THE FUNCTION $\sum_{n=1}^{\infty} n^r z^n$ AND ASSOCIATED POLYNOMIALS

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1. *Introduction.* A transform method for the solution of linear difference equations, analogous to the method of the Laplace transform in the field of linear differential equations, has been described by Stone (1). The transform $u(z)$ of a sequence u_n is defined by the equation

$$u(z) = \sum_{n=1}^{\infty} u_n z^{-n}.$$

A knowledge of the transform of n^r, where r is a positive integer, is of assistance in many practical applications of the method. The present paper develops properties of the function $\psi_r(z)$, where

$$\psi_r(z) = \sum_{n=1}^{\infty} n^r z^n, \tag{1}$$

which have been found useful by the author in his application of the method to the theory of sampling servomechanisms.

It is well known that we may write

$$\psi_r(z) = \frac{S_r(z)}{(1-z)^{r+1}}, \tag{2}$$

$S_r(z)$ being a polynomial of the rth degree. $S_r(z)$ will be referred to as the *associated polynomial.*

2. *Elementary properties.* $\psi_r(z)$ is defined by (1) for $|z| < 1$ and by (2) throughout the z-plane with the exception of the point $z = 1$.

Differentiation of (1) provides us with the recurrence relationship

$$z\psi_r'(z) = \psi_{r+1}(z), \tag{3}$$

which permits a proof by induction of the identity

$$\psi_r(z^{-1}) = (-1)^{r+1}\psi_r(z), \tag{4}$$

where $r \geqslant 1$. Since ψ_r has a zero at $z = 0$, (4) shows that it also has one at infinity.

Writing

$$\sum_{q=1}^{r} s_q^{(r)} z^q = S_r(z) = (1-z)^{r+1} \sum_{n=1}^{\infty} n^r z^n,$$

we obtain an explicit expression for the coefficient $s_q^{(r)}$ in the form

$$s_q^{(r)} = (r+1)! \sum_{m=0}^{q-1} \frac{(-1)^m (q-m)^r}{m!(r-m+1)!}. \tag{5}$$

The first six associated polynomials are:

$$S_0 = z, \qquad S_3 = z + 4z^2 + z^3,$$
$$S_1 = z, \qquad S_4 = z + 11z^2 + 11z^3 + z^4,$$
$$S_2 = z + z^2, \qquad S_5 = z + 26z^2 + 66z^3 + 26z^4 + z^5.$$

Finally in this section we observe that

$$\psi_r(z) + \psi_r(-z) = 2^{r+1}\psi_r(z^2). \tag{6}$$

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3. *Generation of the associated polynomials.* It is easy to show that, provided $|z|<1$ and w is sufficiently small,

$$\sum_{r=0}^{\infty}\frac{S_r(z)}{r!}w^r=\frac{z(z-1)}{z-\exp\{w(z-1)\}}. \tag{7}$$

This provides us with a generating function for the associated polynomials. An alternative way of generating these polynomials will next be described.

From (3)

$$\psi_r(e^t)=\left(\frac{d}{dt}\right)^r\psi_0(e^t)=\left(\frac{d}{dt}\right)^r\sum_{s=0}^{\infty}(-1)^{s+1}B_s t^{s-1}/s!=\frac{(-1)^{r+1}r!}{t^{r+1}}+\sum_{s=0}^{\infty}a_s t^s, \tag{8}$$

the B_s being Bernoulli's numbers.

Putting $e^t=1+x$, we find that (8) is equivalent to

$$S_r(1+x)=r!\left\{\frac{x}{\log(1+x)}\right\}^{r+1}+(-x)^{r+1}\sum_{s=0}^{\infty}a_s\log^s(1+x).$$

Since $S_r(1+x)$ is a polynomial of degree r in x, it may therefore be generated by expansion of

$$r!\left\{\frac{x}{\log(1+x)}\right\}^{r+1}$$

in a series of ascending powers of x as far as the term in x^r.

