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Fully admissible binary relations in topology

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A fully admissible binary relation (3) is an operator $\mathscr{L}$, other than the *equality* operator $\mathscr{I}$ and *universal* operator $\mathscr{S}$, which assigns to each space $|S, \tau|$ a reflexive, symmetric, binary relation $\mathscr{L}^\tau$, and which is such that for any continuous mapping $f:|S,\tau| \to |S',\tau'|$, $x\mathscr{L}^\tau y$ implies $f(x)\,\mathscr{L}^{\tau'} f(y)$. With each such relation $\mathscr{L}$, we associate a 'separation axiom' $T_{\mathscr{L}}$, as well as '$\mathscr{L}$-regularity' and '$\mathscr{L}$-connectedness', where $T_{\mathscr{L}} \equiv \mathscr{L}$-regularity $+ T_0$, and $\mathscr{L}$-regularity $+ \mathscr{L}$-connectedness $\equiv$ indiscreteness.

Given binary relations $\mathscr{A}_1$, $\mathscr{A}_2$ among the elements of the same set S, we write $\mathscr{A}_1 \leqslant \mathscr{A}_2$ if and only if $x\mathscr{A}_1 y$ implies $x\mathscr{A}_2 y$, for any $x, y \in S$. A partial ordering of fully admissible relations is achieved by writing, for any pair $\mathscr{L}_1, \mathscr{L}_2$, $\mathscr{L}_1 \leqslant \mathscr{L}_2$ if and only if, for every space $|S, \tau|$, $\mathscr{L}_1^\tau \leqslant \mathscr{L}_2^\tau$.

Our aim in this paper is to establish certain facts concerning fully admissible relations, such as the existence of a greatest, and to apply this information in a systematic study of the associated axioms.

1. *The relations* $\mathscr{R}, \mathscr{B}, \mathscr{C}, \mathscr{M}$, and $\mathscr{N}$. Fully admissible relations $\mathscr{R}, \mathscr{B}, \mathscr{C}, \mathscr{M}$, and $\mathscr{N}$ (three of which we recall from (3)) are defined by writing, for any elements x, y of any space $|S, \tau|$: $x\mathscr{R}^\tau y$ if and only if $(\bar{x}) = (\bar{y})$; $x\mathscr{B}^\tau y$ if and only if $x \in (\bar{y})$ or $y \in (\bar{x})$; $x\mathscr{C}^\tau y$ if and only if $(\bar{x}) \cap (\bar{y}) \neq \varnothing$; $x\mathscr{M}^\tau y$ if and only if $h(x) = h(y)$ for each continuous, real-valued function h on S; $x\mathscr{N}^\tau y$ if and only if each open-and-closed set which contains x contains y also.

1·1. Lemma. *If* $\mathscr{L}$ *is any fully admissible relation, then every discrete space is a* $T_{\mathscr{L}}$ *space.*

Proof. As $\mathscr{L} \neq \mathscr{S}$, there exists a space $|S_1, \tau_1|$ with elements a, b such that $a\mathscr{L}^{\tau_1} b$ is not true. If x, y are distinct elements of any discrete $|S, \tau|$, a continuous mapping $f: S \to S_1$ can be defined such that $f(x) = a$ and $f(y) = b$. Thus $x\mathscr{L}^\tau y$ is not true.

1·2. Theorem. (*a*) *If* $\mathscr{L}$ *is any fully admissible relation, then* $\mathscr{R} \leqslant \mathscr{L} \leqslant \mathscr{N}$.
(*b*) *If* $\mathscr{L}$ *is a fully admissible relation other than* $\mathscr{R}$, *then* $\mathscr{B} \leqslant \mathscr{L}$.
(*c*) *If* $\mathscr{L}$ *is a fully admissible equivalence relation other than* $\mathscr{R}$, *then* $\mathscr{C} \leqslant \mathscr{L}$.

