Math. Proc. Camb. Phil. Soc. (1983), **93**, 511–518
Printed in Great Britain

S.I.P. measures on metrizable groups

BY C. KARANIKAS
University of Crete, Heraclion, Greece

(*Received* 1 *June* 1982)

1. *Introduction*

Let G be a metrizable non-discrete locally compact group. Let $M(G)$ be the convolution measure algebra of G. We shall denote by L_x and R_x, $x \in G$, the left and the right translation operation on $M(G)$, respectively. A measure $\mu \in M(G)$ has *strongly independent powers* or μ is a *s.i.p. measure*, if, for $x \in G$ and $n, m \in \mathbf{N}$,

$$\|L_x \mu^n - \mu^m\| = \|\mu\|^n + \|\mu\|^m = \|\mu^n R_x - \mu^m\|, \tag{1}$$

whenever $n \neq m$ or $x \neq e$, where e is the identity of G. If a measure μ satisfies (1) for $x = e$ and $n \neq m$, we say that μ has *independent powers*, or μ is an *i.p. measure*. Notice that the powers μ^n and μ^m of μ are convolution powers.

If G is a locally compact non-discrete abelian group, it is well known (see for example (3)) that $M(G)$ has s.i.p. measures. In (1) we have seen i.p. measures in $M(G)$, where G is a locally compact non-discrete group.

In Section 2 we construct by a Cantor-type method thin sets with suitable independence properties in a similar way as in (1). In Section 3 we show that continuous measures concentrated on these sets are s.i.p. measures. Finally we discuss s.i.p. measures in $M_0(G)$.

2. *Thin sets of type Y*

Let S be any finite subset of G; it can be denoted as

$$\{x_1, \ldots, x_k, x_{k+1}, x_{k+1}^{-1}, \ldots, x_t, x_t^{-1}\}, \tag{*}$$

where $x_1, \ldots, x_k$ are elements of order 2 (one of them could be of order 1, if it is the unit e of G), and $x_i^{-1} \neq x_i$ for $k+1 \leqslant i \leqslant t$. We shall say that S is generated by $x_1, \ldots, x_t$.

By a word in the elements of S we mean, as usual, any finite product. It is clear that if an element $x \in S$ has the position k (say) in a word in the elements of S, then x^{-1} cannot have the position $(k-1)$ or $(k+1)$ in this word. A word in the elements of S is called *restricted* if it has at most one entry from each set $\{x, x^{-1}\}$, where $x \in S$.

A condition preventing different words in the elements of S from being identical to the same element of G is called an *independence property*. For example, the condition $x_i x_j \neq x_j x_i$, $i \neq j$, $1 \leqslant i, j \leqslant t$, in the elements of S, is an independence property. It is clear that if S is a subset of an abelian subgroup of G this property cannot hold. Another independence property is the following: We say that S has *Property X* if there is no restricted word in the elements of S equal to a shorter word (see (1)).

An independence property *agrees with the group structure* of G if for any finite subset S of G (as (*)) and any neighbourhood N of e, there exist $y_i \in x_i N$ $(i = 1, \ldots, t)$ such that the set generated by $y_1, \ldots, y_t$ has this independence property. In Proposition 3·3 in (1) we have seen that Property X agrees with the structures of non-discrete groups.

This research was supported by the National Research Council of Greece.

https://doi.org/10.1017/S0305004100060825 Published online by Cambridge University Press

Definition 2·1. We say that a finite subset S of G has *Property* Y if:

(i) S has *Property* X_1, i.e. there is no restricted word (in the elements of S) equal to a shorter word or to a non-restricted word of the same length, and

(ii) There is no pair of equal restricted words.

In this section we aim to establish the following Proposition:

PROPOSITION 2·2. *Let G be a non-discrete group. Then either G has an open abelian subgroup, or the independence Property Y agrees with the structure of G.*

We shall proceed by using, instead of condition (ii) of Definition 2·1, the following weaker condition:

(ii*)

(Y_1): there is no pair w_1, w_2 of equal restricted words (in the elements of S) where x has an entry in w_1 and w_2 has no entry from the set $\{x, x^{-1}\}$, and

(Y_2): there is no pair w_1, w_2 of equal restricted words, where x has an entry in w_1 and x^{-1} has an entry in w_2, and

(Y_3): there is no pair w_1, w_2 of equal restricted words, where w_1 is a permutation of the elements of w_2.

