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GENERALIZED HILBERT MATRIX OPERATORS ACTING ON WEIGHTED SEQUENCE SPACES

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Abstract In this paper, we introduce and study a new kind of generalized Hilbert matrix operators, induced by a positive finite Borel measure on $(0,1)$, acting on weighted sequence spaces. We establish a sufficient and necessary condition for the boundedness of these operators. These results extend some related ones obtained recently in [*Bull. Lond. Math. Soc.* 55(6) (2023), 2598–2610].

Keywords: generalized Hilbert matrix operator; boundedness of operator; norm of operator

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1. Introductions and main results

For $1 < p < \infty$, we denote the Hölder conjugate of p by q, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Let l^p be the space of sequences of complex numbers, i.e.,

$$l^p := \{a = \{a_n\}_{n=0}^{\infty} : \|a\|_p = (\sum_{n=0}^{\infty} |a_n|^p)^{\frac{1}{p}} < \infty\}.$$

For $a = \{a_n\}_{n=0}^{\infty} \in l^p$, $b = \{b_n\}_{n=0}^{\infty} \in l^q$, the famous Hilbert's inequality is stated as

$$\Big| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m b_n}{m+n+1} \Big| \leq \pi \csc(\frac{\pi}{p}) \|a\|_p \|b\|_q,$$

where the constant $\pi \csc(\frac{\pi}{p})$ is the best possible. Hilbert's inequality has an equivalent form as follows.

$$\sum_{n=0}^{\infty} \Big| \sum_{m=0}^{\infty} \frac{a_m}{m+n+1} \Big|^p \leq \Big[\pi \csc(\frac{\pi}{p})\Big]^p \|a\|_p^p, \tag{1.1}$$

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where the constant $\left[\pi \csc(\frac{\pi}{p})\right]^p$ is the best possible, see [19, Theorem 6] or [20, Theorem 323].

The inequality (1.1) can be restated in the language of operator theory. For the Hilbert kernel $\frac{1}{m+n+1}$, we define the Hilbert matrix operator $\mathbf{H}$ as

$$\mathbf{H}(a)(n) := \sum_{m=0}^{\infty} \frac{a_m}{m+n+1},\ a = \{a_m\}_{m=0}^{\infty},\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Then (1.1) can be restated as

Theorem 1.1. *Let $1 < p < \infty$. Then $\mathbf{H}$ is bounded on l^p and the norm of $\mathbf{H}$ is $\pi \csc(\frac{\pi}{p})$.*

The Hilbert matrix operator is important in analysis and there have been many results about this operator and its analogues and generalizations. The history and classical results of this operator can be found in the famous monograph [20]. Modern results about generalizations of the Hilbert matrix operator and their analogues can be found in the survey [30] and Yang's book [31]. Some very new results about the Hilbert matrix operator have been obtained in [9]. In [11, 12], Diamantopoulos and Siskakis initiated the study of the Hilbert matrix operator acting on the analytic function spaces. Later, there have been a good number of researchers studying the Hilbert matrix operator and its generalizations on various spaces of analytic functions, see, for example, [6, 8, 10, 13–17, 21–26, 28, 29, 32]. For more results about these topics, see the recent survey [4].

Note that the Hilbert kernel can be written as

$$\frac{1}{m+n+1} = \binom{n+m}{m} \int_0^1 t^m (1-t)^n \, \mathrm{d}t.$$

Here, for $\gamma \in \mathbb{R}$, $m \in \mathbb{N}_0$, the combinatorial number $\binom{\gamma}{m}$ is defined as

$$\binom{\gamma}{m} := \frac{\gamma(\gamma-1)\cdots(\gamma-m+1)}{m!}.$$

In particular, $\binom{\gamma}{0} = 1$ for any $\gamma \in \mathbb{R}$.

Very recently, inspired by the classical work of Hardy [18], the following generalized Hilbert matrix operator was introduced and studied by Athanasiou in [2]. Let μ be a positive finite Borel measure on $(0,1)$, the operator H_μ is defined as

$$H_\mu(a)(n) := \sum_{m=0}^{\infty} \int_{(0,1)} \binom{n+m}{m} t^m (1-t)^n a_m \, \mathrm{d}\mu(t), a = \{a_m\}_{m=0}^{\infty},\ n \in \mathbb{N}_0.$$

It has been proved in [2] that

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Theorem 1.2. *Let μ be a positive finite Borel measure on $(0,1)$.*
(1) *Let $1 \le p < +\infty$. Then H_μ is bounded on l^p if and only if*

$$\mathcal{C}_\mu(p) = \int_{(0,1)} (1-t)^{\frac{1}{p}-1} t^{-\frac{1}{p}} \, \mathrm{d}\mu(t) < \infty.$$

Moreover, when $H_\mu : l^p \to l^p$ is bounded, the norm of H_μ is $\mathcal{C}_\mu(p)$.
(2) *H_μ is bounded on l^∞ if and only if*

$$\mathcal{C}_\mu(\infty) = \int_{(0,1)} (1-t)^{-1} \, \mathrm{d}\mu(t) < \infty.$$

Moreover, when H_μ is bounded on l^∞, the norm of H_μ is $\mathcal{C}_\mu(\infty)$.

The main purpose of this note is to establish an extension of above results obtained in [2]. To state our results, we first introduce some notations. Let $w(n) \ge 0$ for $n \in \mathbb{N}_0$. For $1 \le p < \infty$, we denote by l^p_w the weighted Lebesgue space of infinite sequences, i.e.,

$$l^p_w := \{a = \{a_n\}_{n=0}^\infty : \|a\|_{p,w} = (\sum_{n=0}^\infty w(n)|a_n|^p)^{\frac{1}{p}} < \infty\}.$$

We define the class of infinite sequences l^∞_w as

$$l^\infty_w := \{a = \{a_n\}_{n=0}^\infty : \|a\|_{\infty,w} = \sup_{n\in\mathbb{N}_0} w(n)|a_n| < \infty\}.$$

When $w(n) \equiv 1$, we will write l^p instead of l^p_w, where $1 \le p \le \infty$.

