds. We also see that f belongs to $\mathfrak{A}$.

7.2 Remarks

In the following, we denote by A_t^{μ} the PCAF in the Revuz correspondence for $\mu \in \mathcal{S}$. If $\mu(dx) = f(x)m(dx)$ for some $f \in \dot{L}^1_{\mathrm{loc}}(E;m) \cap \mathcal{B}_+(E)$, then $\mu \in \mathcal{S}$ and $\mathrm{A}_t^{\mu} = \int_0^t f(X_s)ds$ by Lemma 6.1. In this subsection, we present two cases in which the corresponding PCAF A_t^{μ} can be concretely expressed when $\mu \in \mathcal{S}$ is not necessarily absolutely continuous with respect to m.

7.2.1 ODDP

Let $\mathrm{X} = [\Omega^{\mathrm{X}}, X_t, \mathbb{P}^{\mathrm{X}}_x, \zeta^{\mathrm{X}}]$ be an ODDP on an open interval $I = (l_1, l_2)$ with no killing inside I. It is well known that X is characterized by a scale function s^{X} and a speed measure m^{X}, and a boundary condition (reflecting or absorbing) at l_i if it is regular. s^{X} is a strictly increasing continuous function on I and m^{X} is a positive Radon measure on I with $\mathrm{supp}[m^{\mathrm{X}}] = I$. m^{X} is identified with strictly increasing right continuous function induced by m^{X} (cf. [9]).

We consider the following symmetric bilinear form $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$.

$$\mathcal{E}^{\mathrm{X}}(u,v) = \int_I \frac{du}{ds^{\mathrm{X}}} \frac{dv}{ds^{\mathrm{X}}}\, ds^{\mathrm{X}},$$
$$\mathcal{F}^{\mathrm{X}} = \Big\{ u \in L^2(I^*; m^{\mathrm{X}}) : u \text{ satisfies (7.2.1), (7.2.1), and (7.2.1)} \Big\},$$

where I^* is the interval extended by adding l_i to I if l_i is regular (i.e., $|s^{\mathrm{X}}(l_i)| + |m^{\mathrm{X}}(l_i)| < \infty$) with reflecting boundary condition, and m^{X} is extended to I^* so that $m^{\mathrm{X}}(I^*\setminus I) = 0$:

(7.1) $u(x)$ is absolutely continuous with respect to $s^{\mathrm{X}}(x)$ on I,

(7.2) $$\frac{du}{ds^{\mathrm{X}}} \in L^2(I; s^{\mathrm{X}}), \qquad u(b) - u(a) = \int_{(a,b)} \frac{du}{ds^{\mathrm{X}}}\, ds^{\mathrm{X}} \quad (a, b \in I),$$

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$$\lim_{x\to l_i} u(x) = 0 \text{ if } |s^{\mathrm{X}}(l_i)|$$

(7.3) $< \infty$ but l_i is not regular with reflecting boundary condition $(i = 1, 2)$.

Then $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$ is a regular, strongly local, irreducible Dirichlet form on $L^2(I^*; m^{\mathrm{X}})$ corresponding to the ODDP X, and each one point of I^* has a positive capacity relative to $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$ ([2], [4], [5]). We denote the corresponding items for $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$ by $\mathrm{Cap}^{\mathrm{X}}$, $\mathcal{S}^{\mathrm{X}}$, $\mathcal{S}_0^{\mathrm{X}}$, $\mathcal{S}_{00}^{\mathrm{X}}$, etc. In the same way as in [7, p. 75, Example 2.1.2], we obtain the following:

$$\mathrm{Cap}^{\mathrm{X}}(\{y\}) = 1/g_1^{\mathrm{X}}(y, y)\ (> 0),\ y \in I^*, \tag{7.4}$$

where $g_1^{\mathrm{X}}(x, y)$ is the resolvent kernel corresponding to $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$. We also note that the set $\{u(s^{\mathrm{X}}(\cdot)) : u \in C_c^1(J^*)\}$ is a core of $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$, where $J^* = s^{\mathrm{X}}(I^*)$ and $C_c^1(A)$ is the set of all C^1-functions on A with compact support.

Let μ be a positive Borel measure on I^*. Noting (7.4) and following the same argument as in [7, p. 75, Example 2.1.2], we see the following.