4. *Recurrence relationships.* Making use of the identities

$$(n+1)^r=n^r+\binom{r}{1}n^{r-1}+\ldots+1,$$

$$(n-1)^r=n^r-\binom{r}{1}n^{r-1}+\ldots+(-1)^r,$$

we can show that for $r\geqslant 1$

$$(z^{-1}-1)\psi_r=\binom{r}{1}\psi_{r-1}+\binom{r}{2}\psi_{r-2}+\ldots+\psi_0+1, \tag{9}$$

$$(1-z)\psi_r=\binom{r}{1}\psi_{r-1}-\binom{r}{2}\psi_{r-2}+\ldots+(-1)^{r-1}\psi_0, \tag{10}$$

leading to corresponding recurrence formulae for the associated polynomials. Addition and subtraction of (9) and (10) provides us with shorter formulae.

A non-linear relationship may be obtained from (7) as follows:

Writing

$$V(w,z)=\frac{z(z-1)}{z-\exp\{w(z-1)\}},$$

we find that

$$\frac{\partial V}{\partial w}=V(V-z+1). \tag{11}$$

Substitution of the series for V in (11) and equating coefficients of like powers of w, shows that

$$S_{r+1}=(z+1)S_r+\sum_{s=1}^{r-1}\binom{r}{s}S_sS_{r-s}, \tag{12}$$

with a corresponding relationship for ψ_r.

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5. *Determinantal relationships.* The polynomials $S_r(z)$ are most conveniently exhibited as determinants thus:

Solving for ψ_r from the first r recurrence relationships (10) we calculate that

$$S_r(z) = r!\,z \begin{vmatrix} 1 & 1-z & 0 & \dots & 0 \\ 1/2! & 1 & 1-z & \dots & 0 \\ 1/3! & 1/2! & 1 & \dots & 0 \\ \dots & \dots & \dots & & \dots \\ 1/r! & 1/(r-1)! & 1/(r-2)! & \dots & 1 \end{vmatrix}. \tag{13}$$

Persymmetric determinants whose elements are ψ-functions, possess a number of interesting properties which we shall develop as a series of theorems.

The determinant $|\,\delta_{ij}\,|$ having r rows, where

$$\delta_{ij} = \psi_{i+j-2} \quad (i,j = 1,2,\dots,r),$$

will be denoted by Δ_r, and the determinant obtained from this by replacing the ψ-functions by the corresponding associated polynomials by D_r.

THEOREM 1.

$$\text{(i)} \quad \Delta_r = \frac{\{(r-1)!\,(r-2)!\dots 2!\,1!\}^2}{(1-z)^{r^2}} z^{\frac{1}{2}r(r+1)}, \tag{14}$$

$$\text{(ii)} \quad D_r = \{(r-1)!\,(r-2)!\dots 2!\,1!\}^2 z^{\frac{1}{2}r(r+1)}. \tag{15}$$

We shall proceed to the proof of this theorem by means of a series of lemmas:

LEMMA 1. *If* ${}_0\Delta_r$ *is the determinant obtained from* Δ_r *by the replacement of the leading element by* $1/(1-z)$, *then* $\Delta_r = z^r\,.\,{}_0\Delta_r$.

If $|\,a_{ij}\,|$ be a persymmetric determinant, so that $a_{ij} = a_{i+j-1}$, it is well known (vide e.g. (2)) that $|\,a_{ij}\,| = |\,b_{ij}\,|$, where, in the notation of finite differences, $b_{ij} = \Delta^{i+j-2}a_1$. It follows that

$$\Delta_r = |\,\delta_{ij}\,| = |\,\alpha_{ij}\,|,$$

where $\alpha_{ij} = \Delta^{i+j-2}\psi_0 = z\psi_{i+j-2}$ (by (10)), and $\alpha_{11} = \psi_0$.

Dividing each row of $|\,\alpha_{ij}\,|$ by z, we obtain the stated result.