Let $\alpha, \beta > -1$, $\beta - \alpha > -1$ and let μ be a positive finite Borel measure on $(0,1)$. For a sequence $a = \{a_m\}_{m=0}^\infty$, we define

$$H^{\alpha,\beta}_\mu(a)(n) := \sum_{m=0}^\infty \int_{(0,1)} \frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\alpha+1)\Gamma(n+\beta-\alpha+1)} t^m (1-t)^n a_m \, \mathrm{d}\mu(t), \; n \in \mathbb{N}_0.$$

Here, Γ is the usual Gamma function, see [1]. When $\alpha = 0$, $H^{\alpha,\beta}_\mu$ reduces to the following operator

$$H^{\beta}_\mu(a)(n) := \sum_{m=0}^\infty \int_{(0,1)} \binom{n+m+\beta}{m} t^m (1-t)^n a_m \, \mathrm{d}\mu(t), \; n \in \mathbb{N}_0.$$

For two real numbers s, γ with $\gamma > -1$ and $\gamma + s > -1$, we define

$$(\gamma+1)_s := \frac{\Gamma(\gamma+s+1)}{\Gamma(\gamma+1)}.$$

We shall prove that

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Theorem 1.3. *Let μ be a positive finite Borel measure on $(0,1)$. Let $1 \le p < \infty$, and $\alpha, \beta > -1$, $\beta - \alpha > -1$. Then $H_\mu^{\alpha,\beta}$ is bounded from $l^p_{w_1}$ into $l^p_{w_2}$ if and only if*

$$\mathcal{C}_\mu(\beta,p) := \int_{(0,1)} (1-t)^{-(1-\frac{1}{p})(\beta+1)} t^{-\frac{1}{p}(\beta+1)} \,\mathrm{d}\mu(t) < \infty.$$

Moreover, when $H_\mu^{\alpha,\beta} : l^p_{w_1} \to l^p_{w_2}$ is bounded, the norm $\|H_\mu^{\alpha,\beta}\|$ of $H_\mu^{\alpha,\beta}$ is $\mathcal{C}_\mu(\beta,p)$. Here

$$w_1(m) = (m+1)_\alpha^{-p}(m+1)_\beta,$$

$$w_2(m) = (m+\beta-\alpha+1)_\alpha^{-p}(m+1)_\beta,\ m \in \mathbb{N}_0.$$

Theorem 1.4. *Let μ be a positive finite Borel measure on $(0,1)$. Let $\alpha, \beta > -1$, $\beta - \alpha > -1$. Then $H_\mu^{\alpha,\beta}$ is bounded from $l^\infty_{\overline{w}_1}$ into $l^\infty_{\overline{w}_2}$ if and only if*

$$\mathcal{C}_\mu(\beta,\infty) := \int_{(0,1)} (1-t)^{-\beta-1} \,\mathrm{d}\mu(t) < \infty.$$

Moreover, when $H_\mu^{\alpha,\beta} : l^\infty_{\overline{w}_1} \to l^\infty_{\overline{w}_2}$ is bounded, the norm $\|H_\mu^{\alpha,\beta}\|$ of $H_\mu^{\alpha,\beta}$ is $\mathcal{C}_\mu(\beta,\infty)$. Here

$$\overline{w}_1(m) = (m+1)_\alpha^{-1},\ \overline{w}_2(m) = (m+\beta-\alpha+1)_\alpha^{-1},\ m \in \mathbb{N}_0.$$

When $\alpha = 0$, from Theorems 1.3 and 1.4, we have

Corollary 1.5. *Let μ be a positive finite Borel measure on $(0,1)$. Let $\beta > -1$.*
(1) *Let $1 \le p < \infty$. Then H_μ^β is bounded on l^p_w if and only if*

$$\mathcal{C}_\mu(\beta,p) = \int_{(0,1)} (1-t)^{-(1-\frac{1}{p})(\beta+1)} t^{-\frac{1}{p}(\beta+1)} \,\mathrm{d}\mu(t) < \infty.$$

Moreover, when H_μ^β is bounded on l^p_w, the norm of H_μ^β is $\mathcal{C}_\mu(\beta,p)$. Here,

$$w(m) = (m+1)_\beta,\ m \in \mathbb{N}_0.$$

(2) *H_μ^β is bounded on l^∞ if and only if*

$$\mathcal{C}_\mu(\beta,\infty) = \int_{(0,1)} (1-t)^{-\beta-1} \,\mathrm{d}\mu(t) < \infty.$$

Moreover, when H_μ^β is bounded on l^∞, the norm of H_μ^β is $\mathcal{C}_\mu(\beta,\infty)$.

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Remark 1.6. Theorem 1.2 will follow if we take $\beta = 0$ in Corollary 1.5.

The paper is organized as follows. Some lemmas will be given in the next section. We will first prove Theorem 1.3 in § 3. The proof of Theorem 1.4 will be given in § 4. Final remarks will be presented in § 5.

2. Some lemmas

To prove the main results, in this section, we will list some known lemmas and establish some new ones. For the sake of simplicity, in the rest of the paper, we will write

$$k_{\alpha,\beta}(m,n) = \frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\alpha+1)\Gamma(n+\beta-\alpha+1)}.$$

Lemma 2.1. *Let $m, n \in \mathbb{N}_0$ and let $\alpha, \beta > -1$, $\beta - \alpha > -1$. Then we have*

$$k_{\alpha,\beta}(m,n) = (-1)^m \binom{-n-\beta-1}{m} (m+1)_\alpha^{-1} (n+\beta-\alpha+1)_\alpha$$

and

$$k_{\alpha,\beta}(m,n) = (-1)^n \binom{-m-\beta-1}{n} (n+1)_{\beta-\alpha}^{-1} (m+\alpha+1)_{\beta-\alpha}.$$

Proof. Note that

$$\begin{aligned}\Gamma(n+m+\beta+1) &= \Gamma(n+\beta+1)\prod_{i=1}^{m}(n+\beta+i)\\ &= \Gamma(n+\beta+1)(-1)^m\prod_{i=1}^{m}(-n-\beta-i).\end{aligned}$$

Then

$$\begin{aligned}k_{\alpha,\beta}(m,n) &= \frac{\Gamma(n+m+\beta+1)}{\Gamma(n+\beta-\alpha+1)}\cdot\frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}\cdot\frac{1}{\Gamma(m+1)}\\ &= (-1)^m\frac{\prod_{i=1}^{m}(-n-\beta-i)}{\Gamma(m+1)}\cdot\frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}\cdot\frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta-\alpha+1)}\\ &= (-1)^m\binom{-n-\beta-1}{m}(m+1)_\alpha^{-1}(n+\beta-\alpha+1)_\alpha.\end{aligned}$$

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Similarly, we have

$$\begin{aligned}\Gamma(n+m+\beta+1) &= \Gamma(m+\beta+1)\prod_{i=1}^{n}(m+\beta+i)\\ &= \Gamma(m+\beta+1)(-1)^n\prod_{i=1}^{n}(-m-\beta-i).\end{aligned}$$