(7.5) $\mu \in \mathcal{S}^{\mathrm{X}}$ if and only if μ is a positive Radon measure on I^*.

By virtue of [7, p. 166, Exercise 4.2.2], we see that

$$\mu \in \mathcal{S}_0^{\mathrm{X}} \text{ if and only if } \iint_{I^*\times I^*} g_1^{\mathrm{X}}(x, y)\mu(dx)\mu(dy) < \infty, \tag{7.6}$$

and, in this case,

$$R_1^{\mathrm{X}}\mu(x) := \int_{I^*} g_1^{\mathrm{X}}(x, y)\mu(dy)$$

is a quasi continuous and 1-excessive version of $U_1^{\mathrm{X}}\mu$. By means of (7.5) and (7.6), we get $\mathcal{S}_{00}^{\mathrm{X}} = \mathcal{S}_0^{\mathrm{X}} = \mathcal{S}^{\mathrm{X}}$ if $|s^{\mathrm{X}}(l_i)| + |m^{\mathrm{X}}(l_i)| < \infty$ and l_i is reflecting for $i = 1$ and 2.

We assume that X is conservative, that is, $\zeta^{\mathrm{X}} = \infty$. Let $\mu \in \mathcal{S}^{\mathrm{X}}$. Employing [7, p. 230, Theorem 5.1.4], we have

$$\mathrm{A}_t^{\mu} = \int_{I^*} \ell^{\mathrm{X}}(t, x)\mu(dx), \tag{7.7}$$

where $\ell^{\mathrm{X}}(t, x)$ is the local time of X and it is continuous with respect to $(t, x) \in [0, \infty) \times I^*$ and satisfies $\int_0^t 1_A(X_u)\, du = \int_A \ell^{\mathrm{X}}(t, x)\, m^{\mathrm{X}}(dx)$, $t > 0$, for every measurable set $A \subset I$, where 1_A is the indicator for a set A ([9]). The relation (7.7) for one-dimensional Brownian motion is obtained in [7, p. 238, Example 5.1.1]. Let μ_n, $\mu \in \mathcal{S}^{\mathrm{X}}$ and assume $\mu_n \to \mu$ vaguely as $n \to \infty$. By virtue of (7.7), we obtain the following.

$$\mathbb{P}_x^{\mathrm{X}}\left(\lim_{n\to\infty} \mathrm{A}_t^{\mu_n} = \mathrm{A}_t^{\mu} \text{ locally uniformly in } t \text{ on } [0, \infty)\right) = 1, \quad x \in I^*.$$

7.2.2 Skew product diffusion processes

Let $\mathrm{X} = [\Omega^{\mathrm{X}}, X_t, \mathbb{P}_x^{\mathrm{X}}, \zeta^{\mathrm{X}}]$ be the ODDP as above, where $0 = l_1 < l_2 \le \infty$ and $l_1 = 0$ is entrance in the sense of Feller, that is, $\int_{(0,c)} m^{\mathrm{X}}(dx) \int_{(x,c)} s^{\mathrm{X}}(dy) < \infty$ and $\int_{(0,c)} s^{\mathrm{X}}(dx) \int_{(x,c)} m^{\mathrm{X}}(dy) = \infty$ for $0 < c < l_2$. Let $\Theta = [\Omega^{\Theta}, \Theta_t, \mathbb{P}_{\theta}^{\Theta}]$ be the spherical Brownian motion on $S^{d-1} \subset \mathbb{R}^d$ with generator $\frac{1}{2}\Delta$, Δ being the spherical Laplacian

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on S^{d-1}. The corresponding Dirichlet form $(\mathcal{E}^\Theta, \mathcal{F}^\Theta)$ on $L^2(S^{d-1}, m^\Theta)$ is obtained by Fukushima and Oshima [6], where m^Θ is the spherical measure on S^{d-1}. Let ν be a Radon measure on I and assume that $\mathrm{supp}[\nu] = I := (l_1, l_2)$. Furthermore, we assume that $\int_{(0,c)} |s^X(x)|\nu(dx) = \infty$, and $\int_{(c,l_2)} \nu(dx) < \infty$ whenever $s^X(l_2) + m^X(l_2) < \infty$. We set

$$\mathbf{f}(t) = \begin{cases} \displaystyle\int_I \ell^X(t,x)\,\nu(dx), & t < \zeta^X, \\ \infty, & t \geq \zeta^X. \end{cases}$$