Proof. (*a*) It suffices (by (3), 4·4) to verify that $\mathscr{L} \leqslant \mathscr{N}$. Let x, y be elements of any space $|S, \tau|$ such that $x\mathscr{N}^\tau y$ is not true, and let $|S_1, \tau_1|$ be any two-element discrete space. A continuous mapping $f: S \to S_1$ can be defined such that $f(x) \neq f(y)$. By the preceding lemma it follows that $x\mathscr{L}^\tau y$ is not true.

(*b*) As $\mathscr{L} \neq \mathscr{R}$, there exists (by (*a*)) a space $|S_1, \tau_1|$ with elements a, b such that $a\mathscr{L}^{\tau_1} b$ yet $a\mathscr{R}^{\tau_1} b$ is not true; that is, there exists a τ_1-open set G such that (for example)

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$a \in G$ and $b \notin G$. Now if x, y are distinct elements of a given space $|S, \tau|$ such that $x\mathscr{B}^\tau y$, then (by (3), 4·3) $x\mathscr{B}^{\tau_A} y$, where $A \equiv (x, y)$.

Certainly $x\mathscr{L}^{\tau_A} y$ if $|A, \tau_A|$ is indiscrete. If, on the other hand, the subspace is not indiscrete, we may assume (as $\mathscr{L}$ is symmetric) that (x) is τ_A-open and (y) is not τ_A-open. The mapping $f : S_1 \to A$ such that $f(c) = x$ for each $c \in G$ and $f(c) = y$ for each $c \notin G$ is therefore continuous. Thus $f(a)\mathscr{L}^{\tau_A} f(b)$; that is, $x\mathscr{L}^{\tau_A} y$.

By continuity of the inclusion mapping $A \to S$, we deduce in either case that $x\mathscr{L}^\tau y$.

(*c*) If x, y are elements of any space $|S, \tau|$ such that $x\mathscr{C}^\tau y$, there exists $a \in (\bar{x}) \cap (\bar{y})$. Clearly $x\mathscr{B}^\tau a$ and $a\mathscr{B}^\tau y$. By (*b*), however, $\mathscr{B} \leqslant \mathscr{L}$; so $x\mathscr{L}^\tau y$ by transitivity of $\mathscr{L}^\tau$.

By the following result, the strongest separation axiom, generated by a fully admissible relation, which is satisfied by all metrizable spaces is $T_{\mathscr{M}}$, the property of being functionally Hausdorff.

1·3. PROPOSITION. *For a given fully admissible relation $\mathscr{L}$, the following are equivalent*: (*a*) *any functionally Hausdorff space is a $T_{\mathscr{L}}$ space*; (*b*) *any metrizable space is a $T_{\mathscr{L}}$ space*; (*c*) *the standard space $|R, \tau'|$ of real numbers is a $T_{\mathscr{L}}$ space*; (*d*) $\mathscr{L} \leqslant \mathscr{M}$.

Proof. It suffices to verify that (*c*) implies (*d*). If h is a continuous, real-valued function which distinguishes elements x, y of a given space $|S, \tau|$, $h(x)\mathscr{L}^{\tau'} h(y)$ is not true; so $x\mathscr{L}^\tau y$ is not true.

2. *Perfect separation axioms.* The term *trivial* will be employed to describe a space which contains just one element. A topological property p, other than one which is possessed by all spaces or exclusive to trivial spaces, will be called a *perfect separation axiom* if and only if it is productive, hereditary, and preserved under refinements of topology. Examples of perfect separation axioms are T_0, T_1, T_2, the property of being functionally Hausdorff, and total separatedness. We shall see presently that perfect separation axioms are precisely the separation axioms generated by fully admissible relations.

2·1. PROPOSITION. *If $\mathscr{L}$ is any fully admissible relation, then $T_{\mathscr{L}}$ is a perfect separation axiom.*

Proof. By 1·1, $T_{\mathscr{L}}$ is not exclusive to trivial spaces. As $T_{\mathscr{L}}$ implies T_0, the former is not satisfied by all spaces. $T_{\mathscr{L}}$ was shown in (3) to be productive, hereditary, and preserved under refinements of topology.