It follows that

$$\begin{aligned}k_{\alpha,\beta}(m,n) &= \frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\alpha+1)}\cdot\frac{\Gamma(n+1)}{\Gamma(n+\beta-\alpha+1)}\cdot\frac{1}{\Gamma(n+1)}\\ &= (-1)^n\frac{\prod_{i=1}^{n}(-m-\beta-i)}{\Gamma(n+1)}\cdot\frac{\Gamma(n+1)}{\Gamma(n+\beta-\alpha+1)}\cdot\frac{\Gamma(m+\beta+1)}{\Gamma(m+\alpha+1)}\\ &= (-1)^n\binom{-m-\beta-1}{n}(n+1)^{-1}_{\beta-\alpha}(m+\alpha+1)_{\beta-\alpha}.\end{aligned}$$

This proves the lemma. □

Lemma 2.2. *Let $m,n\in\mathbb{N}_0$ and let $\alpha,\beta>-1$, $\beta-\alpha>-1$. Then we have*

$$\sum_{m=0}^{\infty}k_{\alpha,\beta}(m,n)t^m(m+1)_\alpha = (1-t)^{-n-\beta-1}(n+\beta-\alpha+1)_\alpha$$

and

$$\sum_{n=0}^{\infty}k_{\alpha,\beta}(m,n)(1-t)^n(n+1)_{\beta-\alpha} = t^{-m-\beta-1}(m+\alpha+1)_{\beta-\alpha}.$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}&\sum_{m=0}^{\infty}k_{\alpha,\beta}(m,n)t^m(m+1)_\alpha\\ &\quad= \sum_{m=0}^{\infty}(-1)^m t^m\binom{-n-\beta-1}{m}(n+\beta-\alpha+1)_\alpha\\ &\quad= (1-t)^{-n-\beta-1}(n+\beta-\alpha+1)_\alpha,\end{aligned}$$

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and

$$\begin{aligned}&\sum_{n=0}^{\infty} k_{\alpha,\beta}(m,n)(1-t)^n(n+1)_{\beta-\alpha}\\&\quad=\sum_{n=0}^{\infty}(-1)^n(1-t)^n\binom{-m-\beta-1}{n}(m+\alpha+1)_{\beta-\alpha}\\&\quad=t^{-m-\beta-1}t^m(m+\alpha+1)_{\beta-\alpha}.\end{aligned}$$

The lemma is proved. □

Lemma 2.3. (2, Lemma 1.4) *It holds that*

$$(1-t)(1-t\mathrm{e}^{-x})^{-1}\geq \mathrm{e}^{-t(1-t)^{-1}x},$$

for all $x\geq 0$ *and* $t\in[0,1)$.

Lemma 2.4. (2, Lemma 1.5) *Let* $y,z>0$, *then*

$$y^{-z}=\frac{1}{\Gamma(z)}\int_0^{\infty}\mathrm{e}^{-yx}x^{z-1}\,\mathrm{d}x.$$

Lemma 2.5. (7, Lemma 1) *Let* $n\geq 0$. *Then*

$$(n+1)_s\leq\frac{(n+1)^{s+1}}{n+s+1},$$

for $s\in(-1,0)$, *and*

$$(n+1)_s\leq(n+s+1)^s,$$

for $s\geq 0$.

Lemma 2.6. *Let* $p\geq 1,\beta>-1$. *Let* $\rho\in(0,1)$ *and* $\mathbf{N}$ *be a natural number. For* $\varepsilon\in(0,\frac{1}{p}(\beta+1))$, *we define*

$$a_n(\varepsilon)=(n+\beta+1)^{-1-p\varepsilon},\ n\in\mathbb{N}_0.$$

Then we can take a constant $\varepsilon_1=\varepsilon_1(\rho,\mathbf{N})>0$ *such that*

$$\sum_{n=\mathbf{N}+1}^{\infty}a_n(\varepsilon)\geq(1-\rho)\sum_{n=0}^{\infty}a_n(\varepsilon),$$

for all $\varepsilon\in(0,\varepsilon_1]$.

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Proof. We let $\mathbf{S} = \sum_{n=0}^{\infty} a_n(\varepsilon)$. It is easy to see that

$$\mathbf{S} \geq \int_0^{\infty} (x+\beta+1)^{-1-p\varepsilon}\,\mathrm{d}x = \frac{1}{\varepsilon}\frac{1}{p(\beta+1)^{p\varepsilon}}$$

and

$$\sum_{n=0}^{\mathbf{N}} a_n(\varepsilon) \leq \frac{\mathbf{N}+1}{(\beta+1)^{1+p\varepsilon}}.$$

This means that we can find two constants $C_i = C_i(p,\beta) > 0, i = 1,2$, such that

$$\frac{1}{C_1\varepsilon} \leq \mathbf{S} \text{ and } \sum_{n=0}^{\mathbf{N}} a_n(\varepsilon) \leq C_2\mathbf{N}.$$

It follows that

$$\Big(\sum_{n=\mathbf{N}+1}^{\infty} a_n(\varepsilon)\Big)/\Big(\sum_{n=0}^{\infty} a_n(\varepsilon)\Big) = 1 - \Big(\sum_{n=0}^{\mathbf{N}} a_n(\varepsilon)\Big)/\mathbf{S} \geq 1 - C_1C_2\mathbf{N}\varepsilon.$$

Hence, if $0 < \varepsilon \leq \varepsilon_1 := \frac{\rho}{C_1C_2\mathbf{N}}$, then we have

$$\sum_{n=\mathbf{N}+1}^{\infty} a_n(\varepsilon) \geq (1-\rho)\sum_{n=0}^{\infty} a_n(\varepsilon),$$

for all $\varepsilon \in (0,\varepsilon_1]$. This proves the lemma. □

3. Proof of Theorem 1.3

First we prove the following

Proposition 3.1. *If $\mathcal{C}_\mu(\beta,p) < \infty$, then $H_\mu^{\alpha,\beta} : l^p_{w_1} \to l^p_{w_2}$ is bounded and $\|H_\mu^{\alpha,\beta}\| \leq \mathcal{C}_\mu(\beta,p)$.*

We then prove

Proposition 3.2. *If $H_\mu^{\alpha,\beta} : l^p_{w_1} \to l^p_{w_2}$ is bounded, then $\|H_\mu^{\alpha,\beta}\| \geq \mathcal{C}_\mu(\beta,p)$.*

It is easy to see that Propositions 3.1 and 3.2 imply Theorem 1.3.

