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Hypercyclic, mixing, and chaotic C_0-semigroups induced by semiflows

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Abstract. We characterize when C_0-semigroups induced by semiflows are hypercyclic, topologically mixing, or chaotic both on spaces of integrable functions and on spaces of continuous functions. Furthermore, we give characterizations of transitivity for weighted composition operators on these spaces.

1. *Introduction*

A continuous linear operator T on a separable Banach space E is called *hypercyclic* if there is a hypercyclic vector $x \in E$, which means that $\{T^n x;\ n \in \mathbb{N}\}$ is dense in E. Using Baire's theorem, it can be shown that an operator T on a separable Banach space is hypercyclic if and only if it is *(topologically) transitive*, i.e. for every pair of open, non-empty subsets U, V of E there is a natural number n such that $U \cap T^n(V) \neq \emptyset$ (cf. [**8**, Theorem 1.2]). During the last decade hypercyclicity has been thoroughly investigated by several authors. For a survey on this subject see, e.g., [**9, 10**].

An operator T on E is called *chaotic* if it is hypercyclic and the set of periodic points is dense in E, and T is called *mixing* if for every pair of open, non-empty subsets U, V of E there is a natural number N such that $U \cap T^n(V) \neq \emptyset$ whenever $n \geq N$.

Analogously, a C_0-semigroup $T = (T(t))_{t\geq 0}$ on a Banach space E is called *hypercyclic* if there exists an element $x \in E$ such that $\{T(t)x;\ t \geq 0\}$ is dense in E, and in this case x is again called *hypercyclic vector* for the semigroup T. (By the strong continuity of the semigroup it follows that $\{T(t)x;\ t \geq 0, t \text{ rational}\}$ is a dense subset of E for every hypercyclic vector x, so that a Banach space has to be separable in order to support a hypercyclic semigroup.) The set of all hypercyclic vectors of T will be denoted by $\mathcal{HC}(T)$. As for single operators, it can be shown (cf. [**5**, Theorem 2.2]) that a C_0-semigroup $T = (T(t))_{t\geq 0}$ on a separable Banach space is hypercyclic if and only if it is *(topologically) transitive*, i.e. for every two non-empty, open subsets U, V of E there is $s > 0$ such that $U \cap T(s)(V) \neq \emptyset$. Baire's theorem then implies that $\mathcal{HC}(T)$ is a dense G_δ-set in

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E whenever it is not empty. The first example of a hypercyclic C_0-semigroup was given by Rolewicz in [**16**].

As in the single operator case, we call a C_0-semigroup T *chaotic* if it is hypercyclic and the set of periodic points, i.e. the set $\mathrm{per}(T) := \{x \in E;\ \exists s > 0 :\ T(s)x = x\}$, is dense in E, and T is called *mixing* if for every pair of open, non-empty subsets U, V of E there is a positive number t such that $U \cap T(s)(V) \neq \emptyset$ whenever $s \geq t$.

In [**2**], Bermúdez *et al* showed that every separable infinite-dimensional complex Banach space admits a hypercyclic C_0-semigroup, but that there are separable infinite-dimensional complex Banach spaces admitting no chaotic C_0-semigroup. The existence of a (holomorphic and uniformly continuous) mixing C_0-semigroup on every separable infinite-dimensional complex Banach space was, in particular, shown in [**1**, Theorem 2.4].

For a weight function ρ, Desch *et al* considered in [**5**] the (left) translation semigroup $T(t)f = f(\cdot + t)$ on $L^p_\rho(\mathbb{R})$, $L^p_\rho([0,\infty))$ and $C_{0,\rho}(\mathbb{R})$, $C_{0,\rho}([0,\infty))$ which is generated by the differentiation operator. They showed that the translation semigroup is hypercyclic if and only if for every $x \in \mathbb{R}$ (respectively $x \in [0,\infty)$) there is a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers tending to infinity such that $\lim_{n\to\infty} \rho(x + t_n) = \lim_{n\to\infty} \rho(x - t_n) = 0$ (respectively $\lim_{n\to\infty} \rho(x + t_n) = 0$). Chaoticity of this semigroup on the same spaces was characterized by Matsui *et al* in [**12, 13**], whereas Bermúdez *et al* [**1**, Theorem 4.3] characterized when the translation semigroup is mixing.

Sufficient conditions for hypercyclicity and chaoticity of C_0-semigroups on spaces of continuous and measurable functions generated by certain other linear first-order ordinary differential operators were given by Matsui and Takeo in [**14**], and by Myjak and Rudnicki in [**15**].

The purpose of this article is to investigate systematically hypercyclicity, mixing, and chaoticity of C_0-semigroups which are induced by the solution semiflow φ of an ordinary differential equation $\dot{x} = F(x)$, $x(0) = x_0$ in an open subset Ω of $\mathbb{R}^d$, where we assume that F is locally Lipschitz continuous. The interest in these C_0-semigroups stems from the fact that their generators are first-order partial differential operators.

The paper is organized as follows. In §2, we characterize when an arbitrary family of composition operators on Banach spaces of p-integrable, respectively continuous, functions is transitive. If the Banach space is separable, this gives, in particular, a characterization of hypercyclicity of a single composition operator.

In §3, we apply our results from §2 to describe which C_0-semigroups generated by gradient operators are hypercyclic on spaces of p-integrable, respectively continuous, functions over an open subset Ω of $\mathbb{R}^d$. We give several examples to show the differences in the dynamical behaviour of the left translation semigroup and arbitrary semigroups induced by semiflows.

Allowing non-vanishing zero-order terms in the generator, in §4, we investigate hypercyclicity of C_0-semigroups generated by first-order partial differential operators. To do so, we first study transitivity of arbitrary families of weighted composition operators on Banach spaces of p-integrable, respectively continuous, functions.

Finally, §5 deals with characterizations of mixing, respectively chaoticity, of the C_0-semigroups considered in the previous sections.

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2. *Transitive families of composition operators*

Clearly, the notion of topological transitivity can easily be extended from a single operator or a C_0-semigroup to arbitrary families of continuous linear operators $(T(\iota))_{\iota\in I}$ on a Banach space E, i.e. we call $(T(\iota))_{\iota\in I}$ *(topologically) transitive* if for every pair of non-empty, open subsets U, V of E there is $\iota \in I$ such that $T(\iota)(U) \cap V \neq \emptyset$. In this section we characterize when families of certain composition operators are transitive on spaces of p-integrable and continuous functions, respectively.

For the rest of this section, let X be a locally compact Hausdorff space, $I \neq \emptyset$ a set and let $\varphi : I \times X \to X$ be a mapping such that $\varphi(\iota, \cdot)$ is injective and continuous for all $\iota \in I$. For the applications we have in mind we are mostly interested in $I \in \{\mathbb{N}_0, [0, \infty)\}$.

Furthermore, let $\rho : X \to (0, \infty)$ be upper semicontinuous. We consider spaces of continuous functions $C_{0,\rho}(X, \mathbb{C})$ and $C_{0,\rho}(X, \mathbb{R})$, where $C_{0,\rho}(X, \mathbb{K}) := \{f : X \to \mathbb{K}$ continuous; for all $\varepsilon > 0 : \{x \in X : |f(x)|\rho(x) \geq \varepsilon\}$ is compact$\}$ is equipped with the norm $\|f\| := \sup_{x\in X} |f(x)|\rho(x)$.

Moreover, we deal with spaces of p-integrable functions. In this context we always assume that X not only is locally compact and Hausdorff but also σ-compact. Let μ be a (positive) locally finite Borel measure on X. In particular, μ is σ-finite since now we assume that X is σ-compact. For $1 \leq p < \infty$ let $L^p(\mu, \mathbb{R})$, $L^p(\mu, \mathbb{C})$ be as usual equipped with the norm $\|f\| := \left(\int_X |f|^p \, d\mu\right)^{1/p}$.

Since on most occasions it will not matter whether the considered functions are real or complex valued, we will write $L^p(\mu)$ and $C_{0,\rho}(X)$, respectively, for brevity. Since μ is locally finite, the set of compactly supported, continuous functions $C_c(X)$ is dense in $L^p(\mu)$ (cf. [**17**, Theorem 3.14]), and obviously $C_c(X)$ is dense in $C_{0,\rho}(X)$, too.

We want to define a family of operators on $L^p(\mu)$ and $C_{0,\rho}(X)$ via φ by setting $T_\varphi(\iota) f := f \circ \varphi(\iota, \cdot)$ and our aim is to characterize when $T_\varphi = (T_\varphi(\iota))_{\iota\in I}$ is transitive.

First, let us recall the following well-known theorem stating when composition operators on L^p-spaces are continuous. A proof can be found in [**19**, Theorem 2.1.1].

THEOREM 2.1. *For an arbitrary σ-finite measure space (X, μ) and a measurable function $\psi : X \to X$ the composition operator*

$$T_\psi : (L^p(\mu), \|\cdot\|) \to (L^p(\mu), \|\cdot\|), \quad g \mapsto g \circ \psi$$

is well defined and continuous if and only if the image measure μ^ψ of μ under ψ is absolutely continuous with respect to μ and the μ-density f_ψ of μ^ψ is μ-almost everywhere bounded.

If T_ψ is continuous, then $\|T_\psi\| = \|f_\psi\|_\infty^{1/p}$.

Note that in the context of the above theorem, T_ψ is continuous on $L^p(\mu)$ either for all $1 \leq p < \infty$ or for none.

The corresponding theorem in the case of continuous functions reads as follows. For a proof see, for example, [**19**, Theorem 4.2.4].

THEOREM 2.2. *Let $\psi : X \to X$ be continuous. Then, the following are equivalent.*

(i) *The mapping $T_\psi : C_{0,\rho}(X) \to C_{0,\rho}(X)$, $f \mapsto f \circ \psi$ is well defined and continuous.*

(ii) (a) *There is a constant $C > 0$ such that $\rho(x) \leq C\rho(\psi(x))$ for all $x \in X$.*

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(b) *For every compact subset K of X and every $\delta > 0$ the set $\psi^{-1}(K) \cap \{x \in X;\ \rho(x) \geq \delta\}$ is compact.*

Moreover, if T_ψ is a continuous operator on $C_{0,\rho}(X)$, then we have $\|T_\psi\| = \inf\{C > 0;$ C satisfies condition (ii)(a)*$\}$.*

For the rest of this section we assume that the Borel measure μ and the upper semicontinuous function ρ are always such that $T_\varphi := (T_\varphi(\iota))_{\iota \in I}$ defines a family of continuous linear operators on $L^p(\mu)$ and $C_\rho(X)$, respectively. Instead of $\mu^{\varphi(\iota,\cdot)}$ we write μ_ι for the image measure of μ under $\varphi(\iota, \cdot)$.

Before we characterize topological transitivity of T_φ on $L^p(\mu)$ in terms of μ, we make some observations. From the σ-compactness of X and the continuity of $\varphi(\iota, \cdot)$ it follows that for each closed subset C of X the image $\varphi(\iota, C)$ is an F_σ-set, and in particular, Borel measurable. Since the closed subsets of X generate the Borel σ-algebra over X, the injectivity of $\varphi(\iota, \cdot)$ now implies that $\varphi(\iota, B)$ is a Borel subset of X whenever B is. So, by setting $\mu_{-\iota}(B) := \mu(\varphi(\iota, B))$ for $\iota \in I$ we indeed obtain a well-defined Borel measure on X.

Remark 2.3. Since $\varphi(\iota, \cdot)$ is one-to-one for every $\iota \in I$ it has an inverse mapping from $\varphi(\iota, X)$ to X which we denote by $\varphi(-\iota, \cdot)$. One should note that $\mu_{-\iota}(B) = \mu(\varphi(\iota, B)) = \mu(\varphi(-\iota, \cdot)^{-1}(B) \cap \varphi(\iota, X)) = (\mu_{|\varphi(\iota,X)})^{\varphi(-\iota,\cdot)}(B)$, where $\mu_{|\varphi(\iota,X)} := \mu(\,\cdot\, \cap \varphi(\iota, X))$. In particular, this shows that for a Borel subset B with $B \subseteq \varphi(\iota, X)$ the equation $\mu(B) = (\mu_{-\iota})^{\varphi(\iota,\cdot)}(B)$ holds.

We are now able to formulate our theorem.

THEOREM 2.4. *Under the general assumptions, the following are equivalent.*

(i) *T_φ is topologically transitive on $L^p(\mu)$.*

(ii) *For every compact subset K of X there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that*

$$\lim_{n\to\infty} \mu(K \setminus L_n) = 0$$

and

$$\lim_{n\to\infty} \mu_{\iota_n}(L_n) = \lim_{n\to\infty} \mu_{-\iota_n}(L_n) = 0.$$

Proof. To prove that (i) implies (ii), let $K \subset X$ be a fixed compact subset and let $\varepsilon \in (0, 1/2^p)$ be arbitrary. Since T_φ is transitive, there are $\iota \in I$ and v in $L^p(\mu)$ such that $\|v - \chi_K\|^p < \varepsilon^2$ as well as $\|T_\varphi(\iota)v + \chi_K\|^p < \varepsilon^2$.

By the continuity of the mapping $L^p(\mu, \mathbb{C}) \to L^p(\mu, \mathbb{R})$, $f \mapsto \operatorname{Re} f$ and the fact that T_φ commutes with it, we can assume without loss of generality that v is real-valued.

Furthermore, for measurable subsets $B \subseteq X$ we have $\|T_\varphi(\kappa)(f\chi_B)\| \leq \|T_\varphi(\kappa)f\|$ for arbitrary $\kappa \in I$ and all $f \in L^p(\mu)$. Obviously the mapping $L^p(\mu, \mathbb{R}) \to L^p(\mu, \mathbb{R})$, $f \mapsto f^+$, where $f^+ := \max\{0, f\}$, satisfies $\|(f + g)^+\| \leq \|f^+ + g^+\|$ and commutes with T_φ so that

$$\begin{aligned}\|T_\varphi(\iota)(v^+\chi_B)\| &\leq \|(T_\varphi(\iota)v)^+\| = \|(T_\varphi(\iota)v - (-\chi_K) + (-\chi_K))^+\| \\ &\leq \|(T_\varphi(\iota)v - (-\chi_K))^+\| + \|(-\chi_K)^+\| \\ &= \|(T_\varphi(\iota)v - (-\chi_K))^+\| \leq \|T_\varphi(\iota)v + \chi_K\| < \varepsilon^{2/p}\end{aligned}$$

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and $\|v - \chi_K\|^p < \varepsilon^2$ implies

$$\begin{aligned}\|v^- \chi_B\| &\le \|v^-\| = \|(-v)^+\| = \|(\chi_K - v - \chi_K)^+\| \\ &\le \|\chi_K - v\| + \|(-\chi_K)^+\| = \|\chi_K - v\| < \varepsilon^{2/p},\end{aligned}$$

where $v^- := \max\{0, -v\}$.

In addition, we have

$$\varepsilon^2 > \int_K |1 - v|^p \, d\mu \ge \int_{K \cap \{|1-v|^p > \varepsilon\}} |1 - v|^p \, d\mu \ge \varepsilon \, \mu(K \cap \{|1 - v|^p > \varepsilon\}),$$

that is $\mu(K \cap \{|1 - v|^p > \varepsilon\}) < \varepsilon$. Using a similar argument we also obtain $\mu(K \cap \{|1 + T_\varphi(\iota)v|^p > \varepsilon\}) < \varepsilon$.

If we set $L := K \cap \{|1 - v|^p \le \varepsilon\} \cap \{|1 + T_\varphi(\iota)v|^p \le \varepsilon\}$, we get $\mu(K \backslash L) < 2\varepsilon$, and $v|_L \ge 1 - \varepsilon^{1/p} > 0$ as well as $(T_\varphi(\iota)v)|_L \le \varepsilon^{1/p} - 1 < 0$.

