9

Sharper Edges

In this chapter we briefly comment on phase shift functions having sharper edges than those considered in Chapters 7 and 8. As we shall see, some qualitatively new features appear.

9.1 Quartic Edge

Consider the phase shift function

$$\chi_4(b_x, b_y) = A_4 \exp(-b^4/\beta^4), \tag{9.1}$$

with

$$X_4(b_x) = A_4 \exp(-b_x^4/\beta^4). \tag{9.2}$$

The stationary points will be given by

$$X_4'(b_x) = -4A_4(b_x^3/\beta^4) \exp(-b_x^4/\beta^4) = q. \tag{9.3}$$

Let A_4 be purely imaginary. The loci of stationary points, which satisfy $\operatorname{Im} X_4'(b_x) = 0$, $\operatorname{Re} X_4'(b_x) > 0$, are given in Fig. 9.1. It is instructive to compare with the Gaussian phase shift function of Section 7.1. As was the case with the Gaussian function, when $q \to 0$, the solutions satisfy

$$\operatorname{Re} b_{j\,x} \to \pm\infty, \quad \operatorname{Im} b_{j\,x} \to 0, \quad q \to 0, \tag{9.4}$$

i.e., they come in from infinity, along the real axes. However, as $q \to \infty$, the points of stationary phase move out along the asymptotes

$$b_{\alpha\,x} \sim |b_{\alpha\,x}| \exp(-3i\pi/4), \quad b_{\beta\,x} \sim |b_{\beta\,x}| \exp(-i\pi/4), \quad q \to \infty. \tag{9.5}$$

The reason for this movement is that this is where the argument of the exponential attains a large and positive real part.

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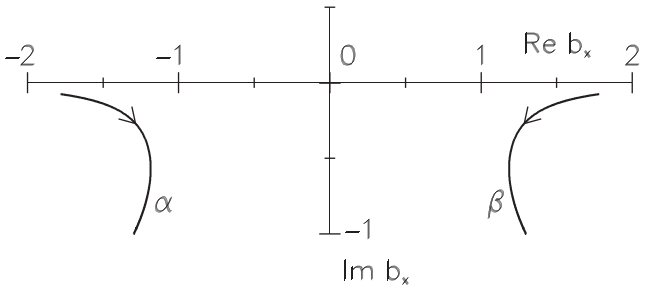


Figure 9.1 Trajectories $b_{\alpha x}$ and $b_{\beta x}$ of stationary impact vectors $\boldsymbol{b}_j = (b_{jx}, 0)$ for the phase shift function Eq. (9.2). The arrows indicate the sense in which the stationary points move as the momentum transfer increases.

This behavior is thus quite different from that observed in Fig. 7.5 for the Gaussian function. As the momentum transfer q increases, starting from some small value, $|\,\mathrm{Re}\, b_{\alpha x}|$ and $|\,\mathrm{Re}\, b_{\beta x}|$ will first decrease, reach a minimum, and then start to grow again. As a consequence, *the corresponding differential cross section will have oscillations with a period that reaches a maximum, and then becomes shorter again.*

However, the change takes place quite slowly with increasing momentum transfer. From Eq. (9.3), it follows that

$$\frac{\mathrm{d}b_x}{\mathrm{d}q} \underset{q\to\infty}{\sim} \frac{\beta^4}{4q|b_x|^3}\, e^{-i\pi/4}, \tag{9.6}$$

or

$$b_x \underset{q\to\infty}{\sim} \beta[\log(q\beta)]^{1/4}\, e^{-i\pi/4}. \tag{9.7}$$

The asymptotic period shortening is so gradual that the "outer" points of the trajectories given in Fig. 9.1 lie at 10 TeV/c.

Another aspect of this sharper edge is the fact that for increasing momentum transfer, the stationary points also move away from the real axis. In the absorptive case, for purely, imaginary A_4, this leads to a steeper envelope of the oscillatory pattern. This is illustrated in Fig. 9.2, where we show the differential cross section for $A_4 = 5i$, $\beta = 1$ fm and a very large range of momentum transfers, namely out to $q = 10$ GeV/c and superimpose the corresponding cross section from $q = 20$ GeV/c to $q = 30$ GeV/c (in steeper curve), multiplied by a scaling factor 10^{50}, in order to better compare the slopes and period of oscillation. The steepening of the envelope is quite pronounced, but in this range of momentum transfers, there is no visible change of the period of oscillation.