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3.1. Proof of Proposition 3.1

For $n \in \mathbb{N}_0$, $a = \{a_m\}_{m=0}^{\infty} \in l_{w_1}^p$, let

$$E_n(t) := \sum_{m=0}^{\infty} k_{\alpha,\beta}(m,n) t^m (1-t)^n |a_m|.$$

First, we show that $E_n(t)$ is a well-defined function on $(0,1)$ for each $n \in \mathbb{N}_0$. When $p = 1$, for $a \in l_{w_1}^1$, we have

$$E_n(t) = \sum_{m=0}^{\infty} k_{\alpha,\beta}(m,n) t^m (1-t)^n [w_1(m)]^{-1} \cdot |a_m| w_1(m).$$

Moreover, we have, for each $t \in (0,1)$,

$$\begin{aligned} k_{\alpha,\beta}(m,n) t^m (1-t)^n [w_1(m)]^{-1} &= k_{\alpha,\beta}(m,n) t^m (1-t)^n (m+1)_\alpha (m+1)_\beta^{-1} \\ &= \frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\beta+1)\Gamma(n+\beta-\alpha+1)} t^m (1-t)^n. \end{aligned}$$

By Stirling's formula, we know that

$$\frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\beta+1)} = (m+\beta+1)^n [1+o(1)], \ \text{ as } \ m \to \infty.$$

Here and later, $o(1)$ denotes some sequence $\{\mathbf{a}_i\}_{i=0}^{\infty}$ with $o(1) = \mathbf{a}_i \to 0$ as $i \to \infty$, which will be different in different places. Hence, for $t \in (0,1)$,

$$\frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\beta+1)} t^m \to 0, \ \text{ as } \ m \to \infty.$$

Then, for each $t \in (0,1)$, there is a constant $M > 0$ such that

$$E_n(t) \le M \frac{(1-t)^n}{\Gamma(n+\beta-\alpha+1)} \sum_{m=0}^{\infty} |a_m| w_1(m).$$

This means that $E_n(t)$ is well-defined on $(0,1)$ for each $n \in \mathbb{N}_0$ when $p = 1$.

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When $p > 1$, by Hölder's inequality, we have

$$\begin{aligned} E_n(t) &= \sum_{m=0}^{\infty} \Big[k_{\alpha,\beta}(m,n)t^m(1-t)^n[w_1(m)]^{-\frac{1}{p}}\Big]\Big[a_m[w_1(m)]^{\frac{1}{p}}\Big] \\ &\le \Big\{\sum_{m=0}^{\infty}\Big[k_{\alpha,\beta}(m,n)t^m(1-t)^n[w_1(m)]^{-\frac{1}{p}}\Big]^q\Big\}^{\frac{1}{q}}\|a\|_{p,w_1} \\ &= \Big\{\sum_{m=0}^{\infty}\Big[\frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\beta+1)}\Big]^q\frac{\Gamma(m+\beta+1)}{\Gamma(m+1)}t^{qm}\Big\}^{\frac{1}{q}} \\ &\quad \frac{(1-t)^n}{\Gamma(n+\beta-\alpha+1)}\|a\|_{p,w_1}. \end{aligned} \tag{3.1}$$

We note that, using Stirling's formula again, we have for $m \ge 1$,

$$\begin{aligned} &\Big\{\Big[\frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\beta+1)}\Big]^q\frac{\Gamma(m+\beta+1)}{\Gamma(m+1)}t^{qm}\Big\}^{\frac{1}{m}} \\ &\quad = t^q\Big[\frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\beta+1)}\Big]^{\frac{q}{m}}\Big[\frac{\Gamma(m+\beta+1)}{\Gamma(m+1)}\Big]^{\frac{1}{m}} \\ &\quad = t^q(m+\beta)^{\frac{nq}{m}}[1+o(1)]^{\frac{q}{m}}m^{\frac{\beta}{m}}[1+o(1)]^{\frac{1}{m}} \\ &\quad\quad \to t^q < 1 \text{ as } m \to \infty. \end{aligned}$$

From the root test for series, we know that, for $t \in (0,1)$, the series

$$\sum_{m=0}^{\infty}\Big[\frac{\Gamma(n+m+\beta+1)}{\Gamma(m+\beta+1)}\Big]^q\frac{\Gamma(m+\beta+1)}{\Gamma(m+1)}t^{qm}$$

is convergent. It then follows from (3.1) that $E_n(t)$ is well-defined on $(0,1)$ for each $n \in \mathbb{N}_0$ when $p > 1$.

We shall give an estimate for $E_n^p(t), p \ge 1$. When $p > 1$, we have

$$\begin{aligned} E_n(t) &= \sum_{m=0}^{\infty}[k_{\alpha,\beta}(m,n)t^m(1-t)^n]^{\frac{1}{p}}|a_m|(m+1)_\alpha^{-\frac{1}{q}} \\ &\quad \times [k_{\alpha,\beta}(m,n)t^m(1-t)^n]^{\frac{1}{q}}(m+1)_\alpha^{\frac{1}{q}} \\ &\le \Big[\sum_{m=0}^{\infty}k_{\alpha,\beta}(m,n)t^m(1-t)^n|a_m|^p(m+1)_\alpha^{1-p}\Big]^{\frac{1}{p}} \\ &\quad \times \Big[\sum_{m=0}^{\infty}k_{\alpha,\beta}(m,n)t^m(1-t)^n(m+1)_\alpha\Big]^{\frac{1}{q}}. \end{aligned}$$

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It follows from Lemma 2.2 that

$$E_n^p(t) \le (1-t)^{-(\beta+1)(p-1)}(n+\beta-\alpha+1)_\alpha^{p-1} \times \Big[\sum_{m=0}^{\infty} k_{\alpha,\beta}(m,n)t^m(1-t)^n|a_m|^p(m+1)_\alpha^{1-p}\Big]. \tag{3.2}$$

When $p=1$, the inequality (3.2) obviously holds.
Hence, for $p \ge 1$, and $t \in (0,1)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} w_2(n)E_n^p(t) &= \sum_{n=0}^{\infty}(n+\beta-\alpha+1)_\alpha^{-p}(n+1)_\beta E_n^p(t)\\ &\le (1-t)^{-(\beta+1)(p-1)}\sum_{n=0}^{\infty}(n+\beta-\alpha+1)_\alpha^{-1}(n+1)_\beta\\ &\quad\times\Big[\sum_{m=0}^{\infty} k_{\alpha,\beta}(m,n)t^m(1-t)^n|a_m|^p(m+1)_\alpha^{1-p}\Big]\\ &= (1-t)^{-(\beta+1)(p-1)}\sum_{m=0}^{\infty}|a_m|^p(m+1)_\alpha^{1-p}\Big[\sum_{n=0}^{\infty} k_{\alpha,\beta}(m,n)t^m(1-t)^n(n+1)_{\beta-\alpha}\Big].\end{aligned}$$