2·2. THEOREM. *If p is any perfect separation axiom, there exists a greatest fully admissible relation $\mathscr{L}_p$ among those fully admissible relations $\mathscr{L}$ such that $p \equiv T_{\mathscr{L}}$. For any space $|S, \tau|$, $\mathscr{L}_p^\tau$ is the smallest equivalence relation among the elements of S for which the corresponding quotient space possesses p.*

Proof. Given any space $|S, \tau|$, the family $\{\mathscr{A}_i\}_I$ of all equivalence relations $\mathscr{A}_i$ on S for which the quotient space $|S/\mathscr{A}_i, \tau(\mathscr{A}_i)|$ possesses p has at least one member, namely $\mathscr{S}^\tau$. Let us denote by $\{P_i\}_I$ the corresponding family of quotient mappings and by $|S^*, \tau^*|$ the product of $\{|S/\mathscr{A}_i, \tau(\mathscr{A}_i)|\}_I$, and let $\mathscr{L}_p^\tau \equiv \bigcap_I \mathscr{A}_i$. For any $a, b \in S$, $a\mathscr{L}_p^\tau b$ if and only if $\psi(a) = \psi(b)$, where ψ is the unique, continuous mapping $S \to S^*$ such that $\pi_i(\psi(x)) = P_i(x)$ for each $i \in I$, $x \in S$. Being therefore homeomorphic to $\psi(S)$, endowed with the quotient topology (which is finer than $\tau^*_{\psi(S)}$), $|S/\mathscr{L}_p^\tau, \tau(\mathscr{L}_p^\tau)|$ is a p-space. (1)

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We now claim that for any $x, y \in S$, $x \mathscr{L}_p^\tau y$ if and only if $f(x) = f(y)$ for each p-space $|S', \tau'|$ and continuous mapping $f: S \to S'$. (2)

For if $|S', \tau'|$ and f are as described, and $\mathscr{A}$ is the equivalence on S such that $a \mathscr{A} b$ if and only if $f(a) = f(b)$, then $|S/\mathscr{A}, \tau(\mathscr{A})|$ is homeomorphic to $f(S)$, the latter being endowed with the quotient topology. Thus $|S/\mathscr{A}, \tau(\mathscr{A})|$ is a p-space and consequently $\mathscr{L}_p^\tau \leqslant \mathscr{A}$. The sufficiency is immediate from (1).

The existence of some space which does not possess p ensures that $\mathscr{L}_p \neq \mathscr{I}$ (where $\mathscr{L}_p$ is the operator which assigns to each space $|S, \tau|$ the equivalence $\mathscr{L}_p^\tau$). On the other hand, as there exists a non-trivial p-space, $\mathscr{L}_p \neq \mathscr{S}$. It is clear from (2) that $\mathscr{L}_p$ fulfils the remaining condition for full admissibility.

Clearly, $T_{\mathscr{L}_p} \equiv p$. Finally, it is easily deduced from (2) that $\mathscr{L} \leqslant \mathscr{L}_p$, for any fully admissible relation $\mathscr{L}$ such that $T_{\mathscr{L}} \equiv p$.

2·3. COROLLARY. *All perfect separation axioms lie in logical strength between total separatedness and T_0. With the exception of T_0, all perfect separation axioms imply T_1; but no such axiom implies T_3.*

Proof. This follows from 1·2 and 2·2, since $T_{\mathscr{N}}$ is total separatedness, which does not imply T_3.

2·4. THEOREM. *Every perfect separation axiom satisfied by the space of real numbers is satisfied by all functionally Hausdorff spaces.*

Proof. From 1·3 and 2·2.

3. *Perfect regularity axioms.* Given topologies τ_1, τ_2 on the same set S, the latter will be said to be a *narrow refinement* of the former, if and only if $\tau_1 \leqslant \tau_2$ and $\mathscr{R}^{\tau_1} = \mathscr{R}^{\tau_2}$. A topological property r will be called a *perfect regularity axiom* if and only if each of the following holds:

(a) r is possessed by all indiscrete spaces but not exclusive to such spaces,

(b) r is preserved under narrow refinements of topology,

(c) if $\{\tau_i\}_I$ is a family of r-topologies on the same set S, then $|S, \tau|$ is an r-space, where τ is the supremum of $\{\tau_i\}_I$, and

(d) if $|S', \tau'|$ is a given r-space, S is any set, and $f: S \to S'$ is any function, then $|S, \tau|$ is an r-space, where τ is the topology induced on S by f.