LEMMA 2·3. *If for some $x \in G$ there is a neighbourhood N of e such that for any $u \in N$*

$$xuxu = x^2 \quad (\text{and so } uxu = x),$$

then G contains an open abelian subgroup.

Proof. Let V be a neighbourhood of e such that $V^2 \subseteq N$; then for $u_1, u_2 \in V$ we have: $u_1 x u_1 = x$ and $u_2 x u_2 = x$, and so

$$u_1(u_2 x u_2)\, u_1 = x.$$

Since $(u_1 u_2) \in N$ and $(u_1 u_2)\, x = x(u_1 u_2)^{-1}$ we have

$$x u_2^{-1} u_1^{-1} u_2 u_1 = x,$$

or

$$u_2 u_1 = u_1 u_2.$$

Thus the elements of V commute and so the group $\cup_{m=1}^{\infty} V^m$ is abelian and open.

Example. Let $\{1, -1\}$ be the two element group, and $\mathbf{R}$ be the additive group of reals. Also $\phi: \{1, -1\} \to \mathrm{Aut}\,\mathbf{R}: \phi(\pm 1) = \pm 1$. Then, if G is the semidirect product $\mathbf{R} \times_{\varphi} \{1, -1\}$ and $x = (0, -1)$, x satisfies (i) of Lemma 2·3, for any $u = (a, 1)$, $a \in \mathbf{R}$.

LEMMA 2·4. *Let $a, b, c, x, y \in G$, $xy \neq e$ and $x, y, \neq e$; suppose that there exists a neighbourhood N of e such that*

$$a(u^{-1}x^{-1})\, b(yv)\, (xu)\, c(v^{-1}y^{-1}) = e, \quad u, v \in N. \tag{1}$$

Then G has an open abelian subgroup.

Proof. By (1) for $v = e$ we have

$$a(u^{-1}x^{-1})\, by(xu)\, cy^{-1} = e. \tag{2}$$

From (1) and (2): $v(xuc) = (xuc)\, v$, and so

$$(x^{-1}vx)\, uc = ucv. \tag{3}$$

https://doi.org/10.1017/S0305004100060825 Published online by Cambridge University Press

In a similar way from $ax^{-1}byxcy^{-1} = ax^{-1}byvxcv^{-1}y^{-1}$, we have $vxc = xcv$, and so

$$(x^{-1}vx)\,c = cv. \tag{4}$$

From (3) and (4), $$(x^{-1}vx)\,u = u(x^{-1}vx). \tag{5}$$

There is a neighbourhood V of e such that $x^{-1}Vx \subseteq N$ and so by (5) the elements of $W = x^{-1}Vx$ commute; thus $\cup_{m=1}^{\infty} W^m$ is an open abelian subgroup of G, and the proof is complete.

Now, let (w_1, w_2) be a pair of equal restricted words, in the elements of S (as in (*)); we consider the following cases:

(1) (w_1, w_2) does not satisfy (Y_1), i.e. $w_1 = w_2$, and there is $x \in S$ which has an entry in w_1, but w_2 has no entry from the set $\{x, x^{-1}\}$. Choose any $y \neq x$ and substitute x by y in w_1; it is easy to verify that the words now obtained from w_1 and w_2 are distinct.

(2) (w_1, w_2) does not satisfy (Y_2), i.e. $w_1 = w_2$, and there is $x \in S$ such that x has an entry in w_1 and x^{-1} has an entry in w_2. Suppose that the group G has no open abelian subgroup. The condition $w_1 = w_2$ can be written as $ax = x^{-1}b$, $a, b \in G$; we see that, for any neighbourhood N of e, there exists $u \in N$ such that $axu \neq u^{-1}x^{-1}b$. In fact, if $axu = u^{-1}x^{-1}b$ for any $u \in N$ we have $(ax)\,u = u^{-1}(ax)$ and so $(ax)\,u(ax)\,u = (ax)^2$; by Lemma 2·3 this contradicts our hypothesis. Thus if G has no open abelian subgroup, we may replace x by an element in xX, so that the words obtained from w_1 and w_2 are distinct.