Using $\varphi(-\iota, \cdot)(\varphi(\iota, K)) = K$ and $\mu_{|_{\varphi(\iota,X)}} = (\mu_{-\iota})^{\varphi(\iota,\cdot)}$ (see Remark 2.3) we conclude from all this and $\varepsilon < 1/2^p$

$$\begin{aligned}\varepsilon^2 &> \|v^- \chi_{\varphi(\iota,K)}\|^p = \int_{\varphi(\iota,K)} |v^-|^p \, d\mu = \int_{\varphi(\iota,K)} |v^-|^p \, d(\mu_{|_{\varphi(\iota,X)}}) \\ &= \int_{\varphi(\iota,K)} |v^-|^p \, d(\mu_{-\iota})^{\varphi(\iota,\cdot)} = \int_{\varphi(-\iota,\cdot)(\varphi(\iota,K))} |v^-(\varphi(\iota,\cdot))|^p \, d\mu_{-\iota} \\ &\ge \int_L |(T_\varphi(\iota)v)^-|^p \, d\mu_{-\iota} \ge (1 - \varepsilon^{1/p})^p \mu_{-\iota}(L) \ge 1/2^p \, \mu_{-\iota}(L),\end{aligned}$$

and

$$\begin{aligned}\varepsilon^2 &> \|T_\varphi(\iota)(v^+ \chi_L)\|^p = \int |(v^+(\varphi(\iota,\cdot)))(\chi_L(\varphi(\iota,\cdot)))|^p \, d\mu \\ &= \int_{\varphi(\iota,\cdot)^{-1}(L)} |v^+(\varphi(\iota,\cdot))|^p \, d\mu = \int_L |v^+|^p \, d\mu_\iota \ge (1 - \varepsilon^{1/p})^p \mu_\iota(L) \\ &\ge 1/2^p \ \mu_\iota(L).\end{aligned}$$

So, we have found $L \subset K$ measurable and $\iota \in I$ with $\mu(K \backslash L) < 2\varepsilon$, $\mu_\iota(L) < 2^p \varepsilon^2$ and $\mu_{-\iota}(L) < 2^p \varepsilon^2$. Since ε was arbitrarily small this shows (ii).

In order to show that (ii) implies (i), let U, V be non-empty open subsets of $L^p(\mu)$. Let f, g be in $C_c(X)$ with $f \in U$ and $g \in V$. We choose a compact subset $K \subset X$ containing both supp f and supp g and take $(L_n)_{n\in\mathbb{N}}$ and $(\iota_n)_{n\in\mathbb{N}}$ as in (ii) for K. Define for $n \in \mathbb{N}$

$$v_n := f\chi_{L_n} + g(\varphi(-\iota_n, \cdot))\chi_{\varphi(\iota_n, L_n)},$$

which is measurable, bounded, and different from zero at most on a subset of the compact set $K \cup \varphi(\iota_n, K)$, hence belongs to $L^p(\mu)$.

Obviously,

$$\|v_n - f\|^p \le \|f\|^p_{\sup} \mu(K \backslash L_n) + \|g\|^p_{\sup} \mu_{-\iota_n}(L_n)$$

and

$$\|T_\varphi(\iota_n)v_n - g\|^p \le \|f\|^p_{\sup} \mu_{\iota_n}(L_n) + \|g\|^p_{\sup} \mu(K \backslash L_n)$$

so that $(v_n)_{n\in\mathbb{N}}$ converges to f and $(T_\varphi(\iota_n)v_n)_{n\in\mathbb{N}}$ converges to g.

Thus, for n sufficiently large $T_\varphi(\iota_n)(U) \cap V \ne \emptyset$, proving (i). □

The question arises immediately of whether in the above theorem L_n can always be chosen to be K. In general, this is not the case, as will be shown in Example 3.19.

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Remark 2.5. One should note that by Theorem 2.4, T_φ is transitive on $L^p(\mu)$ either for all $1 \leq p < \infty$ or for none.

Theorem 2.4 immediately yields a characterization of transitivity for single composition operators T_ψ on $L^p(\mu)$ by setting $\varphi(n, \cdot) := \psi \circ \cdots \circ \psi = \psi^n$. Recall that on separable Banach spaces topological transitivity is equivalent to hypercyclicity. So if $L^p(\mu)$ is separable, the above theorem characterizes hypercyclicity of T_ψ. Moreover, recall that $L^p(\mu)$ is separable for example when the Borel σ-algebra is countably generated, which in turn is the case if X is a second countable topological space.

COROLLARY 2.6. *Let X be a locally compact, σ-compact, second countable Hausdorff space, μ a locally finite Borel measure on X and $\psi : X \to X$ continuous and injective. If the induced composition operator T_ψ on $L^p(\mu)$ is well defined and continuous, the following are equivalent.*

(i) *T_ψ is hypercyclic on $L^p(\mu)$.*

(ii) *For every compact subset K of X there are a sequence of measurable subsets $(L_k)_{k\in\mathbb{N}}$ of K and a sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mu(K\backslash L_k) = 0$ and $\lim_{k\to\infty} \mu(\psi^{-n_k}(L_k)) = \lim_{k\to\infty} \mu(\psi^{n_k}(L_k)) = 0$.*

Proof. This follows immediately from Theorem 2.4 for $I = \mathbb{N}$ and $\varphi(n, \cdot) := \psi \circ \cdots \circ \psi$ (n factors). □

Example 2.7. Let $X \in \{\mathbb{N}, \mathbb{Z}\}$ be equipped with the discrete topology, let $\psi : X \to X$ be injective and $\mu(\{l\}) =: \beta_l > 0$. Instead of $L^p(\mu)$ we write $\ell^p_{\mathbb{N}}(\beta)$ or $\ell^p_{\mathbb{Z}}(\beta)$ respectively, and one checks easily that T_ψ is well defined and continuous if and only if $\sup_{l\in X} \beta_l/\beta_{\psi(l)} < \infty$.

In this special case, the compact subsets K of X are finite, and the sequence $(\mu(K\backslash L_n))_{n\in\mathbb{N}}$ tends to zero if and only if $K = L_n$ for all but finitely many n. Furthermore, one has $\mu_n(K) = \sum_{l\in K\cap\psi^n(X)} \beta_{\psi^{-n}(l)}$ and $\mu_{-n}(K) = \sum_{l\in K} \beta_{\psi^n(l)}$.

When dealing with the special case $\psi(l) := l + 1$, one usually refers to T_ψ as the *unilateral shift* in case of $X = \mathbb{N}$, and in the case of $X = \mathbb{Z}$ the standard name of T_ψ is *bilateral shift*. So, in both cases, one has $\mu_{-n}(K) = \sum_{l\in K} \beta_{l+n}$ for finite subsets K, and in the unilateral case one has $\mu_n(K) = 0$ for $n > \max K$ for every finite set K, whereas $\mu_n(K) = \sum_{l\in K} \beta_{l-n}$ for finite K in the bilateral case.

If $K = \{l_1, \ldots, l_m\}$ with $l_1 < \cdots < l_m$ we get using $\sup_{l\in X} \beta_l/\beta_{l+1} =: M < \infty$ that $\mu_{-n}(K) = \sum_{l\in K} \beta_{l+n} \leq mM^{l_m-l_1}\beta_{l_m+n}$ and in the bilateral case $\mu_n(K) = \sum_{l\in K} \beta_{l-n} \leq mM^{l_m-l_1}\beta_{l_m-n}$.

Using these inequalities and Corollary 2.6, one immediately obtains a new proof of the following well-known characterization of hypercyclicity of weighted backward shifts due to Salas (see [**18**, Theorem 2.1]).

THEOREM 2.8. *We have the following.*

(i) *The bilateral shift on $l^p_{\mathbb{Z}}(\beta)$ is hypercyclic if and only if for every $l \in \mathbb{Z}$ there is a strictly increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \beta_{l+n_k} = \lim_{k\to\infty} \beta_{l-n_k} = 0$.*

(ii) *The unilateral shift on $l^p_{\mathbb{N}}(\beta)$ is hypercyclic if and only if $\liminf_{n\to\infty} \beta_n = 0$.*

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Now we turn to the spaces $C_{0,\rho}(X)$. Remember that in this context we only assume that X is a locally compact Hausdorff space. Before we formulate our theorem we make the following convention. We set $\sup \emptyset = 0$ in the context of sets of positive numbers.

THEOREM 2.9. *In addition to the general hypotheses assume that for all compact subsets K of X we have* $\inf_{x\in K} \rho(x) > 0$. *Then, among the following,* (i) *implies* (ii).

(i) *T_φ is topologically transitive on $C_{0,\rho}(X)$.*

(ii) *For every compact subset K of X there is a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that*

$$\lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} \rho(x) = 0.$$

Furthermore, if $\varphi(\iota, \cdot) : X \to X$ is an open mapping for all $\iota \in I$, then (ii) *also implies* (i).

Proof. Assume that T_φ is transitive. Let K be a compact subset of X and $\varepsilon \in (0, \inf_{x\in K} \rho(x)/2)$. Furthermore, let $f \in C_c(X)$ be positive with $f_{|K} \equiv 1$. From the topological transitivity of T_φ it follows that there are ι and $v \in C_{0,\rho}(X)$ such that $\|v - f\| < \varepsilon$ and $\|T_\varphi(\iota)v + f\| < \varepsilon$ and by the same argument as in the proof of Theorem 2.4 we can assume without loss of generality that v is real valued.

From the positivity of f it follows again that

$$\|T_\varphi(\iota)v^+\| < \varepsilon$$

as well as

$$\|v^-\| < \varepsilon.$$

Moreover, from $\|T_\varphi(\iota)v + f\| < \varepsilon$ we obtain $v(\varphi(\iota, x)) < \varepsilon/\rho(x) - 1 < -1/2$ for every $x \in K$, and $\|v - f\| < \varepsilon$ implies $v(x) > 1 - \varepsilon/\rho(x) > 1/2$ whenever $x \in K$.

Using this, we get

$$\varepsilon > \|T_\varphi(\iota)v^+\| \geq \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} v^+(\varphi(\iota, x))\rho(x) > 1/2 \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} \rho(x)$$

and

$$\varepsilon > \|v^-\| \geq \sup_{x\in\varphi(\iota,K)} |v^-(x)|\rho(x) = \sup_{x\in\varphi(\iota,K)} v^-(x)\rho(x) \geq 1/2 \sup_{x\in\varphi(\iota,K)} \rho(x)$$

which proves (ii).

Now assume that $\varphi(\iota, \cdot) : X \to X$ is an open mapping for all $\iota \in I$. It follows that $\varphi(\iota, X)$ is open and $\varphi(-\iota, \cdot) : \varphi(\iota, X) \to X$ is continuous for all $\iota \in I$.

Let U, V be non-empty open subsets of $C_{0,\rho}(X)$. Choose $f, g \in C_c(X)$ such that $f \in U$, $g \in V$ and let K be a compact subset of X containing supp f and supp g. It follows that the mapping $g \circ \varphi(-\iota, \cdot) : \varphi(\iota, X) \to \mathbb{K}$ is continuous and its support is contained in the compact set $\varphi(\iota, K)$ so that $g \circ \varphi(-\iota, \cdot) \in C_c(\varphi(\iota, X))$. Since, by hypothesis, $\varphi(\iota, X)$ is an open subset of X, we can extend $g \circ \varphi(-\iota, \cdot)$ to a compactly supported continuous function $\tilde{g}_\iota$ on X by setting it equal to zero outside $\varphi(\iota, X)$. Clearly, $T_\varphi(\iota)\tilde{g}_\iota = g$.

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Now, let $(\iota_n)_{n\in\mathbb{N}}$ be as in (ii) for the compact set K. Then, by the above consideration, the functions $v_n := \tilde{g}_{\iota_n}$ belong to $C_c(X)$, hence to $C_{0,\rho}(X)$, and one has

$$\|v_n\| = \sup_{x\in\varphi(\iota_n,K)} |g(\varphi(-\iota_n, x))|\rho(x) \leq \|g\|_{\sup} \sup_{x\in\varphi(\iota_n,K)} \rho(x)$$

so that $\lim_{n\to\infty}(f + v_n) = f$.

On the other hand

$$\|T_\varphi(\iota_n) f\| = \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} |f(\varphi(\iota_n, x))|\rho(x) \leq \|f\|_{\sup} \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} \rho(x)$$

so that $\lim_{n\to\infty} T_\varphi(\iota_n) f = 0$. Thus, $\lim_{n\to\infty} T_\varphi(\iota_n)(f + v_n) = g$ so that $T_\varphi(\iota_n)(U) \cap V \neq \emptyset$, proving (i). □

In the case when $X = \Omega \subset \mathbb{R}^d$ is open, we can use the following deep result from real analysis to obtain a corollary. For a proof of an even more general statement see, for example, [**4**, Theorem I.4.3].

THEOREM 2.10. (Brouwer's theorem) *Let $\Omega \subset \mathbb{R}^d$ be open and $\psi : \Omega \to \mathbb{R}^d$ continuous and injective. Then $\psi(\Omega)$ is an open subset of $\mathbb{R}^d$.*

COROLLARY 2.11. *Let Ω be an open subset of $\mathbb{R}^d$ and ρ a positive, upper semicontinuous function on Ω satisfying $\inf_{x\in K} \rho(x) > 0$ for every compact subset K of Ω.*

Then, the following are equivalent.

(i) *The family T_φ is transitive on $C_{0,\rho}(\Omega)$.*

(ii) *For every compact subset K of Ω there exists a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that*

$$\lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} \rho(x) = 0.$$

Proof. This follows immediately from Brouwer's theorem and Theorem 2.9. □

As in the $L^p(\mu)$ case we obtain the following result for a single composition operator T_ψ on $C_{0,\rho}(X)$ which we want to state explicitly.

COROLLARY 2.12. *Let ρ be a positive, upper semicontinuous function on X satisfying $\inf_{x\in K} \rho(x) > 0$ for every compact subset K of X, and $\psi : X \to X$ injective and continuous such that T_ψ is a well-defined continuous operator on $C_{0,\rho}(X)$.*

Then, among the following, (i) *implies* (ii).

(i) *T_ψ is transitive on $C_{0,\rho}(X)$.*

(ii) *For every compact subset K of X there exists a sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} \sup_{x\in\psi^{n_k}(K)} \rho(x) = \lim_{k\to\infty} \sup_{x\in\psi^{-n_k}(K)} \rho(x) = 0.$$

Furthermore, if $\psi : X \to X$ is an open mapping then the above are equivalent.

Proof. The corollary is an immediate consequence of Theorem 2.9 for $I = \mathbb{N}$ and $\varphi(n, \cdot) := \psi \circ \cdots \circ \psi$ (n factors). □

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Example 2.13. Again, we equip $X \in \{\mathbb{N}, \mathbb{Z}\}$ with the discrete topology. Let $\psi : X \to X$, $n \in \mathbb{N}$, be injective and $\rho := (\rho_l)_{l\in X}$ be a sequence of positive numbers. Instead of $C_{0,\rho}(X)$ we write $c_{0,\rho}(X)$. Because now the pre-image of every finite set is finite, it follows from Theorem 2.2 that the composition operator $c_{0,\rho}(X) \to c_{0,\rho}(X)$, $(x_l)_{l\in X} \mapsto (x_{\psi(l)})_{l\in X}$ is continuous if and only if $\sup_{l\in X} \rho_l/\rho_{\psi(l)} < \infty$.

By using the separability of $c_{0,\rho}(X)$ and $\sup_{l\in X} \rho_l/\rho_{l+1} < \infty$ we again obtain for the special case of $\psi(l) = l + 1$ the following theorem characterizing hypercyclicity of the unilateral and bilateral backwardshift on $c_{0,\rho}(X)$.

