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The Offset Problem

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This paper is concerned with the analysis of locally time-synchronized slot systems for broadcast in packet radio networks. Local synchronization has been proposed in practice as less expensive than global synchronization over very wide areas, or over mobile networks. In the case of two locally coordinated groups of stations, under the assumption that the phase shift on the clocks between the two groups is random, it is shown that the probability of no collision is maximized when occupied slots within each group are chosen consecutively, regardless of the number of total slots, or the number of occupied slots in either group.

1. Introduction

1.1. Decentralized communication networks

Time Division Multiple Access (TDMA) is a well established method to share the same physical channel between a number of users for transmission. The channel is usually divided into slots of equal length, each accommodating control bits and a data packet of a single user. This is the basic idea employed in early communication networks (see [1, 8]).

Later intelligent cruise control systems require a periodic exchange of information between vehicles on just one radio channel, e.g. position, velocity, warnings, and other status reports. A reasonable strategy divides time into periodically repeated frames, each consisting of n slots. Each system, or *station* senses the channel and uses one slot per frame for its own transmission. For the transmissions to be error-free, different stations must coordinate frame and slot starting points, so that they can use different slots in a synchronized manner.

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There are protocols which have been developed to coordinate a group of local stations, so they select different time-slots, e.g. CSAP and its successors [2, 3, 6]. To apply these methods, it is necessary that time be synchronized among these local stations, so that each station in the group applies the same frame and slot starting points. A recent paper [5] suggested overcoming these problems by interpreting transmitters as pulse-coupled oscillators.

However, it is usually not feasible to employ a central station to control an overall global synchronization of time. Even if the above methods could be used so that local groups of stations can be synchronized relatively fast, because of the mobility and the varying network topology, it can happen that two (or more) locally synchronized groups overlap in transmission with a certain offset between their frame beginning times. In this case, messages can be destroyed by superimposition in overlapping slots. If any collision occurs, the synchronization algorithm has to be set up again to achieve synchronized transmission between interfering groups. This of course consumes channel capacity and deteriorates the overall network performance.

Current local implementations of, for example CSAP [6, 2], place stations randomly within each local group. In this case, [4] obtained a closed form representation for the expected throughput of the channel. In this paper, we investigate if this is the best strategy to avoid such superimpositions. The basic quesition is, how should occupied slots be chosen within each group so as to minimize the probability of a collision, when the frame offset between groups is assumed to be uniformly distributed. We call this 'The offset problem', and solve it in the case of two locally synchronized groups.

1.2. The problem

Assume group A consists of m stations with associated packets for broadcast. Another group B contains r stations with associated packets for broadcast. It is assumed that there is a single broadcast channel, and a clock period called the *frame* of fixed duration which is divided into n equal-length slots. In a *station schedule* stations are assigned to unique slots from among $1 \ldots n$, so that each station can broadcast once in his slot in each clock period. The length of the clock period, or frame, we will denote by the interval $[0, 1)$.

While n, and the length of a slot (and thus the frame) is assumed to be known by all stations, the clocks between groups of stations are not assumed to be synchronized. Thus the frame of B will be *offset* by some amount from the frame of A; while A is on slot i, B may be on a different slot j, and the starting points of these slots themselves may be arbitrarily out of phase (see Figure 1). If a transmission from A and a transmission from B overlap in time, then the collision results in both packets being destroyed. In what follows, we assume that the frame starting points occur independently for A and B, and uniformly at random in the interval $[0, 1)$.

Mathar and Cowen [4] calculated the probability of no collision (as a function of n, m, and r), when it was assumed that stations in group A choose a random subset of the m out of n slots to be occupied, and similarly B choose a random r-subset. They were able to give a closed form formula for the throughput of the channel. This is the protocol for slot assignment introduced by [2].