3·1. LEMMA. *If τ is the supremum of a family $\{\tau_i\}_I$ of topologies on the same set S, then*: (i) *for any fully admissible relation $\mathscr{L}$, $\mathscr{L}^\tau \leqslant \bigcap_I \mathscr{L}^{\tau_i}$*; (ii) $\mathscr{R}^\tau = \bigcap_I \mathscr{R}^{\tau_i}$.

3·2. LEMMA. *If $f: S \to |S', \tau'|$ is any function, and τ is the topology induced on S by f, then for any $x, y \in S$, $x \mathscr{R}^\tau y$ if and only if $f(x) \mathscr{R}^{\tau'} f(y)$.*

3·3. PROPOSITION. *If $\mathscr{L}$ is any fully admissible relation, then $\mathscr{L}$-regularity is a perfect regularity axiom.*

Proof. $\mathscr{L}$-regularity is not exclusive to indiscrete spaces by 1·1. Let τ be the supre-

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mum of a given family $\{\tau_i\}_I$ of $\mathscr{L}$-regular topologies on the same set S. By 3·1, then,

$$\mathscr{L}^\tau \leqslant \bigcap_I \mathscr{L}^{\tau_i} = \bigcap_I \mathscr{R}^{\tau_i} = \mathscr{R}^\tau.$$

Thus (by 1·2(*a*)) $\mathscr{L}^\tau = \mathscr{R}^\tau$.

The remainder of the proof is omitted.

3·4. Theorem. *If r is any perfect regularity axiom, there exists a greatest fully admissible relation $\mathscr{L}_r$ among those fully admissible relations $\mathscr{L}$ such that $r \equiv \mathscr{L}$-regularity.*

Proof. Given any space $|S, \tau|$, the family $\{\tau_i\}_I$ of r-topologies on S which are coarser than τ has at least one member, and the supremum τ' of this family is itself an r-topology. Consider, then, the operator $\mathscr{L}_r$ which assigns to each space $|S, \tau|$ the (equivalence) relation $\mathscr{L}_r^\tau \equiv \mathscr{R}^{\tau'}$. As $\mathscr{R}^\tau \leqslant \mathscr{L}_r^\tau$ for all spaces, $\mathscr{L}_r \neq \mathscr{I}$. On the other hand, the non-universality of $\mathscr{L}_r$ is ensured by the existence of an r-space which is not indiscrete ($\equiv \mathscr{R}$-connected).

Let $f : |S_1, \tau_1| \to |S_2, \tau_2|$ be any continuous mapping and let τ_1'' be the topology induced on S_1 by $f : S_1 \to |S_2, \tau_2'|$. As r is a perfect regularity axiom, τ_1'' is an r-topology on S_1, and, of course, $\tau_1'' \leqslant \tau_1$; so in fact $\tau_1'' \leqslant \tau_1'$. Thus f is (τ_1', τ_2')-continuous. By full admissibility of $\mathscr{R}$, it follows that $x \mathscr{L}_r^{\tau_1} y$ implies $f(x) \mathscr{L}_r^{\tau_2} f(y)$, for any $x, y \in S_1$.

Clearly any r-space is $\mathscr{L}_r$-regular. If, conversely, $|S, \tau|$ is an $\mathscr{L}_r$-regular space, then τ is a narrow refinement of τ' and consequently is itself an r-topology.

Suppose, finally, that $\mathscr{L}$ is any fully admissible relation such that $r \equiv \mathscr{L}$-regularity, and let $|S, \tau|$ be any space. By (τ, τ')-continuity of the identity mapping $S \to S$, $\mathscr{L}^\tau \leqslant \mathscr{L}^{\tau'} = \mathscr{R}^{\tau'} = \mathscr{L}_r^\tau$.