By using Lemma 2.2, we get that

$$\begin{aligned}\sum_{n=0}^{\infty} w_2(n)E_n^p(t) &\le (1-t)^{-(\beta+1)(p-1)}t^{-\beta-1}\sum_{m=0}^{\infty}|a_m|^p(m+1)_\alpha^{1-p}(m+\alpha+1)_{\beta-\alpha}\\ &= (1-t)^{-(\beta+1)(p-1)}t^{-\beta-1}\sum_{m=0}^{\infty}|a_m|^p(m+1)_\alpha^{-p}(m+1)_\beta\\ &= (1-t)^{-(\beta+1)(p-1)}t^{-\beta-1}\sum_{m=0}^{\infty}|a_m|^p w_1(m).\end{aligned}$$

We let

$$A_n := \sum_{m=0}^{\infty}\int_{(0,1)} k_{\alpha,\beta}(m,n)t^m(1-t)^n|a_m|\,\mathrm{d}\mu(t).$$

Then we have

$$\begin{aligned}A_n &= \int_{(0,1)}\sum_{m=0}^{\infty} k_{\alpha,\beta}(m,n)t^m(1-t)^n|a_m|\,\mathrm{d}\mu(t)\\ &= \int_{(0,1)} E_n(t)\,\mathrm{d}\mu(t),\end{aligned}$$

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so that

$$\|H_\mu^{\alpha,\beta}(a)\|_{p,w_2} = \Big[\sum_{n=0}^{\infty} w_2(n)|H_\mu^{\alpha,\beta}(a)(n)|^p\Big]^{\frac{1}{p}}$$
$$\le \Big[\sum_{n=0}^{\infty} w_2(n)A_n^p\Big]^{\frac{1}{p}} = \Big[\sum_{n=0}^{\infty} w_2(n)\Big(\int_{(0,1)} E_n(t)\,\mathrm{d}\mu(t)\Big)^p\Big]^{\frac{1}{p}}.$$

It follows from Minkowski's inequality that

$$\|H_\mu^{\alpha,\beta}(a)\|_{p,w_2} \le \int_{(0,1)} \Big(\sum_{n=0}^{\infty} w_2(n)E_n^p(t)\Big)^{\frac{1}{p}}\,\mathrm{d}\mu(t)$$
$$\le \Big[\int_{(0,1)} (1-t)^{-(1-\frac{1}{p})(\beta+1)}t^{-\frac{1}{p}(\beta+1)}\,\mathrm{d}\mu(t)\Big]\Big[\sum_{m=0}^{\infty}|a_m|^p w_1(m)\Big]^{\frac{1}{p}}$$
$$= \mathcal{C}_\mu(\beta,p)\|a\|_{p,w_1}.$$

This proves that $H_\mu^{\alpha,\beta} : l^p_{w_1} \to l^p_{w_2}$ is bounded when $\mathcal{C}_\mu(\beta,p) < \infty$ and $\|H_\mu^{\alpha,\beta}\| \le \mathcal{C}_\mu(\beta,p)$. Proposition 3.1 is proved.

3.2. Proof of Proposition 3.2

Let $\varepsilon \in (0, \frac{1}{p}(\beta+1))$, we define the sequence $a = \{a_m\}_{m=0}^{\infty}$ as

$$a_m := (m+1)_\alpha (m+1)_\beta^{-\frac{1}{p}}(m+\beta+1)^{-(\frac{1}{p}+\varepsilon)},\ m \in \mathbb{N}_0. \tag{3.3}$$

Then

$$\sum_{m=0}^{\infty} w_1(m)a_m^p = \sum_{m=0}^{\infty}(m+\beta+1)^{-1-p\varepsilon} < \infty.$$

To prove Proposition 3.2, we need the following result.

Claim 3.3. *Let $\rho,\eta \in (0,1/2)$. For any $\varepsilon \in (0, \frac{1}{p}(\beta+1))$, we can find a natural number $\mathbf{N}_0 = \mathbf{N}_0(\rho,\eta)$ such that*

$$[w_2(n)]^{\frac{1}{p}}A_n \ge (1-\rho)^2[w_1(n)]^{\frac{1}{p}}a_n \int_\eta^{1-\eta} (1-t)^{-(1-\frac{1}{p})(\beta+1)+\varepsilon}t^{-\frac{1}{p}(\beta+1)-\varepsilon}\,\mathrm{d}\mu(t),$$

for all $n \ge \mathbf{N}_0$.

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Proof of Claim 3.3. We first fix two positive numbers b and τ. We let

$$b := \frac{1}{p}(\beta+1)+\varepsilon$$

and

$$\tau := \tau(\rho) = (1-\rho)^{-\frac{p}{2(\beta+1)}} - 1.$$

By Lemma 2.5, we have

$$a_m \geq (m+1)_\alpha (m+1)^{-\frac{1}{p}(\beta+1)}(m+\beta+1)^{-\varepsilon} \geq (m+1)_\alpha (m+1)^{-\frac{1}{p}(\beta+1)-\varepsilon},$$

for $\beta \in (-1,0)$, and

$$a_m \geq (m+1)_\alpha (m+\beta+1)^{-\frac{1}{p}(\beta+1)-\varepsilon},$$

for $\beta \geq 0$. By Lemma 2.4, we obtain that

$$a_m \geq \frac{(m+1)_\alpha}{\Gamma(b)} \int_0^\infty \mathrm{e}^{-(m+1)x} x^{b-1}\,\mathrm{d}x,$$

for $\beta \in (-1,0)$, and

$$a_m \geq \frac{(m+1)_\alpha}{\Gamma(b)} \int_0^\infty \mathrm{e}^{-(m+\beta+1)x} x^{b-1}\,\mathrm{d}x,$$

for $\beta \geq 0$.