THEOREM 2.14. *We have the following.*

(i) *The bilateral shift on $c_{0,\rho}(\mathbb{Z})$ is hypercyclic if and only if for every $l \in \mathbb{Z}$ there is a strictly increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \rho_{l+n_k} = \lim_{k\to\infty} \rho_{l-n_k} = 0$.*

(ii) *The unilateral shift on $c_{0,\rho}(\mathbb{N})$ is hypercyclic if and only if $\liminf_{n\to\infty} \rho_n = 0$.*

3. *Hypercyclic C_0-semigroups generated by gradient operators*

In this section we use the results from §2 to characterize systematically when C_0-semigroups generated by gradient operators are hypercyclic on spaces of measurable and continuous functions.

In what follows, Ω is an open subset of $\mathbb{R}^d$. We call a continuous function $\varphi : [0, \infty) \times \Omega \to \Omega$ a *semiflow* if $\varphi(0, \cdot) = id_\Omega$, $\varphi(t, \cdot) \circ \varphi(s, \cdot) = \varphi(t + s, \cdot)$ for all $t, s \geq 0$ and if $\varphi(t, \cdot)$ is injective for all $t \geq 0$. Typically, φ will be the solution semiflow of the initial value problem

$$\dot{x} = F(x), \quad x(0) = x_0$$

on Ω, where F is a locally Lipschitz continuous vector field over Ω. Furthermore, let μ be a locally finite Borel measure and ρ a positive, upper semicontinuous function on Ω. We adopt the notation of §2, that is, we write μ_t for the image measure of μ under $\varphi(t, \cdot)$ etc. Remember that by Theorem 2.1 the mapping $T_\varphi(t) : L^p(\mu) \to L^p(\mu)$, $f \mapsto f(\varphi(t, \cdot))$, $1 \leq p < \infty$ is well defined and continuous if and only if μ_t is absolutely continuous with respect to μ and its μ-density f_t is μ-almost everywhere bounded.

Before we give a theorem characterizing when T_φ is a C_0-semigroup on $L^p(\mu)$ we need some properties of semiflows.

PROPOSITION 3.1. *Let φ be a semiflow on Ω. Then, the following hold.*

(i) *For every $s \geq 0$, we have $\lim_{t\to s} \varphi(t, \cdot) = \varphi(s, \cdot)$ locally uniformly.*

(ii) *For every compact subset K of Ω and every $\delta > 0$ satisfying $K + B(0, \delta) := \{x + y : x \in K, y \in B(0, \delta)\} \subset \Omega$, there is $n \in \mathbb{N}$ such that for every $t < 1/n$ we have $K \cap \varphi(t, (K + B(0, \delta))^c) = \emptyset$.*

Proof. (i) Let $s \geq 0$ and K be a compact subset of Ω. From the continuity of φ it follows that $\varphi : [0, s + 1] \times K \to \Omega$ is uniformly continuous. This implies, in particular, that $\lim_{t\to s} \varphi(t, x) = \varphi(s, x)$ uniformly for all $x \in K$.

(ii) Again, let K be a compact subset of Ω and $\delta > 0$ as in (ii). We assume to the contrary that for every $n \in \mathbb{N}$ there are $y_n \in (K + B(0, \delta))^c$ and $t_n \in (0, 1/n)$ such

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that $\varphi(t_n, y_n) \in K$. Without loss of generality we can assume $(t_n)_{n\in\mathbb{N}}$ to be strictly decreasing and $(\varphi(t_n, y_n))_{n\in\mathbb{N}}$ to be convergent to some $z \in K$.

The continuity of φ implies that $(\varphi(t_1 - t_n, \cdot))_{n\in\mathbb{N}}$ is equicontinuous. Using this and $\lim_{n\to\infty} \varphi(t_n, y_n) = z$ we immediately get that $\lim_{n\to\infty} \varphi(t_1, y_n) = \lim_{n\to\infty} \varphi(t_1 - t_n, \varphi(t_n, y_n)) = \varphi(t_1, z)$. By Brouwer's theorem $\varphi(t_1, \cdot)^{-1}$ is continuous, thus $\lim_{n\to\infty} y_n = z \in K$, contradicting $y_n \in (K + B(0, \delta))^c$. □

The next theorem states when T_φ is a C_0-semigroup on $L^p(\mu)$.

THEOREM 3.2. *Let $\Omega \subseteq \mathbb{R}^d$ be open, φ a semiflow, and μ a locally finite Borel measure on Ω. Then, the following are equivalent.*

(i) *The family of mappings $T_\varphi = (T_\varphi(t))_{t\geq 0}$ defined by $T_\varphi(t) : (L^p(\mu), \|\cdot\|) \to (L^p(\mu), \|\cdot\|)$, $g \mapsto g(\varphi(t, \cdot))$ is well defined and a C_0-semigroup.*

(ii) *μ_t has a μ-density $f_t \in L^\infty(\mu)$ and there are constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|f_t\|_\infty \leq Me^{t\omega}$ for all $t \geq 0$.*

Proof. That (i) implies (ii) follows immediately from Theorem 2.1 and the fact that C_0-semigroups grow exponentially (see, e.g., [**7**, Proposition I.5.5]). To show that (ii) implies (i), note that by Theorem 2.1 T_φ is a well-defined family of continuous linear operators on $L^p(\mu)$ satisfying $\|T_\varphi(t)\| = \|f_t\|_\infty^{1/p}$. Clearly, $T_\varphi(t + s) = T_\varphi(t) \circ T_\varphi(s)$. To show strong continuity, let $f \in C_c(\Omega)$ and $K := \operatorname{supp} f$. Let $\delta > 0$ be such that $K + B_\delta := K + B(0, \delta) \subset \Omega$. Using Proposition 3.1(ii) we obtain for sufficiently small $t \geq 0$

$$\int |f(\varphi(t, x)) - f(x)|^p \, d\mu(x) = \int_{K+B_\delta} |f(\varphi(t, x)) - f(x)|^p \, d\mu(x).$$

Thus, from the local finiteness of μ and Lebesgue's dominated convergence theorem we get $\lim_{t\to\infty} \|T_\varphi(t)f - f\| = 0$. Since $C_c(\Omega)$ is dense in $L^p(\mu)$ it follows from $\|T_\varphi(t)\| = \|f_t\|_\infty^{1/p} \leq Me^{t\omega/p}$ that $(T_\varphi(t))_{t\geq 0}$ is a C_0-semigroup on $L^p(\mu)$. □

Definition 3.3. Given a semiflow φ, a Borel measure μ on Ω is called *(L^p)-admissible (for φ)* if it is locally finite and satisfies condition (ii) of the above theorem. Then, the C_0-semigroup T_φ from the above theorem will be referred to as the one *induced by* φ and we will sometimes write T instead of T_φ when it is clear, from the context, which semiflow φ we consider.

The following theorem describes when T_φ is a C_0-semigroup on $C_{0,\rho}(\Omega)$.

THEOREM 3.4. *Let $\Omega \subseteq \mathbb{R}^d$ be open, φ a semiflow, and ρ an upper semicontinuous function on Ω. Then, the following are equivalent.*

(i) *The family of operators $T_\varphi(t) : C_{0,\rho}(\Omega) \to C_{0,\rho}(\Omega)$, $g \mapsto g(\varphi(t, \cdot))$, $t \geq 0$, is well-defined and forms a C_0-semigroup T_φ.*

(ii) (a) *There are constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $\rho(x) \leq Me^{t\omega}\rho(\varphi(t, x))$ for all $x \in \Omega$ and $t \geq 0$.*

(b) *For every compact subset K of Ω and every $\delta > 0$ the set $\varphi(t, \cdot)^{-1}(K) \cap \{x \in X : \rho(x) \geq \delta\}$ is compact for every $t \geq 0$.*

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Proof. Having in mind that the norms of the operators from a C_0-semigroup grow exponentially, Theorem 2.2 immediately shows that (i) implies (ii).

Now, if (ii) holds it follows from Theorem 2.2 that T_φ is a mapping from $[0, \infty)$ into the space of continuous linear operators on $C_{0,\rho}(\Omega)$ satisfying $\|T_\varphi(t)\| \leq Me^{t\omega}$.

Let $f \in C_c(\Omega)$ and $K := \operatorname{supp} f$. Let $\delta > 0$ be such that $K + B_\delta := K + B(0, \delta) \subset \Omega$. Proposition 3.1(ii) gives the existence of $n \in \mathbb{N}$ such that $\varphi(t, y) \notin K$ for every $y \in (K + B_\delta)^c$ and $0 \leq t < 1/n$. We set $L := \sup_{z \in K + \overline{B_\delta}} \rho(z) + 1$ which is positive and finite since ρ is positive and upper semicontinuous.

Now, if $\varepsilon > 0$ by the uniform continuity of f there is $\eta > 0$ such that $|f(x) - f(y)| < \varepsilon/L$ for all $|x - y| < \eta$. Proposition 3.1(i) implies that there is $t_0 < 1/n$ such that $|\varphi(t, x) - x| < \eta$ for all $x \in K + \overline{B_\delta}$ and $0 \leq t \leq t_0$ so that $|f(\varphi(t, x)) - f(x)|\rho(x) < \varepsilon$ for all $x \in K + \overline{B_\delta}$ and $0 \leq t \leq t_0$. Since $f(\varphi(t, x)) = f(x) = 0$ for every $x \in (K + B_\delta)^c$ and $0 \leq t \leq t_0$ this implies $\|T(t)f - f\| \leq \varepsilon$ whenever $0 \leq t \leq t_0$.

Because $C_c(\Omega)$ is dense in $C_{0,\rho}(\Omega)$ it follows from $\|T_\varphi(t)\| \leq Me^{t\omega}$ that T_φ is a C_0-semigroup on $C_{0,\rho}(\Omega)$. □

As in the case of $L^p(\mu)$-spaces we make the following definition.

Definition 3.5. A positive-valued, upper semicontinuous function ρ on Ω is called a *(C_0-) admissible weight function (for the semiflow φ)* if it satisfies conditions (a) and (b) of Theorem 3.4(ii). Then, the C_0-semigroup T_φ from the above theorem will be referred to as the one *induced by* φ and we will sometimes write T instead of T_φ.

The separability of $L^p(\mu)$ (respectively $C_{0,\rho}(\Omega)$) together with Theorem 2.4 (respectively Corollary 2.11) now immediately give a characterization of hypercyclicity of T_φ.

THEOREM 3.6. *Let $\Omega \subseteq \mathbb{R}^d$ be open, φ a semiflow, and μ an L^p-admissible Borel measure on Ω. Then, the following are equivalent.*

(i) *The C_0-semigroup T_φ induced by φ is hypercyclic on $L^p(\mu)$.*

(ii) *For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \mu(K \setminus L_n) = 0$ as well as $\lim_{n\to\infty} \mu_{t_n}(L_n) = \lim_{n\to\infty} \mu_{-t_n}(L_n) = 0$.*

THEOREM 3.7. *Let φ be a semiflow and ρ a C_0-admissible weight function on Ω such that $\inf_{x\in K} \rho(x) > 0$ for all compact subsets K of Ω. Then, the following are equivalent.*

(i) *The C_0-semigroup T induced by φ is hypercyclic on $C_{0,\rho}(\Omega)$.*

(ii) *For every compact subset K of Ω we can find a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that*

$$\lim_{n\to\infty} \sup_{x\in\varphi(t_n,\cdot)^{-1}(K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi(t_n,K)} \rho(x) = 0.$$

To proceed, let $F : \Omega \to \mathbb{R}^d$ be a locally Lipschitz continuous vector field such that the unique solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x),\ x(0) = x_0$ exists for all $t \geq 0$ and all $x_0 \in \Omega$. Clearly, then φ is a semiflow and induces a C_0-semigroup T_φ on $L^p(\mu)$ or $C_{0,\rho}(\Omega)$ if μ (respectively ρ) is admissible and Theorem 3.6 (respectively Theorem 3.7) characterize when this C_0-semigroup is hypercyclic.

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Remark 3.8. If Ω is an open starlike subset of $\mathbb{R}^d$ such that $\partial\Omega = \partial\bar{\Omega}$ is C^1 and if $F \in C(\bar{\Omega}) \cap C^1(\Omega)$ is a vector field such that $\sup_{x\in\Omega} |DF(x)| < \infty$ and $\langle F(z), n(z)\rangle < 0$ for every $z \in \partial\Omega$, where $n(z)$ denotes the outer normal in z, then it is not hard to show that the solution of the initial value problem $y' = F(y)$, $y(0) = x$ exists for all $t \geq 0$ for every $x \in \Omega$.

Theorems 3.6 and 3.7 can be used to describe the dynamical behaviour of solutions of the following Cauchy problem

$$\text{for all } (t, x) \in [0, \infty) \times \Omega: \quad \frac{\partial}{\partial t}u(t, x) = \sum_{i=1}^{d} F_i(x)\frac{\partial}{\partial x_i}u(t, x), \quad u(0, \cdot) = f$$

on $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$.

The connection between this Cauchy problem and the C_0-semigroup T induced by φ is given in the following theorem.

THEOREM 3.9. *Let F be a locally Lipschitz continuous vector field on Ω such that the unique solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists for all $t \geq 0$ and all $x_0 \in \Omega$. Let $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$ where μ and ρ are admissible for φ.*

Then, the generator $(A, D(A))$ of the C_0-semigroup T_φ is an extension of the operator $B : \mathcal{D} \to E$, $f \mapsto \langle F, \nabla f\rangle$, where $\mathcal{D} := \{f \in C^1(\Omega) \cap E; \langle F, \nabla f\rangle \in E\}$.

We therefore call T_φ the solution semigroup of the Cauchy problem

$$\textit{for all } (t, x) \in [0, \infty) \times \Omega: \quad \frac{\partial}{\partial t}u(t, x) = \sum_{i=1}^{d} F_i(x)\frac{\partial}{\partial x_i}u(t, x), \quad u(0, \cdot) = f.$$

Proof. Let $(A, D(A))$ be the generator of the C_0-semigroup T induced by φ. Since T is a C_0-semigroup on E there are constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. So, if $\lambda > \omega$ we have $\lim_{t\to\infty} \|e^{-\lambda t}T(t)f\| = 0$ for all f in E. Using this, the resolvent representation (cf. [**7**, Theorem II.1.10.]) and integration by parts it follows that for sufficiently large λ:

$$\begin{aligned}
\text{for all } f \in \mathcal{D}: \quad & R(\lambda, A)(\lambda f - Bf) \\
&= \int_0^\infty e^{-\lambda t}T(t)(\lambda f - Bf)\, dt \\
&= \int_0^\infty e^{-\lambda t}(\lambda f \circ \varphi(t, \cdot) - \langle F(\varphi(t, \cdot)), (\nabla f) \circ \varphi(t, \cdot)\rangle)\, dt \\
&= \lambda \int_0^\infty e^{-\lambda t} f \circ \varphi(t, \cdot)\, dt - \int_0^\infty e^{-\lambda t}\langle \partial_t\varphi(t, \cdot), (\nabla f) \circ \varphi(t, \cdot)\rangle\, dt \\
&= -e^{-\lambda t}T(t)f|_0^\infty + \int_0^\infty e^{-\lambda t}\frac{d}{dt}[f \circ \varphi(t, \cdot)]\, dt \\
&\quad - \int_0^\infty e^{-\lambda t}\langle \partial_t\varphi(t, \cdot), (\nabla f) \circ \varphi(t, \cdot)\rangle\, dt = f,
\end{aligned}$$

which shows $B \subset A$. □

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Remark 3.10. Under some additional requirements on F in the above theorem, it is possible to show that the generator A of T_φ is not only an extension of a suitable gradient operator but, in fact, its closure. For example, one has the following.

(i) If F is bounded, continuously differentiable, and satisfies $\sup_{x\in\Omega} |DF(x)| < \infty$ then the generator $(A, D(A))$ of T_φ is given by the closure of the operator $B : \mathcal{D} \to E,\ f \mapsto \langle F, \nabla f\rangle$, where $\mathcal{D} := \{f \in C^1(\Omega) \cap E : |\nabla f| \in E\}$.