(3) (w_1, w_2) does not satisfy (Y_3), i.e. $w_1 = w_2$, and w_1 is a permutation of the elements of w_2. Suppose that G has no open abelian subgroup. The condition $w_1 = w_2$ can be written as: $ax^{-1}byxcy^{-1} = e$, where $x, y \in S$ and a, b and c are elements of G. By Lemma 2·4, for any neighbourhood N of e, there are $u, v \subset N$, such that if x, y are substituted by xu and yv, respectively, then the resulting words obtained from w_1, w_2 are distinct.

Now, let w be a word in the elements of S, and N be a symmetric neighbourhood of e. We denote by $w(N)$ the subset of G obtained from w by substituting each element of S in w by a subset of G as follows:

If $x \in S$ has an entry in w and is an element of order 2, we substitute it by the set $xN \cap Nx$.

For each $x \in S$ of order > 2, we substitute x by xN and x^{-1} by Nx^{-1}.

The following lemma is a consequence of the continuity of the mapping $(x, y) \to xy$: $G \times G \to G$ (see proposition 3·1 in (1)).

LEMMA 2·5. *Let Q be a finite collection of pairs of words (w_1, w_2) such that $w_1 \neq w_2$. Then there exists a neighbourhood N of e such that for each $(w_1, w_2) \in Q$*

$$w_1(N) \cap w_2(N) = \varnothing.$$

Proof of Proposition 2·2. We shall work as in (1), proposition 3·3, for each independence property X_1, Y_1, Y_2 and Y_3, separately.

We recall that, for the proof in (1), we considered the set W of all pairs (w_1, w_2), where w_1 is restricted and w_2 is a shorter word and that $Q \subseteq W$ is the set of all (w_1, w_2) such that $w_1 = w_2$.

As in case 2 above and from Lemma 2·5, for each $(w_1, w_2) \in Q$, we can substitute some $x \in S$ by an $x' \in xN$ (N is a given neighbourhood) such that:

(a) all resulting pairs of words obtained from $W \backslash Q$ remain distinct.

(b) the resulting words obtained from w_1, w_2 are distinct.

https://doi.org/10.1017/S0305004100060825 Published online by Cambridge University Press

Since Q is finite, we can repeat the procedure (a finite number of times) and we find a set S', where each $x' \in S'$ is in a neighbourhood of some $x \in S$; therefore S' has Property X.

We suppose that there is no open abelian subgroup in G. Similarly, from Lemma 2·5 and cases (1) and (2), we obtain from S' a set S'' having Properties X_1, Y_1 and Y_2.

Finally for the independence Property Y_3 the procedure of substitution shall be as in case (3), i.e. the elements $x, y \in S''$ will be substituted by x', y', in a neighbourhood of x and y, respectively. Thus we can find a set which has Property Y as we claimed. This completes the proof of Proposition 2·2.

Definitions 2·6. Let S be as in (*) and N be a symmetric neighbourhood of e. We say that $S(N)$ *has Property* Y_1, Y_2, Y_3, X_1 *or* Y, if for each pair of words w_1, w_2, as in (Y_1), (Y_2), (Y_3), (ii*) of 2·1 or (i) and (ii) of 2·1, respectively, $w_1(N) \cap w_2(N) = \varnothing$.

Let S be a finite subset of G with cardinality k and C_0 be a symmetric neighbourhood of e. We say that *the words of* $S(N)$ *are subordinated by* N_0 if, for any word w, in the elements of S, of length $\leqslant 2k$, $w(N) \subseteq wN_0$ and $w(N) \subseteq N_0 w$.

For example, if $x, y \in S$ and $w = xy$, then $(xN)(yN) \subseteq N_0 xy \cap xyN_0$.