Thus, for $\beta \in (-1,0)$, we have

$$\begin{aligned} E_n(t) &= \sum_{m=0}^\infty k_{\alpha,\beta}(m,n) t^m (1-t)^n |a_m| \\ &\geq \frac{(1-t)^n}{\Gamma(b)} \int_0^\infty \mathrm{e}^{-x} x^{b-1} \sum_{m=0}^\infty k_{\alpha,\beta}(m,n) t^m \mathrm{e}^{-mx} (m+1)_\alpha \,\mathrm{d}x. \end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned} E_n(t) &\geq (n+\beta-\alpha+1)_\alpha \frac{(1-t)^n}{\Gamma(b)} \int_0^\infty \mathrm{e}^{-x} x^{b-1} (1-t\mathrm{e}^{-x})^{-n-\beta-1}\,\mathrm{d}x \\ &= (n+\beta-\alpha+1)_\alpha \frac{(1-t)^{-\beta-1}}{\Gamma(b)} \int_0^\infty \mathrm{e}^{-x} x^{b-1} \Big(\frac{1-t}{1-t\mathrm{e}^{-x}}\Big)^{n+\beta+1}\,\mathrm{d}x. \end{aligned}$$

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Then, from Lemma 2.3 and again Lemma 2.4, we obtain that

$$\begin{aligned} E_n(t) &\geq (n+\beta-\alpha+1)_\alpha \frac{(1-t)^{-\beta-1}}{\Gamma(b)} \int_0^\infty \mathrm{e}^{-x} x^{b-1} \mathrm{e}^{-(n+\beta+1)tx(1-t)^{-1}}\, \mathrm{d}x \\ &= (n+\beta-\alpha+1)_\alpha (1-t)^{-\beta-1} \Big[\frac{1+(n+\beta)t}{1-t}\Big]^{-b} := F_n(t), \end{aligned} \tag{3.4}$$

for $\beta \in (-1,0)$. For $\beta \geq 0$, in the same way, we can obtain that

$$E_n(t) \geq (n+\beta-\alpha+1)_\alpha (1-t)^{-\beta-1} \Big[\frac{\beta+1+nt}{1-t}\Big]^{-b} := G_n(t). \tag{3.5}$$

Hereafter we will assume $n \geq 1$. Then, in order to estimate $F_n(t)$, note that when $t \geq \frac{1}{\tau(n+\beta)}$, it holds that

$$\frac{1+(n+\beta)t}{t(n+\beta+1)} = 1 + \frac{1-t}{t(n+\beta+1)} \leq 1 + \frac{1}{t(n+\beta)} \leq 1+\tau.$$

Then it follows from $\varepsilon \in (0, \frac{1}{p}(\beta+1))$ that

$$\Big[\frac{1+(n+\beta)t}{t(n+\beta+1)}\Big]^{-b} \geq (1+\tau)^{-\frac{2}{p}(\beta+1)} = 1-\rho. \tag{3.6}$$

Also, by Stirling's formula, we have

$$(n+1)_\beta^{\frac{1}{p}} (n+\beta+1)^{-\frac{\beta}{p}} \to 1, \ \text{ as } \ n \to \infty.$$

Hence we can take a constant $\mathbf{N}_1 = \mathbf{N}_1(\rho) \in \mathbb{N}$ such that

$$(n+1)_\beta^{\frac{1}{p}} (n+\beta+1)^{-\frac{\beta}{p}} \geq 1-\rho, \tag{3.7}$$

when $n \geq \mathbf{N}_1$. Meanwhile, we can write

$$\begin{aligned} [w_2(n)]^{\frac{1}{p}} F_n(t) &= (1-t)^{-(\beta+1)+b} t^{-b} a_n [w_1(n)]^{\frac{1}{p}} \\ &\quad \times \Big[\frac{1+(n+\beta)t}{t(n+\beta+1)}\Big]^{-b} (n+1)_\beta^{\frac{1}{p}} (n+\beta+1)^{-\frac{\beta}{p}}. \end{aligned} \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we obtain that

$$[w_2(n)]^{\frac{1}{p}} F_n(t) \geq (1-\rho)^2 (1-t)^{-(\beta+1)+b} t^{-b} a_n [w_1(n)]^{\frac{1}{p}}, \tag{3.9}$$

when $n \geq \mathbf{N}_1$ and $t \geq \frac{1}{\tau(n+\beta)}$.

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To estimate $G_n(t)$, note that when $t \geq \frac{(\beta+1)}{\tau(n+\beta)}$, we have

$$\frac{\beta+1+nt}{t(n+\beta+1)} = 1 + \frac{(1-t)(\beta+1)}{t(n+\beta+1)} \leq 1 + \frac{\beta+1}{t(n+\beta)} \leq 1+\tau,$$

so that

$$\Big[\frac{\beta+1+nt}{t(n+\beta+1)}\Big]^{-b} \geq (1+\tau)^{-\frac{2}{p}(\beta+1)} = 1-\rho. \tag{3.10}$$

We write

$$\begin{aligned}[w_2(n)]^{\frac{1}{p}} G_n(t) = (1-t)^{-(\beta+1)+b} t^{-b} a_n [w_1(n)]^{\frac{1}{p}} \\ \times \Big[\frac{\beta+1+nt}{t(n+\beta+1)}\Big]^{-b} (n+1)^{\frac{1}{\beta}\frac{p}{\beta}} (n+\beta+1)^{-\frac{\beta}{p}}.\end{aligned} \tag{3.11}$$

It follows from (3.7), (3.10) and (3.11) that

$$[w_2(n)]^{\frac{1}{p}} G_n(t) \geq (1-\rho)^2 (1-t)^{-(\beta+1)+b} t^{-b} a_n [w_1(n)]^{\frac{1}{p}}, \tag{3.12}$$

when $n \geq \mathbf{N}_1$ and $t \geq \frac{(\beta+1)}{\tau(n+\beta)}$.

Now, for $\beta \in (-1,0)$, note that when

$$n \geq \mathbf{N}_2 = \mathbf{N}_2(\rho,\eta) := \lceil (\tau\eta)^{-1} \rceil + 2,$$

it holds that $\frac{1}{\tau(n+\beta)} \leq \eta$. Here, $\lceil x \rceil$ is the ceiling function. Hence, we obtain from (3.4) and (3.9) that

$$\begin{aligned}[w_2(n)]^{\frac{1}{p}} A_n &= \int_{(0,1)} [w_2(n)]^{\frac{1}{p}} E_n(t)\, \mathrm{d}\mu(t) \\ &\geq (1-\rho)^2 a_n [w_1(n)]^{\frac{1}{p}} \int_{\frac{1}{\tau(n+\beta)}}^{1-\frac{1}{\tau(n+\beta)}} (1-t)^{-(\beta+1)+b} t^{-b}\, \mathrm{d}\mu(t) \\ &\geq (1-\rho)^2 a_n [w_1(n)]^{\frac{1}{p}} \int_{\eta}^{1-\eta} (1-t)^{-(\beta+1)+b} t^{-b}\, \mathrm{d}\mu(t),\end{aligned}$$

when $n \geq \max\{\mathbf{N}_1, \mathbf{N}_2\}$.