(ii) If F is continuously differentiable and such that the unique solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x),\ x(0) = x_0$ exists not only for all $t \geq 0$ but for all $t \in \mathbb{R}$ then the generator $(A, D(A))$ of T_φ is given by the closure of the operator $B : C_c^1(\Omega) \to E,\ f \mapsto \langle F, \nabla f\rangle$.

In both cases, the domain of the operator B is dense in E and it is easily seen that it is invariant under T_φ. The same calculation as in the proof of Theorem 3.9 shows that A extends B so that A is the closure of B by [**7**, Proposition II.1.7].

Before we continue we fix some notation. If the semiflow φ is such that for all $t \geq 0$ the mapping $\varphi(t, \cdot)$ is continuously differentiable we call φ *continuously differentiable* and denote by $D\varphi(t, \cdot)$ the Jacobian of $\varphi(t, \cdot)$. We define $C_t := \{x \in \Omega :\ \det D\varphi(t, \cdot) = 0\}$. Then $\Omega_t := \Omega\backslash C_t$ is an open subset of $\mathbb{R}^d$.

Assume that μ is a locally finite Borel measure on Ω admitting a Lebesgue density. We now want to characterize admissibility for φ of μ in terms of this density. Although this density need not be upper semicontinuous, we denote it by ρ. There will be no danger of confusing it with a C_0-admissible weight function. We need the following result first.

PROPOSITION 3.11. *Let μ be an admissible Borel measure with Lebesgue density ρ and let φ be a continuously differentiable semiflow. Then, a μ-density of μ_t is given by $\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t, \cdot))|\det D\varphi(-t, \cdot)|/\rho$, and a μ-density of μ_{-t} is given by $\rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|/\rho$.*

Proof. By the fact that $\varphi(t, \cdot)$ is one-to-one, $\psi : \Omega_t \to \varphi(t, \Omega_t),\ x \mapsto \varphi(t, x)$ is a C^1-diffeomorphism and Sard's theorem (cf. [**6**, Theorem V.4.8]) states that $\lambda^d(\varphi(t, C_t)) = 0$, where λ^d denotes d-dimensional Lebesgue measure.

Since μ is admissible, μ_t has a μ-density f_t, so that μ_t is absolutely continuous with respect to λ^d, in particular $\mu_t(B \cap \varphi(t, C_t)) = 0$ for every Borel-subset B of Ω. Using that $\varphi(t, \Omega_t) = \psi(\Omega_t)$ is open we get from this for every open subset B of Ω and $t > 0$:

$$\begin{aligned}\mu_t(B) &= \mu_t(B \cap \varphi(t, \Omega_t)) = \mu_t(B \cap \psi(\Omega_t)) = \int_{\psi^{-1}(B\cap\psi(\Omega_t))} \rho\, d\lambda^d \\ &= \int_{B\cap\psi(\Omega_t)} \rho \circ \psi^{-1} |\det D(\psi^{-1})|\, d\lambda^d \\ &= \int_B \chi_{\varphi(t,\Omega_t)} \frac{\rho(\varphi(-t, \cdot))|\det D\varphi(-t, \cdot)|}{\rho}\, d\mu,\end{aligned}$$

where we used substitution in $\mathbb{R}^d$ (see, e.g., [**17**, Theorem 7.26]) for the mapping $\psi^{-1} : \psi(\Omega_t) \to \Omega$ in the fourth equality. Because the open subsets of Ω generate the Borel σ-algebra, we conclude that $\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t, \cdot))|\det D\varphi(-t, \cdot)|/\rho$ is a μ-density of μ_t.

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Furthermore, for open subsets B of Ω we have

$$\mu_{-t}(B) = \mu(\varphi(t, B)) = \int_{\varphi(t,B)} \rho \, d\lambda^d = \int_B \frac{\rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|}{\rho} \, d\mu,$$

where we used substitution in $\mathbb{R}^d$ once more. Again, this implies that μ_{-t} has a μ-density which is given by $\rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|/\rho$. □

Note that under the assumptions of the above proposition a Lebesgue density of μ_t (respectively μ_{-t}) is given by $\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t, \cdot))|\det D\varphi(-t, \cdot)|$ (respectively $\rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|$).

PROPOSITION 3.12. *Let $\rho : \Omega \to (0, \infty)$ be Borel measurable such that $\mu := \rho \, d\lambda^d$ is locally finite and let the semiflow φ be continuously differentiable. Then, the following are equivalent.*

(i) *μ is admissible for φ.*

(ii) *There are $M \geq 1$, $\omega \in \mathbb{R}$ such that for every $t > 0$ the inequality*

$$\rho \leq Me^{t\omega}\rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|$$

holds λ^d-almost everywhere.

Proof. Because ρ is assumed to be positive, μ is equivalent to λ^d.

(i) $\Rightarrow$ (ii): From Theorem 3.2 and Proposition 3.11 it follows that the admissibility of μ is equivalent to $\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t, \cdot))|\det D\varphi(-t, \cdot)|/\rho \leq Me^{t\omega}$ λ^d-almost everywhere for some $M \geq 1$ and $\omega \in \mathbb{R}$, which is equivalent to

$$\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t, \cdot)) \leq \chi_{\varphi(t,\Omega_t)}Me^{t\omega}\rho|\det[(D\varphi(t, \cdot)) \circ \varphi(-t, \cdot)]| \quad \lambda^d\text{-almost everywhere.}$$

So, there is a Borel subset N of Ω such that $\lambda^d(N) = 0$ and

$$\chi_{\varphi(t,\Omega_t)}(x)\rho(\varphi(-t, x)) \leq \chi_{\varphi(t,\Omega_t)}(x)Me^{t\omega}\rho(x)|\det D\varphi(t, \varphi(-t, x))|$$

for every $x \in N^c$, i.e.

$$\chi_{\varphi(t,\Omega_t)}(\varphi(t, y))\rho(y) \leq \chi_{\varphi(t,\Omega_t)}(\varphi(t, y))Me^{t\omega}\rho(\varphi(t, y))|\det D\varphi(t, y)|$$

for every $y \in \varphi(-t, N^c)$.

Because $\lambda^d(\varphi(t, C_t)) = 0$, we have $\lambda^d(N \cup \varphi(t, C_t)) = 0$, so that $0 = \mu_t(N \cup \varphi(t, C_t)) = \mu(\varphi(t, \cdot)^{-1}(N \cup \varphi(t, C_t))$ since $\mu_t \ll \lambda^d$. Now, because of the equivalence of μ and λ^d we get $\lambda^d(\varphi(t, \cdot)^{-1}(N \cup \varphi(t, C_t))) = 0$ and for every $y \in \varphi(t, \cdot)^{-1}((N \cup \varphi(t, C_t))^c) = \varphi(-t, N^c) \cap \Omega_t$ we have

$$\rho(y) = \chi_{\varphi(t,\Omega_t)}(\varphi(t, y))\rho(y) \leq Me^{t\omega}\rho(\varphi(t, y))|\det D\varphi(t, y)|,$$

that is $\rho \leq Me^{t\omega}\rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|$ λ^d-almost everywhere.

This shows that (i) implies (ii).

Using substitution in $\mathbb{R}^d$ it follows immediately that (ii) implies (i). □

Definition 3.13. Let φ be a differentiable semiflow. A Borel measurable function $\rho : \Omega \to (0, \infty)$ is called an *(L^p-)admissible weight function (for the semiflow φ)* if it satisfies condition (ii) of the preceeding theorem and $\mu := \rho \, d\lambda^d$ is locally finite. We write $L^p_\rho(\Omega)$ instead of $L^p(\mu)$.

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We are now able to give a more 'calculable' criterion for hypercyclicity of T_φ, provided that φ is 'nice' and the admissible weight function ρ behaves tamely in a certain sense.

THEOREM 3.14. *Let φ be a continuously differentiable semiflow on the open subset Ω of $\mathbb{R}^d$ such that the mappping $[0, \infty) \to \mathbb{R}$, $t \mapsto \det D\varphi(t, x)$ is continuous for all $x \in \Omega$ and such that $\Omega_t = \Omega$ for every $t > 0$. Assume that one of the following conditions is satisfied.*

(φ1) *For every $t > 0$ we have $\varphi(t, \Omega) = \Omega$.*

(φ2) *For every compact subset K of Ω there is $t_K > 0$ such that $K \cap \varphi(t, \Omega) = \emptyset$ whenever $t > t_K$.*

In the case of (φ1) let $J := \mathbb{R}$ and in case of (φ2) let $J := [0, \infty)$.

Furthermore, let ρ be an L^p-admissible weight function satisfying the following three conditions.

(ρ1) *$\inf_{x \in K} \rho(x) > 0$ for every compact subset K of Ω.*

(ρ2) *For every non-empty compact subset K of Ω there is $x \in K$ such that*

$$\sup_{y \in K, t \in J} \frac{\rho(\varphi(t, y))|\det D\varphi(t, y)|}{\rho(y)\rho(\varphi(t, x))|\det D\varphi(t, x)|} < \infty.$$

(ρ3) *For every $x \in \Omega$ there is a compact subset K of Ω with $x \in K$ and $\lambda^d(K) > 0$ such that*

$$\inf_{y \in K, t \in J} \frac{\rho(\varphi(t, y))|\det D\varphi(t, y)|}{\rho(y)\rho(\varphi(t, x))|\det D\varphi(t, x)|} > 0.$$

Then, the following are equivalent.

(i) *T_φ is hypercyclic on $L^p_\rho(\Omega)$.*

(ii) *For every $x \in \Omega$ there is a sequence of positive numbers $(t_n)_{n \in \mathbb{N}}$ tending to infinity such that*

$$\lim_{n \to \infty} \rho(\varphi(t_n, x)) \det D\varphi(t_n, x) = 0$$

and, if (φ1) holds,

$$\lim_{n \to \infty} \rho(\varphi(-t_n, x)) \det D\varphi(-t_n, x) = 0,$$

too.

Proof. To prove that (i) implies (ii), let $x \in \Omega$ and K be as in (ρ3). Since T_φ is hypercyclic we find a sequence of measurable subsets $(L_n)_{n \in \mathbb{N}}$ of K and a sequence $(t_n)_{n \in \mathbb{N}}$ of positive real numbers as in (ii) of Theorem 3.6. Observe that $\lambda^d(K) > 0$ and $\lim_{n \to \infty} \int_{K \setminus L_n} \rho \, d\lambda^d = 0$ together with the positivity of ρ imply $\int_{L_n} \rho \, d\lambda^d > 0$ for sufficiently large n. Moreover, according to (ρ3) we have

$$M := \inf_{y \in K, t \in J} \frac{\rho(\varphi(t, y))|\det D\varphi(t, y)|}{\rho(y)\rho(\varphi(t, x))|\det D\varphi(t, x)|} > 0.$$

Using Proposition 3.11 we get

$$\begin{aligned} \mu_{-t_n}(L_n) &= \int_{L_n} \frac{\rho(\varphi(t_n, y))|\det D\varphi(t_n, y)|}{\rho(y)} \rho(y) \, d\lambda^d(y) \\ &\geq M\rho(\varphi(t_n, x))|\det D\varphi(t_n, x)| \int_{L_n} \rho(y) \, d\lambda^d(y). \end{aligned}$$

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Since the left-hand side of this inequality tends to zero it follows that $\lim_{n\to\infty} \rho(\varphi(t_n, x))$ $\det D\varphi(t_n, x) = 0$.

Because of $\Omega_{t_n} = \Omega$ we have $\det D\varphi(t_n, x) \neq 0$ for every $n \in \mathbb{N}$. Since $t \mapsto \varphi(t, x)$ and $t \mapsto \det D\varphi(t, x)$ are continuous and $\inf_{x\in\tilde{K}} \rho(x) > 0$ for each compact subset $\tilde{K}$ of Ω we obtain that $(t_n)_{n\in\mathbb{N}}$ tends to infinity.

Now, if condition $(\varphi 1)$ is satisfied, we have, by Proposition 3.11,

$$\begin{aligned}\mu_{t_n}(L_n) &= \int_{L_n} \frac{\rho(\varphi(-t_n, y))|\det D\varphi(-t_n, y)|}{\rho(y)} \rho(y)\, d\lambda^d(y) \\ &\geq M\rho(\varphi(-t_n, x))|\det D\varphi(-t_n, x)| \int_{L_n} \rho(y)\, d\lambda^d(y)\end{aligned}$$

so that $\lim_{n\to\infty} \rho(\varphi(-t_n, x)) \det D\varphi(-t_n, x) = 0$, too, proving (ii).

To show that (ii) implies (i) let $K \subset \Omega$ be compact and non-empty and $x \in K$ as in $(\rho 2)$. According to (ii) there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that $\lim_{n\to\infty} \rho(\varphi(t_n, x)) \det D\varphi(t_n, x) = 0$. Setting

$$N := \sup_{y\in K, t\in J} \frac{\rho(\varphi(t, y))|\det D\varphi(t, y)|}{\rho(y)\rho(\varphi(t, x))|\det D\varphi(t, x)|} < \infty$$

we obtain, with Proposition 3.11,

$$\begin{aligned}\mu_{-t_n}(K) &= \int_K \frac{\rho(\varphi(t_n, y))|\det D\varphi(t_n, y)|}{\rho(y)} \rho(y)\, d\lambda^d(y) \\ &\leq N\rho(\varphi(t_n, x))|\det D\varphi(t_n, x)| \int_K \rho(y)\, d\lambda^d(y)\end{aligned}$$

which shows $\lim_{n\to\infty} \mu_{-t_n}(K) = 0$.

Under condition $(\varphi 1)$, we have

$$\begin{aligned}\mu_{t_n}(K) &= \int_K \frac{\rho(\varphi(-t_n, y))|\det D\varphi(-t_n, y)|}{\rho(y)} \rho(y)\, d\lambda^d(y) \\ &\leq N\rho(\varphi(-t_n, x))|\det D\varphi(-t_n, x)| \int_K \rho(y)\, d\lambda^d(y)\end{aligned}$$

so that $\lim_{n\to\infty} \mu_{t_n}(K) = 0$, too.

If condition $(\varphi 2)$ holds, then

$$\int_K \chi_{\varphi(t,\Omega)}\rho(\varphi(-t, y))|\det D\varphi(-t, y)|\, d\lambda^d(y) = 0$$

for every $t > t_K$, so that $\lim_{n\to\infty} \mu_{-t_n}(K) = 0$, since $(t_n)_{n\in\mathbb{N}}$ tends to infinity. Now, Theorem 3.6 shows that T_φ is hypercyclic on $L^p_\rho(\Omega)$. □

For the $C_{0,\rho}(\Omega)$ case there is an analogous result. Its proof is so similar to that of Theorem 3.14 that we omit it.

THEOREM 3.15. *Let φ be a continuously differentiable semiflow on the open subset Ω of $\mathbb{R}^d$ that satisfies one of the following conditions.*
$(\varphi 1)$ *For every $t > 0$ we have $\varphi(t, \Omega) = \Omega$.*

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($\varphi 2$) *For every compact subset K of Ω there is $t_K > 0$ such that $K \cap \varphi(t, \Omega) = \emptyset$ whenever $t > t_K$.*

In the case of ($\varphi 1$), let $J := \mathbb{R}$ and in case of ($\varphi 2$) let $J := [0, \infty)$.

Furthermore, let ρ be a C_0-admissible weight function satisfying

($\rho 1$) $\inf_{x \in K} \rho(x) > 0$ *for every compact subset K of Ω.*

($\rho 2$) *For every compact subset K of Ω there is $x \in K$ with*

$$\sup_{y \in K, t \in J} \frac{\rho(\varphi(t, y))}{\rho(\varphi(t, x))} < \infty.$$

Then, the following are equivalent.