Let $S(N)$ and $S'(N')$ be two finite collections of subsets of G, as above. Suppose that for each $I \in S'(N')$ there exists some $J \in S(N)$ such that $I \subseteq J$ and each $J \in S(N)$ contains at least two elements of $S'(N')$; then we say that $S'(N')$ is a *refinement* of $S(N)$ and we write $S'(N') < S(N)$.

Now, let $S_n(N_n)$, $n = 1, 2, \ldots$, be a finite collection of subsets of G, such that:

(i) $N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots$ is a basis of neighbourhoods of e,

(ii) the words of $S_{n+1}(N_{n+1})$ are subordinated by N_n, $n = 1, 2, \ldots$,

(iii) $S_{n+1}(N_{n+1}) < S_n(N_n)$, $n = 1, 2, \ldots$,

(iv) $S_n(N_n)$, $n = 1, 2, \ldots$, has Property Y (or X or Y_j, $j = 1, 2, 3$).

If $P = \cap_{n=1}^{\infty} \{\mathrm{cl}\,(I) : I \in S_n(N_n)\}$, we say that P is a *thin set of type* Y (or X or $Y_j, j = 1, 2, 3$).

THEOREM 2·7. *Let G be a non-discrete locally compact group. Suppose that G does not contain an open abelian subgroup. Then G contains a thin set of type Y.*

Proof. Suppose that $S_n(N_n)$ has been constructed; choose at least two points in each $I \in S_n(N_n)$, then apply Proposition 2·2 and Lemma 2·5.

Note 2·8. In the case where G has an open abelian subgroup H of order > 2, one can find a thin set of type X_1, Y_1 and Y_2.

If H is a group of order 2, i.e. there exists a neighbourhood N such that $u^2 = e$, $u \in N$, then there is a thin subset of N which has Properties X_1 and Y_1 (easy).

Furthermore it is well known from commutative harmonic analysis that H contains an independent set.

3. *S.i.p. measures*

The main theorem of this section is the following:

THEOREM 3·1. *Any metrizable non-discrete locally compact group has s.i.p. measures.*

As in (1) we begin with some lemmas of a technical nature.

LEMMA 3·2 (cf. lemma 4·3, (1)). *Let $\mu = \mu_1 + \ldots + \mu_N$ where $\mu_r \geqslant 0$, $\|\mu_r\| < h$, $1 \leqslant r \leqslant N$. Then if m is a given positive integer, there is a constant k_m, depending only on m, such that the sum of the terms in the multinomial expansion of $(\mu_1 + \ldots + \mu_N)^m$ that contain two or more occurrences of the same factor is $< k_m h \|\mu\|$ (we could take, e.g., $k_m = \frac{1}{2}(m!)$).*

https://doi.org/10.1017/S0305004100060825 Published online by Cambridge University Press

We shall denote by $(\mu^m)'$ the sum of the terms in the multinominal expansion of μ^m that are products of m distinct factors μ_i $(1 \leqslant i \leqslant N)$.

Now let us fix q of the factors $\mu_1, \ldots, \mu_N$ $(q < N)$; without loss of generality we consider the factors $\mu_1, \mu_2, \ldots, \mu_q$. We denote by $(\mu^m)''$ the sum of the terms in $(\mu^m)'$ that involve at least one of the factors $\mu_1, \ldots, \mu_q$.

LEMMA 3·3. $\|(\mu^m)''\| \leqslant mq\|\mu\|^{m-1} h$.

Proof. Even if the first factor in each term of $(\mu^m)''$ is one of the fixed factors $\mu_1, \ldots, \mu_q$, we have

$$\begin{aligned}\|(\mu^m)''\| &\leqslant \mu_1(\mu^{m-1})' + \ldots + \mu_q(\mu^{m-1})'\| \\ &\leqslant \|(\mu_1 + \mu_2 + \ldots + \mu_q)\,\mu^{m-1}\| \\ &\leqslant qh\|\mu\|^{m-1}.\end{aligned}$$

Since in each term there are m seats for the fixed factors, the result follows.