Put $\Lambda := I^* \times S^{d-1}$ and let $\Xi = [\Xi_t = (X_t, \Theta_{\mathbf{f}(t)}), \mathbb{P}^\Xi_\xi = \mathbb{P}^X_x \otimes \mathbb{P}^\Theta_\theta,\ \xi = (x,\theta) \in \Lambda, \zeta^\Xi]$ be the skew product of the ODDP X and the spherical Brownian motion Θ with respect to the PCAF $\{\mathbf{f}(t)\}$. Let us consider the following bilinear form.

$$\mathcal{E}^\Xi(f,g) = \int_{S^{d-1}} \mathcal{E}^X(f(\cdot,\theta), g(\cdot,\theta))\, m^\Theta(d\theta) + \int_I \mathcal{E}^\Theta(f(x,\cdot), g(x,\cdot))\,\nu(dx),$$

for $f, g \in \mathcal{C}^\Xi$, where $\mathcal{C}^\Xi$ is the set of all linear combinations of $u^{(1)}(s^X(x))u^{(2)}(\theta)$ with $u^{(1)} \in C_0^1(J^*)$ and $u^{(2)} \in C^\infty(S^{d-1})$, where $J^* = s^X(I^*)$. We put $m^\Xi = m^X \otimes m^\Theta$. By using some results from [6] and [11], we see that the form $(\mathcal{E}^\Xi, \mathcal{C}^\Xi)$ is closable on $L^2(\Lambda; m^\Xi)$, and the closure $(\mathcal{E}^\Xi, \mathcal{F}^\Xi)$ is a regular irreducible Dirichlet form and it is corresponding to the skew product Ξ. We denote the corresponding items for $(\mathcal{E}^\Xi, \mathcal{F}^\Xi)$ by $\mathcal{S}^\Xi$, $\mathcal{S}_0^\Xi$, $\mathcal{S}_{00}^\Xi$, etc. We show the following.

Proposition 7.4 *Let μ^X be a positive Radon measure on I^*. Put $\mu^\Xi(dx, d\theta) = \mu^X(dx)m_o(\theta)m^\Theta(d\theta)$, where m_o is a bounded measurable function on S^{d-1}. Then the following hold*:

(1) *$\mu^\Xi \in \mathcal{S}^\Xi$. In particular, if X is conservative and $m_o = 1$, then*

$$A_t^{\mu^\Xi} = \int_{I^*} \ell^X(t,x)\mu^X(dx), \qquad \mathbb{P}^\Xi_\xi\text{-a.e.} \quad \xi \in \Lambda. \tag{7.8}$$

(2) *If $\mu^X \in \mathcal{S}_0^X$ (resp. $\mathcal{S}_{00}^X$), then $\mu^\Xi \in \mathcal{S}_0^\Xi$ (resp. $\mathcal{S}_{00}^\Xi$).*

We show this by means of the following lemma which can be easily shown from the fact that $\mathcal{C}^\Xi$ is a core of $(\mathcal{E}^\Xi, \mathcal{F}^\Xi)$, so we will omit its proof.

Lemma 7.5 *Let V be a function on Λ Put $v(x) = \int_{S^{d-1}} V(x,\theta)m_o(\theta)m^\Theta(d\theta)$.*

(1) *If $V \in \mathcal{F}^\Xi$ (resp. $C_0(\Lambda) \cap \mathcal{F}^\Xi$), then $v \in \mathcal{F}^X$ (resp. $C_0(I^*) \cap \mathcal{F}^X$), and*

$$\mathcal{E}^X(v,v) \leq c_d \|m_o\|^2_{L^\infty(S^{d-1})} \mathcal{E}^\Xi(V,V), \qquad \|v\|^2_{L^2(I^*,m^X)} \leq c_d \|m_o\|^2_{L^\infty(S^{d-1})} \|V\|^2_{L^2(\Lambda,m^\Xi)},$$

where c_d denotes the surface measure of S^{d-1}.