(i) *T_φ is hypercyclic on $C_{0,\rho}(\Omega)$.*

(ii) *For every $x \in \Omega$ there is a sequence of positive numbers $(t_n)_{n \in \mathbb{N}}$ tending to infinity such that*

$$\lim_{n \to \infty} \rho(\varphi(t_n, x)) = 0$$

and, if ($\varphi 1$) holds,

$$\lim_{n \to \infty} \rho(\varphi(-t_n, x)) = 0,$$

too.

Example 3.16. Let Ω be one of the sets $\mathbb{R}$ or $(0, \infty)$ and $\varphi : [0, \infty) \times \Omega \to \Omega$, $(t, x) \mapsto t + x$. Then we have $D\varphi(t, \cdot) \equiv 1$, so that $\Omega_t = \Omega$ for every $t > 0$. By Proposition 3.12 a measurable weight function ρ is L^p-admissible if and only if there are $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(x) \leq M e^{t\omega} \rho(t + x)$ almost everywhere. One should note that in this special context $T(t) f = f(\cdot + t)$ defines a C_0-semigroup on $C_{0,\rho}(\mathbb{R})$ if and only if $\rho(x) \leq M e^{\omega t} \rho(x + t)$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ even if ρ is not upper semicontinuous.

In both cases we call the induced C_0-semigroup T the *left translation semigroup*. By Remark 3.10 its generator is given by the ordinary differential operator $f \mapsto (d/dx) f$ defined on a suitable domain.

Desch *et al* [**5**] investigated the left translation semigroup on $L^p_\rho([0, \infty))$, $C_{0,\rho}([0, \infty))$, $L^p_\rho(\mathbb{R})$ and $C_{0,\rho}(\mathbb{R})$. Obviously, there is no difference in considering $L^p_\rho((0, \infty))$ or $L^p_\rho([0, \infty))$. However, when dealing with weighted spaces of continuous functions on the positive real axis, it is important to distinguish between $C_{0,\rho}([0, \infty))$ and $C_{0,\rho}((0, \infty))$. Again, $T(t) f = f(\cdot + t)$ defines a C_0-semigroup on $C_{0,\rho}([0, \infty))$ if and only if $\rho(x) \leq M e^{\omega t} \rho(x + t)$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ and all $t \geq 0$ even if ρ is not upper semicontinuous.

We now show that when considering hypercyclicity of the left translation semigroup $C_{0,\rho}([0, \infty))$ is a special case of $C_{0,\rho}((0, \infty))$. To do so, let $\rho : [0, \infty) \to (0, \infty)$ be a fixed measurable function such that $\rho(x) \leq M e^{\omega t} \rho(x + t)$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ and every $t \geq 0$. We define

$$\tilde{\rho} : (0, \infty) \to (0, \infty), x \mapsto \begin{cases} \rho(x - 1), & x \geq 1, \\ x\, \rho(0), & 0 < x < 1. \end{cases}$$

It is not hard to see that $\tilde{\rho}(x) \leq \tilde{M} e^{\omega t} \tilde{\rho}(x + t)$ for some $\tilde{M} \geq 1$. From this it follows that for all $t > 0$ and $f \in C_{0,\tilde{\rho}}((0, \infty))$ we have

$$\lim_{x \to \infty} |f(x + t)| \tilde{\rho}(x) \leq \tilde{M} e^{\omega(t+1)} \lim_{x \to \infty} |f(x + t)| \tilde{\rho}(x + t) = 0$$

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as well as

$$\lim_{x\to 0} |f(x+t)|\tilde{\rho}(x) = |f(t)| \lim_{x\to 0} \tilde{\rho}(x) = 0,$$

so that $T(t)f \in C_{0,\tilde{\rho}}((0,\infty))$, showing that $T(t)$ is a well-defined operator on $C_{0,\tilde{\rho}}((0,\infty))$ satisfying $\|T(t)\| \leq \tilde{M}e^{\omega t}$. Obviously, $T(t+s) = T(t)T(s)$.

Moreover, using $\tilde{\rho}(x) \leq \tilde{M}e^{\omega t}\tilde{\rho}(x+t)$ it is not hard to see that $\tilde{\rho}$ is bounded above on compact intervals of $(0,\infty)$. From this it follows immediately that for $f \in C_c((0,\infty))$ we have $\lim_{t\to 0}\|T(t)f - f\| = 0$. Using $\|T(t)\| \leq \tilde{M}e^{\omega t}$ we finally get that the left translation semigroup is a well-defined C_0-semigroup on $C_{0,\tilde{\rho}}((0,\infty))$, even if ρ, and therefore $\tilde{\rho}$, is not upper semicontinuous.

Now let $\Phi_1 : C_{0,\rho}([0,\infty)) \to C_{0,\tilde{\rho}}((0,\infty))$, $f \mapsto f$, and $\Phi_2 : C_{0,\tilde{\rho}}((0,\infty)) \to C_{0,\rho}([0,\infty))$, $f \mapsto f(\cdot+1)$. For $i = 1, 2$, Φ_i is continuous, linear, has dense image, and satisfies $T(t)\circ\Phi_i = \Phi_i \circ T(t)$ for all $t \geq 0$. So, by the so-called comparison principle (cf. [**1**, Lemma 2.2]) the left translation semigroup is hypercyclic on $C_{0,\rho}([0,\infty))$ if and only if it is on $C_{0,\tilde{\rho}}((0,\infty))$.

Note that, in contrast to the case $\Omega = \mathbb{R}$, for $\Omega = (0,\infty)$ it is not sufficient to only have $\rho(x) \leq Me^{\omega t}\rho(x+t)$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ for the left translation semigroup to be well defined on $C_{0,\rho}((0,\infty))$. Take, for example, $\rho \equiv 1$. Then $T(t)f \in C_{0,\rho}((0,\infty))$ for all $t \geq 0$ if and only if $f \equiv 0$.

Desch *et al* showed in [**5**, Theorems 4.7 and 4.8] that T is hypercyclic on $L^p_\rho([0,\infty))$ (respectively $C_{0,\rho}([0,\infty))$) if and only if $\liminf_{x\to\infty}\rho(x) = 0$, and that T is hypercyclic on $L^p_\rho(\mathbb{R})$ (respectively $C_{0,\rho}(\mathbb{R})$) if and only if for every $x \in \mathbb{R}$ there is a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty}\rho(x+t_n) = \lim_{n\to\infty}\rho(x-t_n) = 0$.

We now derive these results from Theorems 3.14 and 3.15. Let $I \in \{\mathbb{R}, [0,\infty)\}$ and let $\rho : I \to (0,\infty)$ be measurable such that there are $M \geq 1$, $\omega \in \mathbb{R}$ for which $\rho(x) \leq Me^{\omega t}\rho(x+t)$ almost everywhere for all $t \geq 0$, $x \in I$. These functions are called *admissible* by Desch *et al* in [**5**]. In [**5**] they showed the following lemma.

LEMMA 3.17. (Desch *et al* [**5**, Lemma 4.2]) *Let $\rho : I \to (0,\infty)$ be measurable such that there are $M \geq 1$, $\omega \in \mathbb{R}$ satisfying $\rho(x) \leq Me^{\omega t}\rho(x+t)$ almost everywhere for all $t \geq 0$.*

Then for each $l > 0$ there is $M_l > 0$ so that $\rho(x) \leq M_l\rho(y)$ whenever $x, y \in I$, $x \leq y$, $y - x \leq l$.

In particular, the above lemma shows that the measure $\mu := \rho\, d\lambda^d$ is locally finite. Hence, admissible weight functions in the sense of Desch *et al* are L^p-admissible for the semiflow $\varphi(t,x) = t + x$ by Proposition 3.12. Furthermore, we already noted above that the left translation semigroup is a well-defined C_0-semigroup on $C_{0,\rho}(\mathbb{R})$ (respectively $C_{0,\rho}([0,\infty))$) if ρ is admissible in the sense of Desch *et al*.

COROLLARY 3.18. (Desch *et al* [**5**, Theorems 4.7 and 4.8]) *Let ρ be admissible in the sense of Desch et al.*

(a) *For $I = [0,\infty)$, the following are equivalent.*
   (i) *The left translation semigroup is hypercyclic on $L^p_\rho([0,\infty))$ (respectively $C_{0,\rho}([0,\infty))$).*
   (ii) $\liminf_{x\to\infty}\rho(x) = 0$.

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(b) *For $I = \mathbb{R}$, the following are equivalent.*
   (i) *The left translation semigroup is hypercyclic on $L^p_\rho(\mathbb{R})$ (respectively $C_{0,\rho}(\mathbb{R})$).*
   (ii) *For every $x \in \mathbb{R}$ there is a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty} \rho(x + t_n) = \lim_{n\to\infty} \rho(x - t_n) = 0$.*

Proof. We first consider the spaces $L^p_\rho(I)$ and $C_{0,\rho}(\mathbb{R})$. Recall that it does not matter if we consider $L^p_\rho([0, \infty))$ or $L^p_\rho((0, \infty))$ since the spaces are the same. So we can use Theorem 3.14 although $[0, \infty)$ is not open.

Since $D\varphi(t, x) = 1$ we have $I_t = I$ for all $t > 0$. Obviously, if $I = \mathbb{R}$ condition $(\varphi 1)$ of Theorem 3.14 (respectively Theorem 3.15) is satisfied while condition $(\varphi 2)$ holds in the case of $I = [0, \infty)$. Lemma 3.17 implies that $\inf_{x\in K} \rho(x) > 0$ for every compact subset K of I, so that ρ satisfies condition $(\rho 1)$ of Theorem 3.14 (respectively Theorem 3.15). For a compact subset K of I take $x := \max K$ and let $a := \min K$. Using Lemma 3.17 gives for $y \in K$

$$\frac{\rho(y + t)}{\rho(y)\rho(x + t)} \le \frac{M_x}{\rho(y)} \le \frac{M_x^2}{\rho(a)} < \infty$$

for suitable $M_x > 0$. This shows that condition $(\rho 2)$ of Theorem 3.14 (respectively Theorem 3.15) also holds.

Finally, let $x \in I$ and $y \in [x, x + 1]$. It follows from Lemma 3.17 that

$$\frac{\rho(y + t)}{\rho(y)\rho(x + t)} \ge \frac{1}{\rho(y)M_1} \ge \frac{1}{\rho(x + 1)M_{x+1}M_1} > 0$$

for suitable constants $M_1, M_{x+1} > 0$, so that ρ also satisfies condition $(\rho 3)$ of Theorem 3.14. So, in the case of $L^p_\rho(I)$ and $C_{0,\rho}(\mathbb{R})$ the theorem follows from Theorem 3.14 (respectively Theorem 3.15).

For $C_{0,\rho}([0, \infty))$ let $\tilde\rho$ be as in Example 3.16, i.e.

$$\tilde\rho : (0, \infty) \to (0, \infty), x \mapsto \begin{cases} \rho(x - 1), & x \ge 1, \\ x\,\rho(0), & 0 < x < 1. \end{cases}$$

In Example 3.16 we have seen that there are $M > 1, \omega \in \mathbb{R}$ such that $(*)$ $\tilde\rho(x + t) \le Me^{t\omega}\tilde\rho(x)$ for all $x \in (0, \infty)$, $t \ge 0$ and that the left translation semigroup is a well-defined C_0-semigroup on $C_{0,\tilde\rho}((0, \infty))$ which is hypercyclic if and only if it is hypercyclic on $C_{0,\rho}([0, \infty))$. Using the inequality $(*)$ it follows that $\tilde\rho$ satisfies $(\rho 1)$ and $(\rho 2)$ of Theorem 3.15 so that the left translation semigroup is hypercyclic on $C_{0,\rho}([0, \infty))$ if and only if $0 = \liminf_{x\to\infty} \tilde\rho(x) = \liminf_{x\to\infty} \rho(x)$. □

Let us note that the dynamical behaviour of the left translation semigroup is the same on spaces of continuous and p-integrable functions for all $p \ge 1$.

Example 3.19. We consider the semiflow $\varphi : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d, (t, x) \mapsto e^t x$. Then $\det D\varphi(t, \cdot) \equiv e^t$ for all $t > 0$, so that $\Omega_t = \Omega$ for all $t > 0$. So, a weight function ρ is L^p-admissible if and only if there is $M \ge 1$ and $\omega \in \mathbb{R}$ such that $\rho(x) \le Me^{t\omega}\rho(e^t x)$ almost everywhere and $\mu := \rho\, d\lambda^d$ is locally finite.

For example, $\rho : \mathbb{R}^d \to (0, \infty), x \mapsto (1 + |x|^2)^{-1}$ is admissible, because $\rho(e^t x) = (1 + e^{2t}|x|^2)^{-1} \ge e^{-2t}(1 + |x|^2)^{-1} = e^{-2t}\rho(x)$. So, one may take $M = 1$ and $\omega = 2$.

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If $K \subset \mathbb{R}^d$ is compact and $t > 0$ we have

$$\mu_t(K) = e^{-t} \int_K \rho(e^{-t}x)\, d\lambda^d(x) \leq e^{-t}\lambda^d(K),$$

which tends to zero as t tends to infinity. On the other hand, if K does not contain zero, we have $\min_{x\in K} |x|^2 > \delta > 0$ for some suitable constant δ, so that

$$\mu_{-t}(K) = e^{t} \int_K \rho(e^{t}x)\, d\lambda^d(x) \leq e^{t}(1 + e^{2t}\delta)^{-1}\lambda^d(K).$$

This last expression tends to zero as t tends to infinity.

Since for every compact subset K of $\mathbb{R}^d$ and every $\varepsilon > 0$ we can find a compact subset $0 \notin L \subset K$ such that $\int_{K\setminus L} \rho\, d\lambda^d < \varepsilon$, Theorem 3.6 yields that the solution C_0-semigroup of the Cauchy problem

$$\frac{\partial}{\partial t}u(t, x) = \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i}u(t, x), \quad u(0, x) = f_0(x)$$

in $L^p_\rho(\mathbb{R}^d)$ is hypercyclic, where $\rho : \mathbb{R}^d \to (0, \infty)$, $x \mapsto 1/(1 + |x|^2)$.

One should note, that for compact subsets K with $0 \in \overset{\circ}{K}$ it follows from the monotone convergence theorem, that $\lim_{t\to\infty} \mu_{-t}(K) = \lim_{t\to\infty} \int_{e^t K} \rho(x)\, dx = \int_{\mathbb{R}^d} \rho(x)\, dx \geq (\pi/4)\sigma_d$, where σ_d is the surface of the d-dimensional unit sphere. So, this shows that, in general, the measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K in Theorem 2.4 are different from K.

Clearly, $\rho(x) = (1 + |x|^2)^{-1}$ is also C_0-admissible. However, because of $\varphi(t, 0) = 0$ for all $t \geq 0$ we see that T_φ is not hypercyclic on $C_{0,\rho}(\mathbb{R}^d)$. This stands in complete contrast to the left translation semigroup which is hypercyclic on $L^p_\rho(I)$ if and only if it is on $C_{0,\rho}(I)$, as observed above.

In the preceding examples $\rho \equiv 1$ is an admissible weight function for the considered semiflows φ, but the induced C_0-semigroups are not hypercyclic. In the latter example, this follows from $\mu_t(B) = e^t\lambda^d(B)$. We now present an example where $\rho \equiv 1$.

Example 3.20. Let $\Omega = (0, 1)$ and $\varphi : [0, \infty) \times (0, 1) \to (0, 1)$, $(t, x) \mapsto x(x + (1 - x)e^{-t})^{-1}$. One checks that φ is well defined and the solution (semi)flow of the ordinary differential equation $\dot{x} = x(1 - x)$.

Since $D\varphi(t, x) = e^{-t}(x + (1 - x)e^{-t})^{-2}$ we have $\Omega_t = \Omega$ for each $t > 0$ and one easily shows that $\varphi(t, \Omega) = \Omega$ for all $t > 0$.