LEMMA 3·4. *Let $\mu = \mu_1 + \ldots + \mu_N$ where $\|\mu\| = 1$, $\mu_r \geqslant 0$, $\|\mu_r\| < h$, $1 \leqslant r \leqslant N$. Then if m, n and q are given positive integers, $q < N$ and $y \in G$, we have:*

(i) $\|L_y\mu^n - \mu^m\| \geqslant \|L_y(\mu^n)' - (\mu^m)'\| - (k_n + k_m)\,h$,

(ii) $\|L_y\mu^n - \mu^m\| \geqslant L_y(\mu^n)^1 - (\mu^m)^1\| - (nq + mq + k_n + k_m)\,h$,

where $(\mu^r)^1 = (\mu^r)' - (\mu^r)''$, $r = n, m$.

Proof. By Lemma 3·2 we have:

$$\|(L_y(\mu^n)' - (\mu^m)') - (L_y\mu^n - \mu^m)\| \leqslant (k_n + k_m)\,h$$

and this implies (i). From Lemmas 3·2 and 3·3 we obtain:

$$\|(L_y\mu^n - \mu^m) - (L_y(\mu^n)^1 - (\mu^m)^1)\| \leqslant \|\mu^n - (\mu^n)'\| + \|\mu^m - (\mu^m)'\| + \|(\mu^n)''\| + \|(\mu^m)''\|,$$

and this implies (ii).

PROPOSITION 3·5. *Let P be a thin set of type Y in a group G, as in Theorem 2·7. If μ is a continuous positive measure concentrated on P, then μ is a s.i.p. measure.*

Proof. Let $P = \cap_{n=1}^{\infty} \mathrm{cl}\,(I) : I \in S_n(N_n)$ be as in Theorem 2·7. For any $I \in S_n(N_n)$ we denote by μ_I the restriction of μ on I.

Let $h > 0$ be given, and suppose that $\max(\|\mu_I\| : I \in S_n(N_n)) > h$, $n = 1, 2, \ldots$. Since each $I \in S_n(N_n)$ contains at least two elements of S_{n+1} (N_{n+1}), there exists a sequence $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ such that $I_n \in S_n(N_n)$ and $\|\mu_{I_n}\| > h$, $n = 1, 2, \ldots$. We observe that there exist finitely many sequences as above and that, for some $x \in P$, $\cap_{n=1}^{\infty} I_n = \{x\}$. This contradicts the continuity of μ.

Thus, given $h > 0$, there exists a positive integer n_0 such that for any $n \geqslant n_0$ and $I \in S_n(N_n)$, $\|\mu_I\| \leqslant h$.

Now let n, m be two given positive integers, $h > 0$ and $y \in G$; we write $\mu = \Sigma\mu_I : I \in S_{n_0}(N_{n_0})$, where n_0 is as above.

First we suppose that $y \neq e$; without loss of generality suppose also that $y \notin N_{n_0-1}$.

Let $T_1, T_2, \ldots, T_n$, $Z_1, Z_2, \ldots, Z_m \in S_{n_0}(N_{n_0})$ such that the words $T_1, T_2, \ldots, T_n$ and $Z_1 Z_2 \ldots Z_m$ are restricted words.

If there is no pair of restricted words (as above) such that

$$y \in (Z_1 \ldots Z_m)\,(T_1 \ldots T_n)^{-1}, \tag{1}$$

https://doi.org/10.1017/S0305004100060825 Published online by Cambridge University Press

then $\operatorname{supp}(L_y(\mu^n)') \cap \operatorname{supp}(\mu^m)' = \varnothing$. Hence by Lemma 3·4 (i) and as in Lemma 3·2 we obtain:

$$\begin{aligned}\|L_y\mu^n - \mu^m\| &\geqslant \|L_y(\mu^n)' + (\mu^m)'\| - (k_n + k_m)h \\ &\geqslant \|\mu\|^n + \|\mu\|^m - 2(k_n + k_m)h.\end{aligned}$$