(2) *Let $u \in \mathcal{F}^X$. Then $U(x,\theta) := u(x) \in \mathcal{F}^\Xi$ and $\mathcal{E}_1^\Xi(U,U) = c_d\mathcal{E}_1^X(u,u)$. In particular, if $V \in \mathcal{F}^\Xi$ and $m_o = 1$, then $\mathcal{E}_1^\Xi(U,V) = \mathcal{E}_1^X(u,v)$.*

Proof of Proposition 7.4 (step 1) Let $\mu^X \in \mathcal{S}_0^X$ and put $\mu^\Xi(dx, d\theta) = \mu^X(dx)m_o(\theta)m^\Theta(d\theta)$. By means of Lemma 7.5, we get

$$\int_\Lambda |V(x,\theta)|\mu^\Xi(dx,d\theta) \leq C\sqrt{\mathcal{E}_1^\Xi(V,V)}, \quad V \in C_0(\Lambda) \cap \mathcal{F}^\Xi,$$

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for some positive constant C, which implies $\mu^{\Xi} \in \mathcal{S}_0^{\Xi}$. Assume that X is conservative and $m_o = 1$. We put $C_t(\omega, \omega') = \int_{I^*} \ell^{\mathrm{X}}(t, x; \omega)\mu^{\mathrm{X}}(dx)$ for $\omega \in \Omega^{\mathrm{X}}$, $\omega' \in \Omega^{\Theta}$. We show

$$\mathbb{P}_{\xi}^{\Xi}(\mathrm{A}_t^{\mu^{\Xi}} = C_t) = 1, \ \xi \in \Lambda. \tag{7.9}$$

Let $\xi = (x, \theta) \in \Lambda$. Noting [9, Section 5.4, (3)],

$$\begin{aligned} E^{\mathbb{P}_{\xi}^{\Xi}}\left[\int_0^{\infty} e^{-t} dC_t\right] &= \int_0^{\infty} e^{-t} E^{\mathbb{P}_{\xi}^{\Xi}}[C_t]\, dt \\ &= \int_0^{\infty} e^{-t} E^{\mathbb{P}_x^{\mathrm{X}}}[C_t]\, dt = \int_0^{\infty} e^{-t} dt \int_{I^*} E^{\mathbb{P}_x^{\mathrm{X}}}[\ell^{\mathrm{X}}(t, y)]\mu^{\mathrm{X}}(dy) \\ &= \int_0^{\infty} e^{-t} dt \int_{I^*} \mu^{\mathrm{X}}(dy) \int_0^t p^{\mathrm{X}}(s, x, y) ds \\ &= \int_{I^*} \mu^{\mathrm{X}}(dy) \int_0^{\infty} e^{-s} p^{\mathrm{X}}(s, x, y) ds \\ &= \int_{I^*} g_1^{\mathrm{X}}(x, y)\mu^{\mathrm{X}}(dy) = (R_1^{\mathrm{X}}\mu^{\mathrm{X}})(x) = (U_1^{\mathrm{X}}\mu^{\mathrm{X}})(x) = U_1^{\Xi}\mu^{\Xi}(\xi), \end{aligned}$$

where the last equality follows from Lemma 7.5. Combining this with [7, p. 226, Theorem 5.1.2], we get (7.9).

(step 2) Let $\mu^{\mathrm{X}} \in \mathcal{S}_{00}^{\mathrm{X}}$. By means of (step 1), $\mu^{\Xi} \in \mathcal{S}_0^{\Xi}$. Then

$$\mu^{\Xi}(\Lambda) \le \mu^{\mathrm{X}}(I^*)\|m_o\|_{L^{\infty}(S^{d-1}; m^{\Theta})} c_d < \infty.$$

Note $U_1^{\Xi}\mu^{\Xi} \le \|m_o\|_{L^{\infty}(\Lambda; m^{\Xi})} U_1^{\Xi}\mu_o^{\Xi}$ m^{Ξ}-a.e., where $\mu_o^{\Xi}(dxd\theta) = \mu^{\mathrm{X}}(dx)m^{\Theta}(d\theta)$. As in (step 1), $(U_1^{\mathrm{X}}\mu^{\mathrm{X}})(x) = (U_1^{\Xi}\mu_o^{\Xi})(x, \theta)$ and hence $\|U_1^{\Xi}\mu^{\Xi}\|_{L^{\infty}(\Lambda; m^{\Xi})} \le \|m_o\|_{L^{\infty}(\Lambda; m^{\Xi})}\|U_1^{\mathrm{X}}\mu^{\mathrm{X}}\|_{L^{\infty}(I^*; m^{\mathrm{X}})} < \infty$. Therefore $\mu^{\Xi} \in \mathcal{S}_{00}^{\Xi}$.