In this paper, we show that a random assignment of occupied slots is not the best

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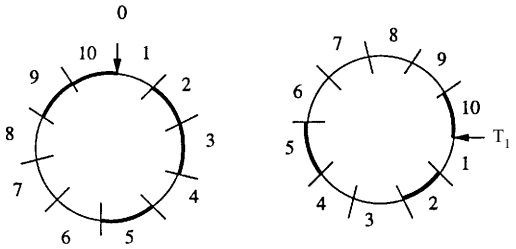


Figure 1 A slot distribution for two groups. n= 10, m=5, r=3. T_1, the frame starting time for the second group, is offset about 25 %.

strategy to maximize the probability of no collision. Instead, we show that the strategy of assigning the m occupied slots for group A in a continuous interval, and the r occupied slots for group B similarly, results in a probability of no collision which is better than (or at least as good as) any other method for assigning the occupied slots in each group (again under the assumption that the offset between the clocks of the two groups is random). Since it is easy to pick slot placement within each coordinated group any way we please, this results in a substantial increase in throughput from the random distribution that would come from a local implementation of, for instance, CSAP [6, 2].

2. Main result

Without loss of generality, we take a clock period of one time unit, and set the frame starting time for group A to 0. Then let T_1 be the frame starting time for group B, which we assume is chosen randomly on the interval (0,1). Let x mod 1 denote $x-\lfloor x \rfloor$ for $x \in \mathbb{R}$. We prove the following:

Theorem 2.1. *Let S be a (possibly randomized) strategy for choosing m occupied slots for A and r occupied slots for B, out of n, resulting in occupied slot distribution $(A_1, \ldots, A_m)$ and $(B_1, \ldots B_r)$. Then if T_1, the offset of the frame starting time for station B is selected uniformly at random from $[0, 1)$,*

$$P(\textit{no collision}) \leq \frac{n-m-r}{n}$$

independent of the distributions of $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_r)$.

Proof. First note that if either m or r is 0, then regardless of the placement of occupied slots in the other group the probability of no collision is 1. Thus assume each group has at least one packet to transmit. Denote by $a_1, \ldots, a_m$ the m occupied slots chosen for A and $b_1, \ldots, b_r$ the r occupied slots chosen for B. Then $(\frac{1}{n}b_1 + T_1, \ldots, \frac{1}{n}b_r + T_1)$ mod 1 represents the end points of occupied blocks in a randomly shifted frame on the unit circle. (x mod 1 denotes $x - \lfloor x \rfloor$, and it is applied componentwise.)

For a fixed $s \in [0, 1)$, we consider the s-shifted slot pattern, $a_1(s), \ldots a_m(s)$, where

$$a_j(s) = \left(\frac{a_j - 1}{n} + s, \frac{a_j}{n} + s \right] \bmod 1$$

for $j = 1 \ldots r$.

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A collision between the two groups occurs if and only if any of the endpoints $b_1, \ldots, b_r$ falls into a slot of the first group which is occupied by some of the m members or is a successor of such a slot. Thus the number j of blocked out slots by the first group is $j = m + \ell$ where ℓ denotes the number of sequences of consecutive slots in the first group. In the example shown in Figure 1, the blocked out slots for the first group are $1, 2, 3, 4, 5, 6, 9, 10$.

For fixed $s \in (0, 1)$ and a fixed combination $(a_1, \ldots, a_m)$ we have

$$P\left(\left(\frac{1}{n}B_1 + T_1\right) \bmod 1, \ldots, \left(\frac{1}{n}B_r + T_1\right) \bmod 1 \in \bigcup_{j=1}^{m} a_j(s)\right) =$$

$$\int P\left(\left(\frac{1}{n}j_1 + T_1\right) \bmod 1, \ldots, \left(\frac{1}{n}j_r + T_1\right) \bmod 1 \in \bigcup_{j=1}^{m} a_j(s)\right) dP^{(B_1,\ldots,B_r)}(j_1, \ldots, j_r).$$

To obtain an upper bound of the probability under the integral sign we transform the problem to a discrete problem, where there will in general be fewer collisions in the new discrete problem than in the above. A lower bound on the number of collisions in the discrete problem is obtained by combinatorial methods; this is then used to bound the number of collisions with ℓ blocks in the true continuous model.