Since h is arbitrarily small and μ is a positive measure we have:

$$\|L_y\mu^n - \mu^m\| = \|\mu\|^n + \|\mu\|^m. \tag{2}$$

Let y be in a set as in (1); we shall denote by M the set $\{Z_1, Z_1^{-1}, \ldots, Z_m, Z_m^{-1}, T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}\}$ which is considered fixed. Let $P_1P_2 \ldots P_n$, $Q_1Q_2 \ldots Q_m$ be another pair of restricted words in the elements of S_{n_0} (N_{n_0}) such that

$$y \in (Q_1Q_2 \ldots Q_m)(P_1P_2 \ldots P_n)^{-1}. \tag{3}$$

From (1) and (3) we have

$$(T_1 \ldots T_n)^{-1}(P_1 \ldots P_n) \cap (Z_1 \ldots Z_m)^{-1}(Q_1 \ldots Q_m) \neq \varnothing. \tag{4}$$

We consider the cases:

(1) $n = m$, we suppose that the word $P_1P_2 \ldots P_n$ has no entry from the set M. Then the word $(T_1 \ldots T_n)^{-1}(P_1 \ldots P_n)$ is restricted. S_{n_0} (N_{n_0}) has Property Y and so has Property X_1. This implies that the word $(Z_1 \ldots Z_m)^{-1}(Q_1 \ldots Q_m)$ is restricted. By condition (ii) of Definition 2·1 we have:

$$T_i = Z_i, \quad i = 1, \ldots, n = m. \tag{5}$$

Since S_{n_0} (N_{n_0}) is subordinated by N_{n_0-1}, from (1) and (5) we obtain $y \in N_{n_0-1}$, a contradiction.

(2) $n > m$, we suppose that the word $P_1P_2 \ldots P_n$ has no entry from the set M. The word $(T_1 \ldots T_n)^{-1}(P_1 \ldots P_n)$ is restricted and the word $(Z_1 \ldots Z_m)^{-1}(Q_1 \ldots Q_m)$ is shorter than this. Thus (4) contradicts Property X.

Therefore in both cases, we have $\operatorname{supp}(L_y(\mu^n)^1) \cap \operatorname{supp}(\mu^m)' = \varnothing$, where $(\mu^r)^1 = (\mu^r)' - (\mu^r)''$, $r = n, m$, $(\mu^r)''$ is the sum of the terms in $(\mu^r)'$ that involve at least one of the factors from the set M. We apply Lemma 3·4 (ii) and obtain

$$\begin{aligned}\|L_y\mu^n - \mu^m\| &\geqslant \|L_y(\mu^n)^1 - (\mu^m)^1\| - c(n,m)h \\ &= \|L_y(\mu^n)^1\| + \|(\mu^m)^1\| - c(n,m)h \\ &\geqslant \|\mu\|^n + \|\mu\|^m - c_1(n,m)h,\end{aligned}$$

where $c(n,m)$ and $c_1(n,m)$ are constants depending only on n and m, and so this implies (2).

The remaining case is when $y = e$ and $n \neq m$. We observe that

$$\operatorname{supp}(\mu^n) \cap \operatorname{supp}(\mu^m) = \varnothing$$

and (2) follows as in the first part of the proof.

4. *On the algebra* $M_0(G)$

Let $G^\wedge$ be the set of all equivalence classes of continuous unitary irreducible representations U^π of G in Hilbert spaces H_π. Given $U^\pi \in G^\wedge$ and $\mu \in M(G)$, let

$$\pi(\mu) = \int_G U^\pi(t)\, d\mu(t),$$

$\pi(\mu)$ is a bounded operator on H_π; as in (1), let

$$\|\mu\|' = \sup \|\pi(\mu)\|,$$

where the supremum is taken over all $U^\pi \in G^\wedge$; $\|\cdot\|'$ is a norm on $M(G)$.

https://doi.org/10.1017/S0305004100060825 Published online by Cambridge University Press

We shall denote by $M_0(G)$ the subset of $M(G)$ consisting of measures that can be approximated arbitrarily closely in the $\|\cdot\|'$ norm by elements of $L_1(G)$. An alternative equivalent definition of $M_0(G)$ is given in (2), theorems 3 and 4:

$$M_0(G) = \{\mu \in M(G) : \|L_t\mu - \mu\|' \to 0 \text{ as } t \to e\}.$$

In the abelian case, $G^\wedge$ consists of 1-dimensional representations and $M_0(G)$ consists of measures whose Fourier–Stieltjes transform tends to zero at infinity.