(step 3) Let $\mu^{\mathrm{X}} \in \mathcal{S}^{\mathrm{X}}$. By means of [7, p. 85, Theorem 2.2.4], there exists a generalized compact nest $\{F_n\}$ satisfying $\mu^{\mathrm{X}}(I^* \setminus \bigcup_{n=1}^{\infty} F_n) = 0$ and $\mu_n^{\mathrm{X}} := 1_{F_n}\mu^{\mathrm{X}} \in \mathcal{S}_{00}^{\mathrm{X}}$ for each n. Put $F_n^{\Xi} = F_n \times S^{d-1}$. Then $\{F_n^{\Xi}\}$ is a generalized compact nest and $\mu^{\Xi}(\Lambda \setminus \bigcup_{n=1}^{\infty} F_n^{\Xi}) = 0$. By (step 2), $\mu_n^{\Xi}(dxd\theta) := 1_{F_n^{\Xi}}(x, \theta)\mu^{\Xi}(dxd\theta) = 1_{F_n}(x)\mu^{\mathrm{X}}(dx)m_o(\theta)\mu^{\Theta}(d\theta) \in \mathcal{S}_{00}^{\Xi}$. Therefore $\mu^{\Xi} \in \mathcal{S}^{\Xi}$ by means of [7, p. 85, Theorem 2.2.4].

Assume that X is conservative and $m_0 = 1$. We show (7.8). We first note that, by (step 1) and (step 2),

$$\mathrm{A}_t^{\mu_n^{\Xi}} = \int_{F_n} \ell^{\mathrm{X}}(t, x)\, \mu^{\mathrm{X}}(dx) \to C_t := \int_{I^*} \ell^{\mathrm{X}}(t, x)\mu^{\mathrm{X}}(dx),$$

uniformly in $t \in [0, T]$, $T \in (0, \infty)$ as $n \to \infty$ $\mathbb{P}_{\xi}^{\Xi}$-a.s., $\xi \in \Lambda$. Let f be a bounded continuous function on Λ and $h \in \mathcal{B}^+(\Lambda)$.

$$\begin{aligned} \int_{\Lambda} h(\xi)m^{\Xi}(d\xi)\left[\int_0^t f(\Xi_s) dC_s\right] &= \lim_{n\to\infty} \int_{\Lambda} h(\xi)m^{\Xi}(d\xi)\left[\int_0^t f(\Xi_s) dA_s^{\mu_n^{\Xi}}\right] \\ &= \lim_{n\to\infty} \int_0^t \langle f \cdot \mu_n^{\Xi}, p_s^{\Xi} h\rangle ds = \int_0^t \langle f \cdot \mu^{\Xi}, p_s^{\Xi} h\rangle ds. \end{aligned}$$

This shows (7.8). ∎

By means of (7.8), we get the following.

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Proposition 7.6 *Let μ_n^X, μ^X be positive Radon measures on I^* and put $\mu_n^\Xi(dx, d\theta) = \mu_n^X(dx)m^\Theta(d\theta)$, $\mu^\Xi(dx, d\theta) = \mu^X(dx)m^\Theta(d\theta)$. Assume that X is conservative and $\mu_n^X \to \mu^X$ vaguely as $n \to \infty$. Then*

$$\mathbb{P}_\xi^\Xi\left(\lim_{n\to\infty} A_t^{\mu_n^\Xi} = A_t^{\mu^\Xi} \text{ locally uniformly in } t \text{ on } [0,\infty)\right) = 1, \quad \xi \in \Lambda.$$

Remark 7.7 We stress here that, in cases of the ODDP and the skew product diffusion process, without taking a subsequence, the full sequence of PCAFs converge, on account of the fact that the PCAFs are expressed directly as the integrals of the local time with respect to their smooth measures.

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