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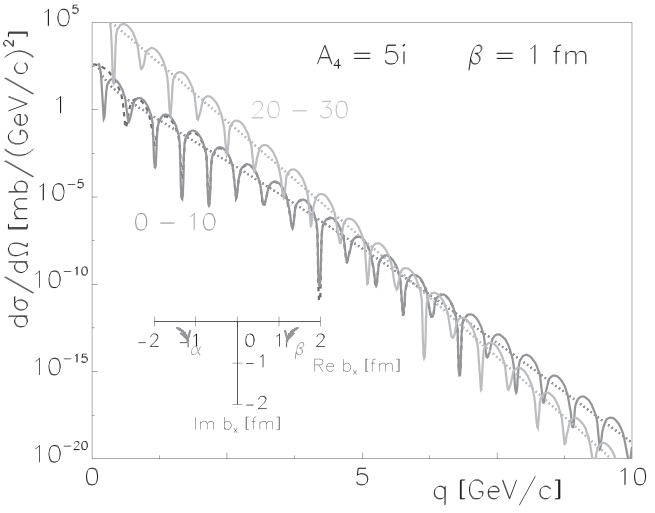


Figure 9.2 Differential cross section for the phase shift function Eq. (9.2). The inserts show the trajectories of stationary points for 0.15 GeV/c $\leq q \leq$ 30 GeV/c (dotted). The parts that correspond to the cross section plots, 0.2 GeV/c $\leq q \leq$ 10 GeV/c are shown as solid curves.

For $q < 7$ GeV/c, we also display the differential cross section, as determined from a numerical integration of the diffraction integral, Eq. (2.2). The agreement is excellent, apart from angles very close to the forward direction.

9.2 Sharper Edges

As an example of an even sharper edge, we consider

$$\chi_6(b_x, b_y) = A_6 \exp(-b^6/\beta^6), \tag{9.8}$$

with

$$X_6(b_x) = A_6 \exp(-b_x^6/\beta^6). \tag{9.9}$$

The stationary points given by

$$X_6'(b_x) = -6A_6(b_x^5/\beta^6) \exp(-b_x^6/\beta^6) = q, \tag{9.10}$$

are shown in Fig. 9.3 for purely imaginary A_6. By an obvious generalization of Eqs. (9.6) and (9.7), we find that the asymptotic period shortening is extremely slow,

$$b_x \underset{q\to\infty}{\sim} \beta[\log(q\beta)]^{1/6}\, e^{-i\pi/6}. \tag{9.11}$$

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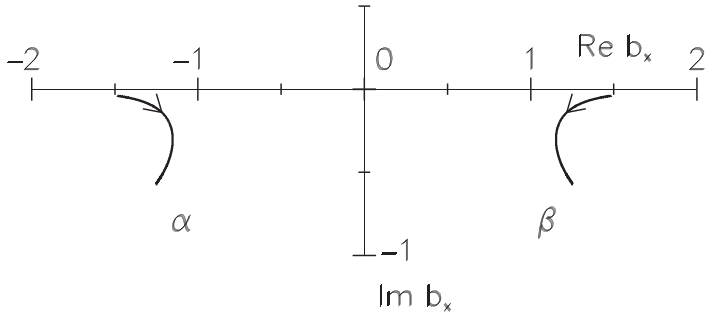


Figure 9.3 Trajectories $b_{\alpha x}$ and $b_{\beta x}$ of stationary impact vectors $\boldsymbol{b}_j = (b_{jx}, 0)$ for the phase shift function Eq. (9.9). The arrows indicate the sense in which the stationary points move as the momentum transfer increases.

In conclusion, we note that these sharper edges can lead to qualitatively different features of the oscillatory pattern of the differential cross section. In particular, the period may reach a maximum, and then decrease again. However, in order to see such phenomena, very large momentum transfers might be required.

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