We introduce binary strings A and B of length n such that:

A has exactly m 1's arranged in k blocks;
B has $s = n - r$ 0's arranged in ℓ blocks.

Here, we assume that both m and r are greater than 0. The blocks are considered cyclically continued (i.e. slot $n-1$ is adjacent to slot 0). Without loss of generality, we assume $\ell \leq k$ (otherwise switch the roles of A and B). We say that A and B collide if there exists at least one index i such that A and B both have a 1 in common in position i. If A and B do not collide, we write that they are orthogonal to each other ($A \perp B$). Let $B(j)$ denote the cyclic shift of string B by j places, and

$$c(A, B) = |\{j \in \{0, \ldots, n-1\} \mid A \perp B(j)\}|$$

denote the number of collision free cyclic shifts of B (or A, respectively). Let x_+ denote the positive part of $x \in \mathbb{R}$, i.e., $x_+ = \max\{x, 0\}$.

Lemma 2.1. *The number of collision-free cyclic shifts satisfies*

$$c(A, B) \leq (s - m + \ell)_+$$

Furthermore, this bound is attained if A and B both have just one block of 0's and one block of 1's.

Proof of lemma. If $s < m$ then $c(A, B) = 0$, and the upper bound trivially holds. Hence, assume $s \geq m$, and $s - m + \ell \geq 0$. We show the assertion by induction on k.

The case $k = 1$ (A has just one block of 0's) is easily checked, since $c(A, B) \leq n - m - r + 1 \leq s - m + \ell$ for all $\ell \geq 1$.

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Now, let $g_1 \geq g_2 \geq \cdots \geq g_k$ be the size of the gaps on wheel A, and $b_1 \geq b_2 \geq \cdots \geq b_\ell$ the size of the blocks of 1's on wheel B. To obtain collision free shifts upon rotating, at least b_1 must fit into corresponding gaps. Thus

$$c(A,B) \leq \sum_{i=1}^{k}(g_i - b_1 + 1)_+ \leq \sum_{i=1}^{\ell}(g_i - b_i + 1)_+ + \sum_{i=\ell+1}^{k} g_i.$$

Now, if $g_i \geq b_i$ for all $i = 1,\ldots,\ell$, we get

$$c(A,B) \leq \sum_{i=1}^{\ell}(g_i - b_i + 1) + \sum_{i=\ell+1}^{k} g_i = \sum_{i=1}^{k} g_i - \sum_{i=1}^{\ell} b_i + \ell = n - m - r + \ell = s - m + \ell,$$

and we are finished.

On the other hand, let $p \leq \ell$ be the maximum position such that $b_p > g_p$, i.e.

$$b_p > g_p \text{ and } b_{p+1} \leq g_{p+1},\ldots,b_\ell \leq g_\ell. \tag{1}$$

If $p = \ell$, then $b_\ell > g_\ell$, and nothing fits into g_ℓ. Transform wheel A into wheel A' by filling gap g_ℓ with 1's, yielding $m' > m$ 1's altogether. This reduces the number of blocks of A by one, but does not change the number of collisions. Hence, by induction

$$c(A,B) = c(A',B) \leq s - m' + \ell \leq s - m + \ell,$$

which shows the assertion.