Similarly, for $\beta \geq 0$, when

$$n \geq \mathbf{N}_3 = \mathbf{N}_3(\rho,\eta) := \lceil (\beta+1)(\tau\eta)^{-1} \rceil + 1,$$

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it holds that $\frac{\beta+1}{\tau(n+\beta)} \le \eta$, so that, from (3.5) and (3.12), we have

$$\begin{aligned}[w_2(n)]^{\frac{1}{p}} A_n &= \int_{(0,1)} [w_2(n)]^{\frac{1}{p}} E_n(t)\,\mathrm{d}\mu(t)\\ &\ge (1-\rho)^2 a_n [w_1(n)]^{\frac{1}{p}} \int_{\frac{\beta+1}{\tau(n+\beta)}}^{1-\frac{\beta+1}{\tau(n+\beta)}} (1-t)^{-(\beta+1)+b} t^{-b}\,\mathrm{d}\mu(t)\\ &\ge (1-\rho)^2 a_n [w_1(n)]^{\frac{1}{p}} \int_{\eta}^{1-\eta} (1-t)^{-(\beta+1)+b} t^{-b}\,\mathrm{d}\mu(t),\end{aligned}$$

when $n \ge \max\{\mathbf{N}_1, \mathbf{N}_3\}$.

Then the claim follows if we take $\mathbf{N}_0 = \mathbf{N}_0(\rho,\eta) = \max\{\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$. The proof of the claim is finished. □

We proceed to the proof of Proposition 3.2. We still let $\varepsilon \in (0, \frac{1}{p}(\beta+1))$ and take the same $a = \{a_m\}_{m=0}^{\infty}$ as in (3.3). Let $\rho, \delta \in (0, 1/2)$, by using Claim 3.3, for any $\varepsilon \in (0, \frac{1}{p}(\beta+1))$, we can take a natural number $\widehat{\mathbf{N}}_0 = \widehat{\mathbf{N}}_0(\rho,\delta)$ such that

$$[w_2(n)]^{\frac{1}{p}} A_n \ge (1-\rho)^2 [w_1(n)]^{\frac{1}{p}} a_n \int_{\delta}^{1-\delta} (1-t)^{-(1-\frac{1}{p})(\beta+1)+\varepsilon} t^{-\frac{1}{p}(\beta+1)-\varepsilon}\,\mathrm{d}\mu(t), \quad (3.13)$$

for $n \ge \widehat{\mathbf{N}}_0$. By using Lemma 2.6 with respect to $\rho, \widehat{\mathbf{N}}_0$, we see that there is a constant $\varepsilon_1 = \varepsilon_1(\rho, \widehat{\mathbf{N}}_0(\rho,\delta)) \in (0, \frac{1}{p}(\beta+1))$ such that, for $\varepsilon \in (0, \varepsilon_1]$,

$$\sum_{n=\widehat{\mathbf{N}}_0+1}^{\infty} w_1(n) a_n^p \ge (1-\rho) \sum_{n=0}^{\infty} w_1(n) a_n^p. \quad (3.14)$$

It follows from (3.13) and (3.14) that, for $\varepsilon \in (0, \varepsilon_1]$,

$$\begin{aligned}\|H_\mu^{\alpha,\beta}(a)\|_{p,w_2} &= \Big[\sum_{n=0}^{\infty} w_2(n) A_n^p\Big]^{\frac{1}{p}} \ge \Big[\sum_{n=\widehat{\mathbf{N}}_0+1}^{\infty} w_2(n) A_n^p\Big]^{\frac{1}{p}}\\ &\ge (1-\rho)^2 \int_{\delta}^{1-\delta} (1-t)^{-(1-\frac{1}{p})(\beta+1)+\varepsilon} t^{-\frac{1}{p}(\beta+1)-\varepsilon}\,\mathrm{d}\mu(t) \Big[\sum_{n=\widehat{\mathbf{N}}_0+1}^{\infty} w_1(n) a_n^p\Big]^{\frac{1}{p}}\\ &\ge (1-\rho)^{2+\frac{1}{p}} \int_{\delta}^{1-\delta} (1-t)^{-(1-\frac{1}{p})(\beta+1)+\varepsilon} t^{-\frac{1}{p}(\beta+1)-\varepsilon}\,\mathrm{d}\mu(t) \|a\|_{p,w_1}.\end{aligned}$$

Then, for $\varepsilon \in (0, \varepsilon_1]$,

$$\|H_\mu^{\alpha,\beta}\| \ge (1-\rho)^{2+\frac{1}{p}} \int_{\delta}^{1-\delta} (1-t)^{-(1-\frac{1}{p})(\beta+1)+\varepsilon} t^{-\frac{1}{p}(\beta+1)-\varepsilon}\,\mathrm{d}\mu(t).$$

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Note that on the interval $[\delta, 1-\delta]$, the functions

$$f_\varepsilon(t) = (1-t)^{-(1-\frac{1}{p})(\beta+1)+\varepsilon} t^{-\frac{1}{p}(\beta+1)-\varepsilon}, \varepsilon \in (0, \varepsilon_1],$$

are uniformly bounded above and μ is finite on $[\delta, 1-\delta]$. Then, let $\varepsilon \to 0^+$, by the dominated convergence theorem, we see that

$$\|H_\mu^{\alpha,\beta}\| \geq (1-\rho)^{2+\frac{1}{p}} \int_\delta^{1-\delta} (1-t)^{-(1-\frac{1}{p})(\beta+1)} t^{-\frac{1}{p}(\beta+1)} \, \mathrm{d}\mu(t).$$

Consequently, let $\delta \to 0^+$, we obtain from the monotone convergence theorem that

$$\|H_\mu^{\alpha,\beta}\| \geq (1-\rho)^{2+\frac{1}{p}} \int_0^1 (1-t)^{-(1-\frac{1}{p})(\beta+1)} t^{-\frac{1}{p}(\beta+1)} \, \mathrm{d}\mu(t).$$

Finally, let $\rho \to 0^+$, the boundedness of $\mathcal{C}_\mu(\beta, p)$ follows and $\|H_\mu^{\alpha,\beta}\| \geq \mathcal{C}_\mu(\beta, p)$. This proves Proposition 3.2.