3·5. Corollary. *Any perfect regularity axiom r which is not satisfied by all spaces lies in logical strength between $\mathscr{N}$-regularity and R_0.*

Proof. As r is not satisfied by all spaces, $\mathscr{L}_r \neq \mathscr{R}$; so by 1·2, $\mathscr{B} \leqslant \mathscr{L}_r \leqslant \mathscr{N}$. The axiom of regularity generated by $\mathscr{B}$ is R_0.

4. *Perfect connectedness conditions.* A family $\{A_i\}_I$ of subsets of a given set S will be said to form a *binary covering* of S if and only if, for each pair x, y in S, there exists $i(x, y)$ in I such that $x, y \in A_{i(x,y)}$. A property q of topological spaces will be said to be *b-additive* if and only if any space possessing a binary covering, each member of which (as a subspace) possesses q, is itself a q-space.

A property q will be said to be *continuously closed* if and only if every continuous image of a q-space possesses q.

A property q which is possessed by any indiscrete space but not by all spaces will be called a *perfect connectedness condition* if and only if it is continuously closed and b-additive.

4·1. Proposition. *If $\mathscr{L}$ is any fully admissible relation, then $\mathscr{L}$-connectedness is a perfect connectedness condition.*

Proof. If $|S, \tau|$ is any space possessing a binary covering $\{A_i\}_I$ such that each A_i is $\mathscr{L}$-connected, then for any $x, y \in S$, $x \mathscr{L}^\tau y$ by continuity of the inclusion mapping $A_{i(x,y)} \to S$ and full admissibility of $\mathscr{L}$. Thus $\mathscr{L}$-connectedness is b-additive. The remainder of the proof is omitted.

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4·2. THEOREM. *If q is any perfect connectedness condition there exists a smallest fully admissible relation $\mathscr{L}_q$ among those fully admissible relations $\mathscr{L}$ such that $q \equiv \mathscr{L}$-connectedness.*

Proof. Consider the operator $\mathscr{L}_q$ which assigns to each space $|S, \tau|$ the reflexive, symmetric, binary relation $\mathscr{L}_q^\tau$ such that, for any $x, y \in S$, $x \mathscr{L}_q^\tau y$ if and only if there is a q-subspace of S containing x and y. If $|S,\tau|$ is any space such that $x \mathscr{L}_q^\tau y$ for all $x, y \in S$, then there is a binary covering of S, each member of which possesses q; so $|S, \tau|$ is itself a q-space. The existence, therefore, of some space which does not possess q ensures that $\mathscr{L}_q$ is non-universal. On the other hand, since any indiscrete space possesses q, $\mathscr{L}_q \neq \mathscr{I}$. As q is continuously closed, clearly $x \mathscr{L}_q^\tau y$ implies $f(x) \mathscr{L}_q^{\tau'} f(y)$, for any continuous mapping $f : |S, \tau| \to |S', \tau'|$ and elements x, y of S. Thus $\mathscr{L}_q$ is a fully admissible relation, and $\mathscr{L}_q$-connectedness $\equiv q$.

Let $\mathscr{L}$ be any fully admissible relation such that $\mathscr{L}$-connectedness $\equiv q$, and let x, y be elements of any space $|S, \tau|$ such that $x \mathscr{L}_q^\tau y$; then there is an $\mathscr{L}$-connected subspace A of S containing x and y. By continuity of the inclusion mapping $A \to S$, therefore, $x \mathscr{L}^\tau y$.

4·3. COROLLARY. *If q is a perfect connectedness condition other than indiscreteness, then q lies in logical strength between nestedness and connectedness.*

Proof. As $\mathscr{R}$-connectedness is the property of being indiscrete, $\mathscr{L}_q \neq \mathscr{R}$; so by 1·2, $\mathscr{B} \leqslant \mathscr{L}_q \leqslant \mathscr{N}$. A space is $\mathscr{B}$-connected if and only if the open sets form a chain; that is, if and only if it is nested. A space is $\mathscr{N}$-connected if and only if it is connected.