Obviously, a weight function ρ is admissible for φ if and only if

$$\rho(x)e^t(x + (1 - x)e^{-t})^2 \leq Me^{t\omega}\rho(x(x + (1 - x)e^{-t})^{-1}) \quad \text{almost everywhere}$$

for some $M \geq 1$, $\omega \in \mathbb{R}$ and every $t > 0$. Since for all $t > 0$, $x \in (0, 1)$ always $0 < (x + (1 - x)e^{-t})^2 < 1$ holds, we see that $\rho \equiv 1$ is admissible for φ.

Moreover, for every $0 < a \leq x \leq y \leq b < 1$ and $t \in \mathbb{R}$ we have

$$\frac{D\varphi(t, x)}{D\varphi(t, y)} = \left(\frac{y + (1 - y)e^{-t}}{x + (1 - x)e^{-t}}\right)^2 \leq \left(\frac{b + (1 - a)e^{-t}}{a + (1 - b)e^{-t}}\right)^2$$

and this last expression is bounded in $t \in \mathbb{R}$.

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Furthermore,

$$\frac{D\varphi(t,x)}{D\varphi(t,y)} \geq \left(\frac{a+(1-b)e^{-t}}{b+(1-a)e^{-t}}\right)^2$$

and the last expression is a monotone function in t, so that $D\varphi(t,x)/D\varphi(t,y) \geq \min\{a^2/b^2, (1-b)^2/(1-a)^2\}$.

For $t \in \mathbb{R}$, we therefore have

$$\min\{a^2/b^2, (1-b)^2/(1-a)^2\} \leq D\varphi(t,x)/D\varphi(t,y) \leq \max\{b^2/a^2, (1-a)^2/(1-b)^2\}$$

which shows that the hypotheses of Theorem 3.14 are satisfied. Because of

$$\lim_{t\to\pm\infty} D\varphi(t,x) = \lim_{t\to\pm\infty} e^{-t}(x+(1-x)e^{-t})^{-2} = 0,$$

it follows from Theorem 3.14 that the solution C_0-semigroup of the Cauchy problem

$$\frac{\partial}{\partial t}u(t,x) = x(1-x)\,\frac{\partial}{\partial x}u(t,x), \quad u(0,x) = f_0(x)$$

is hypercyclic on $L^p(0,1)$.

Example 3.21. Let $\Omega = (0,\infty)\times(-1,1)$ and $\varphi : [0,\infty)\times\Omega \to \Omega$, $(t,x,y) \mapsto (x+t, y)$. Then, $|\det D\varphi(t,x,y)| = 1$ and if we choose $\rho(x,y) = \exp(-|y|x)$ we conclude from $\rho(\varphi(t,x,y)) = \rho(x,y)\exp(-|y|t)$ that ρ is L^p-admissible for φ ($M=1$, $\omega=1$).

If K is a compact subset of Ω and $C := \min_{(x,y)\in K}|y|$ we calculate $\mu_{-t}(K) = \int_K \exp(-|y|t)\,d(x,y) \leq e^{-tC}\lambda^2(K)$. For t large enough we have $\varphi(t,\cdot)^{-1}(K) = \emptyset$ showing $\mu_t(K) = 0$, so that T_φ is hypercyclic on $L^p_\rho(\Omega)$ by Theorem 3.6.

Clearly, for every $x \in (0,\infty)$ we have $\lim_{t\to\infty}\rho(\varphi(t,x,0)) = 1$, so that the hypotheses of Theorem 3.14 cannot be satisfied.

4. *Generators with a non-vanishing zero-order term*

In the previous section we investigated C_0-semigroups generated by gradient operators. In order to consider C_0-semigroups generated by first-order partial differential operators with a non-vanishing zero-order term, we have to perturb the generator of our C_0-semigroups by a multiplication operator. The resulting C_0-semigroup will consist of weighted composition operators. Since the transitivity result is valid for arbitrary families of weighted composition operators, we adapt the general assumptions of §2. That is we consider a locally compact Hausdorff space X, an index set $I \neq \emptyset$, and a mapping $\varphi : I\times X \to X$ such that $\varphi(\iota,\cdot)$ is injective and continuous for all $\iota \in I$. Let ρ be a positive, upper semicontinuous function on X and μ a locally finite Borel measure on X. As in §2 we additionally assume that X is σ-compact when considering $L^p(\mu)$-spaces.

Let $w : I\times X \to (0,\infty)$ be such that for every $\iota \in I$ we have $1/w(\iota,\cdot) \in L^\infty_{\mathrm{loc}}(\mu)$. For the rest of this section we assume that for $\iota \in I$ the mapping $T_{w,\varphi}(\iota) : L^p(\mu) \to L^p(\mu)$, $f \mapsto w(\iota,\cdot)f(\varphi(\iota,\cdot))$ is a well-defined continuous operator on $L^p(\mu)$ and if all $w(\iota,\cdot)$ are continuous that the same is true for $T_{w,\varphi}(\iota) : C_{0,\rho}(X) \to C_{0,\rho}(X)$, $f \mapsto w(\iota,\cdot)f(\varphi(\iota,\cdot))$. $T_{w,\varphi}(\iota)$ is called a *weighted composition operator.*

Before we continue, let us present two theorems characterizing when weighted composition operators are well defined and continuous. Their proofs are obvious

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modifications of the proofs of Theorems 2.1 and 2.2, respectively, so that we omit them. Note, that in the $L^p(\mu)$ case we now no longer have continuity on all the spaces $L^p(\mu)$, $1 \leq p < \infty$ in general.

THEOREM 4.1. *Let (X, μ) be an arbitrary σ-finite measure space and let $v : X \to (0, \infty)$, $\psi : X \to X$ be a measurable functions. The weighted composition operator $T_{v,\psi} : (L^p(\mu), \|\cdot\|) \to (L^p(\mu), \|\cdot\|)$, $g \mapsto v(\cdot)\, g(\psi(\cdot))$ is well defined and continuous if and only if the measure $\mu_{v^p,\psi}$ is absolutely continuous with respect to μ and the μ-density $f_{v^p,\psi}$ of $\mu_{v^p,\psi}$ is μ-almost everywhere bounded, where $\mu_{v^p,\psi}(A) = \int_{\psi^{-1}(A)} v^p \, d\mu$.*

If $T_{v,\psi}$ is continuous, then $\|T_{v,\psi}\| = \|f_{v^p,\psi}\|_\infty^{1/p}$.

THEOREM 4.2. *Let X be a locally compact Hausdorff topological space and let $v : X \to (0, \infty)$, $\psi : X \to X$ be continuous. For an upper semicontinuous function ρ, the following are equivalent.*

(i) *The mapping $T_{v,\psi} : C_{0,\rho}(X) \to C_{0,\rho}(X)$, $f \mapsto v(\cdot)\, f(\psi(\cdot))$ is well defined and continuous.*

(ii) (a) *There is a constant $C > 0$ such that $v(x)\rho(x) \leq C\rho(\psi(x))$ for all $x \in X$.*

(b) *For every compact subset K of X and every $\delta > 0$ the set $\psi^{-1}(K) \cap \{x \in X : v(x)\rho(x) \geq \delta\}$ is compact.*

Moreover, if $T_{v,\psi}$ defines a continuous operator on $C_{0,\rho}(X)$ we have $\|T_{v,\psi}\| = \inf\{C > 0;\ C$ satisfies condition (ii)(a)$\}$.

In order to formulate our next results conveniently we introduce the following notation. For $p \in [1, \infty)$ and $\iota \in I$ we define for a Borel subset B of X

$$\nu_{p,\iota}(B) := \int_{\varphi(\iota,\cdot)^{-1}(B)} w(\iota, \cdot)^p \, d\mu$$

and

$$\nu_{p,-\iota}(B) := \int_{\varphi(\iota,B)} 1/w(\iota, \varphi(-\iota, \cdot))^p \, d\mu.$$

Note that $\varphi(\iota, B)$ is a Borel measurable subset of X whenever B is, as mentioned at the beginning of §2. So it follows that $\nu_{p,-\iota}$ is a well-defined Borel measure on X. The same is obviously true for $\nu_{p,\iota}$.

By standard arguments one shows that for positive, measurable functions f on X we have

$$\int f \, d\nu_{p,\iota} = \int w(\iota, \cdot)^p f(\varphi(\iota, \cdot)) \, d\mu$$

and

$$\int f \, d\nu_{p,-\iota} = \int \chi_{\varphi(\iota,X)} f(\varphi(-\iota, \cdot))/w(\iota, \varphi(-\iota, \cdot))^p \, d\mu.$$

Observe that in case of $w(\iota, \cdot) \equiv 1$ for all $\iota \in I$ we have $\nu_{p,\iota} = \mu_\iota$ and $\nu_{p,-\iota} = \mu_{-\iota}$ for all $p \geq 1$, $\iota \in I$.

Now, we will characterize when $T_{w,\varphi}$ is transitive on $L^p(\mu)$ (respectively $C_{0,\rho}(X)$). As before, X is a locally compact Hausdorff space when considering $C_{0,\rho}(X)$ (respectively X is a locally compact, σ-compact Hausdorff space when considering $L^p(\mu)$).

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THEOREM 4.3. *Under the general assumptions, the following are equivalent.*

(i) *$T_{w,\varphi}$ is transitive on $L^p(\mu)$.*

(ii) *For every compact subset K of X there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that*

$$\lim_{n\to\infty} \mu(K\backslash L_n) = 0$$

as well as

$$\lim_{n\to\infty} v_{p,\iota_n}(L_n) = \lim_{n\to\infty} v_{p,-\iota_n}(L_n) = 0.$$

Proof. (i) $\Rightarrow$ (ii): Exploiting the crucial fact that $w(\iota, \cdot)$ is positive, the proof works by the same arguments as for the corresponding implication of Theorem 2.4.

(ii) $\Rightarrow$ (i): The proof works again by the same arguments as for the corresponding implication of Theorem 2.4. This time one only has to define

$$v_n := f\chi_{L_n} + \left(\frac{g(\varphi(-\iota_n, \cdot))}{w(\iota_n, \varphi(-\iota_n, \cdot))}\right)\chi_{\varphi(\iota_n, L_n)},$$

which is measurable and because of $1/w(\iota_n, \cdot) \in L^\infty_{\mathrm{loc}}(m)$ in $L^p(m)$. □

By setting $\varphi(n, \cdot) := \psi \circ \cdots \circ \psi = \psi^n$ and $w(n, \cdot) := \prod_{i=0}^{n-1} v(\psi^i(\cdot))$ for $n \in \mathbb{N}$ the above theorem yields a characterization of transitivity of a single weighted composition operator $T_{v,\psi}$ on $L^p(\mu)$. Recall that on separable Banach spaces transitivity is equivalent to hypercyclicity and that $L^p(\mu)$ is separable if X is second countable.

COROLLARY 4.4. *Let X be a locally compact, σ-compact, second countable Hausdorff space, μ a locally finite Borel measure on X, $\psi : X \to X$ continuous and injective, and $v : X \to (0, \infty)$ measurable such that $1/v \in L^\infty_{\mathrm{loc}}(X)$. If the corresponding weighted composition operator $T_{v,\psi}$ on $L^p(\mu)$ is well defined and continuous, the following are equivalent.*

(i) *$T_{v,\psi}$ is hypercyclic on $L^p(\mu)$.*

(ii) *For every compact subset K of X there are a sequence of measurable subsets $(L_k)_{k\in\mathbb{N}}$ of K and a sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mu(K\backslash L_k) = 0$,*

$$\lim_{k\to\infty} \int_{\psi^{-n_k}(L_k)} \prod_{i=0}^{n_k-1} v^p(\psi^i(\cdot))\, d\mu = 0$$

and

$$\lim_{k\to\infty} \int_{\psi^{n_k}(L_k)} \prod_{i=0}^{n_k-1} v^{-p}(\psi^{i-n_k}(\cdot))\, d\mu = 0.$$

The analogue of Theorem 4.3 for the $C_{0,\rho}(X)$ case is the following.

THEOREM 4.5. *Under the general hypotheses and the additional assumption that for all compact subsets K of X we have $\inf_{x\in K} \rho(x) > 0$, among the following,* (i) *implies* (ii).

(i) *$T_{w,\varphi}$ is topologically transitive on $C_{0,\rho}(X)$*

(ii) *For every compact subset K of X we can find a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that*

$$\lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} w(\iota_n, x)\rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,K)} \frac{\rho(x)}{w(\iota_n, \varphi(-\iota_n, x))} = 0.$$

Moreover, if $\varphi(\iota, \cdot) : X \to X$ is an open mapping for all $\iota \in I$, then (ii) *also implies* (i).

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Proof. The proof is an obvious modification of the proof of Theorem 2.9. □

As in the L^p-case the above theorem gives a characterization of hypercyclicity of a single composition operator on $C_{0,\rho}(\Omega)$.

COROLLARY 4.6. *Let Ω be an open subset of $\mathbb{R}^d$, ρ a positive, upper semicontinuous function on Ω such that for all compact subsets K of X we have $\inf_{x\in K}\rho(x) > 0$. Moreover, let $\psi : \Omega \to \Omega$ be continuous and injective, and $v : \Omega \to (0,\infty)$ continuous such that the weighted composition operator $T_{v,\psi}$ is well defined and continuous on $C_{0,\rho}(\Omega)$. Then, the following are equivalent.*

(i) *$T_{v,\psi}$ is hypercyclic on $C_{0,\rho}(\Omega)$.*

(ii) *For every compact subset K of Ω we can find a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that*

$$\lim_{k\to\infty}\sup_{x\in\psi^{-n_k}(K)}\prod_{i=0}^{n_k-1}v(\psi^i(x))\rho(x) = \lim_{k\to\infty}\sup_{x\in\psi^{n_k}(K)}\frac{\rho(x)}{\prod_{i=0}^{n_k-1}v(\psi^{i-n_k}(x))} = 0.$$

Proof. This is a direct consequence of the separability of $C_{0,\rho}(\Omega)$, Brouwer's theorem, and Theorem 4.5 applied to $I = \mathbb{N}$, $\varphi(n,\cdot) = \psi\circ\cdots\circ\psi$, and $w(n,\cdot) = \prod_{i=0}^{n} v(\psi^i(\cdot))$. □

From now on, let φ again be a continuous semiflow on an open subset Ω of $\mathbb{R}^d$ and $h : \Omega \to \mathbb{R}$ be continuous. Let $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$, where μ is a locally finite Borel measure on Ω and ρ a positive, upper semicontinuous function.

If we set $h_t : \Omega \to \mathbb{R}$, $x \mapsto \exp(\int_0^t h(\varphi(r,x))\,dr)$ for $t \geq 0$, then $h_{s+t}(x) = h_s(x)h_t(\varphi(s,x))$ and h_t as well as $1/h_t$ are continuous, hence in $L^\infty_{\mathrm{loc}}(\mu)$, and positive for every $t \geq 0$. Instead of the measures μ_t and μ_{-t}, we will need the following Borel measures on Ω

$$\nu_{p,t}(B) := \int_{\varphi(t,\cdot)^{-1}(B)} h_t^p\,d\mu$$

and

$$\nu_{p,-t}(B) := \int_{\varphi(t,B)} 1/h_t(\varphi(-t,\cdot))^p\,d\mu.$$

Note that $\nu_{p,t}$ and μ_t are equivalent measures because of $h_t > 0$. Recall that for measurable $f : \Omega \to [0,\infty)$ we have

$$\int f\,d\nu_{p,t} = \int h_t(\cdot)^p f(\varphi(t,\cdot))\,d\mu$$

and

$$\int f\,d\nu_{p,-t} = \int \chi_{\varphi(t,\Omega)} f(\varphi(-t,\cdot))/h_t(\varphi(-t,\cdot))^p\,d\mu.$$

We assume that $S(t) : E \to E$, $f \mapsto h_t(\cdot)\,f(\varphi(t,\cdot))$ is a well-defined operator for $t \geq 0$ and that $S = (S(t))_{t\geq 0}$ is a C_0-semigroup on E. The following theorem is proved by using the same arguments as in the proof of Theorem 3.2, using Theorem 4.1 instead of Theorem 2.1, so we omit it.