Last, assume $p < \ell$. Then, by equation 1 we have

$$b_1 \geq \cdots \geq b_p > g_p \geq g_{p+1} \geq \cdots \geq g_\ell.$$

In this case, construct wheel A' by filling gaps $g_{p+1},\ldots,g_\ell$ with 1's, and wheel B' by emptying blocks $b_{p+1},\ldots,b_\ell$. This reduces the number of blocks on either wheel. Let m' and s' denote the number of 1's on A', and of 0's on B', respectively. By induction it follows that

$$c(A,B) \;\leq\; c(A',B') \;\leq\; s' - m' + \ell - (\ell - p) \;\leq\; s' - m' + \ell \;\leq\; s - m + \ell.$$

This proves the lemma. □

We now complete the proof of the theorem. Since T_1 is uniformly distributed,

$$\int P\left(\left(\frac{1}{n}j_1 + T_1\right) \bmod 1,\ldots,\left(\frac{1}{n}j_r + T_1\right) \bmod 1 \in \bigcup_{j=1}^{m} a_j(s)\right) dP^{(B_1,\ldots,B_r)}(j_1,\ldots,j_r) \leq$$

$$\frac{c(A,B) - \ell}{n} \leq \frac{s - m + \ell - \ell}{n} = \frac{n - r - m}{n},$$

since $s = n - r$. To determine the probability of no collision we have to integrate with respect to $P^{(A_1,\ldots,A_m)}$. So

$$P(\text{no collision}) \leq \int \frac{n - m - r}{n} dP^{(A_1,\ldots,A_m)} = \frac{n - m - r}{n}$$

whatever the distributions of $(A_1,\ldots,A_m)$ and $(B_1,\ldots,B_r)$ are.

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It is clear that the upper bound is attained if slots are chosen consecutively within both groups, e.g. $A_1 = 1, \ldots, A_m = m$ and $(B_1 = 1, \ldots, B_r = r)$. □

3. Discussion and open problems

We remark that the $-\ell$ term in the lemma is truly necessary: in the discrete case it is *not* always true that fewer blocks are better. For a simple example, look at the case where n is even, and each wheel has $n/2$ occupied slots. Then alternating between occupied and unoccupied slots on both wheels gives only $n/2$ collisions, which is the best possible; any configuration with fewer blocks will have more collisions. Of course, the probability of a collision in the continuous case for this example is 1, as the blocks will not line up exactly for a randomly chosen offset, and so there will always be some overlap, and hence a collision.

The obvious open problem is the extension to the case of k groups, for $k > 2$. We conjecture that a similar result holds in the k-station case, namely that the optimal placement to optimize the probability of no collision among groups $A_1, \ldots A_k$ with $m_1 \ldots m_k$ occupied slots, respectively, in the case where all offsets are assumed to be mutually independent among groups, is to place all occupied slots in continuous intervals in each group. This is currently open even for the case $k = 3$.

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References

[1] Abramson, N. (1970) The ALOHA System– Another Alternative For Computer Communications. *American Federation of Information Processing Societies Conference Proceedings* **36** 295–298.

[2] Form, P., Grotheer, U., Hoff, S., Hubner, D. and Reichert, F. (1990) Transponder Technique – An Approach to Integrate Relative Position Finding and Inter-Vehicle-Communication into a Single System, *Technical Report*, Aachen University of Technology.

[3] Kaltwasser, J., Reichert, F., Hellmich, T. and Walke, B. (1989) A Mobile Network Media Access Protocol. *Proc. of the 2nd PROMETHEUS Workshop*, Stockholm, Sweden, 30–31.

[4] Mathar, R. and Cowen, L. (1993) Concurrence probabilities for a locally slotted packet radio network by combinatorial methods. *Performance Evaluation* **17** 43–51.

[5] Mathar, R. and Mattfeldt, J. (1996) Pulse-coupled decentral synchronization. *SIAM J. Applied Mathematics* (to appear).

[6] Mann, A. and Rückert, J. (1988) Concurrent slot assignment protocol for packet radio networks. In D. Khakhar and V. B. Iversen (eds.), *Information Network and Data Communication, II*, North–Holland, 219–232.

[7] Tannenbaum, A. S. (1988) *Computer Networks (2nd ed.)*, Prentice–Hall.

[8] Roberts, L. G. (1975) ALOHA packet systems with and without slots and capture. *Comput. Commun. Rev.* **5** 28–42.

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