LEMMA 2. $$\Delta_r(z) = (-z)^r\,\Delta_r(z^{-1}).$$

It follows from (4) that $$\Delta_r(z^{-1}) = (-1)^r\,.\,{}_0\Delta_r(z).$$

Lemma 2 can now be deduced from Lemma 1.

LEMMA 3. $$D_r(z) = z^{r(r+1)}D_r(z^{-1}).$$

Since $$D_r(z) = (1-z)^{r^2}\,\Delta_r(z), \tag{16}$$

this lemma may be derived from the preceding lemma.

We can now proceed to a proof of Theorem 1.

By expansion of each element of the determinant $\Delta_r(z)$ in a series of powers of z, we may expand $\Delta_r(z)$ in such a series, the coefficient of z^n in this latter expansion being

$$\Sigma\, s_1^0 s_2^1 \dots s_r^{r-1} \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ s_1 & s_2 & s_3 & \dots & s_r \\ s_1^2 & s_2^2 & s_3^2 & \dots & s_r^2 \\ \dots & \dots & \dots & & \dots \\ s_1^{r-1} & s_2^{r-1} & s_3^{r-1} & \dots & s_r^{r-1} \end{vmatrix}, \tag{17}$$

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where $s_1, s_2, \ldots, s_r$ are distinct non-zero positive integers and the summation is carried out with respect to these integers over all possible values satisfying $s_1 + s_2 + \ldots + s_r = n$.

Choosing integers $t_1 < t_2 < \ldots < t_r$ whose sum is n, and letting the variables $s_1, s_2, \ldots s_r$ range over all permutations of the set $(t_1 t_2 \ldots t_r)$, we find that the corresponding contribution to the sum (17) is $T \sum_{\pi} \epsilon_\pi s_1^0 s_2^1 \ldots s_r^{r-1}$,

where $T = \prod_{i>j} (t_i - t_j)$, π is a permutation of $(t_1 t_2 \ldots t_r)$ and ϵ_π is $+1$ or -1 accordingly as π is even or odd.

But $$\sum_{\pi} \epsilon_\pi s_1^0 s_2^1 \ldots s_r^{r-1} = T,$$

so that (17) may be expressed in the form

$$\Sigma \prod_{i>j} (t_i - t_j)^2, \tag{18}$$

the summation being with respect to all sets $(t_1 t_2 \ldots t_r)$.

If $n < \frac{1}{2}r(r+1)$, (18) is evidently zero. If $n = \frac{1}{2}r(r+1)$, (18) is

$$\prod_{i>j} (i-j)^2 = \{(r-1)!\,(r-2)! \ldots 2!\,1!\}^2,$$

and we have therefore proved that

$$\Delta_r(z) = \{(r-1)!\,(r-2)! \ldots 2!\,1!\}^2 z^{\frac{1}{2}r(r+1)} + \ldots. \tag{19}$$

From (16) and (19) it follows that if the terms of D_r are arranged in ascending order of powers of z, the first term will be

$$\{(r-1)!\,(r-2)! \ldots 2!\,1!\}^2 z^{\frac{1}{2}r(r+1)}.$$

In view of Lemma 3, however, we can write

$$D_r(z) = d_0 + d_1 z + \ldots + d_1 z^{r(r+1)-1} + d_0 z^{r(r+1)},$$

and it therefore follows that

$$D_r(z) = \{(r-1)!\,(r-2)! \ldots 2!\,1!\}^2 z^{\frac{1}{2}r(r+1)}.$$

The remaining part of Theorem 1 is a consequence of (16).