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THEOREM 4.7. *Under the general assumptions the following are equivalent.*

(i) *The family of mappings $S(t) : L^p(\mu) \to L^p(\mu)$, $g \mapsto h_t(\cdot)\, g(\varphi(t, \cdot))$, $t \geq 0$ is well defined and a C_0-semigroup.*

(ii) *$\nu_{p,t}$ has a μ-density $f_{p,t} \in L^\infty(\mu)$ and there are constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|f_{p,t}\|_\infty \leq Me^{t\omega}$ for all $t \geq 0$.*

One should note that the property of S of being a C_0-semigroup on $L^p(\mu)$ now depends on p in general. In the $C_{0,\rho}(\Omega)$ case the corresponding theorem is proved in the same way as Theorem 3.4.

THEOREM 4.8. *Under the general assumptions the following are equivalent.*

(i) *The family of mappings $S(t) : C_{0,\rho}(X) \to C_{0,\rho}(X)$, $g \mapsto h_t(\cdot)\, g(\varphi(t, \cdot))$ is well defined and a C_0-semigroup.*

(ii) (a) *There are constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $h_t(x)\rho(x) \leq Me^{t\omega}\rho(\varphi(t, x))$ for all $x \in \Omega$ and $t \geq 0$.*

(b) *For every compact subset K of X and every $\delta > 0$ the set $\varphi(t, \cdot)^{-1}(K) \cap \{x \in X : h_t(x)\rho(x) \geq \delta\}$ is compact for every $t \geq 0$.*

Note that as a direct consequence of the above two theorems, S is always a C_0-semigroup if μ (respectively ρ) are admissible for the semiflow φ and h is bounded above.

Definition 4.9. We define the following.

(i) The locally finite Borel measure μ on Ω is called *L^p-admissible for φ and h* if the measures $(\nu_{p,t})_{t\geq 0}$ satisfy condition (ii) of Theorem 4.7.

(ii) ρ is called *C_0-admissible for φ and h* if it satisfies condition (ii) of Theorem 4.8.

Using Theorem 4.3 and the separability of $L^p(\mu)$ we obtain a characterization of when S is hypercyclic.

THEOREM 4.10. *Under the general assumptions the following are equivalent.*

(i) *The C_0-semigroup S is hypercyclic on $L^p(\mu)$.*

(ii) *For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \mu(K \setminus L_n) = 0$ as well as*

$$\lim_{n\to\infty} \nu_{p,t_n}(L_n) = \lim_{n\to\infty} \nu_{p,-t_n}(L_n) = 0.$$

Proof. The theorem follows directly from Theorem 4.3 for $I = [0, \infty)$ and $w(t, \cdot) = h_t$. □

THEOREM 4.11. *Let ρ be a C_0-admissible weight function for φ and h such that $\inf_{x\in K} \rho(x) > 0$ for every compact subset K of Ω. Then, under the general assumptions, the following are equivalent.*

(i) *The C_0-semigroup S is hypercyclic on $C_{0,\rho}(\Omega)$.*

(ii) *For every compact subset K of Ω there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that*

$$\lim_{n\to\infty} \sup_{x\in\varphi(t_n,\cdot)^{-1}(K)} h_{t_n}(x)\rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi(t_n,K)} \frac{\rho(x)}{h_{t_n}(\varphi(-t_n, x))} = 0.$$

Proof. The theorem follows immediately from Theorem 4.5, the separability of $C_{0,\rho}(\Omega)$, and Brouwer's theorem. □

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Again, if φ is the solution semiflow of the ordinary differential equation $\dot{x} = F(x)$ one shows as in Theorem 3.9 that the generator $(A, D(A))$ of the C_0-semigroup S is an extension of the densely defined operator $B : C_c^1(\Omega) \to E$, $f \mapsto \langle F, \nabla f\rangle + hf$, where $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$. As in Remark 3.10 the generator of S is given by the closure of $(B, C_c^1(\Omega))$ if F and h are continuously differentiable and F is such that the solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ is defined on the whole line $\mathbb{R}$ for every $x_0 \in \Omega$. In this case, we therefore again call S the *solution semigroup of the Cauchy problem*

$$\text{for all } t \geq 0,\ x \in \Omega: \quad \frac{\partial}{\partial t}u(t, x) = \langle F(x), \nabla_x u(t, x)\rangle + h(x)u(t, x),$$
$$u(0, x) = u_0(x), \quad u_0 \in E.$$

As in the case of $h \equiv 0$ treated in §3, one shows the following proposition by using the same kind of arguments as used in the proofs of Propositions 3.11 and 3.12, where one has to use that $\nu_{p,t}$ and μ_t are equivalent for all $t \geq 0$ to show that (i) implies (ii) in part (b).

PROPOSITION 4.12. *We have the following.*

(a) *Let μ be a locally finite Borel measure, L^p-admissible for φ and h. Assume that φ is continuously differentiable and that μ has a Lebesgue density ρ. Then, a μ-density of $\nu_{p,t}$, respectively $\nu_{p,-t}$, is given by*

$$\chi_{\varphi(t,\Omega_t)} \frac{h_t^p(\varphi(-t, \cdot))\rho(\varphi(-t, \cdot))|\det D\varphi(-t, \cdot)|}{\rho},$$

respectively

$$\frac{\rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|}{h_t^p \rho}.$$

(b) *Let $\rho : \Omega \to (0, \infty)$ be Borel measurable such that $\mu := \rho\, d\lambda^d$ is locally finite. If φ is continuously differentiable, the following are equivalent.*

(i) *μ is admissible for φ and h.*

(ii) *There are $M \geq 1$ and $\omega \in \mathbb{R}$ such that for every $t \geq 0$ the inequality*

$$h_t^p \rho \leq M e^{t\omega} \rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|$$

holds λ^d-almost everywhere.

Example 4.13. Consider on $\Omega = \mathbb{R}$ the semiflow $\varphi(t, x) := t + x$ and the weight function $\rho(x) := e^x \chi_{(-\infty,0)}(x) + \chi_{[0,\infty)}(x)$. Clearly, ρ is L^p-admissible and the C_0-semigroup T induced by φ is a semigroup of contractions on $L_\rho^p(\mathbb{R})$, in particular, it is not hypercyclic.

Now, we perturb its generator by $h \equiv c$, where $c \in \mathbb{R}$. The resulting operators form a C_0-semigroup on $L_\rho^p(\mathbb{R})$ for every $1 \leq p < \infty$ because of the remark following Theorem 4.7.

We have $h_t \equiv e^{tc}$ and for every compact subset K of $\mathbb{R}$ and every measurable subset L of K we have $\int_L e^{tcp}\rho(x - t)\, dx = e^{t(cp-1)} \int_L e^x dx$ for sufficiently large t. Now, provided that $\lambda(L) > 0$, this expression tends to zero as t tends to infinity if and only if $c < 1/p$.

On the other hand, for every compact subset K of $\mathbb{R}$ and every measurable subset L of K we have $\int_L e^{-tcp}\rho(x + t)\, dx = e^{-tcp}\lambda(L)$ for sufficiently large t. In the case of

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$\lambda(L) > 0$, this converges to zero as t tends to infinity if and only if $c > 0$. Theorem 4.10 yields that the solution C_0-semigroup of the Cauchy problem

$$\frac{\partial}{\partial t}u(t, x) = \frac{\partial}{\partial x}u(t, x) + cu(t, x), \quad u(0, x) = f_0(x)$$

is hypercyclic on $L^p_\rho(\mathbb{R})$ for $\rho(x) := e^x \chi_{(-\infty,0)}(x) + \chi_{[0,\infty)}(x)$ if and only if $c \in (0, 1/p)$.

Note that this example is interesting for another reason apart from the above theorem. As already noted, the unperturbed semigroup T generated by $f \mapsto f'$ is not hypercyclic on $L^p_\rho(\mathbb{R})$. The same is true for the C_0-semigroup generated by the bounded operator $f \mapsto cf$. However, the semigroup generated by $f \mapsto f' + cf$ is hypercyclic. It is even mixing, as will immediately follow from Theorem 5.1. Moreover, it is chaotic as will be seen at the end of Example 5.5.

5. *The topologically mixing and chaotic cases*

Throughout this section, let φ be a semiflow on an open subset Ω of $\mathbb{R}^d$ and let $h : \Omega \to \mathbb{R}$ be continuous. Moreover, let μ be an L^p-admissible Borel measure on Ω for φ and h and let ρ be a C_0-admissible positive, upper semicontinuous function on Ω.

We adopt the notation from §4, i.e. $h_t : \Omega \to \mathbb{R}$, $x \mapsto \exp(\int_0^t h(\varphi(r, x))\, dr)$ for $t \geq 0$,

$$\nu_{p,t}(B) = \int_{\varphi(t,\cdot)^{-1}(B)} h_t^p \, d\mu$$

and

$$\nu_{p,-t}(B) = \int_{\varphi(t,B)} 1/h_t(\varphi(-t, \cdot))^p \, d\mu.$$

Recall that $h_{s+t}(x) = h_s(x)h_t(\varphi(s, x))$ and that for measurable $f : \Omega \to [0, \infty)$ we have

$$\int f \, d\nu_{p,t} = \int h_t(\cdot)^p f(\varphi(t, \cdot)) \, d\mu$$

and

$$\int f \, d\nu_{p,-t} = \int \chi_{\varphi(t,\Omega)} f(\varphi(-t, \cdot))/h_t(\varphi(-t, \cdot))^p \, d\mu.$$

Again we denote by S the C_0-semigroup defined via $S(t)f = h_t(\cdot)\, f(\varphi(t, \cdot))$. In this section we investigate when S is mixing or chaotic on $L^p(\mu)$, respectively $C_{0,\rho}(\Omega)$.

As already mentioned in the introduction, we call a C_0-semigroup T on a Banach space E *topologically mixing* if for every non-empty, open subsets U, V of E there is $t_0 > 0$ such that $T(t)(U) \cap V \neq \emptyset$ for $t > t_0$. Obviously, topologically mixing C_0-semigroups are topologically transitive.

Bermúdez *et al* showed in [**1**] that the translation-semigroup is mixing on $L^p_\rho(I)$ or $C_{0,\rho}(I)$, respectively, where $I \in \{[0, \infty), \mathbb{R}\}$, if and only if $\lim_{t\to\pm\infty} \rho(t) = 0$. Then, considering Theorems 4.10 and 4.11, the following result is no surprise.

THEOREM 5.1. *We have the following.*

(a) *Under the general assumptions of this section the following are equivalent.*

(i) *The C_0-semigroup S is topologically mixing on $L^p(\mu)$.*

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(ii) *For every compact subset K of Ω there exists a family of measurable subsets $(L_t)_{t\geq 0}$ of K such that $\lim_{t\to\infty} \mu(K\setminus L_t) = 0$ and*

$$\lim_{t\to\infty} \nu_{p,-t}(L_t) = \lim_{t\to\infty} \nu_{p,t}(L_t) = 0.$$

(b) *In addition to the general hypotheses, let ρ be such that $\inf_{x\in K} \rho(x) > 0$ for all compact subsets K of Ω. Then, the following are equivalent.*

(i) *The C_0-semigroup S is topologically mixing on $C_{0,\rho}(\Omega)$.*

(ii) *For every compact subset K of Ω we have*

$$\lim_{t\to\infty} \sup_{x\in\varphi(t,\cdot)^{-1}(K)} h_t(x)\rho(x) = \lim_{t\to\infty} \sup_{x\in\varphi(t,K)} \frac{\rho(x)}{h_t(\varphi(-t,\, x))} = 0.$$

Proof. S is topologically mixing if and only if the sequence $(S(t_n))_{n\in\mathbb{N}}$ is topologically transitive for each strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ tending to infinity (cf. [**3**, Teorema 6.14]). Thus, (a) follows immediately from Theorem 4.3 and (b) is a direct consequence of Theorem 4.5 and Brouwer's theorem. □

Remark 5.2. (a) It follows from the above theorem that in all of the examples of §§3 and 4 the semigroups are not only hypercyclic but topologically mixing.

(b) If X is a locally compact Hausdorff space (respectively a locally compact, σ-compact Hausdorff space) characterizations of when a sequence of bounded weighted composition operators on $C_{0,\rho}(X)$, $c_{0,\beta}(X)$ (respectively $L^p(\mu)$, $\ell^p_\beta(X)$) is topologically mixing are obtained in an analogous way.

(c) Note that the C_0-semigroup T_φ induced by φ is either mixing on $L^p(\mu)$ for all values of $p \in [1, \infty)$ or for none.

We now turn our attention to chaoticity. Recall that a C_0-semigroup T on a Banach space E is *chaotic* if it is hypercyclic and if the set of periodic points, i.e. the set $\mathrm{per}(T) := \{x \in E : \exists\, t > 0 :\ T(t)x = x\}$, is dense in E.

Taking into account the characterizations of chaoticity for the left translation semigroup on $L^p_\rho(I)$ given in [**12, 13**], and the characterization of chaoticity for weighted shifts given in [**11**], one might guess that a summability condition will enter. The next theorem shows that this is indeed the case, but we have to impose an extra condition on the semiflow φ.

THEOREM 5.3. *In addition to the general assumptions of this section let φ be such that for every compact subset K of Ω there is $t_K > 0$ such that $\varphi(t,\, K) \cap K = \emptyset$ for every $t > t_K$. Then, the following are equivalent.*

(i) *The C_0-semigroup S is chaotic on $L^p(\mu)$.*

(ii) *$\mathrm{per}(S)$ is dense in $L^p(\mu)$.*

(iii) *For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a strictly increasing sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that $\lim_{n\to\infty} \mu(K\setminus L_n) = 0$, and $\lim_{n\to\infty} s_n = 0$, where*

$$s_n := \sum_{l=1}^{\infty} \nu_{p,lt_n}(L_n) + \sum_{l=1}^{\infty} \nu_{p,-lt_n}(L_n).$$

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Proof. That (i) implies (ii) follows directly from the definition.

(ii) $\Rightarrow$ (iii): Let K be a compact subset of Ω. Since $\mathrm{per}(S)$ is dense in $L^p(\mu)$, we can find a sequence $(v_n)_{n\in\mathbb{N}}$ of periodic points of S satisfying $\|\chi_K - v_n\| < 1/4^n$. Let $(t_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ be such that $S(t_n)v_n = v_n$, where we can assume without loss of generality that $t_{n+1} > t_n$ and $\varphi(rt_n, K) \cap \varphi(st_n, K) = \emptyset$ for all $n \in \mathbb{N}$ and $r, s \in \mathbb{Z}$, $r \neq s$.