5. *Concluding remarks.* It is interesting to note that in the notations of 2·2 and 3·4, $\mathscr{L}_p$ and $\mathscr{L}_r$ are fully admissible *equivalence* relations. However, the fully admissible relation $\mathscr{L}_p$ of 4·2 need not be an equivalence. Consider, for example, the property q of being nested. As the following result shows, there is *no* fully admissible equivalence relation $\mathscr{L}$ such that $\mathscr{L}$-connectedness $\equiv q$. (An ultraconnected space need not be nested.)

5·1. PROPOSITION. *If $\mathscr{L}$ is a fully admissible equivalence relation other than $\mathscr{R}$, then any ultraconnected space is $\mathscr{L}$-connected.*

Proof. By 1·2(*c*), $\mathscr{C} \leqslant \mathscr{L}$. $\mathscr{C}$-connectedness is the property of being ultraconnected.

5·2. LEMMA. *If $\mathscr{L}$ is a fully admissible equivalence relation and $\{E_i\}_I$ is a family of $\mathscr{L}$-connected subspaces of a given space $|S, \tau|$ such that $E_i \cap E_j \neq \varnothing$ for any i, j in I, then $E \equiv \bigcup_I E_i$ is $\mathscr{L}$-connected also.*

Proof. If $x, y \in E$ there exist $i(x), i(y) \in I$ such that $x \in E_{i(x)}$ and $y \in E_{i(y)}$. Choosing $a \in E_{i(x)} \cap E_{i(y)}$, we observe that $x \mathscr{L}^{\tau_E} a$ and $a \mathscr{L}^{\tau_E} y$, by continuity of the inclusion mappings $E_{i(x)} \to E$ and $E_{i(y)} \to E$ respectively. Thus $x \mathscr{L}^{\tau_E} y$.

5·3. THEOREM. *For a given property q of topological spaces, the following are equivalent*: (*a*) *the smallest fully admissible relation $\mathscr{L}$ such that $q \equiv \mathscr{L}$-connectedness is an equivalence*; (*b*) *there is a fully admissible equivalence $\mathscr{L}$ such that $q \equiv \mathscr{L}$-connectedness*; (*c*) *q is a continuously closed property, possessed by any indiscrete space but not by all spaces, and if*

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$\{E_i\}_I$ is a family of q-subspaces of a given space $|S, \tau|$ such that $E_i \cap E_j \neq \varnothing$ for any i, j in I, then $\bigcup_I E_i$ is also a q-subspace of S.

Proof. It suffices to verify that (*c*) implies (*a*). If (*c*) holds, and $\{A_i : i \in I\}$ is a binary covering of a space $|S, \tau|$, then for any $a \in S$, $\{A_i : i \in I, a \in A_i\}$ covers S. Hence q is certainly a perfect connectedness condition, and we can define $\mathscr{L}_q$ as in 4·2. In this case, however, if x, y, z are elements of a given space $|S, \tau|$ and A, B are q-subspaces containing x, y and y, z respectively, then $A \cup B$ is a q-subspace of S containing x and z. Thus $\mathscr{L}_q$ is a equivalence relation.

There is an obvious one–one correlation between properties satisfying the conditions of 5·3 and those *connection properties*, in the sense of (2), which are possessed by any indiscrete space, but not by all spaces and which are such that $E \in C(X)$ if and only if, giving E the subspace topology from X, $E \in C(E)$.

While the approach adopted in (1) is quite different from ours, it is not difficult to see directly that any *connectedness* (*disconnectedness*) in the sense of (1) is $\mathscr{L}$-connectedness ($T_{\mathscr{L}}$) for some fully admissible relation $\mathscr{L}$. Thus our Theorems 2·2, 4·2 may be considered as generalizations of 2·12, 3·10 of (1). But T_2, though a perfect separation axiom, is not a disconnectedness, for every non-trivial disconnectedness contains the class of totally disconnected spaces. Moreover, the property of being nested, though a perfect connectedness condition, is not a connectedness, for every connectedness is productive. The product of just two nested spaces need not be nested.

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