As in (1), the basic procedure in the construction of a singular measure (w.r.t. left Haar measure) in $M_0(G)$ is as follows:

Let (μ_s) be a sequence of measures and $(\mathscr{F}_s, \tau_s)$, $s = 1, 2, \ldots$, be a sequence of *templates*, i.e. each $(\mathscr{F}_s, \tau_s)$ is a pair consisting of a collection $\mathscr{F}$ of open sets in G with disjoint compact closures and of a function $\tau_s : \mathscr{F}_s - \mathbf{R}^+$. We suppose:

(i) $(\mathscr{F}_{s+1}, \tau_{s+1})$ is a refinement of $(\mathscr{F}_s, \tau_s)$, $s = 1, 2, \ldots$ i.e. for each $I \in \mathscr{F}_{s+1}$ there exists $J \in \mathscr{F}_s : I \subseteq J$ and $\tau_s(J) = \sum_{I \subset J} \tau_{s+1}(I)$.

(ii) $\max(\tau_s(I) : I \in \mathscr{F}_s) \to 0$ as $s \to \infty$.

(iii) $\max(\text{diameter } I : I \in \mathscr{F}_s) \to 0$ as $s \to \infty$.

(iv) For each $s = 1, 2, \ldots, \mu_s$ is the canonical measure of $(\mathscr{F}_s, \tau_s)$ i.e.

$$\mu_s = \sum_{I \in \mathscr{F}_s} \tau_s(I)/\sigma_s(I)\, \sigma_s,$$

where $d\sigma_s = \Delta^{\frac{1}{2}} d\lambda$, Δ is the modular function, and λ is the left Haar measure.

(v) $\|\mu_s - \mu_t\|' \to 0$ as $s, t \to \infty$.

It is shown in (1) that the weak* limit μ of (μ_s) is a singular measure in $M_0(G)$. Next we shall arrange $(\mathscr{F}_s, \tau_s)$ such that μ is a s.i.p. measure.

The refinement process for $(\mathscr{F}_1, \tau_1)$ will be in stages, each stage comprising many sub-stages. Thus we can write $s = s(n, m)$ or simply (n, m) for the sub-stage m of the stage n. In this procedure we refine each sub-collection Q_m consisting of n elements of $\mathscr{F}_{n,1}$ in some sub-stage m, leaving the remaining sets of $\mathscr{F}_{n,m-1}$ unchanged. In this manner we write:

$$\mathscr{F}_{n,m-1} = \mathscr{F}^{(1)}_{n,m-1} \cup \mathscr{F}^{(2)}_{n,m-1},$$

where $\mathscr{F}^{(1)}_{n,m-1}$ is the collection of all sets $I \in \mathscr{F}_{n,m-1}$ such that for each I there exists a $T \in Q_m$ and $I \subseteq T$; $\mathscr{F}^{(2)}_{n,m-1}$ is the complement of $\mathscr{F}^{(1)}_{n,m-1}$. Now if $\mathscr{F}^{(3)}_{n,m-1}$ is the refinement of $\mathscr{F}^{(1)}_{n,m-1}$ we write:

$$\mathscr{F}_{n,m} = \mathscr{F}^{(3)}_{n,m-1} \cup \mathscr{F}^{(2)}_{n,m-1}.$$

We can arrange the sequence of templates and the sequence of measures to satisfy (i)–(v), see theorem 5·6 in (1). From the proof of Proposition 2·2 we may claim:

(vi) $\mathscr{F}^{(3)}_{n,m}$ has Property Y whenever G has not an open abelian subgroup. (In the case where G has an open abelian subgroup H, one can construct a sequence of templates in H such that $\mathscr{F}^{(3)}_{n,m}$ has suitable independence properties, see Note 2·8).

We shall examine the first case: the other case is similar.