From $\|\chi_K - v_n\| < 1/4^n$ it follows that for $L_n := K \cap \{|1 - v_n| \le 1/2^n\}$ we have $\mu(K\setminus L_n) \le 1/2^{np}$, so that $\lim_{n\to\infty} \mu(K\setminus L_n) = 0$. Since $S(t_n)v_n = v_n$ we obtain

$$
\begin{aligned}
1/4^{np} > \|\chi_K - v_n\|^p &= \int |\chi_K - v_n|^p \, d\mu \ge \int_{\Omega\setminus K} |v_n|^p \, d\mu \\
&\ge \sum_{l=1}^{\infty} \int_{\varphi(-lt_n,K)} |v_n|^p \, d\mu + \sum_{l=1}^{\infty} \int_{\varphi(lt_n,K)} |v_n|^p \, d\mu \\
&= \sum_{l=1}^{\infty} \int_{\varphi(-lt_n,K)} |S(lt_n)v_n|^p \, d\mu \\
&\quad + \sum_{l=1}^{\infty} \int_{\varphi(lt_n,K)} \left| \frac{h_{lt_n}(\varphi(-lt_n,\cdot))}{h_{lt_n}(\varphi(-lt_n,\cdot))} v_n(\varphi(lt_n,\varphi(-lt_n,\cdot))) \right|^p d\mu \\
&= \sum_{l=1}^{\infty} \int h_{lt_n}^p |v_n(\varphi(lt_n,\cdot))|^p \chi_K(\varphi(lt_n,\cdot)) \, d\mu \\
&\quad + \sum_{l=1}^{\infty} \int \left| \frac{h_{lt_n}(\varphi(-lt_n,\cdot)) v_n(\varphi(lt_n,\varphi(-lt_n,\cdot)))}{h_{lt_n}(\varphi(-lt_n,\cdot))} \right|^p \chi_K(\varphi(-lt_n,\cdot)) \, d\mu \\
&= \sum_{l=1}^{\infty} \int |v_n|^p \chi_K \, d\nu_{p,lt_n} + \sum_{l=1}^{\infty} \int |h_{lt_n} v_n(\varphi(lt_n,\cdot))|^p \chi_K \, d\nu_{p,-lt_n} \\
&\ge \sum_{l=1}^{\infty} \int_{L_n} |v_n|^p \, d\nu_{p,lt_n} + \sum_{l=1}^{\infty} \int_{L_n} |S(t_n)v_n|^p \, d\nu_{p,-lt_n} \\
&\ge (1-1/2^n)^p \sum_{l=1}^{\infty} \nu_{p,lt_n}(L_n) + (1-1/2^n)^p \sum_{l=1}^{\infty} \nu_{p,-lt_n}(L_n),
\end{aligned}
$$

where we used in the last inequality that $1/2^n \ge |1 - v_n|$ on L_n, that is, $|v_n| \ge v_n \ge 1 - 1/2^n$, so that (iii) follows.

(iii) $\Rightarrow$ (i): It follows from Theorem 4.10 that S is hypercyclic, so it remains to prove that $\mathrm{per}(S)$ is dense in $L^p(m)$. To this end, we choose $f \in C_c(X)$ and a compact subset K of Ω containing $\mathrm{supp}\, f$. Let $(L_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be as in (iii) for K, where we can assume without loss of generality, that $t_1 > t_K$.

Because of $t_1 > t_K$ we have that $(f(\varphi(lt_n,\cdot))\chi_{\varphi(-lt_n,L_n)})_{l\in\mathbb{Z}}$ is a locally finite sequence of functions, thus

$$
f_n := f\chi_{L_n} + \sum_{l=1}^{\infty} h_{lt_n} f(\varphi(lt_n,\cdot))\chi_{\varphi(-lt_n,L_n)} + \sum_{l=1}^{\infty} \frac{f(\varphi(-lt_n,\cdot))\chi_{\varphi(lt_n,L_n)}}{h_{lt_n}(\varphi(-lt_n,\cdot))}
$$

is a well-defined, measurable function.

From

$$
\int_{\varphi(-lt_n,L_n)} |h_{lt_n} f(\varphi(lt_n,\cdot))|^p \, d\mu = \int_{L_n} |f|^p \, d\nu_{p,lt_n} \le \|f\|_{\sup}^p \nu_{p,lt_n}(L_n)
$$

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and

$$\int_{\varphi(lt_n,L_n)} \left| \frac{f(\varphi(-lt_n,\cdot))}{h_{lt_n}(\varphi(-lt_n,\cdot))} \right|^p d\mu = \int_{L_n} |f|^p \, d\nu_{p,-lt_n} \leq \|f\|_{\sup}^p \nu_{p,-lt_n}(L_n),$$

it follows using the properties of $(L_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ that $f_n \in L^p(m)$ and $\lim_{n\to\infty} f_n = f$.

It remains to show that $f_n \in \operatorname{per}(S)$. By the definition of h_t, it follows immediately that $h_s(\cdot)h_t(\varphi(s,\cdot)) = h_{t+s}$ for every $s, t \geq 0$. Having this in mind, it is a straightforward calculation to show that $S(t_n)f_n = f_n$. □

Remark 5.4. If N is a closed subset of Ω with $\varphi(t, N) = N$ for every $t > 0$, and $\mu(N) = 0$, then clearly $L^p(\Omega,\mu) = L^p(\Omega\backslash N,\mu)$. So, we can weaken the extra condition $\varphi(t,K)\cap K = \emptyset$ for every compact subset K of Ω and sufficiently large t in Theorem 5.3 to: there is a closed μ-zero set N with $\varphi(t,N) = N$ for every $t > 0$ such that for every compact subset K of $\Omega\backslash N$ and sufficiently large t $\varphi(t,K)\cap K = \emptyset$.

Example 5.5. In Example 3.19 we have seen that the C_0-semigroup T induced by the semiflow

$$\varphi : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d, \quad (t,x)\mapsto e^t x$$

is hypercyclic in $L^p_\rho(\mathbb{R}^d)$ with $\rho(x) := (1+|x|^2)^{-1}$. For all $0 < r < |x| < R$ and $t \in \mathbb{R}$ it follows that

$$\frac{\rho(\varphi(t,x))|\det D\varphi(t,x)|}{\rho(x)} \leq \frac{1+R^2}{e^{-2t}+r^2}e^{-t}.$$

Thus, for every compact subset K of $\Omega\backslash\{0\}$ there is a constant C_K such that

$$\sup_{x\in K} \frac{\rho(\varphi(t,x))}{\rho(x)}|\det D\varphi(t,x)|$$

and

$$\sup_{x\in K} \frac{\rho(\varphi(-t,x))}{\rho(x)}|\det D\varphi(-t,x)|$$

are bounded by $C_K e^{-t}$ for all $t > 0$.

From this and Proposition 3.11 we conclude that $\mu_t(K) \leq C_K e^{-t}$ and $\mu_{-t}(K) \leq C_K e^{-t}$ for all $t > 0$. Choosing $L_n := K$ and $t_n := n$, we find that

$$s_n := \sum_{l\in\mathbb{Z}\backslash\{0\}} \mu_{ln}(K) \leq 2C_K \frac{e^{-n}}{1-e^{-n}}$$

which converges to zero as n tends to infinity. Now, since for every $x \in \Omega\backslash\{0\}$ it follows that $\lim_{t\to\infty}|\varphi(t,x)| = 1$, we see that $\varphi(t,K)\cap K = \emptyset$ for sufficiently large t. From Theorem 5.3 and Remark 5.4 we conclude that T is a chaotic C_0-semigroup on $L^p_\rho(\Omega)$. A similar argument shows the chaoticity of Examples 3.20 and 4.13.

In order to characterize chaoticity of S on $C_{0,\rho}(\Omega)$, we have to provide the following lemma.

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LEMMA 5.6. *Let the semiflow φ be such that for every compact subset K of Ω there is a positive number s_K satisfying $\varphi(s, K) \cap K = \emptyset$ for every $s > s_K$.*

Then, for every compact subset K of Ω and every $f \in C_{0,\rho}(\Omega)$ we have

$$\lim_{t\to\infty} \sup_{x\in\varphi(t,K)} |f(x)|\rho(x) = \lim_{t\to\infty} \sup_{x\in\varphi(t,\cdot)^{-1}(K)} |f(x)|\rho(x) = 0.$$

Proof. Let K be a compact subset of Ω, $f \in C_{0,\rho}(\Omega)$, and $\varepsilon > 0$. Then, the set $K_\varepsilon := \{x \in \Omega : |f(x)|\rho(x) \geq \varepsilon\}$ is compact, as is $K \cup K_\varepsilon$. By the hypotheses, there is a positive number s_ε such that for every $s > s_\varepsilon$ $\varphi(s, K \cup K_\varepsilon) \cap (K \cup K_\varepsilon) = \emptyset$, which is equivalent to $\varphi(s, \cdot)^{-1}(K \cup K_\varepsilon) \cap (K \cup K_\varepsilon) = \emptyset$. In particular, for every $x \in \varphi(s, K) \cup \varphi(s, \cdot)^{-1}(K)$ we have $x \notin K_\varepsilon$ whenever $s > s_\varepsilon$, that is $|f(x)|\rho(x) < \varepsilon$. □

THEOREM 5.7. *In addition to the general assumptions of this section let ρ be such that $\inf_{x\in K} \rho(x) > 0$ for every compact subset K of Ω. Furthermore, let φ be such that for every compact subset K of Ω there is a positive number s_K satisfying $\varphi(s, K) \cap K = \emptyset$ for every $s > s_K$.*

Then, the following are equivalent.

(i) *The C_0-semigroup S is chaotic on $C_{0,\rho}(\Omega)$.*

(ii) $\mathrm{per}(S)$ *is dense in $C_{0,\rho}(\Omega)$.*

(iii) *For every compact subset K of Ω there is $p > 0$ such that*

$$\lim_{n\to\infty} \sup_{x\in\varphi(np,K)} \frac{\rho(x)}{h_{np}(\varphi(-np, x))} = \lim_{n\to\infty} \sup_{x\in\varphi(np,\cdot)^{-1}(K)} h_{np}(x)\rho(x) = 0.$$

Proof. Trivially (i) implies (ii).

(ii) ⇒ (iii): Let K be a compact subset of Ω and $\varepsilon \in (0, \inf_{x\in K} \rho(x)/2)$. Let U_K be a relatively compact, open neighbourhood of K and $t > 0$ be such that $\varphi(s, \overline{U_K}) \cap \overline{U_K} = \emptyset$ for all $s \geq t$. Let $f \in C_c(\Omega)$ be positive with $f_{|K} \equiv 1$ and $f_{|\Omega\setminus U_K} \equiv 0$.

Since $\mathrm{per}(S)$ is dense in $C_{0,\rho}(\Omega)$ we can find $v \in C_{0,\rho}(\Omega)$ and $p > 0$ such that $S(p)v = v$ and $\varepsilon > \|f - v\|$, where we can assume without loss of generality that $p > t$.

From $\varepsilon > \|f - v\|$ and $\varepsilon \in (0, \inf_{x\in K} \rho(x)/2)$ it follows that $v(x) \geq 1/2$ for all $x \in K$. Using $S(np)v = h_{np}v(\varphi(np, \cdot)) = v$ this implies

$$\text{for all } n \in \mathbb{N},\ x \in \varphi(np, K): \quad 1/2 \leq h_{np}(\varphi(-np, x))v(x)$$

$$\text{for all } n \in \mathbb{N},\ x \in \varphi(np, \cdot)^{-1}(K): \quad 1/2 \leq v(\varphi(np, x)) = \frac{v(x)}{h_{np}(x)}.$$

Thus, we have, for every $n \in \mathbb{N}$,

$$\begin{aligned}\sup_{x\in\varphi(np,\cdot)^{-1}(\overline{U_K})} |v(x)|\rho(x) &\geq \sup_{x\in\varphi(np,\cdot)^{-1}(K)} |v(x)|\rho(x)\\ &\geq 1/2 \sup_{x\in\varphi(np,\cdot)^{-1}(K)} h_{np}(x)\rho(x)\end{aligned}$$

and

$$\begin{aligned}\sup_{x\in\varphi(np,\overline{U_K})} |v(x)|\rho(x) &\geq \sup_{x\in\varphi(np,K)} |v(x)|\rho(x)\\ &\geq 1/2 \sup_{x\in\varphi(np,K)} \frac{\rho(x)}{h_{np}(\varphi(-np, x))}.\end{aligned}$$

Lemma 5.6 applied to $\overline{U_K}$ now yields (iii).

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(iii) $\Rightarrow$ (i): Theorem 4.11 implies that S is hypercyclic. To show that $\operatorname{per}(S)$ is dense in $C_{0,\rho}(\Omega)$, let $f \in C_c(\Omega)$ and $K := \operatorname{supp} f$. Let p be as in (iii) for K where without loss of generality we can assume $p > s_K$. Using Brouwer's theorem, we can understand $f(\varphi(-np,\cdot))$ as an element of $C_c(\Omega)$ for every $n \in \mathbb{N}_0$ so that by $p > s_K$ the family $(f(\varphi(np,\cdot)))_{n\in\mathbb{Z}}$ is, in particular, locally finite and in $C(\Omega)$. Therefore,

$$v_m := f + \sum_{l=1}^{\infty} h_{lmp} f(\varphi(lmp,\cdot)) + \sum_{l=1}^{\infty} \frac{f(\varphi(-lmp,\cdot))}{h_{lmp}(\varphi(-lmp,\cdot))}$$

is a continuous function on Ω and using $h_s(\cdot)h_t(\varphi(s,\cdot)) = h_{t+s}$ one calculates $S(mp)v_m = v_m$.

Because of $\varphi(lmp, K) \cap \varphi(lnp, K) = \emptyset$ for every $n, m \in \mathbb{Z}$, $n \neq m$, we have

$$\begin{aligned}
&\sup_{x\in\Omega} |f(x) - v_m(x)|\rho(x) \\
&\quad = \sup_{x\in\Omega} \left| \sum_{l=1}^{\infty} h_{lmp}(x) f(\varphi(lmp,x)) + \sum_{l=1}^{\infty} \frac{f(\varphi(-lmp,x))}{h_{lmp}(\varphi(-lmp,x))} \right| \rho(x) \\
&\quad = \max \left\{ \sup_{l\in\mathbb{N}} \sup_{x\in\varphi(lmp,\cdot)^{-1}(K)} |h_{lmp}(x) f(\varphi(lmp,x))|\rho(x), \right. \\
&\qquad\qquad \left. \sup_{l\in\mathbb{N}} \sup_{x\in\varphi(lmp,K)} \left| \frac{f(\varphi(-lmp,x))}{h_{lmp}(\varphi(-lmp,x))} \right| \rho(x) \right\} \\
&\quad \leq \|f\|_{\sup} \left\{ \sup_{l\in\mathbb{N}} \sup_{x\in\varphi(lmp,\cdot)^{-1}(K)} h_{lmp}(x)\rho(x) + \sup_{l\in\mathbb{N}} \sup_{x\in\varphi(lmp,K)} \frac{\rho(x)}{h_{lmp}(\varphi(-lmp,x))} \right\}
\end{aligned}$$

which converges to zero as m tends to infinity by our hypothesis (iii).

So, all that remains is to show that v_m belongs to $C_{0,\rho}(\Omega)$. Note that for $l \in \mathbb{N}$ we have $f(\varphi(-lmp,\cdot))(h_{lmp}(\varphi(-lmp,\cdot)))^{-1} \in C_c(\Omega)$ and $h_{lmp}(\cdot) f(\varphi(lmp,\cdot)) = S(lmp)f$, i.e. all summands in the series defining v_m belong to $C_{0,\rho}(\Omega)$. We show that the series converges in $C_{0,\rho}(\Omega)$, which then implies $v_m \in C_{0,\rho}(\Omega)$.

As above, we have

$$\begin{aligned}
&\sup_{x\in\Omega} \left| \sum_{l=N}^{\infty} h_{lmp}(x) f(\varphi(lmp,x)) + \sum_{l=N}^{\infty} \frac{f(\varphi(-lmp,x))}{h_{lmp}(\varphi(-lmp,x))} \right| \rho(x) \\
&\qquad \leq \|f\|_{\sup} \left(\sup_{l\geq N} \sup_{x\in\varphi(lmp,\cdot)^{-1}(K)} h_{lmp}(x)\rho(x) + \sup_{l\geq N} \sup_{x\in\varphi(lmp,K)} \frac{\rho(x)}{h_{lmp}(\varphi(-lmp,x))} \right),
\end{aligned}$$

which tends to zero as N tends to infinity by (iii), so that v_m belongs to $C_{0,\rho}(\Omega)$. □

Remark 5.8. Obviously, analogous results of Theorems 5.3 and 5.7 are true for single weighted composition operators.

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