Now, the proof of Theorem 1.3 is done.

4. Proof of Theorem 1.4

If $\mathcal{C}_\mu(\beta, \infty) < \infty$, for $a = \{a_m\}_{m=0}^\infty \in l^\infty_{\overline{w}_1}$, we have

$$\begin{aligned}
\|H_\mu^{\alpha,\beta}(a)\|_{\infty,\overline{w}_2} &\leq \sup_{n\in\mathbb{N}_0} [\overline{w}_2(n)] A_n \\
&= \sup_{n\in\mathbb{N}_0} [\overline{w}_2(n)] \int_{(0,1)} \sum_{m=0}^\infty k_{\alpha,\beta}(m,n) t^m (1-t)^n |a_m| \, \mathrm{d}\mu(t) \\
&= \sup_{n\in\mathbb{N}_0} [\overline{w}_2(n)] \int_{(0,1)} \sum_{m=0}^\infty k_{\alpha,\beta}(m,n) t^m (1-t)^n (m+1)_\alpha \cdot |a_m| [\overline{w}_1(m)] \, \mathrm{d}\mu(t).
\end{aligned}$$

Here A_n is the same as in the proof of Theorem 1.3. Then, it follows from Lemma 2.2 that

$$\|H_\mu^{\alpha,\beta}(a)\|_{\infty,\overline{w}_2} \leq \int_{(0,1)} (1-t)^{-\beta-1} \, \mathrm{d}\mu(t) \|a\|_{\infty,\overline{w}_1} = \mathcal{C}_\mu(\beta,\infty) \|a\|_{\infty,\overline{w}_1}.$$

This proves that $H_\mu^{\alpha,\beta}$ is bounded from $l^\infty_{\overline{w}_1}$ into $l^\infty_{\overline{w}_2}$, and $\|H_\mu^{\alpha,\beta}\| \leq \mathcal{C}_\mu(\beta,\infty)$.

To prove $\|H_\mu^{\alpha,\beta}\| = \mathcal{C}_\mu(\beta,\infty)$. We take $a = \{a_m\}_{m=0}^\infty$ as

$$a_m = (m+1)_\alpha, \; m \in \mathbb{N}_0. \tag{4.1}$$

Then $\|a\|_{\infty,w_1} = 1$, and by Lemma 2.2 again, we have

$$\|H_\mu^{\alpha,\beta}(a)\|_{\infty,\overline{w}_2} = \int_{(0,1)} (1-t)^{-\beta-1} \, \mathrm{d}\mu(t) \|a\|_{\infty,\overline{w}_1} = \mathcal{C}_\mu(\beta,\infty) \|a\|_{\infty,\overline{w}_1}. \tag{4.2}$$

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This implies that $\|H_\mu^{\alpha,\beta}\| = \mathcal{C}_\mu(\beta,\infty)$. Also, if $H_\mu^{\alpha,\beta} : l^\infty_{\overline{w}_1} \to l^\infty_{\overline{w}_2}$ is bounded, take the sequence a as in (4.1), and by (4.2), we can obtain that $\mathcal{C}_\mu(\beta,\infty) < \infty$. This completes the proof of Theorem 1.4.

5. Final remarks

In this section, we consider the generalized Hilbert matrix operator acting on the spaces of analytic functions in the unit disk $\mathbb{D}$. We denote by $\mathcal{H}(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let $\beta > -1$, μ be a positive finite Borel measure on $(0,1)$. For $f = \sum_{m=0}^\infty a_m z^m \in \mathcal{H}(\mathbb{D})$, we formally define

$$H_\mu^\beta(f)(z) := \sum_{n=0}^\infty \Big[\sum_{m=0}^\infty \int_{(0,1)} \binom{n+m+\beta}{m} t^m (1-t)^n a_m \, \mathrm{d}\mu(t) \Big] z^n, \ z \in \mathbb{D}.$$

When $\beta = 0$, the operator H_μ^0 has been studied in [3] and [5], where the authors characterize the measure μ such that the operator H_μ^0 is bounded on Hardy spaces and Bergman spaces. We will consider the operator H_μ^β acting on the Dirichlet-type space $\mathcal{D}_\lambda$.

For $\lambda \in \mathbb{R}$, the Dirichlet-type space $\mathcal{D}_\lambda$ is defined as

$$\mathcal{D}_\lambda = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathcal{D}_\lambda} := (\sum_{n=0}^\infty (n+1)^{1-\lambda} |a_n|^2)^{\frac{1}{2}} < \infty\}.$$

When $\lambda = 0$, $\mathcal{D}_0$ coincides the classic Dirichlet space $\mathcal{D}$, and when $\lambda = 1$, $\mathcal{D}_1$ becomes the Hardy space H^2. More generalized Dirichlet-type spaces can be found in [27].

Note that, for $n \geq 0, s > -1$, there exist two positive constants C_1, C_2, which are independent of the number n, such that $C_1(n+1)^s \leq (n+1)_s \leq C_2(n+1)^s$. Then, we see from (1) of Corollary 1.5 that

Theorem 5.1. *Let $\beta > -1$ and let μ be a positive finite Borel measure on $(0,1)$. Let H_μ^β be as above. Then H_μ^β is bounded on $\mathcal{D}_{1-\beta}$ if and only if*

$$\mathcal{C}_\beta(\mu) := \int_{(0,1)} (1-t)^{-\frac{1}{2}(\beta+1)} t^{-\frac{1}{2}(\beta+1)} \, \mathrm{d}\mu(t) < \infty.$$

Moreover, if $\beta = 1$, and

$$\mathcal{C}_1(\mu) = \int_{(0,1)} (1-t)^{-1} t^{-1} \, \mathrm{d}\mu(t) < \infty,$$

then H_μ^1 is bounded on $\mathcal{D}$ and the norm of H_μ^1 is $\mathcal{C}_1(\mu)$, and if $\beta = 0$,

$$\mathcal{C}_0(\mu) = \int_{(0,1)} (1-t)^{-\frac{1}{2}} t^{-\frac{1}{2}} \, \mathrm{d}\mu(t) < \infty,$$

then H_μ^0 is bounded on H^2 and the norm of H_μ^0 is $\mathcal{C}_0(\mu)$.

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We end the paper with the following general problem:

Problem 5.2. *Characterize the measure μ such that H^β_μ is bounded from one analytic function space X to another one Y in $\mathbb{D}$.*

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