THEOREM 4·1. *Let G be a metrizable non-discrete group, which has no open abelian subgroups. Let $(\mathscr{F}_s, \tau_s)$ and (μ_s) be a sequence of templates and measures (respectively), satisfying* (i)–(vi). *Let μ be the weak* limit of (μ_s). Then μ is a s.i.p. measure in $M_0(G)$.*

Proof. We have seen that μ is in $M_0(G)$. For any $s = s(n, 1)$ we can write $\mu = \Sigma\mu_I : I \in \mathscr{F}_s$, where μ_I is the restriction of μ on I. We shall show that, for any $y \in G$ and $p, q \in \mathbf{N}$,

$$\|L_y\mu^p - \mu^q\| = \|\mu\|^p + \|\mu\|^q.$$

https://doi.org/10.1017/S0305004100060825 Published online by Cambridge University Press

Let $(\mu^j)', j = p, q$, be the sum of the terms in the multinomial expansion of μ^j that are products of j distinct factors $\mu_I: I \in \mathscr{F}_s$. We fix a pair $\mu_0(j)$, of terms in $(\mu^j)', j = p, q$, such that $y \in (\operatorname{supp} \mu_0(q))(\operatorname{supp} \mu(p))^{-1}$.

We consider another pair $\mu(p), \mu(q)$ in $(\mu^p)'$, and $(\mu^q)'$ (respectively) such that

$$(\operatorname{supp} \mu_0(q))(\operatorname{supp} \mu_0(p))^{-1}(\operatorname{supp} \mu(p)) \cap (\operatorname{supp} \mu(q)) \neq \varnothing. \tag{2}$$

Suppose that $n > 2(p+q)$ (n appears in $s = s(n+1)$). Then all the elements of $\mathscr{F}_s$ in the support of $\mu(p)$, $\mu(q)$, $\mu_0(p)$ and $\mu_0(q)$ are contained in $\mathscr{F}^{(3)}_{n,m}$ for some sub-stage m of the stage n. By (vi), $\mathscr{F}^{(3)}_{n,m}$ has Property Y; thus as in (4) of Proposition 3·5, (2) holds whenever $\mu(\rho)$ has at least a factor of $\mu_0(p)$ and $\mu_0(q)$. Hence as in (6) of Proposition 3·5, we have

$$\|L_y \mu^p - \mu^q\| \geqslant \|\mu\|^p + \|\mu\|^q - Ch,$$

where C is a constant and $h = \max(\|\mu_I\| : I \in \mathscr{F}_s)$, and this implies (1).

We close this work with one application and two problems.

Application. Let G be a metrizable non-discrete locally compact group. If $k > 0$ is given, there exists $a_\nu \equiv M(G)$ such that $\|\nu\| = 1$ and $\|\nu\|' < k$.

In fact, let μ be as in Theorem 4·1 with $\|\mu\| = \frac{1}{2}$ and $\nu = L_y\mu - \mu$, $y \in G$; clearly $\|\nu\| = 1$ for any $y \neq e$ and since $\mu \in M_0(G)$, there exists a $y \in G/\{e\}$ such that $\|\nu\|' < k$.

Problem 1. Let G be a non-discrete non-metrizable group. Can a s.i.p. measure in $M(G)$ be constructed?

Problem 2. Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in G$, such that either $n \neq m$, or $x_1 \ldots x_n (y_1 y_2 \ldots y_m)^{-1} \neq e$. Can a positive measure μ be found such that:

$$\|L_{x_1}\mu L_{x_2}\mu \ldots L_{x_n}\mu - \mu R_{y_1}\mu R_{y_2} \ldots \mu R_{y_m}\| = \|\mu\|^n + \|\mu\|^m ?$$

REFERENCES

(1) BROWN, G., KARANIKAS, C. and WILLIAMSON, J. The asymmetry of $M_0(G)$. *Math. Proc. Cambridge Philos. Soc.* **91** (1982), 407–433.

(2) DUNKL, C. and RAMIREZ, D. Translations in measure algebras and the correspondence to Fourier transforms vanishing at infinity. *Michigan Math. J.* **17** (1970), 311–319.

(3) GRAHAM, C. and MCGEHEE, O. *Essays in commutative harmonic analysis*. Grundlehren der math. Wissenschaften no 238 (Springer-Verlag, 1979).

https://doi.org/10.1017/S0305004100060825 Published online by Cambridge University Press