THEOREM 2.

$$\Lambda_r(w,z) = \begin{vmatrix} \psi_0(w) & \psi_1(z) & \ldots & \psi_{r-1}(z) \\ \psi_1(w) & \psi_2(z) & \ldots & \psi_r(z) \\ \ldots & \ldots & & \ldots \\ \psi_{r-1}(w) & \psi_r(z) & \ldots & \psi_{2r-2}(z) \end{vmatrix} = \Delta_r(z) \left\{1 - \left(\frac{w-z}{wz-z}\right)^r\right\}.$$

$\Lambda_r(w,z)$ is a regular function of w in the neighbourhood of $w = z$, provided $z \neq 1$, and in view of (3) we have

$$\left(\frac{\partial \Lambda_r}{\partial w}\right)_{w=z} = \left(\frac{\partial^2 \Lambda_r}{\partial w^2}\right)_{w=z} = \ldots = \left(\frac{\partial^{r-1} \Lambda_r}{\partial w^{r-1}}\right)_{w=z} = 0.$$

It follows that $$\Lambda_r(w,z) = \Delta_r(z) + (w-z)^r \Theta_r(w,z), \tag{20}$$

where Θ_r is a regular function of w in the neighbourhood of $w = z$.

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But it is evident that $\Lambda_r(w,z) = A_r(w,z)\,(w-1)^{-r}$, where A_r is a polynomial in w of degree r, so that from (20) we deduce that

$$\Theta_r(w,z) = B_r(w,z)\,(w-1)^{-r}\,(w-z)^{-r},$$

where B_r is another polynomial in w of degree r. Since Θ_r is regular in w near $w=z$, B_r is divisible by $(w-z)^r$, and hence $\Theta_r(w,z) = C_r(z)\,(w-1)^{-r}$.

Equation (20) can now be written

$$\Lambda_r(w,z) = \Delta_r(z) + C_r(z)\left(\frac{w-z}{w-1}\right)^r, \tag{21}$$

and this result having been proved true for w in the neighbourhood of $w=z$, may be extended analytically over the whole w-plane with the exception of $w=1$.

Since $\Delta_r(0,z) = 0$, by putting $w=0$ in (21) we obtain $C_r(z)$ and establish the theorem.

THEOREM 3.

$$\Gamma_r(z) = \begin{vmatrix} \psi_1(z) & \psi_2(z) & \dots & \psi_r(z) \\ \psi_2(z) & \psi_3(z) & \dots & \psi_{r+1}(z) \\ \dots & \dots & & \dots \\ \psi_r(z) & \psi_{r+1}(z) & \dots & \psi_{2r-1}(z) \end{vmatrix} = \frac{r!\{(r-1)!\dots 2!\,1!\}^2 z^{\frac{1}{2}r(r+1)}}{(1-z)^{r(r+1)}}.$$

We have

$$\lim_{w\to 1}(1-w)^{r+1}\Lambda_{r+1}(w,z) = (-1)^r r!\,\Gamma_r(z). \tag{22}$$

By Theorem 2

$$(1-w)^{r+1}\Lambda_{r+1}(w,z) = \Delta_{r+1}(z)\{(1-w)^{r+1} - (1-wz^{-1})^{r+1}\},$$

so that

$$\lim_{w\to 1}(1-w)^{r+1}\Lambda_{r+1}(w,z) = -\Delta_{r+1}(z)\,(1-z^{-1})^{r+1}, \tag{23}$$

and the theorem follows from (22), (23) and Theorem 1.

Replacing the ψ-functions in the determinant Γ_r by the corresponding associated polynomials, we obtain a determinant $G_r(z)$. As a corollary to Theorem 3, we have the result

$$G_r(z) = r!\{(r-1)!\dots 2!\,1!\}^2 z^{\frac{1}{2}r(r+1)}. \tag{24}$$

THEOREM 4.

$$\Phi_r(z) = \begin{vmatrix} \psi_2(z) & \psi_3(z) & \dots & \psi_{r+1}(z) \\ \psi_3(z) & \psi_4(z) & \dots & \psi_{r+2}(z) \\ \dots & \dots & & \dots \\ \psi_{r+1}(z) & \psi_{r+2}(z) & \dots & \psi_{2r}(z) \end{vmatrix}$$
$$= \{r!\,(r-1)!\dots 2!\,1!\}^2 z^{\frac{1}{2}r(r+1)}(1-z^{r+1})/(1-z)^{(r+1)^2}.$$

We have

$$\lim_{w\to\infty}\Lambda_{r+1}(w,z) = -\Phi_r(z),$$

and also from Theorem 2

$$\lim_{w\to\infty}\Lambda_{r+1}(w,z) = \Delta_{r+1}(z)\,(1-z^{-r-1}).$$

The theorem follows immediately.

Replacing the ψ-functions in the determinant Φ_r by the corresponding associated polynomials we define a determinant $F_r(z)$. As a corollary to Theorem 4 we may prove that

$$F_r(z) = \{r!(r-1)!\dots 2!\,1!\}^2 z^{\frac{1}{2}r(r+1)}(1-z^{r+1})/(1-z). \tag{25}$$

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6. *Zeros of* $S_r(z)$. It may be verified that the coefficients of all terms in the first few associated polynomials are positive and it follows from (12) by induction that this is true generally. Accordingly, S_r cannot have any zeros on the positive real axis.

$z = 0$ is a zero.

From (4) we deduce that if $z_0 \neq 0$ is a zero, so is z_0^{-1} and if r be even $z = -1$ is zero.

THEOREM 5. *$S_r(z)$ has all its zeros on the negative real axis.*

We shall assume the theorem true for $S_r(z)$ and then prove it for $S_{r+1}(z)$.

Let $0 > x_1 > x_2 > \ldots > x_{r-1}$ be the zeros of S_r, where $x_{r-1} = 1/x_1$, $x_{r-2} = 1/x_2$, etc. Then S_r' has $(r-1)$ real distinct zeros x_i' where

$$0 > x_1' > x_2' > \ldots > x_{r-1}', x_{s-1} > x_s' > x_s \quad (s = 1, 2, \ldots, r-1).$$

Since $S_r'(0) = 1$, $S_r(x)$ is negative for small enough negative x and therefore is negative throughout the interval $0 > x > x_1$ and in particular at $x = x_1'$. But from (3)

$$S_{r+1}(x_1') = (r+1)\, x_1' S_r(x_1') > 0, \tag{26}$$

whereas for small negative x, S_{r+1} like S_r, is negative. Hence S_{r+1} has a zero in the interval $0 > x > x_1'$.

S_r has only one zero in the interval $x_1' > x > x_2'$ and since $S_r(x_1') < 0$, this implies that $S_r(x_2') > 0$ and hence from (3) that

$$S_{r+1}(x_2') = (r+1)\, x_2' S_r(x_2') < 0. \tag{27}$$

Equations (26) and (27) together show that S_{r+1} has a zero in the interval $x_1' > x > x_2'$.

Arguing in this manner we can prove that S_{r+1} has a zero in each interval

$$x_s' > x > x_{s+1}' \quad (s = 1, 2, \ldots, r-2).$$

Finally, if α be the zero of S_{r+1} in the interval $(x_1', 0)$, $1/\alpha$ is also a real negative zero of S_{r+1}. Since $\alpha > x_1$, $1/\alpha < 1/x_1 = x_{r-1}$ and hence $1/\alpha$ is not included in the set of zeros already isolated.

$(r+1)$ zeros of S_{r+1} have now been isolated and the theorem is proved.

COROLLARY. *$\psi_r(z)$ is analytic having but one singularity, a pole of order $(r+1)$ at $z = 1$, and having $(r+1)$ zeros, one at infinity, one at the origin, and the remainder on the negative real axis.*

REFERENCES

(1) STONE, W. M. Note on a paper by N. J. Durant. *Phil. Mag.* 39 (1948), 988–91.
(2) AITKEN, A. C. *Determinants and matrices* (Edinburgh, 1942), pp. 121–2.

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